Miscellaneous Exercise Page No: 191

Differentiate with respect to x the functions in Exercises 1 to 11.

1. $\left(3x^2-9x+5\right)^9$

Solution: Consider $y=\left(3x^2-9x+5\right)^9$

$$\frac{dy}{dx}=9\left(3x^2-9x+5\right)^8\frac{d}{dx}\left(3x^2-9x+5\right)$$

$$\left[\because \frac{d}{dx}\{f(x)\}^n=n\{f(x)\}^{n-1}\frac{d}{dx}f(x)\right]$$

$$\frac{dy}{dx}=9\left(3x^2-9x+5\right)^8\left[3(2x)-9(1)+0\right]$$

$$\frac{dy}{dx}=27\left(3x^2-9x+5\right)^8\left[2x-3\right]$$

2. $\sin^3 x+\cos^6 x$

Solution: Consider $y=\sin^3 x+\cos^6 x$

or y = $(\sin x)^3+(\cos x)^6$

$$\frac{dy}{dx}=3(\sin x)^2\frac{d}{dx}\sin x+6(\cos x)^5\frac{d}{dx}\cos x$$

$$\frac{dy}{dx}=3\sin^2 x\cos x-6\cos^5 x\sin x$$

= $3\sin x\cos x\left(\sin x-2\cos^4 x\right)$

3. $(5x)^{3\cos 2x}$

Solution: Consider $y=(5x)^{3\cos 2x}$

Taking log both the sides, we get

$$\log y=\log(5x)^{3\cos 2x}$$

$$\log y = 3\cos 2x \log(5x)$$

Derivate above function:

$$\frac{d}{dx}\log y = 3\frac{d}{dx}\{\cos 2x \log(5x)\}$$

$$\frac{1}{y}\frac{dy}{dx} = 3\left[\cos 2x \frac{d}{dx}\log(5x) + \log(5x)\frac{d}{dx}\cos 2x\right]$$

$$\frac{1}{y}\frac{dy}{dx} = 3\left[\cos 2x \frac{1}{5x}\frac{d}{dx}5x + \log(5x)(-\sin 2x)\frac{d}{dx}2x\right]$$

$$\frac{1}{y}\frac{dy}{dx} = 3\left[\cos 2x \frac{1}{5x}.5 - 2\sin 2x \log(5x)\right]$$

$$\frac{dy}{dx} = 3y\left[\frac{\cos 2x}{x} - 2\sin 2x \log(5x)\right]$$

$$\frac{dy}{dx} = 3(5x)^{3\cos 2x}\left[\frac{\cos 2x}{x} - 2\sin 2x \log(5x)\right]$$ (using value of y)

4. $\sin^{-1}\left(x\sqrt{x}\right), 0 \le x \le 1$

Solution: Consider $y = \sin^{-1}\left(x\sqrt{x}\right)$

or y = $\sin^{-1}\left(x^{\frac{3}{2}}\right)$

Apply derivation:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(x^{\frac{3}{2}}\right)^2}}\frac{d}{dx}x^{\frac{3}{2}}$$

$$= \frac{1}{\sqrt{1-x^3}}.\frac{3}{2}x^{\frac{1}{2}}$$

$= \frac{3}{2}\sqrt{\frac{x}{1-x^3}}$

5. $\frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}, -2 < x < 2$

Solution: Consider $y = \frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}$

Apply derivation:

$$\frac{dy}{dx} = \frac{\sqrt{2x+7}\frac{d}{dx}\cos^{-1}\frac{x}{2} - \cos^{-1}\frac{x}{2}\frac{d}{dx}\sqrt{2x+7}}{\left(\sqrt{2x+7}\right)^2}$$

