

Exercise 4.1

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Evaluate the following determinants in Exercise 1 and 2.

1. $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$

Solution:

$$\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix} = 2(-1) - 4(-5) = 18$$

2.

(i) $\begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix}$

(ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

Solution:

(i) $\begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix} = \cos \theta \times \cos \theta - (-\sin \theta) \times \sin \theta = 1$

(ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix} = (x^2 - x + 1)(x + 1) - (x + 1)(x - 1) = x^3 - x^2 + 2$

3. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$ then show that $|2A| = 4|A|$.

Solution:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$$

$$2A = \begin{bmatrix} 2 & 4 \\ 8 & 4 \end{bmatrix}$$

$$\text{L.H.S.} = |2A| = \begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix} = 8 - 32 = -24$$

$$\text{R.H.S.} = 4|A| = 4 \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = 4(2 - 8) = -24$$

$$\text{LHS} = \text{RHS}$$

4. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$ then show that $|3A| = 27|A|$

Solution:

$$3A = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{bmatrix}$$

LHS:

$$|3A| = \begin{vmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{vmatrix}$$

$$= 3 \times 36 = 108$$

RHS

$$27|A| = 27 \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix}$$

$$= 27(4) = 108$$

$$\text{LHS} = \text{RHS}$$

5. Evaluate the determinants

(i) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

(iii) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$ (iv) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

Solution:

(i)

$$\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix} = 3 \begin{vmatrix} 0 & -1 \\ -5 & 0 \end{vmatrix} - (-1) \begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix} + (-2) \begin{vmatrix} 0 & 0 \\ 3 & -5 \end{vmatrix}$$

$$= -15 + 3 - 0 = -12$$

(ii)

$$\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 3 \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} - (-4) \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + 5 \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= 3(7) + 4(5) + 5(1)$$

$$= 46$$

(iii)

$$\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix} = 0 \begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix} - 1 \begin{vmatrix} -1 & -3 \\ 0 & -2 \end{vmatrix} + 2 \begin{vmatrix} -1 & 0 \\ -2 & 3 \end{vmatrix}$$

$$= 0 - 1(-6) + 2(-3-0)$$

$$= 0$$

(iv)

$$\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix} = 2 \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} - (-1) \begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} + (-2) \begin{vmatrix} 0 & 2 \\ 3 & -5 \end{vmatrix}$$

$$= 2(-5) + (0+3) - 2(0-6)$$

$$= 5$$

6. If $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$ find $|A|$.

Solution:

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{vmatrix} \\ &= 1 \begin{vmatrix} 1 & -3 \\ 4 & -9 \end{vmatrix} - 1 \begin{vmatrix} 2 & -3 \\ 5 & -9 \end{vmatrix} + (-2) \begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix} \\ &= 1(-9 + 12) - (-18 + 15) - 2(8 - 5) \\ &= 0 \end{aligned}$$

7. Find values of x , if

(i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

Solution:

(i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

$$2 - 20 = 2x^2 - 24$$

$$2x^2 = 6$$

$$x^2 = 3$$

$$\text{or } x = \pm\sqrt{3}$$

(ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

$$10 - 12 = 5x - 6x$$

$$x = 2$$

8. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$ then x is equal to

- (A) 6 (B) ± 6 (C) -6 (D) 0

Solution:

Option (B) is correct.

Explanation:

$$\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$$

$$x^2 - 36 = 36 - 32$$

$$x^2 = 36$$

$$x = \pm 6$$

Exercise 4.2

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Using the property of determinants and without expanding in Exercises 1 to 7, prove that:

1.
$$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$$

Solution:

L.H.S.

$$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix}$$

Applying: $C_1 + C_2$

$$\begin{vmatrix} x+a & a & x+a \\ y+b & b & y+b \\ z+c & c & z+c \end{vmatrix}$$

Elements of Column 1 and Column 2 are same. So determinant value is zero as per determinant properties.

$$= 0$$

$$= \text{RHS}$$

Proved.

2.
$$\begin{vmatrix} a-b & b-c & a-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

Solution:

$$\begin{vmatrix} a-b & b-c & a-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

Applying: $C_1 \rightarrow C_1 + C_2 + C_3$

$$= \begin{vmatrix} 0 & b-c & a-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix}$$

$$= 0$$

All entries of first column are zero. (As per determinant properties)

3. $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$

Solution:

$$\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$$

Applying: $C_3 \rightarrow C_3 - C_1$

$$\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 9 \begin{vmatrix} 2 & 7 & 7 \\ 3 & 8 & 8 \\ 5 & 9 & 9 \end{vmatrix}$$

Elements of 2 columns are same, so determinant is zero.

= 0

Proved.

4.

$$\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$$

Solution:

$$\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$$

Applying: $C_3 \rightarrow C_3 + C_2$

$$\begin{vmatrix} 1 & bc & ab+ab+ac \\ 1 & ca & ab+ab+ac \\ 1 & ab & ab+ab+ac \end{vmatrix}$$

$(ab + ab + ac)$ is a common element in 3rd row.

$$= (ab+ab+ac) \begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix}$$

Two columns are identical, so determinant is zero.

$$= 0$$

5. Prove that

$$\begin{vmatrix} (b+c) & q+r & y+z \\ (c+a) & r+p & z+x \\ (a+b) & p+q & x+y \end{vmatrix} = 2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

Solution:

LHS:

Applying: $R_1 \rightarrow R_1 + R_2 + R_3$

$$\begin{vmatrix} b+c+c+a+a+b & q+r+r+p+p+q & y+z+z+x+x+y \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

$$= 2 \begin{vmatrix} (a+b+c) & (p+q+r) & (x+y+z) \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

Applying:

$$R_1 \rightarrow R_1 - R_2$$

and $R_3 \rightarrow R_3 - R_1$

$$2 \begin{vmatrix} b & q & y \\ c+a & r+p & z+x \\ a & p & x \end{vmatrix}$$

Again, $R_2 \rightarrow R_2 - R_3$

$$2 \begin{vmatrix} b & q & y \\ c & r & z \\ a & p & x \end{vmatrix}$$

Interchanging rows, we have

$$2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

= RHS

Proved.

6. Prove that

$$\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$$

Solution:

Let $\Delta = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$

Taking (-1) common from all the 3 rows. Again, interchanging rows and columns, we have

$$\Delta = - \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

$$\Delta = -\Delta$$

Which shows that, $2\Delta = 0$ or $\Delta = 0$. Proved.

7. Prove that

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution: LHS:

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$$

Taking a, b, c from row 1, row and row 3 respectively,

$$= abc \begin{vmatrix} -a & a & a \\ a & -b & b \\ a & b & -c \end{vmatrix}$$

$$R_1 \rightarrow R_1 + R_2$$

$$= abc \begin{vmatrix} 0 & 0 & 2c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

$$= abc(2c) \begin{vmatrix} a & -b \\ a & b \end{vmatrix}$$

$$= 2abc^2 (ab + ab)$$

$$= 4a^2b^2c^2$$

$$= \text{RHS}$$

Proved.

By using properties of determinants, in Exercises 8 to 14, show that:
8.

$$(i) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

$$(ii) \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

Solution:

(i)LHS:

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

Expanding 1st column,

$$= 1 \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$$

Taking (b-a) common from first row,

$$= (b-a)(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

Simplifying above expression, we have

$$= (b-c)(c-a)(c-b)$$

$$= (a-b)(b-c)(c-a)$$

$$= \text{RHS}$$

Proved.

(ii) LHS

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$$C_2 \rightarrow C_2 - C_1 \text{ and } C_3 \rightarrow C_3 - C_1$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix}$$

Expanding first row

$$= 1 \begin{vmatrix} b-a & c-a \\ (b-a)(b^2+a^2+ab) & (c-a)(c^2+a^2+ac) \end{vmatrix}$$

$$= (b-a)(c-a) \begin{vmatrix} 1 & 1 \\ b^2+a^2+ab & c^2+a^2+ac \end{vmatrix}$$

$$= (b-a)(c-a)(c^2+a^2+ac-b^2-a^2-ab)$$

$$= (b-a)(c-a)(c^2-b^2+ac-ab)$$

$$= (b-a)(c-a)[(c-b)(c+b)+a(c-b)]$$

$$= (b-a)(c-a)(c-b)(c+b+a)$$

$$= (a-b)(b-c)(c-a)(a+b+c)$$

=RHS

Proved

9. Prove that

$$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$$

Solution: LHS

$$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}$$

Multiplying R_1, R_2, R_3 by x, y, z respectively

$$\begin{vmatrix} x^2 & x^3 & xyz \\ y^2 & y^3 & xyz \\ z^2 & z^3 & xyz \end{vmatrix}$$

$$= \frac{xyz}{xyz} \begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix} = \begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix}$$

$$R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$$

$$= \begin{vmatrix} x^2 & x^3 & 1 \\ y^2 - x^2 & y^3 - x^3 & 0 \\ z^2 - x^2 & z^3 - x^3 & 0 \end{vmatrix}$$

$$= 1 \begin{vmatrix} y^2 - x^2 & y^3 - x^3 \\ z^2 - x^2 & z^3 - x^3 \end{vmatrix}$$

$$= \begin{vmatrix} (y-x)(y+x) & (y-x)(y^2+x^2+yx) \\ (z-x)(z+x) & (z-x)(z^2+x^2+zx) \end{vmatrix}$$

$$= (y-x)(z-x) \begin{vmatrix} y+x & y^2+x^2+yx \\ z+x & z^2+x^2+zx \end{vmatrix}$$

$$= (y-x)(z-x) [yz^2 + yx^2 + xyz + xz^2 + x^3 + x^2z - zy^2 - zx^2 - xyz - xy^2 - x^3 - x^2y]$$

$$= (y-x)(z-x) [yz^2 - zy^2 + xz^2 - xy^2]$$

$$= (y-x)(z-x) [yz(z-y) + x(z^2 - y^2)]$$

$$= (y-x)(z-x) [yz(z-y) + x(z-y)(z+y)]$$

$$= (y-x)(z-x)(z-y) [yz + x(z+y)]$$

$$= (x-y)(y-z)(z-x)(xy + yz + zx)$$

RHS(Proved)

