

Exercise 2.2

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Prove the following

1.

$$3 \sin^{-1} x = \sin^{-1} (3x - 4x^3), x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

Solution:

$$3 \sin^{-1} x = \sin^{-1} (3x - 4x^3), x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

(Use identity: $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$)

Let $x = \sin \theta$. then

$$\theta = \sin^{-1} x$$

Now, RHS

$$= \sin^{-1} (3x - 4x^3)$$

$$= \sin^{-1} (3 \sin \theta - 4 \sin^3 \theta)$$

$$= \sin^{-1} (\sin 3\theta)$$

$$= 3\theta$$

$$= 3 \sin^{-1} x$$

$$= \text{LHS}$$

Hence Proved

2.

$$3 \cos^{-1} x = \cos^{-1} (4x^3 - 3x), x \in \left[\frac{1}{2}, 1\right]$$

Solution:

$$3 \cos^{-1} x = \cos^{-1} (4x^3 - 3x), x \in \left[\frac{1}{2}, 1 \right]$$

Using identity: $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$

Put $x = \cos \theta$

$$\theta = \cos^{-1} (x)$$

Therefore, $\cos 3\theta = 4x^3 - 3x$

RHS:

$$\begin{aligned} & \cos^{-1} (4x^3 - 3x) \\ &= \cos^{-1} (\cos 3\theta) \end{aligned}$$

$$= 3\theta$$

$$= 3 \cos^{-1} (x)$$

$$= \text{LHS}$$

Hence Proved.

3.

$$\tan^{-1} \frac{2}{11} + \tan^{-1} \frac{7}{24} = \tan^{-1} \frac{1}{2}$$

Solution:

$$\tan^{-1} \frac{2}{11} + \tan^{-1} \frac{7}{24} = \tan^{-1} \frac{1}{2}$$

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$$

Using identity:

$$\text{LHS} = \tan^{-1} \frac{2}{11} + \tan^{-1} \frac{7}{24}$$

$$= \tan^{-1} \frac{\frac{2}{11} + \frac{7}{24}}{1 - \frac{2}{11} \times \frac{7}{24}}$$

$$= \tan^{-1} \frac{48+77}{264-14}$$

$$= \tan^{-1} (125/250)$$

$$= \tan^{-1} (1/2)$$

$$= \text{RHS}$$

Hence Proved

4.

$$2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{31}{17}$$

Solution:

Use identity: $2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2}$

LHS

$$= 2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7}$$

$$= \tan^{-1} \frac{2 \times \frac{1}{2}}{1 - \left(\frac{1}{2}\right)^2} + \tan^{-1} \frac{1}{7}$$

$$= \tan^{-1}(4/3) + \tan^{-1}(1/7)$$

Again using identity:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$$

We have,

$$\tan^{-1} \frac{\frac{4}{3} + \frac{1}{7}}{1 - \frac{4}{3} \times \frac{1}{7}}$$

$$= \tan^{-1} \left(\frac{28+3}{21-4} \right)$$

$$= \tan^{-1} (31/17)$$

RHS

Write the following functions in the simplest form:

5. $\tan^{-1} \frac{\sqrt{1+x^2}-1}{x}, x \neq 0$

Solution:

Let's say $x = \tan \theta$ then $\theta = \tan^{-1} x$

We get,

$$\tan^{-1} \frac{\sqrt{1+x^2}-1}{x} = \tan^{-1} \left(\frac{\sqrt{1+\tan^2 \theta}-1}{\tan \theta} \right)$$

$$\begin{aligned}
 &= \tan^{-1} \left(\frac{\sec \theta - 1}{\tan \theta} \right) \\
 &= \tan^{-1} \left(\frac{1 - \cos \theta}{\sin \theta} \right) \\
 &= \tan^{-1} \left(\frac{2 \sin^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \right) \\
 &= \tan^{-1} \left(\tan \frac{\theta}{2} \right) = \frac{\theta}{2} = \frac{1}{2} \tan^{-1} x
 \end{aligned}$$

This is simplest form of the function.