[Using Quotient Rule]

$$\frac{dy}{dx} = \frac{\sqrt{2x+7}\left(\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\right)\frac{d}{dx}\frac{x}{2} - \left(\cos^{-1}\frac{x}{2}\right)\frac{1}{2}(2x+7)^{\frac{-1}{2}}\frac{d}{dx}(2x+7)}{\left(\sqrt{2x+7}\right)^2}$$

$$\frac{dy}{dx} = \frac{-\sqrt{2x+7}.\frac{2}{\sqrt{4-x^2}}.\frac{1}{2} - \frac{1}{2}.\frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}}{(2x+7)}$$

$$= \frac{-\left[\frac{\sqrt{2x+7}}{\sqrt{4-x^2}} + \frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}\right]}{(2x+7)}$$

$$\frac{dy}{dx} = -\left[\frac{2x+7+\sqrt{4-x^2}\cos^{-1}\frac{x}{2}}{\sqrt{4-x^2}\sqrt{2x+7}\,(2x+7)}\right]$$

$$= -\left[\frac{2x+7+\sqrt{4-x^2}\cos^{-1}\frac{x}{2}}{\sqrt{4-x^2}\,(2x+7)^{\frac{3}{2}}}\right]$$

6. $\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right], 0 < x < \frac{\pi}{2}$

Solution: Consider $y = \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right), 0 < x < \frac{\pi}{2}$ ……….(i)

Reduce the functions into simplest form,

$$\sqrt{1+\sin x} = \sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2} = \cos\frac{x}{2}+\sin\frac{x}{2}$$

And $\sqrt{1-\sin x} = \sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}$

$$= \sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2} = \cos\frac{x}{2}-\sin\frac{x}{2}$$

Now, we are available with the equation below:

$$y = \cot^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}}\right)$$

$= \cot^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right)$

$y = \cot^{-1}\left(\cot\frac{x}{2}\right)$

$= \frac{x}{2}$

Apply derivation:

$$\frac{dy}{dx} = \frac{1}{2}(1) = \frac{1}{2}$$

7. $(\log x)^{\log x}, x > 1$

Solution: Consider $y = (\log x)^{\log x}, x > 1$ ……….(i)

Taking log both sides:

$\log y = \log(\log x)^{\log x} = \log x \log(\log x)$

Apply derivation:

$$\frac{d}{dx}(\log y) = \frac{d}{dx}\left(\log x \log(\log x)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \log x \frac{d}{dx}\log(\log x) + \log(\log x)\frac{d}{dx}\log x$$

$$\frac{1}{y}\frac{dy}{dx} = \log x \frac{1}{\log x}\frac{d}{dx}(\log x) + \log(\log x)\frac{1}{x}$$

$= \frac{1}{x} + \frac{\log(\log x)}{x}$

$$\frac{dy}{dx} = y\left(\frac{1+\log(\log x)}{x}\right)$$

= $(\log x)^{\log x}\left(\frac{1+\log(\log x)}{x}\right)$

8. $\cos(a\cos x+b\sin x)$ **for some constants a and b.**

Solution: Consider $y=\cos(a\cos x+b\sin x)$ for some constants a and b.
Apply derivation:

$$\frac{dy}{dx}=-\sin(a\cos x+b\sin x)\frac{d}{dx}(a\cos x+b\sin x)$$

$$\frac{dy}{dx}=-\sin(a\cos x+b\sin x)(-a\sin x+b\cos x)$$

$$\frac{dy}{dx}=-(-a\sin x+b\cos x)\sin(a\cos x+b\sin x)$$

$$\frac{dy}{dx}=(-a\sin x-b\cos x)\sin(a\cos x+b\sin x)$$

9. $(\sin x-\cos x)^{\sin x-\cos x}, \frac{\pi}{4}<x<\frac{3\pi}{4}$

Solution: Consider $y=(\sin x-\cos x)^{\sin x-\cos x}$ ……….(i)
Apply log both sides:

$$\log y=\log(\sin x-\cos x)^{\sin x-\cos x}$$

= $(\sin x-\cos x)\log(\sin x-\cos x)$

Apply derivation:

$$\frac{d}{dx}\log y=(\sin x-\cos x)\frac{d}{dx}(\sin x-\cos x)+\log(\sin x-\cos x)\frac{d}{dx}(\sin x-\cos x)$$

$$\frac{1}{y}\frac{dy}{dx}=(\sin x-\cos x)\frac{1}{(\sin x-\cos x)}\frac{d}{dx}(\sin x-\cos x)+\log(\sin x-\cos x).(\cos x+\sin x)$$

