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Using the property of determinants and without expanding in Exercises 1 to 7, prove that:

1. $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$

Solution:

L.H.S.

$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix}$

Applying: C_1 + C_2

$\begin{vmatrix} x+a & a & x+a \\ y+b & b & y+b \\ z+c & c & z+c \end{vmatrix}$

Elements of Column 1 and Column 2 are same. So determinant value is zero as per determinant properties.

= 0

= RHS

Proved.

2. $\begin{vmatrix} a-b & b-c & a-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

Solution:

$\begin{vmatrix} a-b & b-c & a-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

Applying: $C_1 \rightarrow C_1 + C_2 + C_3$

$$= \begin{vmatrix} 0 & b-c & a-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix}$$

= 0

All entries of first column are zero. (As per determinant properties)

3. $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$

Solution:

$$\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$$

Applying: $C_3 \to C_3 - C_1$

$$\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 72 \\ 5 & 9 & 81 \end{vmatrix} = 9\begin{vmatrix} 2 & 7 & 7 \\ 3 & 8 & 8 \\ 5 & 9 & 9 \end{vmatrix}$$

Elements of 2 columns are same, so determinant is zero.

= 0

Proved.

4.

$$\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$$

Solution:

$$\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$$

Applying: C_3 -> $C_3 + C_2$

$$\begin{vmatrix} 1 & bc & ab+ab+ac \\ 1 & ca & ab+ab+ac \\ 1 & ab & ab+ab+ac \end{vmatrix}$$

(ab + ab + ac) is a common element in 3rd row.

$$= (ab+ab+ac)\begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix}$$

Two columns are identical, so determinant is zero.

= 0

5. Prove that

$$\begin{vmatrix} (b+c) & q+r & y+z \\ (c+a) & r+p & z+x \\ (a+b) & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

Solution:

LHS:

Applying: $R_1 \to R_1 + R_2 + R_3$

$$\begin{vmatrix} b+c+c+a+a+b & q+r+r+p+p+q & y+z+z+x+x+y \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} (a+b+c) & (p+q+r) & (x+y+z) \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

Applying:

$R_1 \to R_1 - R_2$

and $R_3 \to R_3 - R_1$

$$2\begin{vmatrix} b & q & y \\ c+a & r+p & z+x \\ a & p & x \end{vmatrix}$$

Again, $R_2 \to R_2 - R_3$

$$2\begin{vmatrix} b & q & y \\ c & r & z \\ a & p & x \end{vmatrix}$$

Interchanging rows, we have

$$2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

= RHS

Proved.

6. Prove that

$$\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$$

Solution:

Let $\Delta = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$

Taking (-1) common from all the 3 rows. Again, interchanging rows and columns, we have

$$\Delta = -\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

$\Delta = -\Delta$

Which shows that, $2\Delta = 0$ or $\Delta = 0$. Proved.

7. Prove that

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution: LHS:

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$$

Taking a, b, c from row 1, row and row 3 respectively,

$$= abc\begin{vmatrix} -a & a & a \\ a & -b & b \\ a & b & -c \end{vmatrix}$$

$R_1 \to R_1 + R_2$

$$= abc\begin{vmatrix} 0 & 0 & 2c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

$$= abc(2c)\begin{vmatrix} a & -b \\ a & b \end{vmatrix}$$

= 2abc^2 (ab + ab)

= $4a^2b^2c^2$

= RHS

Proved.

By using properties of determinants, in Exercises 8 to 14, show that:
8.

(i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

Solution:

(i)LHS:

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 1 & c-a & c^2-a^2 \end{vmatrix}$$

Expanding 1st column,

$$= 1\begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$$

Taking (b-a) common from first row,

$$= (b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

Simplifying above expression, we have

= (b-c)(c-a)(c-b)

= (a – b)(b – c)(c – a)

= RHS

Proved.