10.

$$(i) \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$$

$$(ii) \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$$

Solution:

(i) LHS

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$$[R_1 \rightarrow R_1 + R_2 + R_3]$$

$$= \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$$= (5x+4) \begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$$C_2 \rightarrow C_2 - C_1 \text{ and } C_3 \rightarrow C_3 - C_1$$

$$= (5x+4) \begin{vmatrix} 1 & 0 & 0 \\ 2x & 4-x & 0 \\ 2x & 0 & 4-x \end{vmatrix}$$

$$= (5x+4) \cdot 1 \begin{vmatrix} 4-x & 0 \\ 0 & 4-x \end{vmatrix}$$

$$= (5x+4)(4-x)^2$$

= RHS (Proved)

(ii) LHS

$$\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & 2x & y+k \end{vmatrix}$$

$$C_1 \rightarrow C_1 + C_2 + C_3$$

$$= \begin{vmatrix} 3y+k & y & y \\ 3y+k & y+k & y \\ 3y+k & y & y+k \end{vmatrix}$$

$$= (3y+k) \begin{vmatrix} 1 & y & y \\ 1 & y+k & y \\ 1 & y & y+k \end{vmatrix}$$

$$C_2 \rightarrow C_2 - C_1 \text{ and } C_3 \rightarrow C_3 - C_1$$

$$= (3y+k) \begin{vmatrix} 1 & y & y \\ 0 & k & 0 \\ 0 & 0 & k \end{vmatrix}$$

$$= (3y+k) \cdot 1 \begin{vmatrix} k & 0 \\ 0 & k \end{vmatrix}$$

$$= k^2(3y+k)$$

RHS (Proved)

11. Prove that,

$$(i) \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

$$(ii) \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$$

Solution: LHS

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$R_1 \rightarrow R_1 + R_2 + R_3$$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$C_2 \rightarrow C_2 - C_1 \text{ and } C_3 \rightarrow C_3 - C_1$$

$$= (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} -b-c-a & 0 \\ 0 & -c-a-b \end{vmatrix}$$

$$= (a+b+c) [-(b+c+a)] [-(c+a+b)]$$

$$= (a+b+c)^3$$

(ii) LHS

$$\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$$

$$C_1 \rightarrow C_1 + C_2 + C_3$$

$$= \begin{vmatrix} 2(x+y+z) & x & y \\ 2(x+y+z) & y+z+2x & y \\ 2(x+y+z) & x & z+x+2y \end{vmatrix}$$

Taking $2(x+y+z)$ common from first column. Then apply operations:

$$R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1$$

$$\begin{aligned}
 &= 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 0 & x+y+z & 0 \\ 0 & 0 & x+y+z \end{vmatrix} \\
 &= 2(x+y+z)(1) \begin{vmatrix} x+y+z & 0 \\ 0 & x+y+z \end{vmatrix} \\
 &= 2(x+y+z) [(x+y+z)^2 - 0] \\
 &= 2(x+y+z)^3 \\
 &= \text{RHS (Proved)}
 \end{aligned}$$

12. Prove that

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$$

Solution:

LHS

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

$$R_1 \rightarrow R_1 + R_2 + R_3$$

$$= \begin{vmatrix} 1+x+x^2 & 1+x+x^2 & 1+x+x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

$$= (1+x+x^2) \begin{vmatrix} 1 & 1 & 1 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

$$C_2 \rightarrow C_2 - C_1 \text{ and } C_3 \rightarrow C_3 - C_1$$

$$= (1+x+x^2) \begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1-x^2 & x-x^2 \\ x & x^2-x & 1-x \end{vmatrix}$$

$$\begin{aligned}
 &= (1+x+x^2) \begin{vmatrix} 1-x^2 & x-x^2 \\ x^2-x & 1-x \end{vmatrix} \\
 &= (1+x+x^2) \begin{vmatrix} (1-x)(1+x) & x(1-x) \\ -x(1-x) & 1-x \end{vmatrix} \\
 &= (1+x+x^2) [(1-x)^2(1+x) + x^2(1-x)^2] \\
 &= (1+x+x^2)^2 (1-x)^2 \\
 &= (1-x+x-x^2+x^2-x^3)^2 \\
 &= (1-x^3)^2
 \end{aligned}$$

RHS

Proved.

13. Prove that

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$

Solution:

LHS

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

$$C_1 \rightarrow C_1 - b C_3 \text{ and } C_2 \rightarrow C_2 + a C_3$$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

$$R_3 \rightarrow R_3 - b R_1$$

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & -a & 1-a^2+b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 2a \\ -a & 1-a^2+b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^2 (1-a^2+b^2+2a^2)$$

$$= (1+a^2+b^2)^3$$

RHS

Proved

14. Prove that

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$

Solution: LHS

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$$

Multiply, C_1, C_2, C_3 by a, b, c respectively

Then divide the determinant by abc

$$= \frac{1}{abc} \begin{vmatrix} a(a^2+1) & ab^2 & ac^2 \\ a^2b & b(b^2+1) & bc^2 \\ a^2c & b^2c & c(c^2+1) \end{vmatrix}$$

$$= \frac{abc}{abc} \begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

$$C_1 \rightarrow C_1 + C_2 + C_3$$

$$= \frac{abc}{abc} \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix}$$

$$= (1+a^2+b^2+c^2) \begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

$$R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1$$

$$= (1+a^2+b^2+c^2) \begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (1+a^2+b^2+c^2)(1)(1-0)$$

$$= 1+a^2+b^2+c^2$$

LHS
(Proved)

Choose the correct answer in Exercises 15 and 16

15. Let A be a square matrix of order 3×3 , then $|kA|$ is equal to

(A) $k|A|$ (B) $k^2|A|$ (C) $k^3|A|$ (D) $3k|A|$

Solution:

Option (C) is correct.

16. Which of the following is correct

- (A) Determinant is a square matrix.
- (B) Determinant is a number associated to a matrix.
- (C) Determinant is a number associated to a square matrix.
- (D) None of these

Solution:

Option (C) is correct.

Exercise 4.3

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1. Find area of the triangle with vertices at the point given in each of the following:

(i) (1, 0), (6, 0), (4, 3)

(ii) (2, 7), (1, 1), (10, 8)

(iii) (-2, -3), (3, 2), (-1, -8)

Solution:

Formula for Area of triangle:

$$\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

(i)

$$\frac{1}{2} \begin{vmatrix} 1 & 0 & 1 \\ 6 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2} [1(0-3) - 0(6-4) + 1(18-0)]$$

$$= \frac{15}{2} \text{ sq. units}$$

(ii)

$$\text{Area} = \frac{1}{2} \begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$$

$$= \frac{1}{2} [2(1-8) - 7(1-10) + 1(8-10)]$$

$$= \frac{47}{2} \text{ sq. unit}$$

(iii)

$$\text{Area} = \frac{1}{2} \begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix}$$

$$= \frac{1}{2} [-2(10) + 3(4) - 22]$$

$$= 15 \text{ sq. Units}$$

2. Show that points: A (a, b + c), B (b, c + a), C (c, a + b) are collinear.

Solution:

Points are collinear if area of triangle is equal to zero.

i.e. Area of triangle = 0

$$\begin{aligned}\text{Area of Triangle} &= \frac{1}{2} \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix} \\ &= \frac{1}{2} [a(c+a-a-b) - (b+c)(b-c) + 1\{b(a+b) - c(c+a)\}] \\ &= \frac{1}{2} (ac - ab - b^2 + c^2 + ab + b^2 - c^2 - ac) \\ &= 0\end{aligned}$$

Therefore, points are collinear.

3. Find values of k if area of triangle is 4 sq. units and vertices are

(i) (k, 0), (4, 0), (0, 2)

(ii) (-2, 0), (0, 4), (0, k)

Solution:

(i)

Area of triangle = ± 4 (Given)

$$\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 4$$

$$\frac{1}{2} [k(0-2) - 0 + 1(8-0)] = 4$$

$$\frac{1}{2}(-2k + 4) = 4$$

$$-k + 4 = 4$$

$$\text{Now: } -k + 4 = \pm 4$$

$$-k + 4 = 4 \text{ and } -k + 4 = -4$$

$$k = 0 \text{ and } k = 8$$

$$(ii) \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 4$$

$$\frac{1}{2} \begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix} = 4$$

$$\frac{1}{2}(-8+2k) = 4$$

$$\text{or } -k + 4 = 4$$

$$\text{Now: } -k + 4 = \pm 4$$

$$-k + 4 = 4 \text{ and } -k + 4 = -4$$

$$k = 0 \text{ and } k = 8$$

4. (i) Find equation of line joining (1, 2) and (3, 6) using determinants.