$$\frac{1}{y}\frac{dy}{dx}=(\cos x+\sin x)+(\cos x+\sin x)\log(\sin x-\cos x)$$

$$\frac{1}{y}\frac{dy}{dx}=(\cos x+\sin x)\left[1+\log(\sin x-\cos x)\right]$$

$$\frac{dy}{dx}=y(\cos x+\sin x)\left[1+\log(\sin x-\cos x)\right]$$

$$\frac{dy}{dx}=(\sin x-\cos x)^{\sin x-\cos x}(\cos x+\sin x)\left[1+\log(\sin x-\cos x)\right]$$

10. $x^x+x^a+a^x+a^a$, **for some fixed a> 0 and x>0.**

Solution: Consider $y=x^x+x^a+a^x+a^a$

Apply derivation:

$$\frac{dy}{dx}=\frac{d}{dx}x^x+\frac{d}{dx}x^a+\frac{d}{dx}a^x+\frac{d}{dx}a^a$$

$$=\frac{d}{dx}x^x+ax^{a-1}+a^x\log a+0 \quad \text{.......(i)}$$

First term from equation (i) :

$\frac{d}{dx}(x^x)$, Consider $u=x^x$

$\log u=\log x^x = x\log x$

$$\frac{d}{dx}\log u=\frac{d}{dx}(x\log x)$$

$$\frac{1}{u}\frac{du}{dx}=x\frac{d}{dx}\log x+\log x\frac{d}{dx}x$$

$$\frac{1}{u}\frac{du}{dx}=x\frac{1}{x}+\log x.1$$

$$=1+\log x$$

This implies,

$$\frac{du}{dx}=u(1+\log x)$$

Substitute value of u back:

$$\frac{d}{dx}x^x = x^x(1+\log x) \quad \text{...(ii)}$$

Using equation (ii) in (i), we have

$$\frac{dy}{dx} = x^x(1+\log x)ax^{a-1} + a^x \log a$$

11. $x^{x^2-3} + (x-3)^{x^2}$ **for x>3.**

Solution: Consider $y = x^{x^2-3} + (x-3)^{x^2}$ for x>3.

Put $u = x^{x^2-3}$ and $v = (x-3)^{x^2}$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{..........(i)}$$

Now $u = x^{x^2-3}$

$$\log u = \log x^{x^2-3} = (x^2-3)\log x$$

$$\frac{1}{u}\frac{du}{dx} = (x^2-3)\frac{d}{dx}\log x + \log x \frac{d}{dx}(x^2-3)$$

$$= (x^2-3)\frac{1}{x} + \log x(2x-0)$$

$$\frac{1}{u}\frac{du}{dx} = \frac{x^2-3}{x} + 2x\log x$$

$$\frac{du}{dx} = u\left(\frac{x^2-3}{x} + 2x\log x\right)$$

$$\frac{du}{dx} = x^{x^2-3}\left(\frac{x^2-3}{x} + 2x\log x\right) \quad \text{..........(ii)}$$

Again $v=(x-3)^{x^2}$

$\log v=\log(x-3)^{x^2}$

$= x^2\log(x-3)$

$\frac{1}{v}\frac{dv}{dx}=x^2\frac{d}{dx}\log(x-3)+\log(x-3)\frac{d}{dx}x^2$

$= x^2\frac{1}{x-3}\frac{d}{dx}(x-3)+\log(x-3)2x$

$\frac{1}{v}\frac{dv}{dx}=\frac{x^2}{x-3}+2x\log(x-3)$

$\frac{dv}{dx}=v\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$

$\frac{dv}{dx}=(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$ ……….(iii)

Using equation (ii) and (iii) in eq. (i), we have

$\frac{dy}{dx}=x^{x^2-3}\left(\frac{x^2-3}{x}+2x\log x\right)+(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$

