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1. Determine whether or not each of the definition of ∗ given below gives a binary operation. In the event that ∗ is not a binary operation, give justification for this.

(i) On Z^+, define ∗ by a ∗ b = a – b

(ii) On Z^+, define ∗ by a ∗ b = ab

(iii) On R, define ∗ by a ∗ b = ab^2

(iv) On Z^+, define ∗ by a ∗ b = | a – b |

(v) On Z+, define ∗ by a ∗ b = a

Solution:

(i) On Z^+, define ∗ by a ∗ b = a – b

On Z^+ = {1, 2,3 , 4, 5,…….}

Let a = 1 and b = 2

Therefore, a ∗ b = a – b = 1 – 2 = -1 $\notin Z^+$

operation * is not a binary operation on Z^+ .

(ii) On Z^+, define ∗ by a ∗ b = ab

On Z^+ = {1, 2,3 , 4, 5,…….}

Let a = 2 and b = 3

Therefore, a ∗ b = a b = 2 * 3 = 6 $\in Z^+$

operation * is a binary operation on Z^+

(iii) On R, define ∗ by a ∗ b = ab^2

R = { - ∞, ……, -1, 0, 1, 2,……, ∞}

Let a = 1.2 and b = 2

Therefore, a ∗ b = ab^2 = (1.2) x 2^2 = 4.8 ∈ R

Operation * is a binary operation on R.

(iv) On Z^+, define ∗ by a ∗ b = | a – b |

On Z^+ = {1, 2,3 , 4, 5,…….}

Let a = 2 and b = 3

Therefore, a ∗ b = a b = 2 * 3 = 6 ∈ Z^+

operation * is a binary operation on Z^+

(v) On Z+, define ∗ by a ∗ b = a

On Z^+ = {1, 2, 3, 4, 5,…….}

Let a = 2 and b = 1

Therefore, a ∗ b = 2 ∈ Z^+

Operation * is a binary operation on Z^+ .

2. For each operation ∗ defined below, determine whether ∗ is binary, commutative or associative.

(i) On Z, define a ∗ b = a – b

(ii) On Q, define a ∗ b = ab + 1

(iii) On Q, define a ∗ b = ab/2

(iv) On Z^+, define a ∗ b = 2^{ab}

(v) On Z^+, define a ∗ b = a^b

(vi) On R – {– 1}, define a ∗ b = a/(b+1)

Solution:

(i) On Z, define a ∗ b = a – b

Step 1: Check for commutative

Consider ∗ is commutative, then

a ∗ b = b * a

Which means, a – b = b – a (not true)

Therefore, ∗ is not commutative.

Step 2: Check for Associative.

Consider ∗ is associative, then

(a ∗ b)* c = a * (b * c)

LHS = (a ∗ b)* c = (a – b) * c

= a – b – c

RHS = a * (b * c) = a - (b- c)

= a – (b – c)

= a – b + c

This implies LHS ≠ RHS

Therefore, ∗ is not associative.

(ii) On Q, define a ∗ b = ab + 1

Step 1: Check for commutative

Consider ∗ is commutative, then

a ∗ b = b * a

Which means, ab + 1 = ba + 1

or ab + 1 = ab + 1 (which is true)

a ∗ b = b * a for all a, b ∈ Q

Therefore, ∗ is commutative.

Step 2: Check for Associative.

Consider ∗ is associative, then

(a ∗ b)* c = a * (b * c)

LHS = (a ∗ b) * c = (ab + 1) * c

= (ab + 1)c + 1

= abc + c + 1

RHS = a * (b * c) = a * (bc + 1)

= a(bc + 1) + 1

= abc + a + 1

This implies LHS ≠ RHS

Therefore, ∗ is not associative.

(iii) On Q, define a ∗ b = ab/2

Step 1: Check for commutative

Consider ∗ is commutative, then

a ∗ b = b * a

Which means, ab/2 = ba/2

or ab/2 = ab/2 (which is true)

a ∗ b = b * a for all a, b ∈ Q

Therefore, ∗ is commutative.

Step 2: Check for Associative.

