Miscellaneous Exercise

Find the value of the following:

1. $\cos^{-1}(\cos\frac{13\pi}{6})$

Solution:

First solve for, $\cos\frac{13\pi}{6} = \cos(2\pi + \frac{\pi}{6}) = \cos\frac{\pi}{6}$

Now: $\cos^{-1}(\cos\frac{13\pi}{6}) = \cos^{-1}(\cos\frac{\pi}{6}) = \frac{\pi}{6} \in [0, \pi]$

[As $\cos^{-1}\cos(x) = x$ if $x \in [0, \pi]$]

So the value of $\cos^{-1}(\cos\frac{13\pi}{6})$ is $\frac{\pi}{6}$.

2. $tan^{-1}(\tan\frac{7\pi}{6})$

Solution:

First solve for, $\tan\frac{7\pi}{6} = \tan(\pi + \frac{\pi}{6}) = \tan\frac{\pi}{6}$

Now: $tan^{-1}(\tan\frac{7\pi}{6}) = \tan^{-1}(\tan\frac{\pi}{6}) = \frac{\pi}{6} \in (-\pi/2, \pi/2)$

[As $\tan^{-1}\tan(x) = x$ if $x \in (-\pi/2, \pi/2)$]

So the value of $tan^{-1}(\tan\frac{7\pi}{6})$ is $\frac{\pi}{6}$.

3. Prove that $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$

Solution:
Step 1: Find the value of cos x and tan x

Let us consider $\sin^{-1}\frac{3}{5} = x$, then sin x = 3/5

So, cos x = $\sqrt{1-\sin^2 x} = \sqrt{1-\left(\frac{3}{5}\right)^2}$ = 4/5 tan x = sin x/ cos x = ¾

Therefore, x = $\tan^{-1}$ (3/4), substitute the value of x,

⇨ $\sin^{-1}\frac{3}{5} = \tan^{-1}\left(\frac{3}{4}\right)$ …..(1)

Step 2: Solve LHS

$2\sin^{-1}\frac{3}{5} = 2\tan^{-1}\frac{3}{4}$

Using identity: 2$\tan^{-1}$ x = $\tan^{-1} = \tan^{-1}(\frac{2x}{1-x^2})$, we get

= $\tan^{-1}(\frac{2(\frac{3}{4})}{1-\left(\frac{3}{4}\right)^2})$

= $\tan^{-1}$(24/7)

= RHS

Hence Proved.

4. Prove that $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$

Solution:

Let $\sin^{-1}\left(\frac{8}{17}\right) = x$ then sin x = 8/17

Again, cos x = $\sqrt{1-\sin^2 x}$ = $\sqrt{1-\frac{64}{289}}$ = 15/17

And tan x = sin x / cos x = 8/ 15

Again,

Let $\sin^{-1}\left(\frac{3}{5}\right) = y$ then sin y = 3/5

Again, cos y = $\sqrt{1-\sin^2 y}$ = $\sqrt{1-\frac{9}{25}}$ = 4/5

And tan y = sin y / cos y = ¾

Solve for tan(x + y), using below identity,

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$= \frac{\frac{8}{15}+\frac{3}{4}}{1-\frac{8}{15}\times\frac{3}{4}}$$

$= \frac{32+45}{60-24}$

= 77/36

This implies x + y = $\tan^{-1}$(77/36)

Resubstituting the values, we have

$\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$ (Proved)

5. Prove that $\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$

Solution:

Let $\cos^{-1}\frac{4}{5} = \theta$

$$\cos\theta = \frac{4}{5}$$

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

$$= \sqrt{1-\frac{16}{25}}$$

$$= \frac{3}{5}$$

Let $\cos^{-1}\frac{12}{13} = \phi$

$$\cos\phi = \frac{12}{13}$$

$$\sin\phi = \sqrt{1-\cos^2\phi}$$

$$= \sqrt{1-\frac{144}{169}}$$

$$= \frac{5}{13}$$

Solve the expression, Using identity: cos (θ + ϕ) = cosθ cos ϕ - sinθ sin ϕ

= 4/5 x 12/13 – 3/5 x 5/13

= (48-15)/65

= 33/65

This implies cos (θ + ϕ) = 33/65

or θ + ϕ = $\cos^{-1}$ (33/65)

Putting back the value of θ and ϕ, we get

$$\cos^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{12}{13}\right)=\cos^{-1}\left(\frac{33}{65}\right)$$

Hence Proved.