12. Find $\frac{dy}{dx}$ **if** $y=12(1-\cos t)$ **and** $x=10(t-\sin t), -\frac{\pi}{2}<t<\frac{\pi}{2}.$

Solution: Given expressions are $y=12(1-\cos t)$ and $x=10(t-\sin t)$

$\frac{dy}{dt}=12\frac{d}{dt}(1-\cos t)=12(0+\sin t)=12\sin t$

and $\frac{dx}{dt}=10\frac{d}{dt}(1-\cos t)$

$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{12\sin t}{10(1-\cos t)}$

= $\frac{6}{5}.\frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}}$

= $\frac{6}{5}.\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}$ = $\frac{6}{5}\cot\frac{t}{2}$

13. Find $\frac{dy}{dx}$ **if** $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}, -1\le x\le 1.$

Solution: Given expression is $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}$

Apply derivation:

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}\frac{d}{dx}\sqrt{1-x^2}$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}\frac{1}{2}(1-x^2)^{\frac{-1}{2}}\frac{d}{dx}(1-x^2)$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-1+x^2}}\frac{1}{2\sqrt{1-x^2}}(-2x)$$

= $\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{x^2}}\left(\frac{-x}{\sqrt{1-x^2}}\right)$

Which implies:

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-\frac{x}{x\sqrt{1-x^2}}$$

= $\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}$ = 0

Therefore, dy/dx = 0

14.

If $x\sqrt{1+y}+y\sqrt{1+x}=0,$ **for** $-1<x<1,$

Prove that $\frac{dy}{dx}=\frac{-1}{(1+x)^2}.$

Solution: Given expression is $x\sqrt{1+y}+y\sqrt{1+x}=0$

$x\sqrt{1+y}=-y\sqrt{1+x}$

Squaring both sides:

$x^2(1+y)=y^2(1+x)$

$x^2+x^2y=y^2+y^2x$

$x^2-y^2=-x^2y+y^2x$

$(x-y)(x+y)=-xy(x-y)$

$x+y=-xy$

=> $y(1+x)=-x$

=> $y=\frac{-x}{1+x}$

Apply derivation:

$$\frac{dy}{dx}=-\frac{(1+x)\frac{d}{dx}x-x\frac{d}{dx}(1+x)}{(1+x)^2}$$

= $-\frac{(1+x).1-x.1}{(1+x)^2}$

= $-\frac{1}{(1+x)^2}$

Hence Proved.

15.

If $(x-a)^2+(y-b)^2=c^2,$ **for some c>0, prove that**

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

is a constant independent of a and b.

Solution: Given expression is $(x-a)^2+(y-b)=c^2$ ……….(1)

Apply derivation:

$$2(x-a)+2(y-b)\frac{dy}{dx}=0$$

$$2(x-a)=-2(y-b)\frac{dy}{dx}$$

$$\frac{dy}{dx}=-\left(\frac{x-a}{y-b}\right)$$ ……….(2)

Again $$\frac{d^2y}{dx^2}=\frac{-\left[(y-b).1-(x-a)\frac{dy}{dx}\right]}{(y-b)^2}$$

$$\frac{d^2y}{dx^2}=\frac{-\left[(y-b).1-(x-a)\left(\frac{-(x-a)}{y-b}\right)\right]}{(y-b)^2}$$ [Using equation (2)]

$$\frac{d^2y}{dx^2}=\frac{-\left[(y-b)+\left(\frac{(x-a)^2}{y-b}\right)\right]}{(y-b)^2}$$

$$=\frac{-\left[(y-b)^2+(x-a)^2\right]}{(y-b)^3}$$

$$=\frac{-c^2}{(y-b)^3}$$ ……….(3)

Put values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given, we get

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{\frac{-c^2}{(y-b)^3}}$$

$$= \frac{\left[(y-b)^2+(x-a)^2\right]^{\frac{3}{2}}}{(y-b)^3} \times \frac{(y-b)^3}{-c^2} = \frac{(c^2)^{\frac{3}{2}}}{-c^2} = -c \text{ (Constant value)}$$

Which is a constant and is independent of a and b.

16. If $\cos y = x\cos(a+y)$ **with** $\cos a \neq \pm 1$, **prove that** $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$.