(ii) Find equation of line joining (3, 1) and (9, 3) using determinants.

Solution:

Let A(x, y) be any vertex of a triangle.

All points are on one line if area of triangle is zero.

$$\frac{1}{2} \begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$\frac{1}{2} [x(2-6) - y(1-3) + 1(6-6)] = 0$$

$$-4x + 2y = 0$$

$$y = 2x$$

Which is equation of line.

(ii) Let A(x, y) be any vertex of a triangle.

All points are on one line if area of triangle is zero.

$$\frac{1}{2} \begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ 9 & 3 & 1 \end{vmatrix} = 0$$

$$\frac{1}{2} [x(1-3) - y(3-9) + 1(9-9)] = 0$$

$$-2x + 6y = 0$$

$$x - 3y = 0$$

Which is equation of line.

12. If area of triangle is 35 sq units with vertices (2, -6), (5, 4) and (k, 4). Then k is

(A) 12 (B) -2 (C) -12, -2 (D) 12, -2

Solution:

Option (D) is correct.

Explanation:

$$\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 35$$

$$\frac{1}{2} [2(4-4) - (-6)(5-k) + 1(20-4k)] = 35$$

Solving above expression, we have

$$25 - 5k = \pm 35$$

$$25 - 5k = 35 \text{ and } 25 - 5k = -35$$

$$k = -2 \text{ and } k = 12.$$

Exercise 4.4

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Write Minors and Cofactors of the elements of following determinants:

1.

$$(i) \begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix} \quad (ii) \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

Solution:

Find Minors of elements:

Say, M_{ij} is minor of element a_{ij}

M_{11} = Minor of element $a_{11} = 3$

M_{12} = Minor of element $a_{12} = 0$

M_{21} = Minor of element $a_{21} = -4$

M_{22} = Minor of element $a_{22} = 2$

Find cofactor of a_{ij}

Let cofactor of a_{ij} is A_{ij} , which is $(-1)^{i+j} M_{ij}$

$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (0) = 0$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (-4) = 4$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (2) = 2$

(ii)

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

Solution:

Find Minors of elements:

Say, M_{ij} is minor of element a_{ij}

M_{11} = Minor of element $a_{11} = d$

M_{12} = Minor of element $a_{12} = b$

M_{21} = Minor of element $a_{21} = c$

M_{22} = Minor of element $a_{22} = a$

Find cofactor of a_{ij}

Let cofactor of a_{ij} is A_{ij} , which is $(-1)^{i+j} M_{ij}$

$$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (d) = d$$

$$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (b) = -b$$

$$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (c) = -c$$

$$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (a) = a$$

2.

$$(i) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$(ii) \begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -5 \\ 0 & 2 & 2 \end{vmatrix}$$

Solution:

$$(i) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Find Minors and cofactors of elements:

Say, M_{ij} is minor of element a_{ij} and A_{ij} is cofactor of a_{ij}

$$M_{11} = \text{Minor of element } a_{11} = \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} = 0 - 0 = 0 \quad \text{and } A_{11} = 1$$

$$M_{12} = \text{Minor of element } a_{12} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1 \quad \text{and } A_{12} = 0$$

$$M_{13} = \text{Minor of element } a_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0 \quad \text{and } A_{13} = 0$$

$$M_{21} = \text{Minor of element } a_{21} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 - 0 = 0 \quad \text{and } A_{21} = 0$$

$$M_{22} = \text{Minor of element } a_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1 \quad \text{and } A_{22} = 1$$

$$M_{23} = \text{Minor of element } a_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0 \quad \text{and } A_{23} = 0$$

$$M_{31} = \text{Minor of element } a_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0 - 0 = 0 \quad \text{and } A_{31} = 0$$

$$M_{32} = \text{Minor of element } a_{32} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0 \quad \text{and } A_{32} = 0$$

$$M_{33} = \text{Minor of element } a_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1 \quad \text{and } A_{33} = 1$$

(ii) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -5 \\ 0 & 2 & 2 \end{vmatrix}$

Find Minors and cofactors of elements:

Say, M_{ij} is minor of element a_{ij} and A_{ij} is cofactor of a_{ij}

$$M_{11} = \text{Minor of element } a_{11} = \begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 10 - (-1) = 11 \quad \text{and } A_{11} = 11$$

$$M_{12} = \text{Minor of element } a_{12} = \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6 - 0 = 6 \quad \text{and } A_{12} = -6$$

$$M_{13} = \text{Minor of element } a_{13} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3 \quad \text{and } A_{13} = 3$$

$$M_{21} = \text{Minor of element } a_{21} = \begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = 0 - 4 = -4 \quad \text{and } A_{21} = 4$$

$$M_{22} = \text{Minor of element } a_{22} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2 \quad \text{and } A_{22} = 2$$

$$M_{23} = \text{Minor of element } a_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1 \quad \text{and } A_{23} = -1$$

$$M_{31} = \text{Minor of element } a_{31} = \begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix} = 0 - 20 = -20 \quad \text{and } A_{31} = -20$$

$$M_{32} = \text{Minor of element } a_{32} = \begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} = -1 - 12 = -13 \quad \text{and } A_{32} = 13$$

$$M_{33} = \text{Minor of element } a_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 5 - 0 = 5 \quad \text{and } A_{33} = 5$$

3. Using Cofactors of elements of second row, evaluate Δ .

$$\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

Solution:

Find Cofactors of elements of second row:

$$A_{21} = \text{Cofactor of element } a_{21} = (-1)^{2+1} \begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} = (-1)^3 (9 - 16) = 7$$

$$A_{22} = \text{Cofactor of element } a_{22} = (-1)^{2+2} \begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} = (-1)^4 (15 - 8) = 7$$

$$A_{23} = \text{Cofactor of element } a_{23} = (-1)^{2+3} \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = (-1)^5 (10 - 3) = -7$$

$$\text{Now, } \Delta = a_{21} A_{21} + a_{22} A_{22} + a_{23} A_{23} = 14 + 0 - 7 = 7$$

4. Using Cofactors of elements of third column, evaluate Δ .

$$\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$$

Solution:

Find Cofactors of elements of third column:

$$A_{13} = \text{Cofactor of element } a_{13} = (-1)^{1+3} \begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix} = (-1)^4 (z - y) = z - y$$

$$A_{23} = \text{Cofactor of element } a_{23} = (-1)^{2+3} \begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix} = (-1)^5 (z - x) = x - z$$

$$A_{33} = \text{Cofactor of element } a_{33} = (-1)^{3+3} \begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix} = (-1)^6 (y - x) = y - x$$

Now, $\Delta = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$

$$= yz(z - y) + zx(x - z) + xy(y - x)$$

$$= (yz^2 - y^2z) + (xy^2 - xz^2) + (xz^2 - x^2y)$$

$$= (y - z)[-yz + x(y + z) - x^2]$$

$$= (y - z)[-y(z - x) + x(z - x)]$$

$$= (x - y)(y - x)(z - x)$$

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

5. If Δ and A_{ij} is cofactor of a_{ij} then value of Δ is given by:

(A) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$

(B) $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$

(C) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$

(D) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

Solution: Option (D) is correct.

Exercise 4.5

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Find adjoint of each of the matrices in Exercises 1 and 2.

1.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

2.

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$$

Solution:

1. Let $A =$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Cofactors of the above matrix are

$$A_{11} = 4$$

$$A_{12} = -3$$

$$A_{21} = -2$$

$$A_{22} = 1$$

$$\text{adj. } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

2.

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$$

Cofactors of the above matrix are

$$A_{11} = + \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 \quad A_{21} = - \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \quad A_{31} = + \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -11$$

$$A_{12} = - \begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} = -12 \quad A_{22} = + \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 5 \quad A_{32} = - \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -1$$

$$A_{13} = + \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 6 \quad A_{23} = - \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = 2 \quad A_{33} = + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5$$

Therefore,

$$\text{adj. } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$$

Verify $A (\text{adj } A) = (\text{adj } A) A = |A| I$ in Exercises 3 and 4

3. $\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$ 4. $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

Solution:

3.

$$\text{Let } A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$$

$$\text{adj. } A = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$$

$$\begin{aligned} A(\text{adj. } A) &= \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix} \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

$$\text{Again, } |A| = \begin{vmatrix} 2 & 3 \\ -4 & -6 \end{vmatrix} = -12 + 12 = 0$$

$$|A|I = (0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

LHS = RHS

Verified.

4.

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

Cofactors of A,

$$A_{11} = + \begin{vmatrix} 0 & -2 \\ 0 & 3 \end{vmatrix} = 0 \quad A_{21} = - \begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} = 3 \quad A_{31} = + \begin{vmatrix} -1 & 2 \\ 0 & -2 \end{vmatrix} = 2$$

$$A_{12} = - \begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} = -11 \quad A_{22} = + \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1 \quad A_{32} = - \begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = 8$$

$$A_{13} = + \begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} = 0 \quad A_{23} = - \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -1 \quad A_{33} = + \begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} = 3$$

Now,

$$\text{adj. } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

Now, Verify $A (\text{adj } A) = (\text{adj } A) A = |A| I$

$$A(\text{adj. } A) = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$(\text{adj. } A)A = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{vmatrix} = 1(0) - (-1)(11) + 2(0) = 11$$

$$|A|I = 11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

Verified.