6. Prove that $\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(\frac{56}{65}\right)$

Solution:

Let $\cos^{-1}\frac{12}{13}=\theta$

So $\cos\theta=\frac{12}{13}$

$\sin\theta=\sqrt{1-\cos^2\theta}$

$=\sqrt{1-\frac{144}{169}}$

$=\frac{5}{13}$

Let $\sin^{-1}\frac{3}{5}=\phi$

So $\sin\phi=\frac{3}{5}$

$\cos\phi=\sqrt{1-\sin^2\phi}$

$=\sqrt{1-\frac{9}{25}}$

$=\frac{4}{5}$

Solve the expression, Using identity: sin (θ + ϕ) = sin θ cos ϕ + cos θ sin ϕ

= 12/13 x 3/5 + 12/13 x 3/5

= (20+36)/65

= 56/65

or sin (θ + ϕ) = 56/65

or θ + ϕ) = sin $^{-1}$ 56/65

Putting back the value of θ and ϕ, we get

$$\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(\frac{56}{65}\right)$$

Hence Proved.

7. Prove that $\tan^{-1}\left(\frac{63}{16}\right)=\sin^{-1}\left(\frac{5}{13}\right)+\cos^{-1}\left(\frac{3}{5}\right)$

Solution:

Let $\sin^{-1}\frac{5}{13}=\theta$

so $\sin\theta=\frac{5}{13}$

$\cos\theta=\sqrt{1-\sin^2\theta}$

$=\sqrt{1-\frac{25}{169}}$

$=\frac{12}{13}$

$\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{5}{12}$

Let $\cos^{-1}\frac{3}{5}=\phi$

so $\cos\phi=\frac{3}{5}$

$\sin\phi=\sqrt{1-\cos^2\phi}$

$=\sqrt{1-\frac{9}{25}}$

$=\frac{4}{5}$

$\tan\phi=\frac{\sin\phi}{\cos\phi}=\frac{4}{3}$

Solve the expression, Using identity:

$$\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$$

$$=\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\times\frac{4}{3}}$$

= 63/16

(θ + ϕ) = tan^{-1} (63/16)

Putting back the value of θ and ϕ, we get

$$\tan^{-1}\left(\frac{63}{16}\right) = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$$

Hence Proved.

8. Prove that $\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$

Solution:

LHS = $(\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right)) + (\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right))$

Solve above expressions, using below identity:

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right)$$

After simplifying, we have

= $\tan^{-1}(6/17) + \tan^{-1}(11/23)$

Again, applying the formula, we get

$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right)$$

After simplifying,

= $\tan^{-1}(325/325)$

= $\tan^{-1}(1)$

= π/4

9. Prove that $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\frac{1-x}{1+x}$, x ∈ (0, 1)

Solution:

Let $\tan^{-1}\sqrt{x}$ = θ , then $\sqrt{x}$ = tan θ

Squaring both the sides

$\tan^2\theta = x$

Now, substitute the value of x in $\frac{1}{2}\cos^{-1}\frac{1-x}{1+x}$, we get

$$= \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

= ½ cos-1 (cos 2 θ)

= ½ (2 θ)

= θ

$= \tan^{-1}\sqrt{x}$

10. Prove that $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}$, x ∈ (0, π/4)

Solution:

We can write 1+ sin x as,

$$1+\sin x = \cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\cos\frac{x}{2}\sin\frac{x}{2} = \left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2$$

And

$$1-\sin x = \cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\cos\frac{x}{2}\sin\frac{x}{2} = \left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2$$

LHS:

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$$

$$=\cot^{-1}\left[\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right]$$

$= \cot^{-1}(\frac{2\cos(\frac{x}{2})}{2\sin(\frac{x}{2})})$

$= \cot^{-1}(\cot(x/2)$

$= x/2$

11. Prove that $\tan^{-1}(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}) = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x,\ -\frac{1}{\sqrt{2}} \le x \le 1$
[Hint: Put x = cos 2 θ]

Solution:

Put $x=\cos 2\theta$ so, $\theta=\frac{1}{2}\cos^{-1}x$

$$\text{LHS} = \tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)$$

Divide each term by $\sqrt{2}\cos\theta$

$$= \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right)$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4}-\theta\right)$$

$$= \frac{\pi}{4}-\theta$$

$$= \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$$

= RHS

Hence proved

12. Prove that $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}=\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

Solution:

LHS = $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}$

$$= \frac{9}{4}\left(\frac{\pi}{2}-\sin^{-1}\frac{1}{3}\right)$$

$$= \frac{9}{4}\cos^{-1}\frac{1}{3}$$

…….(1)

(Using identity: $\sin^{-1}\theta+\cos^{-1}\theta=\frac{\pi}{2}$.)