(ii) LHS

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix}$$

Expanding first row

$$= 1\begin{vmatrix} b-a & c-a \\ (b-a)(b^2+a^2+ab) & (c-a)(c^2+a^2+ac) \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & 1 \\ (b^2+a^2+ab) & (c^2+a^2+ac) \end{vmatrix}$$

$$= (b-a)(c-a)(c^2+a^2+ac-b^2-a^2-ab)$$
$$= (b-a)(c-a)(c^2-b^2+ac-ab)$$
$$= (b-a)(c-a)[(c-b)(c+b)+a(c-b)]$$
$$= (b-a)(c-a)(c-b)(c+b+a)$$
$$= (a-b)(b-c)(c-a)(a+b+c)$$

=RHS

Proved

9. Prove that

$$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$$

Solution: LHS

$$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}$$

Mulitiplying R_1, R_2, R_3 by x, y, z respectively

$$\begin{vmatrix} x^2 & x^3 & xyz \\ y^2 & y^3 & xyz \\ z^2 & z^3 & xyz \end{vmatrix}$$

$$= \frac{xyz}{xyz}\begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix} = \begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix}$$

$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$= \begin{vmatrix} x^2 & x^3 & 1 \\ y^2 - x^2 & y^3 - x^3 & 0 \\ z^2 - x^2 & z^3 - x^3 & 0 \end{vmatrix}$$

$$= 1\begin{vmatrix} y^2 - x^2 & y^3 - x^3 \\ z^2 - x^2 & z^3 - x^3 \end{vmatrix}$$

$$= \begin{vmatrix} (y-x)(y+x) & (y-x)(y^2+x^2+yx) \\ (z-x)(z+x) & (z-x)(z^2+x^2+zx) \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} y+x & y^2+x^2+yx \\ z+x & z^2+x^2+zx \end{vmatrix}$$

$$= (y-x)(z-x)\left[yz^2 + yx^2 + xyz + xz^2 + x^3 + x^2z - zy^2 - zx^2 - xyz - xy^2 - x^3 - x^2y\right]$$

$$= (y-x)(z-x)\left[yz^2 - zy^2 + xz^2 - xy^2\right]$$

$$= (y-x)(z-x)\left[yz(z-y) + x(z^2-y^2)\right]$$

$$= (y-x)(z-x)\left[yz(z-y) + x(z-y)(z+y)\right]$$

$$= (y-x)(z-x)(z-y)\left[yz + x(z+y)\right]$$

$$= (x-y)(y-z)(z-x)(xy+yz+zx)$$

RHS(Proved)

10.

(i) $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$

(ii) $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$

Solution:
(i) LHS

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$[R_1 \to R_1 + R_2 + R_3]$

$$= \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$$= (5x+4)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$= (5x+4)\begin{vmatrix} 1 & 0 & 0 \\ 2x & 4-x & 0 \\ 2x & 0 & 4-x \end{vmatrix}$$

$$= (5x+4).1\begin{vmatrix} 4-x & 0 \\ 0 & 4-x \end{vmatrix}$$

$$= (5x+4)(4-x)^2$$

= RHS (Proved)

(ii)LHS

$$\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & 2x & y+k \end{vmatrix}$$

$C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 3y+k & y & y \\ 3y+k & y+k & y \\ 3y+k & y & y+k \end{vmatrix}$$

$$= (3y+k)\begin{vmatrix} 1 & y & y \\ 1 & y+k & y \\ 1 & y & y+k \end{vmatrix}$$

$C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$= (3y+k)\begin{vmatrix} 1 & y & y \\ 0 & k & 0 \\ 0 & 0 & k \end{vmatrix}$$

$$= (3y+k).1\begin{vmatrix} k & 0 \\ 0 & k \end{vmatrix}$$

$$= k^2(3y+k)$$

RHS (Proved)

11. Prove that,

(i) $$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

(ii) $$\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$$

Solution: LHS

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix}$$

$$= (a+b+c)(1)\begin{vmatrix} -b-c-a & 0 \\ 0 & -c-a-b \end{vmatrix}$$

$$= (a+b+c)\left[-(b+c+a)\right]\left[-(c+a+b)\right]$$

$$= (a+b+c)^3$$

(ii) LHS

$$\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$$

$C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 2(x+y+z) & x & y \\ 2(x+y+z) & y+z+2x & y \\ 2(x+y+z) & x & z+x+2y \end{vmatrix}$$