Find the inverse of each of the matrices (if it exists) given in Exercises 5 to 11.

5.

$$\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$$

Solution:

$$\text{Let } A = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -2 \\ 4 & 3 \end{vmatrix} = 14 \neq 0$$

Since determinant of the matrix is not zero, so inverse of this matrix is possible.

As we know, formula to find matrix inverse is:

$$A^{-1} = \frac{1}{|A|} \text{adj.} A$$

$$\text{adj. } A = \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$$

This implies,

$$A^{-1} = \frac{1}{14} \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$$

6.

$$\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$$

Solution:

$$\text{Let } A = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} -1 & 5 \\ -3 & 2 \end{vmatrix} = 13 \neq 0$$

Since determinant of the matrix is not zero, so inverse of this matrix is possible.

As we know, formula to find matrix inverse is:

$$A^{-1} = \frac{1}{|A|} \text{adj.} A$$

$$\text{adj. } A = \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$$

This implies,

$$A^{-1} = \frac{1}{13} \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$$

7.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

Solution:

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{vmatrix} = 1(10) - 2(0) + 3(0) = 10 \neq 0$$

Therefore,

A^{-1} exists

Find adj A:

$$A_{11} = + \begin{vmatrix} 2 & 4 \\ 0 & 5 \end{vmatrix} = 10 \quad A_{21} = - \begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = -10 \quad A_{31} = + \begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} = 2$$

$$A_{12} = - \begin{vmatrix} 0 & 4 \\ 0 & 5 \end{vmatrix} = 0 \quad A_{22} = + \begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = 5 \quad A_{32} = - \begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = -4$$

$$A_{13} = + \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 0 \quad A_{23} = - \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0 \quad A_{33} = + \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2$$

As we know, formula to find matrix inverse is:

$$A^{-1} = \frac{1}{|A|} \text{adj. } A$$

$$A^{-1} = \frac{1}{10} \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

8.

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$$

Solution:

$$\text{Let } A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{vmatrix} = 1(-3) - 0 + 0 = -3 \neq 0$$

Therefore,

A^{-1} exists

Find adj A:

$$A_{11} = + \begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} = -3 \quad A_{21} = - \begin{vmatrix} 0 & 0 \\ 2 & -1 \end{vmatrix} = 0 \quad A_{31} = + \begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix} = 0$$

$$A_{12} = - \begin{vmatrix} 3 & 0 \\ 5 & -1 \end{vmatrix} = 3 \quad A_{22} = + \begin{vmatrix} 1 & 0 \\ 5 & -1 \end{vmatrix} = -1 \quad A_{32} = - \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} = 0$$

$$A_{13} = + \begin{vmatrix} 3 & 3 \\ 5 & 2 \end{vmatrix} = -9 \quad A_{23} = - \begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} = -2 \quad A_{33} = + \begin{vmatrix} 1 & 0 \\ 3 & 3 \end{vmatrix} = 3$$

$$\text{adj. } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

As we know, formula to find matrix inverse is:

$$A^{-1} = \frac{1}{|A|} \text{adj.} A$$

$$A^{-1} = \frac{-1}{3} \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

9.

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$$

Solution:

$$\text{Let } A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{vmatrix}$$

$$= 2(-1) - 1(4) + 3(1) = -3 \neq 0$$

Therefore,

A^{-1} exists

Find adj A:

$$A_{11} = + \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = -1 \quad A_{21} = - \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 5 \quad A_{31} = + \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} = 3$$

$$A_{12} = - \begin{vmatrix} 4 & 0 \\ -7 & 1 \end{vmatrix} = -4 \quad A_{22} = + \begin{vmatrix} 2 & 3 \\ -7 & 1 \end{vmatrix} = 23 \quad A_{32} = - \begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix} = 12$$

$$A_{13} = + \begin{vmatrix} 4 & -1 \\ -7 & 2 \end{vmatrix} = 1 \quad A_{23} = - \begin{vmatrix} 2 & 1 \\ -7 & 2 \end{vmatrix} = -11 \quad A_{33} = + \begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix} = -6$$

$$\text{adj. } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

As we know, formula to find matrix inverse is:

$$A^{-1} = \frac{1}{|A|} \text{adj. } A$$

$$A^{-1} = \frac{-1}{3} \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

10.

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$$

Solution:

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{vmatrix}$$

$$= 1(2) + 1(9) + 2(-6) = -1$$

$$\neq 0$$

Therefore,

A^{-1} exists

Find adj A:

$$A_{11} = + \begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} = 2 \quad A_{21} = - \begin{vmatrix} -1 & 2 \\ -2 & 4 \end{vmatrix} = 0 \quad A_{31} = + \begin{vmatrix} -1 & 2 \\ 2 & -3 \end{vmatrix} = -1$$

$$A_{12} = - \begin{vmatrix} 0 & -3 \\ 3 & 4 \end{vmatrix} = -9 \quad A_{22} = + \begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = -2 \quad A_{32} = - \begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} = 3$$

$$A_{13} = + \begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix} = -6 \quad A_{23} = - \begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} = -1 \quad A_{33} = + \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = 2$$

$$\text{adj. } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$

As we know, formula to find matrix inverse is:

$$A^{-1} = \frac{1}{|A|} \text{adj. } A$$

$$A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

11.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & \sin \alpha & -\cos \alpha \end{bmatrix}$$

Solution:

$$\text{Let } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & \sin \alpha & -\cos \alpha \end{bmatrix}$$

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & \sin \alpha & -\cos \alpha \end{vmatrix} \\ &= (-\cos^2 \alpha - \sin^2 \alpha) - 0 + 0 \\ &= -(\cos^2 \alpha + \sin^2 \alpha) = -1 \neq 0 \end{aligned}$$

Therefore,

A^{-1} exists

Find adj A:

$$\begin{aligned}
 A_{11} &= + \begin{vmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & -\cos \alpha \end{vmatrix} = -1 & A_{21} &= - \begin{vmatrix} 0 & 0 \\ \sin \alpha & -\cos \alpha \end{vmatrix} = 0 & A_{31} &= + \begin{vmatrix} 0 & 0 \\ \cos \alpha & \sin \alpha \end{vmatrix} = 0 \\
 A_{12} &= - \begin{vmatrix} 0 & \sin \alpha \\ 0 & -\cos \alpha \end{vmatrix} = 0 & A_{22} &= + \begin{vmatrix} 1 & 0 \\ 0 & -\cos \alpha \end{vmatrix} = -\cos \alpha & A_{32} &= - \begin{vmatrix} 1 & 0 \\ 0 & \sin \alpha \end{vmatrix} = -\sin \alpha \\
 A_{13} &= + \begin{vmatrix} 0 & \cos \alpha \\ 0 & \sin \alpha \end{vmatrix} = 0 & A_{23} &= - \begin{vmatrix} 1 & 0 \\ 0 & \sin \alpha \end{vmatrix} = \sin \alpha & A_{33} &= + \begin{vmatrix} 1 & 0 \\ 0 & \cos \alpha \end{vmatrix} = \cos \alpha
 \end{aligned}$$

$$\text{adj. } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos \alpha & -\sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix}$$

As we know, formula to find matrix inverse is:

$$A^{-1} = \frac{1}{|A|} \text{adj. } A$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & \sin \alpha & -\cos \alpha \end{bmatrix}$$

12. Let $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$ verify that $(AB)^{-1} = B^{-1} A^{-1}$.

Solution:

$$A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & 7 \\ 2 & 5 \end{vmatrix} = 1 \neq 0$$

$$A^{-1} = \frac{1}{|A|} \text{adj. } A$$

$$A^{-1} = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

Again,

$$B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 6 & 8 \\ 7 & 9 \end{vmatrix} = -2 \neq 0$$

$$B^{-1} = \frac{1}{-2} \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$$

Now Multiply A and B,

$$AB = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$$

Find determinant of AB:

$$|AB| = \begin{vmatrix} 67 & 87 \\ 47 & 61 \end{vmatrix} = 4087 - 4089 = -2 \neq 0$$

Now, Verify $(AB)^{-1} = B^{-1} A^{-1}$

LHS:

$$|AB| = \begin{vmatrix} 67 & 87 \\ 47 & 61 \end{vmatrix} = 4087 - 4089 = -2 \neq 0$$

and

$$(AB)^{-1} = \frac{1}{-2} \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$$

RHS:

$$B^{-1}A^{-1} = \frac{1}{-2} \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix} \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

$$= \frac{1}{-2} \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$$

This implies, LHS = RHS (Verified)

13. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence find A^{-1} .

