Exercise 2.1   Page No: 41

Find the principal values of the following:

1. $\sin^{-1}\left(-\frac{1}{2}\right)$

2. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

3. Cosec $^{-1}$ (2)

4. $\tan^{-1}\left(-\sqrt{3}\right)$

5. $\cos^{-1}\left(\frac{-1}{2}\right)$

6. tan^{-1}(-1)

7. $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

8. $\cot^{-1}\left(\sqrt{3}\right)$

9. $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

10. $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$

Solution 1: Consider $y = \sin^{-1}\left(-\frac{1}{2}\right)$

Solve the above equation, we have

sin y = -1/2

We know that sin π/6 = ½

So, sin y = - sin π/6

$\sin y = \sin\left(-\frac{\pi}{6}\right)$

Since range of principle value of sin^{-1} is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

Principle value of $\sin^{-1}\left(-\dfrac{1}{2}\right)$ is - π/6.

Solution 2:

Let y = $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$

Cos y = cos π/6 (as cos π/6 = $\sqrt{3}$ / 2)

y = π/6

Since range of principle value of cos^{-1} is [0, π]

Therefore, Principle value of $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$ is π/6

Solution 3: Cosec $^{-1}$ (2)

Let y = Cosec $^{-1}$ (2)

Cosec y = 2

We know that, cosec π /6 = 2

So Cosec y = cosec π /6

Since range of principle value of cosec^{-1} is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

Therefore, Principle value of Cosec $^{-1}$ (2) is Π/6.

Solution 4: $\tan^{-1}\left(-\sqrt{3}\right)$

Let y = $\tan^{-1}\left(-\sqrt{3}\right)$

tan y = - tan π/3

or tan y = tan (-π/3)

Since range of principle value of tan⁻¹ is $\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$

Therefore, Principle value of $\tan^{-1}\left(-\sqrt{3} \right)$ is – π/3.

Solution 5: $\cos^{-1}\left(\frac{-1}{2} \right)$

y = $\cos^{-1}\left(\frac{-1}{2} \right)$

cos y = -1/2

$\cos y = -\cos \frac{\pi}{3}$

cos y = cos(π – π/3) = cos (2π/3)

Since principle value of cos⁻¹ is [0, π]

Therefore, Principle value of $\cos^{-1}\left(\frac{-1}{2} \right)$ is 2π/3.

Solution 6: tan⁻¹(-1)

Let y = tan⁻¹(-1)

tan (y) = -1

tan y = -tan π/4

$\tan y = \tan\left(-\frac{\pi}{4} \right)$

Since principle value of tan⁻¹ is $\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$

Therefore, Principle value of tan⁻¹(-1) is - π/4.

Solution 7: $\sec^{-1}\left(\dfrac{2}{\sqrt{3}}\right)$

$y = \sec^{-1}\left(\dfrac{2}{\sqrt{3}}\right)$

$\sec y = 2/\sqrt{3}$

$\sec y = \sec\dfrac{\pi}{6}$

Since principle value of $\sec^{-1}$ is [0, π]

Therefore, Principle value of $\sec^{-1}\left(\dfrac{2}{\sqrt{3}}\right)$ is π/6

Solution 8: $\cot^{-1}\left(\sqrt{3}\right)$

$y = \cot^{-1}\left(\sqrt{3}\right)$

$\cot y = \sqrt{3}$

$\cot y = \pi/6$

Since principle value of $\cot^{-1}$ is [0, π]

Therefore, Principle value of $\cot^{-1}\left(\sqrt{3}\right)$ is π/6.

Solution 9: $\cos^{-1}\left(\dfrac{-1}{\sqrt{2}}\right)$

Let $y = \cos^{-1}\left(\dfrac{-1}{\sqrt{2}}\right)$

$$\cos y = -\frac{1}{\sqrt{2}}$$

$$\cos y = -\cos\frac{\pi}{4}$$

$$\cos y = \cos\left(\pi - \frac{\pi}{4}\right) = \cos\frac{3\pi}{4}$$

Since principle value of cos⁻¹ is [0, π]

Therefore, Principle value of $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ is 3 π/ 4.