Taking 2(x + y + z) common from first column. Then apply operations:

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & x+y+z & 0 \\ 0 & 0 & x+y+z \end{vmatrix}$$

$$= 2(x+y+z)(1)\begin{vmatrix} x+y+z & 0 \\ 0 & x+y+z \end{vmatrix}$$

$$= 2(x+y+z)\left[(x+y+z)^2 - 0\right]$$

= 2(x + y + z)^3

= RHS (Proved)

12. Prove that

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$$

Solution:

LHS

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} 1+x+x^2 & 1+x+x^2 & 1+x+x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

$$= (1+x+x^2)\begin{vmatrix} 1 & 1 & 1 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

$C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$= (1+x+x^2)\begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1-x^2 & x-x^2 \\ x & x^2-x & 1-x \end{vmatrix}$$

$$= \left(1+x+x^2\right) \begin{vmatrix} 1-x^2 & x-x^2 \\ x^2-x & 1-x \end{vmatrix}$$

$$= \left(1+x+x^2\right) \begin{vmatrix} (1-x)(1+x) & x(1-x) \\ -x(1-x) & 1-x \end{vmatrix}$$

$$= \left(1+x+x^2\right)\left[(1-x)^2(1+x)+x^2(1-x)^2\right]$$

$$= \left(1+x+x^2\right)^2(1-x)^2$$

$$= \left(1-x+x-x^2+x^2-x^3\right)^2$$

$$= \left(1-x^3\right)^2$$

RHS

Proved.

13. Prove that

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = \left(1+a^2+b^2\right)^3$$

Solution:

LHS

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

$C_1 \to C_1 - b\, C_3$ and $C_2 \to C_2 + a\, C_3$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b\left(1+a^2+b^2\right) & -a\left(1+a^2+b^2\right) & 1-a^2-b^2 \end{vmatrix}$$

$$= \left(1+a^2+b^2\right)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

$R_3 \to R_3 - b\, R_1$

$$= \left(1+a^2+b^2\right)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & -a & 1-a^2+b^2 \end{vmatrix}$$

$$= \left(1+a^2+b^2\right)^2 \begin{vmatrix} 1 & 2a \\ -a & 1-a^2+b^2 \end{vmatrix}$$

$$= \left(1+a^2+b^2\right)^2 \left(1-a^2+b^2+2a^2\right)$$

$$= \left(1+a^2+b^2\right)^3$$

RHS

Proved

14. Prove that

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$

Solution: LHS

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$$

Multiply, C_1, C_2, C_3 by a, b , c respectively

Then divide the determinant by abc

$$= \frac{1}{abc} \begin{vmatrix} a\left(a^2+1\right) & ab^2 & ac^2 \\ a^2b & b\left(b^2+1\right) & bc^2 \\ a^2c & b^2c & c\left(c^2+1\right) \end{vmatrix}$$

$$= \frac{abc}{abc} \begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

$C_1 \to C_1 + C_2 + C_3$

$$= \frac{abc}{abc}\begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix}$$

$$= \left(1+a^2+b^2+c^2\right)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$.

$$= \left(1+a^2+b^2+c^2\right)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \left(1+a^2+b^2+c^2\right)(1)(1-0)$$

$$= 1+a^2+b^2+c^2$$

LHS

(Proved)

Choose the correct answer in Exercises 15 and 16

15. Let A be a square matrix of order 3 × 3, then | kA| is equal to

(A) k | A | (B) k^2 | A | (C) k^3 | A | (D) 3k | A |

Solution:
Option (C) is correct.

16. Which of the following is correct
(A) Determinant is a square matrix.
(B) Determinant is a number associated to a matrix.
(C) Determinant is a number associated to a square matrix.
(D) None of these

Solution:
Option (C) is correct.