Solution: Given expression is $\cos y = x\cos(a+y)$

$$x = \frac{\cos y}{\cos(a+y)}$$

Apply derivative w.r.t. y

$$\frac{dx}{dy} = \frac{d}{dy}\left(\frac{\cos y}{\cos(a+y)}\right)$$

$$\frac{dx}{dy} = \frac{\cos(a+y)\frac{d}{dy}\cos y - \cos y\frac{d}{dy}\cos(a+y)}{\cos^2(a+y)}$$

$$\frac{dx}{dy}=\frac{\cos(a+y)(-\sin y)-\cos y\{-\sin(a+y)\}}{\cos^2(a+y)}$$

$$=\frac{-\cos(a+y)\sin y+\sin(a+y)\cos y}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dx}{dy}=\frac{\sin(a+y-y)}{\cos^2(a+y)}$$

$$=\frac{\sin a}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$$ [Take reciprocal]

17. If $x=a(\cos t+t\sin t)$ **and** $y=a(\sin t-t\cos t)$, **find** $\frac{d^2y}{dx^2}$.

Solution: Given expressions are $x=a(\cos t+t\sin t)$ and $y=a(\sin t-t\cos t)$

$x=a(\cos t+t\sin t)$

Differentiating both sides w.r.t. t

$$\frac{dx}{dt}=a\left(-\sin t+\frac{d}{dt}t\sin t\right)$$

$$\frac{dx}{dt}=a\left(-\sin t+t\frac{d}{dt}\sin t+\sin t\frac{d}{dt}t\right)$$

$$\frac{dx}{dt}=a(-\sin t+t\cos t+\sin t)$$

$$\Rightarrow \frac{dx}{dt}=at\cos t$$

And:

$y = a(\sin t - t\cos t)$

Differentiating both sides w.r.t. t

$$\frac{dy}{dt} = a\left(\cos t - \frac{d}{dt} t\cos t\right)$$

$$\frac{dy}{dt} = a\left(\cos t - \left(t\frac{d}{dt}\cos t + \cos t\frac{d}{dt}t\right)\right)$$

$$\frac{dy}{dt} = a(\cos t - (-t\sin t + \cos t))$$

$$\frac{dy}{dt} = at\sin t$$

Now $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{at\sin t}{at\cos t} = \frac{\sin t}{\cos t} = \tan t$

Again $\frac{d^2y}{dx^2} = \frac{d}{dx}\tan t$ = $\sec^2 t\frac{d}{dx}t$

= $\sec^2 t\frac{dt}{dx}$ = $\sec^2 t\frac{1}{at\cos t}$

= $\sec^2 t\frac{\sec t}{at} = \frac{\sec^3 t}{at}$

18. If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x and find it.

Solution: Given expression is $f(x) = |x^3|$ = $\begin{cases} x^3, & \text{if} \quad x \geq 0 \\ (-x^3), & \text{if} \quad x < 0 \end{cases}$

Step 1: when x < 0

f(x) = -x^3

Differentiate w.r.t. to x,

f'(x) = -3x^2

Differentiate w.r.t. to x,

f''(x) = -6x, exist for all values of x < 0.

Step 2: When x ≥ 0

f(x) = x^3

Differentiate w.r.t. to x,

f'(x) = 3x^2

Differentiate w.r.t. to x,

f''(x) = 6x, exist for all values of x > 0.

Step 3: When x = 0

$$\lim_{h\to 0^-} \frac{f(0)-f(0+h)}{h} = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = f'(c)$$

$$f'(x) = \begin{cases} 3x^2, & \text{if} \quad x \geq 0 \\ -3x^2, & \text{if} \quad x < 0 \end{cases}$$

Now, Check differentiability at x = 0

L.H.D. at x = 0

$$\lim_{h\to 0^-} \frac{f'(0)-f'(0+h)}{h}$$

$$= \lim_{h\to 0^-} \frac{3(0)-(-3(-h)^2)}{h}$$

$$= \lim_{h\to 0^-} \frac{3h^2}{h}$$

As h = 0,

= 0

And R.H.D. at x = 0

$$\lim_{h\to 0^+} \frac{f'(0+h)-f'(0)}{h}$$

$$= \lim_{h\to 0^+} \frac{f'(h)-f'(0)}{h}$$

$= \lim_{h\to 0^+} \frac{3(h)^2-3(0)^2}{h}$

$= \lim_{h\to 0^+} 3h = 0$ (at h = 0)

Again L.H.D. at $x = 0$ = R.H.D. at x = 0.