Solution 10. $\csc^{-1}\left(-\sqrt{2}\right)$

Ley y = $\csc^{-1}\left(-\sqrt{2}\right)$

$$\csc y = -\sqrt{2}$$
$$\csc y = \csc\frac{-\pi}{4}$$

Since principle value of cosec⁻¹ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Therefore, Principle value of $\csc^{-1}\left(-\sqrt{2}\right)$ is – π/4

Find the values of the following:

11. $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

12. $\cos^{-1}\frac{1}{2} + 2\sin^{-1}\frac{1}{2}$

13. If sin⁻¹ x = y, then

(A) $0 \le y \le \pi$

(B) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(C) $0 < y < \pi$

(D) $-\frac{\pi}{2} < y < \frac{\pi}{2}$

14. tan⁻¹ ($\sqrt{3}$) - sec ⁻¹ (-2) is equal to

(A) π

(B) - π/3

(C) π/3

(D) 2 π/3

Solution 11.
$$\tan^{-1}(1) + \cos^{-1}\left(\frac{-1}{2}\right) + \sin^{-1}\left(\frac{-1}{2}\right)$$

$$= \tan^{-1}\tan\frac{\pi}{4} + \cos^{-1}\left(-\cos\frac{\pi}{3}\right) + \sin^{-1}\left(-\sin\frac{\pi}{6}\right)$$

$$= \frac{\pi}{4} + \cos^{-1}\left(\pi - \frac{\pi}{3}\right) + \sin^{-1}\sin\left(-\frac{\pi}{6}\right)$$

$$= \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6}$$

$$= \frac{3\pi + 8\pi - 2\pi}{12}$$

$$= \frac{9\pi}{12} = \frac{3\pi}{4}$$

Solution 12:

Let $\cos^{-1}\left(\dfrac{1}{2}\right) = x$. Then, $\cos x = \dfrac{1}{2} = \cos\left(\dfrac{\pi}{3}\right)$

$\cos^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3}$

Let $\sin^{-1}\left(\dfrac{1}{2}\right) = y$. Then, $\sin y = \dfrac{1}{2} = \sin\left(\dfrac{\pi}{6}\right)$

$\sin^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{6}$

Now,

$$\cos^{-1}\left(\dfrac{1}{2}\right) + 2\sin^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3} + \dfrac{2\pi}{6}$$

$$= \dfrac{\pi}{3} + \dfrac{\pi}{3}$$

$$= \dfrac{2\pi}{3}$$

Solution 13: Option (B) is correct.

Given $\sin^{-1} x = y$,

The range of the principle value of $\sin^{-1}$ is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

Therefore, $-\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}$

Solution 14:

Option (B) is correct.

$\tan^{-1}\left(\sqrt{3}\right) - \sec^{-1}(-2) = \tan^{-1}(\tan \pi/3) - \sec^{-1}(-\sec \pi/3)$

$= \pi/3 - \sec^{-1}(\sec(\pi - \pi/3))$

$= \pi/3 - 2\pi/3 = -\pi/3$

Exercise 2.2   Page No: 47

Prove the following

1.

$$3\sin^{-1}x = \sin^{-1}\left(3x - 4x^3\right), x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

Solution:

$$3\sin^{-1}x = \sin^{-1}\left(3x - 4x^3\right), x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

(Use identity: $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$)

Let $x = \sin\theta$. then

$$\theta = \sin^{-1}x$$

Now, RHS

$$= \sin^{-1}(3x - 4x^3)$$

$$= \sin^{-1}(3\sin\theta - 4\sin^3\theta)$$

$$= \sin^{-1}(\sin 3\theta)$$

$$= 3\theta$$

$$= 3\sin^{-1}x$$

= LHS

Hence Proved

2.

$$3\cos^{-1}x = \cos^{-1}\left(4x^3 - 3x\right), x \in \left[\frac{1}{2}, 1\right]$$

Solution:

$$3\cos^{-1} x = \cos^{-1}\left(4x^3 - 3x\right), x \in \left[\frac{1}{2}, 1\right]$$

Using identity: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Put x = cos θ

θ = cos⁻¹ (x)

Therefore, cos 3 θ = 4x³ – 3x

RHS:

$\cos^{-1}\left(4x^3 - 3x\right)$

= cos⁻¹ (cos 3 θ)

= 3 θ

= 3 cos⁻¹ (x)

= LHS

Hence Proved.