Solution:

$$A^2 = AA$$

$$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$\text{LHS} = A^2 - 5A + 7I$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5 \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

= RHS. (Proved)

To Find A^{-1}

Multiply $A^2 - 5A + 7I$ by A^{-1} , we have
(Consider I is 2×2 matrix)

$$A^2 A^{-1} - 5A A^{-1} + 7I A^{-1} = O A^{-1}$$

$$A - 5I + 7A^{-1} = O$$

$$7A^{-1} = -A + 5I$$

$$= \begin{bmatrix} -3 & -1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

14. For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = O$.

Solution:

$$A^2 = AA = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$$

Since $A^2 + aA + bI = O$

$$\begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + a \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + b \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = O$$

$$\begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 3a & 2a \\ a & a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 11+3a+b & 8+2a+0 \\ 4+a+0 & 3+a+b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equate corresponding elements, we get

$$11 + 3a + b = 0 \dots(1)$$

$$8 + 2a = 0 \Rightarrow a = -4$$

Substitute the value of a in equation (1),

$$11 + 3(-4) + b = 0$$

$$11 - 12 + b = 0$$

$$b = 1.$$

15. For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$. Show that $A^3 - 6A^2 + 5A + 11I = O$. Hence, find A^{-1} .

Solution:

$$A^2 = AA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

Now, LHS = $A^3 - 6A^2 + 5A + 11I$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 8-24+5+11 & 7-12+5 & 1-6+5 \\ -23+18+5 & 27-48+10+11 & -69+84-15 \\ 32-42+10 & -13+18-5 & 58-84+15+11 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

= 0

RHS (Proved)

Now, find A^{-1}

Multiply $A^3 - 6A^2 + 5A + 11I$ by A^{-1} , we have
(Consider I is 3×3 matrix)

$$A^3 A^{-1} - 6A^2 A^{-1} + 5A A^{-1} + 11I A^{-1} = 0 A^{-1}$$

$$A^2 - 6A + 5I + 11A^{-1} = 0$$

$$11A^{-1} = 6A - 5I - A^2$$

$$\begin{aligned} 11A^{-1} &= 6 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} \\ &= \begin{bmatrix} 6-5-4 & 6-2 & 6-1 \\ 6+3 & 12-5-8 & -18+14 \\ 12-7 & -6+3 & 18-5-14 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix} \end{aligned}$$

Therefore,

$$A^{-1} = \frac{1}{11} \begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

$$16. \text{ If } A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

Verify that $A^3 - 6A^2 + 9A - 4I = O$ and hence find A^{-1} .

Solution:

$$A^2 = AA$$

$$\begin{bmatrix} 4+1+1 & -2-2-1 & 1+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

Again, $A^3 = A^2A$

$$\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \\ = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Now, $A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ = \begin{bmatrix} 22-36 & -21+30 & 21-30 \\ -21+30 & 22-36 & -21+30 \\ 21-30 & -21+30 & 22-36 \end{bmatrix} + \begin{bmatrix} 18-4 & -9-0 & 9-0 \\ -9-0 & 18-4 & -9-0 \\ 9-0 & -9-0 & 18-4 \end{bmatrix} \\ = \begin{bmatrix} -14+14 & 9-9 & -9+9 \\ 9-9 & -14+14 & 9-9 \\ -9+9 & 9-9 & -14+14 \end{bmatrix} \\ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= 0$ (RHS)

Multiply $A^3 - 6A^2 + 9A - 4I = O$ by A^{-1} , (here I is 3×3 matrix)

$$A^3 A^{-1} - 6A^2 A^{-1} + 9A A^{-1} - 4I A^{-1} = O A^{-1}$$

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

$$4A^{-1} = A^2 - 6A + 9I$$

Now Placing all the matrices,

$$4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$4A^{-1} = \begin{bmatrix} 6-12+9 & -5+6+0 & 5-6+0 \\ -5+6+0 & 6-12+9 & -5+6+0 \\ 5-6+0 & -5+6+0 & 6-12+9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Inverse of the matrix is :

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

17. Let A be a non-singular matrix of order 3 x 3. Then |adj. A| is equal to:

- (A) |A| (B) |A|² (C) |A|³ (D) 3|A|

Solution:

Option (B) is correct.

Explanation:

$$|\text{adj. } A| = |A|^{n-1} = |A|^2 \quad (\text{for } n = 3)$$

18. If A is an invertible matrix of order 2, then det (A⁻¹) is equal to:

- (A) det A (B) 1/ det A (C) 1 (D) 0

Solution:

Option (B) is correct.

Explanation:

$$A A^{-1} = I$$

$$\det (A A^{-1} = I)$$

$$\det(A) \det(A^{-1}) = 1$$

$$\det(A^{-1}) = 1/\det A$$

Exercise 4.6

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Examine the consistency of the system of equations in Exercises 1 to 6.

1. $x + 2y = 2$ and $2x + 3y = 3$

Solution:

Given set of equations is : $x + 2y = 2$ and $2x + 3y = 3$

This set of equation can be written in the form of matrix as $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ and}$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

So, $AX = B$ is

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1 \neq 0$$

Inverse of matrix exists. So system of equations is consistent.

2. $2x - y = 5$ and $x + y = 4$

Solution:

Given set of equations is : $2x - y = 5$ and $x + y = 4$

This set of equation can be written in the form of matrix as $AX = B$, where

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

So, $AX = B$ is

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3 \neq 0$$

Inverse of matrix exists. So system of equations is consistent.

3. $x + 3y$ and $2x + 6y = 8$

Solution:

Given set of equations is : $x + 3y$ and $2x + 6y = 8$

This set of equation can be written in the form of matrix as $AX = B$.

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

Where,

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 0$$

$$\text{adj. } A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$$

And

$$(\text{adj. } A)B = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \neq 0$$

The given equations are inconsistent.

4. $x + y + z = 1$; $2x + 3y + 2z = 2$ and $ax + ay + 2az = 4$

Solution:

Given set of equations is : $x + y + z = 1$; $2x + 3y + 2z = 2$ and $ax + ay + 2az = 4$

This set of equation can be written in the form of matrix as $AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{vmatrix} = 1(6a - 2a) - 1(4a - 2a) + 1(2a - 3a) \\ = 4a - 2a - a = a \neq 0$$

System of equations is consistent.

5. $3x - y - 2z = 2$; $2y - z = -1$ and $3x - 5y = 3$

Solution:

Given set of equations is : $3x - y - 2z = 2$; $2y - z = -1$ and $3x - 5y = 3$

This set of equation can be written in the form of matrix as $AX = B$

$$\begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$$

$$= 3(-5) + (3) - 2(-6)$$

$$= 15 - 15$$

$$= 0$$

Now,

$$(\text{adj. } A) = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}$$

$$(\text{adj. } A)B = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ -6 \end{bmatrix} \neq 0$$

6. Given set of equations is :

$$5x - y + 4z = 5$$

$$2x + 3y + 5z = 2$$

$$5x - 2y + 6z = -1$$

Solution:

Given set of equations is: $5x - y + 4z = 5$; $2x + 3y + 5z = 2$; $5x - 2y + 6z = -1$

This set of equation can be written in the form of matrix as $AX = B$

$$\begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{vmatrix}$$

$$= 5(18 + 10) + 1(12 - 25) + 4(-4 - 15)$$

$$= 140 - 13 - 76$$

$$= 140 - 89$$

$$= 51$$

$$\neq 0$$

System of equations is consistent.

Solve system of linear equations, using matrix method, in Exercises 7 to 14.

7. $5x + 2y = 4$ and $7x + 3y = 5$

Solution:

Given set of equations is : $5x + 2y = 4$ and $7x + 3y = 5$

This set of equation can be written in the form of matrix as $AX = B$

$$\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

Where,

$$A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$$

And $|A| = 1 \neq 0$

System is consistent.

Now,

$$X = A^{-1}B = \frac{1}{|A|}(\text{adj. } A)B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 12-10 \\ -28+25 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\Rightarrow x = 2 \text{ and } y = -3$$

8. $2x - y = -2$ and $3x + 4y = 3$

Solution:

Given set of equations is : $2x - y = -2$ and $3x + 4y = 3$

This set of equation can be written in the form of matrix as $AX = B$

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$$

$$|A| = 11 \neq 0$$

System is consistent.

So,

$$X = A^{-1}B = \frac{1}{|A|}(\text{adj. } A)B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} -8+3 \\ 6+6 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} -5 \\ 12 \end{bmatrix}$$

Therefore, $x = -5/11$ and $y = 12/11$

9. $4x - 3y = 3$ and $3x - 5y = 7$

Solution:

Given set of equations is : $4x - 3y = 3$ and $3x - 5y = 7$

This set of equation can be written in the form of matrix as $AX = B$

$$\begin{bmatrix} 4 & -3 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

Where,

$$A = \begin{bmatrix} 4 & -3 \\ 3 & -5 \end{bmatrix}$$

And $|A| = -20 + 9 = -11 \neq 0$

System is consistent.

So,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-11} \begin{bmatrix} -5 & 3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$= \frac{1}{-11} \begin{bmatrix} -15+21 \\ -9+28 \end{bmatrix} = \frac{1}{-11} \begin{bmatrix} 6 \\ 19 \end{bmatrix}$$

Therefore, $x = 6/-11$ and $y = 19/-11$

10. $5x + 2y = 3$ and $3x + 2y = 5$

Solution:

Given set of equations is : $5x + 2y = 3$ and $3x + 2y = 5$

This set of equation can be written in the form of matrix as $AX = B$

$$\begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Where

$$A = \begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}$$

And $|A| = 4 \neq 0$

System is consistent.