Let θ = $\cos^{-1}$ (1/3), so cos θ = 1/3

As

$$\sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-\frac{1}{9}}=\sqrt{\frac{8}{9}}=\frac{2\sqrt{2}}{3}$$

Using equation (1), $\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

Which is right hand side of the expression.

Solve the following equations:

13. 2tan^{-1} (cos x) = tan^{-1} (2 cosec x)

Solution:

$$2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$$

$$\tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}\left(\frac{2}{\sin x}\right)$$

$$\frac{2\cos x}{1-\cos^2 x} = \frac{2}{\sin x}$$

$$\frac{\cos x}{\sin x} = 1$$

Cot x = 1

x = π/4

14. Solve $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1} x, (x > 0)$

Solution:

Put x = tan θ

$$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1} x$$

This implies

$$\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x$$

$$\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)=\frac{1}{2}\tan^{-1}\tan\theta$$

$$\tan^{-1}\left(\frac{\tan\frac{\pi}{4}-\tan\theta}{\tan\frac{\pi}{4}+\tan\theta}\right)=\frac{1}{2}\theta$$

$$\tan^{-1}\tan\left(\frac{\pi}{4}-\theta\right)=\frac{\theta}{2}$$

π/4 – θ = θ /2

or 3θ /2 = π/4

θ = π/6

Therefore, x = tan θ = tan π/6 = $1/\sqrt{3}$

15. $\sin\left(\tan^{-1}x\right), |x|<1$ **is equal to**

(A) $\frac{x}{\sqrt{1-x^2}}$ (B) $\frac{1}{\sqrt{1-x^2}}$

(C) $\frac{1}{\sqrt{1+x^2}}$ (D) $\frac{x}{\sqrt{1+x^2}}$

Solution:

Option (D) is correct.

Explanation:

Let $\theta=\tan^{-1}x$ so, $x=\tan\theta$

Again, Let's say

$$\sin\left(\tan^{-1}x\right)=\sin\theta$$

This implies,

$$\sin\left(\tan^{-1}x\right)=\frac{1}{\cos ec\theta}=\frac{1}{\sqrt{1+\cot^2\theta}}$$

Put $\cot\theta=\frac{1}{\tan\theta}=\frac{1}{x}$

Which shows,

$$\sin\left(\tan^{-1}x\right)=\frac{1}{\sqrt{1+\frac{1}{x^2}}}=\frac{x}{\sqrt{x^2+1}}$$

16. $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$ **then x is equal to**

(A) 0, ½ (B) 1, ½ (C) 0 (D) ½

Solution:

Option (C) is correct.

Explanation:

Put $\sin^{-1}x=\theta$ So, $x=\sin\theta$

Now,

$$\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$$

$$\sin^{-1}(1-x)-2\theta=\frac{\pi}{2}$$
$$\sin^{-1}(1-x)=\frac{\pi}{2}+2\theta$$
$$1-x=\sin\left(\frac{\pi}{2}+2\theta\right)$$
$$1-x=\cos 2\theta$$
$$1-x=1-2x^2$$

(As $x = \sin\theta$)

After simplifying, we get

$x(2x - 1) = 0$

$x = 0$ or $2x - 1 = 0$

$x = 0$ or $x = ½$

Equation is not true for $x = ½$. So the answer is $x = 0$.

17. $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)$ **is equal to**

(A) π/2 (B) π/3 (C) π/4 (D) -3 π/4

Solution:

Option (C) is correct.

Explanation:
Given expression can be written as,

$$= \tan^{-1}\left[\frac{\frac{x}{y}-\left(\frac{x-y}{x+y}\right)}{1+\frac{x}{y}\left(\frac{x-y}{x+y}\right)}\right]$$

$$= \tan^{-1}\left[\frac{x(x+y)-y(x-y)}{y(x+y)+x(x-y)}\right]$$

$$= \tan^{-1}\left(\frac{x^2+xy-xy+y^2}{xy+y^2+x^2-xy}\right)$$

$$= \tan^{-1}\left(\frac{x^2+y^2}{x^2+y^2}\right)$$

= $\tan^{-1}(1)$

= $\pi/4$