This implies, f''(x) exists and differentiable at all real values of x.

19. Using mathematical induction, prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n.

Solution: Consider p(n) be the given statement.

$p(n) = \frac{d}{dx}(x^n) = nx^{n-1}$ …..(1)

Step 1: Result is true at n = 1

$p(1) = \frac{d}{dx}(x^1) = (1)x^{1-1}$ = $(1)x^0 = 1$,

which is true as $\frac{d}{dx}(x) = 1$

Step 2: Suppose $p(m)$ is true.

$p(m) = \frac{d}{dx}(x^m) = mx^{m-1}$ …….(2)

Step 3: Prove that result is true for n = m+1.

$p(m+1) = \frac{d}{dx}(x^{m+1}) = (m+1)x^{m+1-1}$

$x^{m+1} = x^1 + x^m$

$\frac{d}{dx}x^{m+1} = \frac{d}{dx}(x.x^m)$

= $x.\frac{d}{dx}x^m + x^m\frac{d}{dx}x$

= $x.mx^{m-1} + x^m(1)$

Therefore, $mx^m + x^m = x^m(m+1)$

$(m+1)x^m = (m+1)x^m$

$(m+1)x^{(m+1)-1}$

Therefore, $p(m+1)$ is true if $p(m)$ is true but $p(1)$ is true.

Thus, by Principal of Induction p(n) is true for all n $\in$ N.

20. Using the fact that $\sin(A+B) = \sin A \cos B + \cos A \sin B$ and the differentiation, obtain the sum formula for cosines.

Solution: Given expression is $\sin(A+B) = \sin A \cos B + \cos A \sin B$
Consider A and B as function of t and differentiating both sides w.r.t. x,

$$\cos(A+B)\left(\frac{dA}{dt}+\frac{dB}{dt}\right) = \sin A(-\sin B)\frac{dB}{dt} + \cos B\left(\cos A\frac{dA}{dt}\right) + \cos A\cos B\frac{dB}{dt} + \sin B(-\sin A)\frac{dA}{dt}$$

$$\Rightarrow \cos(A+B)\left(\frac{dA}{dt}+\frac{dB}{dt}\right) = (\cos A\cos B - \sin A\sin B)\left(\frac{dA}{dt}+\frac{dB}{dt}\right)$$

$$\Rightarrow \cos(A+B) = (\cos A\cos B - \sin A\sin B)$$

21. Does there exist a function which is continuous everywhere but not differentiable at exactly two points?

Solution: Consider us consider the function $f(x) = |x| + |x-1|$
f is continuous everywhere but it is not differentiable at x = 0 and x = 1.

22. If $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, prove that

$$\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}.$$

Solution: Given expression is

$$y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Apply derivative:

$$\frac{dy}{dx} = \begin{vmatrix} \frac{d}{dx}f(x) & \frac{d}{dx}g(x) & \frac{d}{dx}h(x) \\ l & m & n \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ 0 & 0 & 0 \\ a & b & c \end{vmatrix} + y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ 0 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

23. If $y = e^{a\cos^{-1}x}, -1 \le x \le 1,$ **show that** $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0.$

Solution: Given expression is $y = e^{a\cos^{-1}x}$

$$\frac{dy}{dx} = e^{a\cos^{-1}x}.\frac{d}{dx}a\cos^{-1}x$$

$$= e^{a\cos^{-1}x}.a\left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$$= \frac{-ay}{\sqrt{1-x^2}}$$

This implies,

$$\left(\frac{dy}{dx}\right)^2 = \frac{a^2y^2}{1-x^2}$$

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

Differentiating both sides with respect to x, we have

$$\left(1-x^2\right)2.\frac{dy}{dx}.\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\left(-2x\right)=2a^2y\frac{dy}{dx}$$

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}=a^2y$$

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-a^2y=0$$

Hence Proved.