So,

$$X = A^{-1}B = \frac{1}{|A|}(\text{adj. } A)B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Therefore, $x = -1$ and $y = 4$.

11. $2x + y + z = 1$ and $x - 2y - z = 3/2$ and $3y - 5z = 9$

Solution:

Given set of equations is : $2x + y + z = 1$ and $x - 2y - z = 3/2$ and $3y - 5z = 9$

This set of equation can be written in the form of matrix as $AX = B$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \\ 9 \end{bmatrix}$$

Where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}$$

And

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{vmatrix}$$

$$= 34 \neq 0$$

System is consistent.

So,

$$X = A^{-1}B = \frac{1}{|A|}(\text{adj. } A)B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34} \begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \\ 9 \end{bmatrix}$$

$$= \frac{1}{34} \begin{bmatrix} 13+12+9 \\ 5-15+27 \\ 3-9-45 \end{bmatrix} = \frac{1}{34} \begin{bmatrix} 34 \\ 17 \\ -51 \end{bmatrix}$$

Therefore, $x = 1$, $y = \frac{1}{2}$ and $z = \frac{3}{2}$

12. $x - y + z = 4$ and $2x + y - 3z = 0$ and $x + y + z = 2$

Solution:

Given set of equations is : $x - y + z = 4$ and $2x + y - 3z = 0$ and $x + y + z = 2$

This set of equation can be written in the form of matrix as $AX = B$

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

Where,

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$$

And

$$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 10 \neq 0$$

System is consistent.

So,

$$X = A^{-1}B = \frac{1}{|A|}(\text{adj. } A)B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 16+0+4 \\ -20+0+10 \\ 4-0+6 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Therefore, $x = 2$, $y = -1$ and $z = 1$

13.

$$2x + 3y + 3z = 5$$

$$x - 2y + z = -4$$

$$3x - y - 2z = 3$$

Solution:

Given set of equations is :

$$2x + 3y + 3z = 5$$

$$x - 2y + z = -4$$

$$3x - y - 2z = 3$$

This set of equation can be written in the form of matrix as $AX = B$

$$\begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

Where

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

And,

$$|A| = \begin{vmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{vmatrix}$$

$$= 40 \neq 0$$

System is consistent.

So,

$$X = A^{-1}B = \frac{1}{|A|}(\text{adj. } A)B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{40} \begin{bmatrix} 25 - 12 + 27 \\ 25 + 52 + 3 \\ 25 - 44 - 21 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Therefore, $x = 1$, $y = 2$ and $z = -1$.

14.

$$x - y + 2z = 7$$

$$3x + 4y - 5z = -5$$

$$2x - y + 3z = 12$$

Solution:

Given set of equations is :

$$x - y + 2z = 7$$

$$3x + 4y - 5z = -5$$

$$2x - y + 3z = 12$$

This set of equation can be written in the form of matrix as $AX = B$

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

Where,

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}$$

And

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 4 \neq 0$$

System is consistent.

So,

$$X = A^{-1}B = \frac{1}{|A|}(\text{adj. } A)B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix} \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 49 - 5 - 36 \\ -133 + 5 + 132 \\ -77 + 5 + 84 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Therefore, $x = 2$, $y = 1$ and $z = 3$.

15. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1} . Using A^{-1} solve the system of equations.

$$2x - 3y + 5z = 11$$

$$3x + 2y - 4z = -5$$

$$x + y - 2z = -3$$

Solution:

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$$

$= -1 \neq 0$; Inverse of matrix exists.

Find the inverse of matrix:

Cofactors of matrix:

$$A_{11} = 0, A_{12} = 2, A_{13} = 1$$

$$A_{21} = -1, A_{22} = -9, A_{23} = -5$$

$$A_{31} = 2, A_{32} = 23, A_{33} = 13$$

$$\text{adj. } A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

So,

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

Now, matrix of equation can be written as:

$$AX = B$$

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

And, $X = A^{-1} B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Therefore, $x = 1$, $y = 2$ and $z = 3$.

16. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is Rs. 60. The cost of 2 kg onion, 4 kg wheat and 2 kg rice is Rs. 90. The cost of 6 kg onion, 2 kg wheat and 3 kg rice is Rs. 70. Find cost of each item per kg by matrix method.

Solution:

Let x , y and z be the per kg. prices of onion, wheat and rice respectively.

According to given statement, we have following equations,

$$4x + 3y + 2z = 60$$

$$2x + 4y + 6z = 90$$

$$6x + 2y + 3z = 70$$

The above system of equations can be written in the form of matrix as, $AX = B$

$$\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

Where,

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}$$

And

$$|A| = \begin{vmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{vmatrix}$$

$$= 4(0) - 3(-30) + 2(-20)$$

$$= 50 \neq 0$$

System is consistent, and $X = A^{-1} B$

First find invers of A.

Cofactors of all the elements of A are:

$$A_{11} = 0, A_{12} = 30, A_{13} = -20$$

$$A_{21} = -5, A_{22} = 0, A_{23} = 10$$

$$A_{31} = 10, A_{32} = -20, A_{33} = 10$$

$$\text{adj. } A = \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

Again,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50} \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix} \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$= \frac{1}{50} \begin{bmatrix} -450 + 700 \\ 1800 - 1400 \\ -1200 + 900 + 700 \end{bmatrix}$$

$$= \frac{1}{50} \begin{bmatrix} 250 \\ 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

Therefore, $x = 5$, $y = 8$ and $z = 8$.

The cost of onion, wheat and rice per kg are Rs. 5, Rs. 8 and Rs. 8 respectively.

Miscellaneous Examples

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1. Prove that the determinant $\begin{vmatrix} x & \sin \theta & \cos \theta \\ -\sin \theta & -x & 1 \\ \cos \theta & 1 & x \end{vmatrix}$ is independent of θ .

Solution:

$$\text{Let } \Delta = \begin{vmatrix} x & \sin \theta & \cos \theta \\ -\sin \theta & -x & 1 \\ \cos \theta & 1 & x \end{vmatrix}$$

$$\begin{aligned} \Delta &= x \begin{vmatrix} -x & 1 \\ 1 & x \end{vmatrix} - \sin \theta \begin{vmatrix} -\sin \theta & 1 \\ \cos \theta & x \end{vmatrix} + \cos \theta \begin{vmatrix} -\sin \theta & -x \\ \cos \theta & 1 \end{vmatrix} \\ &= x(-x^2 - 1) - \sin \theta(-x \sin \theta - \cos \theta) + \cos \theta(-\sin \theta + x \cos \theta) \\ &= -x^3 - x + x(\sin^2 \theta + \cos^2 \theta) \\ &= -x^3 \end{aligned}$$

Which is independent of θ (Proved)

2. Without expanding the determinant, prove that

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

Solution:

Start with LHS:

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$$

Multiplying R1 by a R2 by b and R3 by c, we have

$$\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix}$$

Taking out common elements

$$\frac{abc}{abc} \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$$

Interchanging C_1 and C_3

$$= \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & a^3 & a^2 \\ 1 & b^3 & b^2 \\ 1 & c^3 & c^2 \end{vmatrix}$$

Interchanging C_2 and C_3

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

=RHS (proved)

3. Evaluate

$$\begin{vmatrix} \cos \alpha \cos \beta & \cos \alpha \sin \beta & -\sin \alpha \\ -\sin \beta & \cos \beta & 0 \\ \sin \alpha \cos \beta & \sin \alpha \sin \beta & \cos \alpha \end{vmatrix}$$

Solution:

$$\begin{aligned} & \begin{vmatrix} \cos \alpha \cos \beta & \cos \alpha \sin \beta & -\sin \alpha \\ -\sin \beta & \cos \beta & 0 \\ \sin \alpha \cos \beta & \sin \alpha \sin \beta & \cos \alpha \end{vmatrix} \\ &= \cos \alpha \cos \beta (\cos \alpha \cos \beta - 0) - \cos \alpha \sin \beta (-\cos \alpha \sin \beta - 0) - \sin \alpha (-\sin \alpha \sin^2 \beta - \sin \alpha \cos^2 \beta) \\ &= \cos^2 \alpha \cos^2 \beta + \cos^2 \alpha \sin^2 \beta + \sin^2 \alpha (\sin^2 \beta + \cos^2 \beta) \\ &= \cos^2 \alpha (\cos^2 \beta + \sin^2 \beta) + \sin^2 \alpha (\sin^2 \beta + \cos^2 \beta) \\ &= \cos^2 \alpha + \sin^2 \alpha = 1 \end{aligned}$$

4. If a , b and c are real numbers, and

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

Show that either $a + b + c = 0$ or $a = b = c$

Solution:

$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Applying: $R_1 \rightarrow R_1 + R_2 + R_3$

$$\begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$2(a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Since $\Delta = 0$

This implies,

Either $2(a + b + c) = 0$ or

$$\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

Case 1: If $2(a + b + c) = 0$

Then $(a + b + c) = 0$

Case 2:

$$\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

Applying: $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$

$$\begin{vmatrix} 1 & 0 & 0 \\ c+a & a+b-c-a & b+c-c-a \\ a+b & b+c-a-b & c+a-a-b \end{vmatrix} = 0$$

$$\text{or } \begin{vmatrix} b-c & b-a \\ c-a & c-b \end{vmatrix} = 0$$

$$\Rightarrow (b-c)(c-b) - (b-a)(c-a) = 0$$

$$\Rightarrow bc - b^2 - c^2 + bc - bc + ab + ac - a^2 = 0$$

$$\Rightarrow -a^2 - b^2 - c^2 + ab + bc + ca = 0$$

$$\Rightarrow a^2 + a^2 + b^2 + b^2 + c^2 + c^2 - 2ab - 2bc - 2ca = 0$$

$$\Rightarrow (a^2 + b^2 - 2ab) + (b^2 + c^2 - 2bc) + (a^2 + c^2 - 2ca) = 0$$

$$\Rightarrow (a-b)^2 + (b-c)^2 + (c-a)^2 = 0$$

Above expression only possible, if $(a-b) = 0$ and $(b-c) = 0$ and $(c-a) = 0$

That is $a = b$ and $b = c$ and $c = a$

Therefore, we have result, either $a+b+c = 0$ or $a=b=c$.