6. $\tan^{-1} \frac{1}{\sqrt{x^2 - 1}}, |x| > 1$

Solution:

Let us consider, $x = \sec \theta$, then $\theta = \sec^{-1} x$

$$\begin{aligned}
 \tan^{-1} \frac{1}{\sqrt{x^2 - 1}} &= \tan^{-1} \frac{1}{\sqrt{\sec^2 \theta - 1}} \\
 &= \tan^{-1} \frac{1}{\sqrt{\tan^2 \theta}} \\
 &= \tan^{-1} \left(\frac{1}{\tan \theta} \right)
 \end{aligned}$$

$$= \tan^{-1}(\cot \theta)$$

$$= \tan^{-1} \tan(\pi/2 - \theta)$$

$$= (\pi/2 - \theta)$$

$$= \pi/2 - \sec^{-1} x$$

This is simplest form of the given function.

7. $\tan^{-1} \left(\sqrt{\frac{1 - \cos x}{1 + \cos x}} \right), 0 < x < \pi$

Solution:

$$\begin{aligned} \tan^{-1} \left(\sqrt{\frac{1 - \cos x}{1 + \cos x}} \right) &= \tan^{-1} \left(\sqrt{\frac{2 \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}}} \right) \\ &= \tan^{-1} \left(\frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \right) \\ &= \tan^{-1} \left(\tan \frac{x}{2} \right) = \frac{x}{2} \end{aligned}$$

8. $\tan^{-1} \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right), -\frac{\pi}{4} < x < \frac{3\pi}{4}$

Solution:

Divide numerator and denominator by $\cos x$, we have

$$\begin{aligned} \tan^{-1} \left(\frac{\frac{\cos(x)}{\cos(x)} - \frac{\sin(x)}{\cos(x)}}{\frac{\cos(x)}{\cos(x)} + \frac{\sin(x)}{\cos(x)}} \right) \\ &= \tan^{-1} \left(\frac{1 - \frac{\sin(x)}{\cos(x)}}{1 + \frac{\sin(x)}{\cos(x)}} \right) \\ &= \tan^{-1} \left(\frac{1 - \tan x}{1 + \tan x} \right) \end{aligned}$$

$$\tan^{-1} \left(\frac{\tan \frac{\pi}{4} - \tan x}{1 + \tan \frac{\pi}{4} \tan x} \right)$$

$$= \tan^{-1} \tan(\pi/4 - x)$$

$$= \pi/4 - x$$

9. $\tan^{-1} \frac{x}{\sqrt{a^2 - x^2}}, |x| < a$

Solution:

Put $x = a \sin \theta$, which implies $\sin \theta = x/a$ and $\theta = \sin^{-1}(x/a)$

Substitute the values into given function, we get

$$\tan^{-1} \frac{x}{\sqrt{a^2 - x^2}} = \tan^{-1} \left(\frac{a \sin \theta}{\sqrt{a^2 - a^2 \sin^2 \theta}} \right)$$

$$= \tan^{-1} \left(\frac{a \sin \theta}{a \sqrt{1 - \sin^2 \theta}} \right)$$

$$= \tan^{-1} \left(\frac{a \sin \theta}{a \cos \theta} \right)$$

$$= \tan^{-1}(\tan \theta)$$

$$= \theta$$

$$= \sin^{-1}(x/a)$$

10. $\tan^{-1} \left(\frac{3a^2x - x^3}{a^3 - 3ax^2} \right), a > 0; \frac{-a}{\sqrt{3}} < x < \frac{a}{\sqrt{3}}$

Solution:

After dividing numerator and denominator by a^3 we have

$$\tan^{-1} \left(\frac{3 \left(\frac{x}{a} \right) - \left(\frac{x}{a} \right)^3}{1 - 3 \left(\frac{x}{a} \right)^2} \right)$$

Put $x/a = \tan \theta$ and $\theta = \tan^{-1}(x/a)$

$$= \tan^{-1} \left(\frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} \right)$$

$$= \tan^{-1} (\tan 3 \theta)$$

$$= 3 \theta$$

$$= 3 \tan^{-1}(x/a)$$

Find the values of each of the following:

11. $\tan^{-1} \left[2 \cos \left(2 \sin^{-1} \frac{1}{2} \right) \right]$

Solution:

$$= \tan^{-1} \left[2 \cos \left(2 \sin^{-1} \sin \frac{\pi}{6} \right) \right]$$

$$= \tan^{-1} \left[2 \cos \left(2 \times \frac{\pi}{6} \right) \right]$$

$$= \tan^{-1} (2 \cos \pi/3)$$

$$= \tan^{-1} (2 \times \frac{1}{2})$$

$$= \tan^{-1} (1)$$

$$= \tan^{-1} (\tan (\pi/4))$$

$$= \pi/4$$

12. $\cot (\tan^{-1}a + \cot^{-1}a)$

Solution:

$$\cot (\tan^{-1}a + \cot^{-1}a) = \cot \pi/2 = 0$$

Using identity: $\tan^{-1}a + \cot^{-1}a = \pi/2$

13.

$$\tan \frac{1}{2} \left[\sin^{-1} \frac{2x}{1+x^2} + \cos^{-1} \frac{1-y^2}{1+y^2} \right], |x| < 1, y > 0 \text{ and } xy < 1$$

Solution:

Put $x = \tan \theta$ and $y = \tan \phi$, we have

$$\tan \frac{1}{2} \left[\sin^{-1} \frac{2 \tan \theta}{1 + \tan^2 \theta} + \cos^{-1} \frac{1 - \tan^2 \phi}{1 + \tan^2 \phi} \right]$$

$$= \tan \frac{1}{2} [\sin^{-1} \sin 2\theta + \cos^{-1} \cos 2\phi]$$

$$= \tan \frac{1}{2} [2\theta + 2\phi]$$

$$= \tan (\theta + \phi)$$

$$= \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi}$$

$$= (x+y) / (1-xy)$$

14. If $\sin \left(\sin^{-1} \frac{1}{5} + \cos^{-1} x \right) = 1$, then find the value of x.

Solution:

We know that, $\sin 90 \text{ degrees} = \sin \pi/2 = 1$

So, given equation turned as,

$$\sin^{-1} \frac{1}{5} + \cos^{-1} x = \frac{\pi}{2}$$

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} \frac{1}{5}$$

Using identity: $\sin^{-1} t + \cos^{-1} t = \pi/2$

$$\cos^{-1} x = \cos^{-1} \frac{1}{5}$$

We have,

Which implies, the value of x is $1/5$.

15. If $\tan^{-1} \frac{x-1}{x-2} + \tan^{-1} \frac{x+1}{x+2} = \frac{\pi}{4}$, then find the value of x .

Solution:

We have reduced the given equation using below identity:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$$

$$\tan^{-1} \frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)} = \frac{\pi}{4}$$

or

$$\tan^{-1} \frac{(x-1)(x+2) + (x+1)(x-2)}{(x-2)(x+2) - (x-1)(x+1)} = \frac{\pi}{4}$$

or

$$\tan^{-1} \frac{x^2 + 2x - x - 2 + x^2 - 2x + x - 2}{x^2 - 4 - (x^2 - 1)} = \frac{\pi}{4}$$

$$\text{or } \frac{2x^2 - 4}{x^2 - 4 - x^2 + 1} = \tan\left(\frac{\pi}{4}\right)$$

$$\text{or } (2x^2 - 4)/-3 = 1$$

$$\text{or } 2x^2 = 1$$

$$\text{or } x = \pm \frac{1}{\sqrt{2}}$$

The value of x is either $\frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$

Find the values of each of the expressions in Exercises 16 to 18.