Consider $*$ is associative, then

$(a * b) * c = a * (b * c)$

LHS = $(a * b) * c = (ab/2) * c$

$= \frac{\frac{ab}{2} \times c}{2}$

= abc/4

RHS = $a * (b * c) = a * (bc/2)$

$= \frac{a \times \frac{bc}{2}}{2}$

= abc/4

This implies LHS = RHS

Therefore, $*$ is associative binary operation.

(iv) On Z^+, define $a * b = 2^{ab}$

Step 1: Check for commutative

Consider $*$ is commutative, then

$a * b = b * a$

Which means, $2^{ab} = 2^{ba}$

or $2^{ab} = 2^{ab}$ (which is true)

$a * b = b * a$ for all $a, b \in Z^+$

Therefore, $*$ is commutative.

Step 2: Check for Associative.

Consider $*$ is associative, then

$(a * b) * c = a * (b * c)$

LHS = $(a * b) * c = (2^{ab}) * c$

$= 2^{2^{ab}c}$

RHS = $a * (b * c) = a * 2^{bc}$

$= 2^{2^{bc}a}$

This implies LHS ≠ RHS

Therefore, $*$ is not associative binary operation.

(v) On Z^+, define $a * b = a^b$

Step 1: Check for commutative

Consider $*$ is commutative, then

$a * b = b * a$

Which means, $a^b = b^a$

Which is not true

$a * b = b * a$ for all $a, b \in Z^+$

Therefore, $*$ is not commutative.

Step 2: Check for Associative.

Consider $*$ is associative, then

$(a * b) * c = a * (b * c)$

LHS = $(a^b) * c$

$= (a^b)^c$

RHS = $a * (b * c) = a * (b^c)$

$= a^{b^c}$

This implies LHS $\neq$ RHS

Therefore, $*$ is not associative.

(vi) On R – {– 1}, define a ∗ b = a/(b+1)

Step 1: Check for commutative

Consider $*$ is commutative, then

a ∗ b = b * a

Which means, a/(b+1) = b/(a+1)

Which is not true

Therefore, $*$ is commutative.

Step 2: Check for Associative.

Consider $*$ is associative, then

(a ∗ b)* c = a * (b * c)

LHS = (a ∗ b) * c = (a/(b+1)) * c

$= \frac{\frac{a}{b+1}}{c}$

= a/(c(b+1)

RHS = a * (b * c) = a * (b/(c + 1))

$= \frac{\frac{a}{b}}{c+1}$

= a(c+1)/b

This implies LHS $\neq$ RHS

Therefore, $*$ is not associative binary operation.

3. Consider the binary operation ∧ on the set {1, 2, 3, 4, 5} defined by a ∧ b = min {a, b}. Write the operation table of the operation ∧ .

Solution:

The binary operation ∧ on the set, say A = {1, 2, 3, 4, 5} defined by a ∧ b = min {a, b}.
the operation table of the operation ∧ as follow:

^	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

4. Consider a binary operation ∗ on the set {1, 2, 3, 4, 5} given by the following multiplication table (Table 1.2).

(i) Compute (2 ∗ 3) ∗ 4 and 2 ∗ (3 ∗ 4)
(ii) Is ∗ commutative?
(iii) Compute (2 ∗ 3) ∗ (4 ∗ 5).
(Hint: use the following table)

Table 1.2

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

Solution:

(i) Compute (2 ∗ 3) ∗ 4 and 2 ∗ (3 ∗ 4)

From table: (2 ∗ 3) = 1 and (3 ∗ 4) = 1

(2 ∗ 3) ∗ 4 = 1 * 4 = 1 and

2 ∗ (3 ∗ 4) = 2 * 1 = 1

(ii) Is ∗ commutative?

Consider 2 * 3, we have 2 * 3 = 1 and 3 * 2 = 1

Therefore, * is commutative.

(iii) Compute (2 ∗ 3) ∗ (4 ∗ 5).

From table: (2 ∗ 3) = 1 and (4 ∗ 5) = 1

So (2 ∗ 3) ∗ (4 ∗ 5) = 1 * 1 = 1

5. Let ∗′ be the binary operation on the set {1, 2, 3, 4, 5} defined by a ∗′ b = H.C.F. of a and b. Is the operation ∗′ same as the operation ∗ defined in Exercise 4 above? Justify your answer.