5. Solve the equation

$$\Delta = \begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0, a \neq 0$$

Solution:

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Applying: $R_1 \rightarrow R_1 + R_2 + R_3$

$$\begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

$$(3x+a) \begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Case 1: Either $3x + a = 0$

then $x = -a/3$

Case 2: or

$$\begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Applying:

$C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$

$$\begin{vmatrix} 1 & 0 & 0 \\ x & a & 0 \\ x & 0 & a \end{vmatrix} = 0$$

$$a^2 = 0$$

or $a = 0$

Not possible, as we are given $a \neq 0$.

So, $x = -a/3$ is only the solution.

6. Prove that

$$\Delta = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution:

LHS:

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

Taking a, b, and c from all the row1, row 2 and row 3 respectively.

$$= abc \begin{vmatrix} a & c & (a+c) \\ (a+b) & b & a \\ b & (b+c) & c \end{vmatrix}$$

$$R_1 \rightarrow R_1 - R_2 - R_3$$

$$= abc \begin{vmatrix} a-a-b-b & c-b-b-c & a+c-a-c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

$$C_2 \rightarrow C_2 - C_1$$

$$= abc \begin{vmatrix} -2b & -2b & 0 \\ a+b & b & a \\ b & b+c & c \end{vmatrix} = abc \begin{vmatrix} -2b & 0 & 0 \\ a+b & -a & a \\ b & c & c \end{vmatrix}$$

$$= abc(-2b)(-ac-ac) = 4a^2b^2c^2$$

RHS (Proved)

$$A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

7. If **and B = find (AB)⁻¹.**

Solution:

As we know, $(AB)^{-1} = B^{-1}A^{-1} \dots(1)$

First find inverse of matrix B.

$$|B| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix}$$

$$= 1(3-0) - 2(-1-0) + (-2)(2-0) = 1 \neq 0 \text{ (Inverse of B is possible)}$$

Find cofactors of B:

$$B_{11} = 3, B_{12} = 1, B_{13} = 2$$

$$B_{21} = 2, B_{22} = 1, B_{23} = 2 \text{ and}$$

$$B_{31} = 6, B_{32} = 2, B_{33} = 5$$

So adj. of B is

$$\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Now,

$$B^{-1} = \frac{1}{|B|} (\text{adj. } B) = \frac{1}{1} \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

From equation (1),

$$(AB)^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

=

$$\begin{bmatrix} 9 & -3 & 5 \\ -2 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix},$$

8. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$, verify that

(i) $(\text{adj. } A)^{-1} = \text{adj. } (A^{-1})$

(ii) $(A^{-1})^{-1} = A$

Solution:

$$|A| = \begin{vmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{vmatrix}$$

$$= -13 \neq 0 \text{ (Inverse of A exists)}$$

Cofactors of A are:

$$A_{11} = 14, A_{12} = 11, A_{13} = -5$$

$$A_{21} = 11, A_{22} = 4, A_{23} = -3$$

$$A_{31} = -5, A_{32} = -3, A_{33} = -1$$

So, adjoint of A is

$$\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

Now, A^{-1}

$$\frac{-1}{13} \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

Again,

$$|B| = \begin{vmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{vmatrix}$$

$$= 169 \neq 0 \text{ (Inverse of B exists)}$$

Cofactors of B are:

$$B_{11} = -13, B_{12} = 26, B_{13} = -13$$

$$B_{21} = 26, B_{22} = -39, B_{23} = -13$$

$$B_{31} = -13, B_{32} = -13, B_{33} = -65$$

Therefore, Inverse of B is

$$\frac{-1}{13} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

Find: $\text{adj } A^{-1}$

$$A^{-1} = \frac{-1}{13} \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$|A^{-1}| = -1/13 \neq 0$ (After solving the determinant we get the value. Try at your own)

Inverse of A^{-1} exists.

Let say cofactors of A^{-1} are represented as C_{ij} , we have

$$C_{11} = \frac{-1}{13}, C_{12} = \frac{2}{13}, C_{13} = \frac{-1}{13}$$

$$C_{21} = \frac{2}{13}, C_{22} = \frac{-3}{13}, C_{23} = \frac{-1}{13} \text{ and}$$

$$C_{31} = \frac{-1}{13}, C_{32} = \frac{-1}{13}, C_{33} = \frac{-5}{13}$$

Therefore:

$$(A^{-1})^{-1} = C^{-1} = \frac{1}{|C|} (\text{adj. } C)$$

Which implies,

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

$$= A$$

Which is again given matrix A.

$$(i) \quad (\text{adj. } A)^{-1} = \text{adj. } (A^{-1})$$

$$\frac{-1}{13} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = \frac{-1}{13} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

$$(ii) \quad (A^{-1})^{-1} = A$$

$$\frac{-1}{13} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

9. Evaluate

$$\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Solution:

Consider,

$$\Delta = \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Operation: $[R_1 \rightarrow R_1 + R_2 + R_3]$

$$\begin{vmatrix} 2(x+y) & 2(x+y) & 2(x+y) \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Taking $2(x+y)$ common from first row

$$2(x+y) \begin{vmatrix} 1 & 1 & 1 \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Operation: $[C_2 \rightarrow C_2 - C_1 \text{ and } C_3 \rightarrow C_3 - C_1]$

$$2(x+y) \begin{vmatrix} 1 & 0 & 0 \\ y & x+y-y & x-y \\ x+y & x-x-y & y-x-y \end{vmatrix}$$

$$2(x+y) \cdot 1 \begin{vmatrix} x & x-y \\ -y & -x \end{vmatrix}$$

$$2(x+y) \{-x^2 + y(x-y)\}$$

$$-2(x+y)(x^2 - xy + y^2)$$

$$=-2(x^3 + y^3)$$

$$\Delta = -2(x^3 + y^3)$$

10. Evaluate

$$\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$$

Solution:

Consider

$$\Delta = \begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$$

Operation: $[R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1]$

$$\begin{vmatrix} 1 & x & y \\ 0 & x+y-x & 0 \\ 0 & 0 & x+y-y \end{vmatrix}$$

$$\begin{vmatrix} 1 & x & y \\ 0 & y & 0 \\ 0 & 0 & x \end{vmatrix}$$

$$=xy$$

$$\Rightarrow \Delta = xy$$

Using properties of determinants in Exercises 11 to 15, prove that:

11.

$$\begin{vmatrix} \alpha & \alpha^2 & \beta + \gamma \\ \beta & \beta^2 & \gamma + \alpha \\ \gamma & \gamma^2 & \alpha + \beta \end{vmatrix} = (\beta - \gamma)(\gamma - \alpha)(\alpha - \beta)(\alpha + \beta + \gamma)$$

Solution:

LHS

$$\begin{vmatrix} \alpha & \alpha^2 & \beta + \gamma \\ \beta & \beta^2 & \gamma + \alpha \\ \gamma & \gamma^2 & \alpha + \beta \end{vmatrix}$$

Operation: $[C_3 \rightarrow C_3 + C_1]$

$$\begin{vmatrix} \alpha & \alpha^2 & \alpha + \beta + \gamma \\ \beta & \beta^2 & \alpha + \beta + \gamma \\ \gamma & \gamma^2 & \alpha + \beta + \gamma \end{vmatrix}$$

$$(\alpha + \beta + \gamma) \begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta & \beta^2 & 1 \\ \gamma & \gamma^2 & 1 \end{vmatrix}$$

Operation: $[R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1]$

$$(\alpha + \beta + \gamma) \begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta - \alpha & \beta^2 - \alpha^2 & 0 \\ \gamma - \alpha & \gamma^2 - \alpha^2 & 0 \end{vmatrix}$$

=

$$(\alpha + \beta + \gamma) \begin{vmatrix} \beta - \alpha & (\beta - \alpha)(\beta + \alpha) \\ \gamma - \alpha & (\gamma - \alpha)(\gamma + \alpha) \end{vmatrix}$$

=

$$(\alpha + \beta + \gamma)(\beta - \alpha)(\gamma - \alpha) \begin{vmatrix} 1 & (\beta + \alpha) \\ 1 & (\gamma + \alpha) \end{vmatrix}$$

$$= (\alpha + \beta + \gamma)(\beta - \alpha)(\gamma - \alpha)(\gamma + \alpha - \beta - \alpha)$$

$$= (\alpha + \beta + \gamma)[-(\alpha - \beta)](\gamma - \alpha)[-(\beta - \gamma)]$$

$$= (\alpha + \beta + \gamma)(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)$$