16. $\sin^{-1}(\sin(\frac{2\pi}{3}))$

Solution:

Given expression is $\sin^{-1}(\sin(\frac{2\pi}{3}))$

First split $\frac{2\pi}{3}$ as $\frac{(3\pi-\pi)}{3}$ or $\pi - \frac{\pi}{3}$

After substituting in given we get,

$$\sin^{-1}(\sin(\frac{2\pi}{3})) = \sin^{-1}(\sin(\pi - \frac{\pi}{3})) = \frac{\pi}{3}$$

Therefore, the value of $\sin^{-1}(\sin(\frac{2\pi}{3}))$ is $\frac{\pi}{3}$

17. $\tan^{-1}(\tan(\frac{3\pi}{4}))$

Solution:

Given expression is $\tan^{-1}(\tan(\frac{3\pi}{4}))$

First split $\frac{3\pi}{4}$ as $\frac{(4\pi-\pi)}{4}$ or $\pi - \frac{\pi}{4}$

After substituting in given we get,

$$\tan^{-1}(\tan(\frac{3\pi}{4})) = \tan^{-1}(\tan(\pi - \frac{\pi}{4})) = -\frac{\pi}{4}$$

The value of $\tan^{-1}(\tan(\frac{3\pi}{4}))$ is $-\frac{\pi}{4}$.

18. $\tan(\sin^{-1}(\frac{3}{5})) + \cot^{-1}(\frac{3}{2})$

Solution:

Given expression is $\tan(\sin^{-1}(\frac{3}{5})) + \cot^{-1}(\frac{3}{2})$

Putting, $\sin^{-1}(\frac{3}{5}) = x$ and $\cot^{-1}(\frac{3}{2}) = y$

Or $\sin(x) = 3/5$ and $\cot y = 3/2$

Now, $\sin(x) = 3/5 \Rightarrow \cos x = \sqrt{1 - \sin^2 x} = 4/5$ and $\sec x = 5/4$

(using identities: $\cos x = \sqrt{1 - \sin^2 x}$ and $\sec x = 1/\cos x$)

Again, $\tan x = \sqrt{\sec^2 x - 1} = \sqrt{\frac{25}{16} - 1} = 3/4$ and $\tan y = 1/\cot(y) = 2/3$

Now, we can write given expression as,

$$\tan(\sin^{-1}(\frac{3}{5}) + \cot^{-1}(\frac{3}{2})) = \tan(x + y)$$

$$= \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{3}{4} \times \frac{2}{3}}$$

$$= 17/6$$

19. $\cos^{-1}(\cos \frac{7\pi}{6})$ is equal to

- (A) $7\pi/6$ (B) $5\pi/6$ (C) $\pi/3$ (D) $\pi/6$

Solution:

Option (B) is correct.

Explanation:

$$\cos^{-1}(\cos \frac{7\pi}{6}) = \cos^{-1}(\cos (2\pi - \frac{7\pi}{6}))$$

(As $\cos (2\pi - A) = \cos A$)

$$\text{Now } 2\pi - \frac{7\pi}{6} = \frac{12\pi - 7\pi}{6} = \frac{5\pi}{6}$$

20. $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$ is equal to

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) 1

Solution:

Option (D) is correct

Explanation:

First solve for: $\sin^{-1}\left(-\frac{1}{2}\right)$

$$\sin^{-1}\left(-\frac{1}{2}\right) = \sin^{-1}\left(-\sin\frac{\pi}{6}\right) = \sin^{-1}\left[\sin\left(-\frac{\pi}{6}\right)\right]$$

$$= -\pi/6$$

Again,

$$\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$$

$$= \sin\left[\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right]$$

$$= \sin\left[\frac{\pi}{3} + \frac{\pi}{6}\right]$$

$$= \sin(\pi/2)$$

$$= 1$$

21. $\tan^{-1} \sqrt{3} - \cot^{-1} (-\sqrt{3})$ is equal to

- (A) π (B) $-\pi/2$ (C) 0 (D) $2\sqrt{3}$

Solution:

Option (B) is correct.

Explanation:

$\tan^{-1} \sqrt{3} - \cot^{-1} (-\sqrt{3})$ can be written as

$$= \tan^{-1} \tan \frac{\pi}{3} - \cot^{-1} \left(-\cot \frac{\pi}{6} \right)$$

$$= \frac{\pi}{3} - \cot^{-1} \left[\cot \left(\pi - \frac{\pi}{6} \right) \right]$$

$$= \frac{\pi}{3} - \left(\pi - \frac{\pi}{6} \right)$$

$$= \frac{\pi}{3} - \frac{5\pi}{6}$$

$$= \frac{-3\pi}{6}$$

$$= -\pi/2$$