Solution: Let A = {1, 2, 3, 4, 5} and a ∗′ b H.C.F. of a and b. Plot a table values, we have

∗′	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

Operation ∗′ same as the operation *.

6. Let ∗ be the binary operation on N given by a ∗ b = L.C.M. of a and b. Find
(i) 5 ∗ 7, 20 ∗ 16
(ii) Is ∗ commutative?
(iii) Is ∗ associative?
(iv) Find the identity of ∗ in N

(v) Which elements of N are invertible for the operation ∗?

Solution:

(i) 5 ∗ 7 = LCM of 5 and 7 = 35

20 ∗ 16 = LCM of 20 and 16 = 80

(ii) Is ∗ commutative?

a ∗ b = L.C.M. of a and b

b ∗ a = L.C.M. of b and a

a ∗ b = b ∗ a

Therefore ∗ is commutative.

(iii) Is ∗ associative?
For a,b, c ∈ N

(a ∗ b) * c = (L.C.M. of a and b) * c = L.C.M. of a, b and c

a ∗ (b * c) = a * (L.C.M. of b and c) = L.C.M. of a, b and c

(a ∗ b) * c = a ∗ (b * c)

Therefore, operation ∗ associative.

(iv) Find the identity of ∗ in N
Identity of * in N = 1

because a * 1 = L.C.M. of a and 1 = a

(v) Which elements of N are invertible for the operation ∗?

Only the element 1 in N is invertible for the operation * because 1 * 1/1 = 1

7. Is ∗ defined on the set {1, 2, 3, 4, 5} by a ∗ b = L.C.M. of a and b a binary operation? Justify your answer.

Solution:

The operation ∗ defined on the set {1, 2, 3, 4, 5} by a ∗ b = L.C.M. of a and b

Suppose, a = 2 and b = 3

2 * 3 = L.C.M. of 2 and 3 = 6

But 6 does not belongs to the set A.
Therefore, given operation * is not a binary operation.

8. Let ∗ be the binary operation on N defined by a ∗ b = H.C.F. of a and b. Is ∗ commutative? Is ∗ associative? Does there exist identity for this binary operation on N?

Solution:

The operation ∗ be the binary operation on N defined by a ∗ b = H.C.F. of a and b

a * b = H.C.F. of a and b = H.C.F. of b and a = b * a

Therefore, operation * is commutative.

Again, (a *b)*c = (HCF of a and b) * c = HCF of (HCF of a and b) and c = a * (b *c)

(a *b)*c = a * (b *c)

Therefore, the operation is associative.

Now, 1 * a = a * 1 ≠ a

Therefore, there does not exist any identity element.

9. Let ∗ be a binary operation on the set Q of rational numbers as follows:
(i) a ∗ b = a – b
(ii) a ∗ b = $a^2 + b^2$
(iii) a ∗ b = a + ab
(iv) a ∗ b = $(a – b)^2$
(v) a ∗ b = ab/4
(vi) a ∗ b = ab^2

Find which of the binary operations are commutative and which are associative.

Solution:
(i) a ∗ b = a – b

a ∗ b = a – b = - (b – a) = -b * c ≠ b * a (Not commutative)

(a * b) * c = (a – b) * c = (a – (b – c) = a – b + c ≠ a * (b *c) (Not associative)

(ii) $a * b = a^2 + b^2$

$a * b = a^2 + b^2 = b^2 + a^2 = b * a$ (operation is commutative)

Check for associative:

$(a * b) * c = (a^2 + b^2) * c^2 = (a^2 + b^2) + c^2$

$a * (b * c) = a * (b^2 + c^2) = a^2 * (b^2 + c^2)^2$

$(a * b) * c \neq a * (b * c)$ (Not associative)

(iii) $a * b = a + ab$

$a * b = a + ab = a(1 + b)$

$b * a = b + ba = b(1+a)$

$a * b \neq b * a$

The operation * is not commutative

Check for associative:

$(a * b) * c = (a + ab) * c = (a + ab) + (a + ab)c$

$a * (b * c) = a * (b + bc) = a + a(b + bc)$

$(a * b) * c \neq a * (b * c)$

The operation * is not associative

(iv) $a * b = (a – b)^2$

$a * b = (a – b)^2$

$b * a = (b – a)^2$

$a * b = b * a$

The operation * is commutative.