= RHS

12.

$$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$$

Solution:

LHS=

$$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix}$$

We have two determinants, say Δ_1 and Δ_2

$$= \Delta_1 + \Delta_2$$

$$\Delta_1 = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$$

Operation: $[R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1]$

$$= \begin{vmatrix} x & x^2 & 1 \\ y-x & y^2-x^2 & 0 \\ z-x & z^2-x^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} y-x & (y-x)(y+x) \\ z-x & (z-x)(z+x) \end{vmatrix}$$

$$= (y-x)(z-x) \begin{vmatrix} 1 & y+x \\ 1 & z+x \end{vmatrix}$$

$$= (y-x)(z-x)(z+x-y-x)$$

$$= (x-y)(y-z)(z-x)$$

Again:

$$\Delta_2 = \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix}$$

$$pxyz\Delta_2 = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= -pxyz \begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix}$$

$$= pxyz \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$$

$$= pxyz \Delta_1$$

Therefore: $\Delta_1 + \Delta_2$

LHS

$$= (y-x)(z-x)(z-y) + pxyz(y-x)(z-x)(z-y)$$

$$= (1+pxyz)(y-x)(z-x)(z-y)$$

= RHS

(Proved)

13. Prove that

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$$

Solution:

LHS

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$$

Operation: $[C_1 \rightarrow C_1 + C_2 + C_3]$

$$\begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$$

$$(a+b+c) \begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$$

Operation: $[R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1]$

$$(a+b+c) \cdot 1 \begin{vmatrix} 2b+a & a-b \\ a-c & 2c+a \end{vmatrix}$$

$$= (a+b+c) [(2b+a)(2c+a) - (a-b)(a-c)]$$

$$= (a+b+c) [4bc + 2ab + a^2 - a^2 + ac + ab - bc]$$

$$= 3(a+b+c)(ab+bc+ac)$$

$$= \text{RHS}$$

Hence Proved.

14. Prove that

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$$

Solution:

LHS

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$$

Operation: $[R_2 \rightarrow R_2 - 2R_1 \text{ and } R_3 \rightarrow R_3 - 3R_1]$

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 3 & 7+3p \end{vmatrix}$$

=

$$1 \begin{vmatrix} 1 & 2+p \\ 3 & 7+3p \end{vmatrix} - 0 + 0$$

$$= 7+3p-6-3p$$

$$= 1$$

$$= \text{RHS}$$

Hence Proved.

15. Prove that

$$\begin{vmatrix} \sin \alpha & \cos \alpha & \cos(\alpha + \delta) \\ \sin \beta & \cos \beta & \cos(\beta + \delta) \\ \sin \gamma & \cos \gamma & \cos(\gamma + \delta) \end{vmatrix} = 0$$

Solution:

LHS

$$\begin{vmatrix} \sin \alpha & \cos \alpha & \cos(\alpha + \delta) \\ \sin \beta & \cos \beta & \cos(\beta + \delta) \\ \sin \gamma & \cos \gamma & \cos(\gamma + \delta) \end{vmatrix}$$

=

$$\begin{vmatrix} \sin \alpha & \cos \alpha & \cos \alpha \cos \delta - \sin \alpha \sin \delta \\ \sin \beta & \cos \beta & \cos \beta \cos \delta - \sin \beta \sin \delta \\ \sin \gamma & \cos \gamma & \cos \gamma \cos \delta - \sin \gamma \sin \delta \end{vmatrix}$$

Operation: $[C_3 \rightarrow C_3 + (\sin \delta)C_1]$

=

$$\begin{vmatrix} \sin \alpha & \cos \alpha & \cos \alpha \cos \delta - \sin \alpha \sin \delta + \sin \alpha \sin \delta \\ \sin \beta & \cos \beta & \cos \beta \cos \delta - \sin \beta \sin \delta + \sin \beta \sin \delta \\ \sin \gamma & \cos \gamma & \cos \gamma \cos \delta - \sin \gamma \sin \delta + \sin \gamma \sin \delta \end{vmatrix}$$

$$= \begin{vmatrix} \sin \alpha & \cos \alpha & \cos \alpha \cos \delta \\ \sin \beta & \cos \beta & \cos \beta \cos \delta \\ \sin \gamma & \cos \gamma & \cos \gamma \cos \delta \end{vmatrix}$$

=

$$\cos \delta \begin{vmatrix} \sin \alpha & \cos \alpha & \cos \alpha \\ \sin \beta & \cos \beta & \cos \beta \\ \sin \gamma & \cos \gamma & \cos \gamma \end{vmatrix}$$

=

Column 2 and column 3 are identical, as per determinant property, value is zero.

= 0

= RHS

16. Solve the system of equations

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4; \quad \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1; \quad \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

Solution:

Let

$$\frac{1}{x} = u, \frac{1}{y} = v, \frac{1}{z} = w$$

We have

$$2u + 3v + 10w = 4;$$

$$4u - 6v + 5w = 1; \text{ and}$$

$$6u + 9v - 20w = 2$$

Below is the matrix from the given equations: $AX = B$

$$\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

Let say

$$A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$$

Then,

$$|A| = 1200 \neq 0$$

 A^{-1} exists.

Cofactors of A are:

$$A_{11} = 75, A_{12} = 110, A_{13} = 72$$

$$A_{21} = 150, A_{22} = -100, A_{23} = 0 \text{ and}$$

$$A_{31} = 75, A_{32} = 30, A_{33} = -24$$

$$\text{adj. } A = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

Inverse of A is

$$A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{1}{1200} \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

Resubstitute the values, to get answer in the form of x, y and z.

Since $AX = B$

$$X = A^{-1} B$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{1200} \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{1200} \begin{bmatrix} 300 + 150 + 150 \\ 440 - 100 + 60 \\ 288 + 0 - 48 \end{bmatrix}$$

=

$$\frac{1}{1200} \begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix}$$

Which means:

$$u = \frac{1}{2}, v = \frac{1}{3} \text{ and } w = \frac{1}{5}$$

This implies:

$$x = \frac{1}{u} = 2$$

$$y = \frac{1}{v} = 3$$

$$z = \frac{1}{w} = 5$$

Answer!

Choose the correct answer in Exercise 17 to 19.

17. If a, b, c are in A.P., then the determinant is $\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$.
- (A) 0 (B) 1 (C) x (D) $2x$

Solution:

Option (A) is correct.

Explanation:

Since a, b, c are in A.P.

So, $b - a = c - b$

Let

$$\Delta = \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

Operation: $[R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1]$

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ 1 & 1 & 2(b-a) \\ 1 & 1 & 2(c-b) \end{vmatrix}$$

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ 1 & 1 & 2(b-a) \\ 1 & 1 & 2(b-a) \end{vmatrix}$$

Row 2 and row 3 are identical, so value is zero.

$= 0$

$$A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

18. If x, y, z are non-zero real numbers, then the inverse of matrix $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is:

(A)

$$\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

(B)

$$xyz \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

(C)

$$\frac{1}{xyz} \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

(D)

$$\frac{1}{xyz} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: Option (A) is correct.

Explanation:

$$\text{Let } A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

$$|A| = xyz \neq 0 ; (A^{-1} \text{ exists})$$

Now: Cofactors of A are:

$$A_{11} = yz, A_{12} = 0, A_{13} = 0$$

$$A_{21} = 0, A_{22} = xz, A_{23} = 0 \text{ and}$$

$$A_{31} = 0, A_{32} = 0, A_{33} = xy$$

Therefore:

$$A^{-1} = \frac{\text{adj.}A}{|A|} = \frac{1}{xyz} \begin{vmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{vmatrix}$$

$$= \begin{vmatrix} \frac{yz}{xyz} & 0 & 0 \\ 0 & \frac{xz}{xyz} & 0 \\ 0 & 0 & \frac{xy}{xyz} \end{vmatrix}$$

$$= \begin{vmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{vmatrix}$$

19. Let $A = \begin{bmatrix} 1 & \sin \theta & 1 \\ -\sin \theta & 1 & \sin \theta \\ -1 & -\sin \theta & 1 \end{bmatrix}$, where $0 \leq \theta \leq 2\pi$. then:

(A) $\text{Det}(A) = 0$

(B) $\text{Det}(A) \in (2, \infty)$

(C) $\text{Det}(A) \in (2, 4)$

(D) $\text{Det}(A) \in [2, 4]$

Solution:

Option (D) is correct.

Explanation:

Let $A = \begin{bmatrix} 1 & \sin \theta & 1 \\ -\sin \theta & 1 & \sin \theta \\ -1 & -\sin \theta & 1 \end{bmatrix}$

$$|A| = 2 + 2 \sin^2 \theta \neq 0 ; (A^{-1} \text{ exists})$$

Since :

$$-1 \leq \sin \theta \leq 1$$

$$0 \leq \sin^2 \theta \leq 1$$

(The value of θ cannot be negative)

$$\text{So, } 0 \leq 2 \sin^2 \theta \leq 2$$

Add 2 in all the expressions:

$$2 \leq 2 + 2 \sin^2 \theta \leq 4$$

Which is equal to

$$2 \leq \text{Det. } A \leq 4$$