Check for associative:

$(a * b) * c = (a - b)^2 * c = ((a – b)^2 – c)^2$

$a * (b *c) = a * (b - c)^2 = (a - (b - c)^2)^2$

$(a * b) * c \neq a * (b *c)$

The operation * is not associative

(v) a ∗ b = ab/4

b * a = ba/2 = ab/2

a ∗ b = b * a

The operation * is commutative.

Check for associative:

(a * b) * c = ab/4 * c = abc/16

a * (b *c) = a * (bc/4) = abc/16

(a * b) * c = a * (b *c)

The operation * is associative.

(vi) $a * b = ab^2$

$b * a = ba^2$

$a * b \neq b * a$

The operation * is not commutative.

Check for associative:

$(a * b) * c = (ab^2) * c = ab^2 c^2$

$a * (b *c) = a * (b c^2) = ab^2 c^4$

$(a * b) * c \neq a * (b *c)$

The operation * is not associative.

10. Find which of the operations given above has identity.

Solution: Let I be the identity.

(i) a * I = a – I ≠ a

(ii) a * I = $a^2 - I^2 \neq a$

(iii) a * I = a + a I ≠ a

(iv) a * I = $(a - I)^2 \neq a$

(v) a * I = aI/4 ≠ a

Which is only possible at I = 4 i.e. a * I = aI/4 = a(4)/4 = a

(vi) a * I = a $I^2 \neq a$

Above identities does not have identity element except (V) at b = 4.

11. Let A = N × N and ∗ be the binary operation on A defined by
(a, b) ∗ (c, d) = (a + c, b + d)
Show that ∗ is commutative and associative. Find the identity element for ∗ on A, if any.

Solution: A = N x N and * is a binary operation defined on A.
(a, b) * (c, d) = (a + c, b + d)

(c, d) * (a, b) = (c + a, d + b) = (a + c, b + d)

The operation * is commutative

Again, ((a, b) * (c, d)) * (e, f) = (a + c, b + d) * (e, f)
= (a + c + e, b + d + f)

(a, b) * ((c, d)) * (e, f)) = (a, b) * (c+e, e+f) = (a+c+e, b+d+f)

=> ((a, b) * (c, d)) * (e, f) = (a, b) * ((c, d)) * (e, f))

The operation * is associative.

Let (e, f) be the identity function, then

(a, b) * (e, f) = (a + e, b + f)

For identity function, a = a + e => e = 0 and b = b + f => f = 0

As zero is not a part of set of natural numbers. So identity function does not exist.

As $0 \notin N$, therefore, identity-element does not exist.

12. State whether the following statements are true or false. Justify.
(i) For an arbitrary binary operation * on a set N, a * a = a ∀ a ∈ N.
(ii) If * is a commutative binary operation on N, then a * (b * c) = (c * b) * a

Solution:
(i) Given: * being a binary operation on N, is defined as a * a = a ∀ a ∈ N

Here operation * is not defined, therefore, the given statement is not true.

(ii) Operation * being a binary operation on N.

c * b = b * c

(c * b) * a = (b * c) * a = a * (b * c)

Thus, a * (b * c) = (c * b) * a, therefore the given statement is true.

13. Consider a binary operation ∗ on N defined as a ∗ b = $a^3 + b^3$. Choose the correct answer.
(A) Is ∗ both associative and commutative?
(B) Is ∗ commutative but not associative?
(C) Is ∗ associative but not commutative?
(D) Is ∗ neither commutative nor associative?
Solution:
A binary operation ∗ on N defined as a ∗ b = $a^3 + b^3$,

Also, a ∗ b = $a^3 + b^3 = b^3 + a^3$ = b * a
The operation * is commutative.

Again, $(a * b)*c = (a^3 + b^3) * c = (a^3 + b^3)^3 + c^3$

$a * (b * c) = a * (b^3 + c^3) = a^3 + (b^3 + c^3)^3$

⇨ $(a * b)*c \neq a * (b * c)$

The operation * is not associative.

Therefore, option (B) is correct.