3.

$$\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$$

Solution:

$$\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$$

Using identity: $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$

LHS = $\tan^{-1}\dfrac{2}{11}+\tan^{-1}\dfrac{7}{24}$

$= \tan^{-1}\dfrac{\dfrac{2}{11}+\dfrac{7}{24}}{1-\dfrac{2}{11}\times\dfrac{7}{24}}$

$= \tan^{-1}\dfrac{48+77}{264-14}$

= tan⁻¹ (125/250)

= tan⁻¹ (1/2)

= RHS

Hence Proved

4.

$2\tan^{-1}\dfrac{1}{2}+\tan^{-1}\dfrac{1}{7}=\tan^{-1}\dfrac{31}{17}$

Solution:

Use identity: $2\tan^{-1}x=\tan^{-1}\dfrac{2x}{1-x^2}$.

LHS

$= 2\tan^{-1}\dfrac{1}{2}+\tan^{-1}\dfrac{1}{7}$

$= \tan^{-1}\dfrac{2\times\dfrac{1}{2}}{1-\left(\dfrac{1}{2}\right)^2}+\tan^{-1}\dfrac{1}{7}$

= tan⁻¹(4/3) + tan⁻¹(1/7)

Again using identity:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$$

We have,

$$\tan^{-1}\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}$$

$= tan^{-1}(\frac{28+3}{21-4})$

= tan⁻¹ (31/17)

RHS

Write the following functions in the simplest form:

5. $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}, x \neq 0$

Solution:

Let's say x = tan θ then θ = tan⁻¹ x

We get,

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x} = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{1 - \cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$$

This is simplest form of the function.

6. $\tan^{-1}\dfrac{1}{\sqrt{x^2 - 1}}$, $|x| > 1$

Solution:

Let us consider, x = sec θ, then θ = sec⁻¹ x

$$\tan^{-1}\frac{1}{\sqrt{x^2 - 1}} = \tan^{-1}\frac{1}{\sqrt{\sec^2\theta - 1}}$$

$$= \tan^{-1}\frac{1}{\sqrt{\tan^2\theta}}$$

$$= \tan^{-1}\left(\frac{1}{\tan\theta}\right)$$

= tan⁻¹(cot θ)

= tan⁻¹ tan(π/2 - θ)

= (π/2 - θ)

= π/2 − sec⁻¹ x

This is simplest form of the given function.

7. $\tan^{-1}\left(\sqrt{\dfrac{1-\cos x}{1+\cos x}}\right), \ 0 < x < \pi$

Solution:

$$\tan^{-1}\left(\sqrt{\dfrac{1-\cos x}{1+\cos x}}\right) = \tan^{-1}\left(\sqrt{\dfrac{2\sin^2 \dfrac{x}{2}}{2\cos^2 \dfrac{x}{2}}}\right)$$

$$= \tan^{-1}\left(\dfrac{\sin\dfrac{x}{2}}{\cos\dfrac{x}{2}}\right)$$

$$= \tan^{-1}\left(\tan\dfrac{x}{2}\right) = \dfrac{x}{2}$$

8. $\tan^{-1}\left(\dfrac{\cos x - \sin x}{\cos x + \sin x}\right), \ \dfrac{-\pi}{4} < x < \dfrac{3\pi}{4}$

Solution:

Divide numerator and denominator by cos x, we have

$$\tan^{-1}\left(\dfrac{\dfrac{cox(x)}{\cos(x)} - \dfrac{sin(x)}{\cos(x)}}{\dfrac{\cos(x)}{\cos(x)} + \dfrac{sin(x)}{\cos(x)}}\right)$$

$$= \tan^{-1}\left(\dfrac{1 - \dfrac{sin(x)}{cos(x)}}{1 + \dfrac{sin(x)}{cos(x)}}\right)$$

$$= \tan^{-1}\left(\dfrac{1-\tan x}{1+\tan x}\right)$$

$$\tan^{-1}\left(\dfrac{\tan\dfrac{\pi}{4}-\tan x}{1+\tan\dfrac{\pi}{4}\tan x}\right)$$

= tan $^{-1}$ tan(π/4 − x)

= π/4 − x

9. $\tan^{-1}\dfrac{x}{\sqrt{a^2-x^2}}$, |x| < a

Solution:

Put x = a sin θ, which implies sin θ = x/a and θ = sin^{-1}(x/a)

Substitute the values into given function, we get

$$\tan^{-1}\dfrac{x}{\sqrt{a^2-x^2}}=\tan^{-1}\left(\dfrac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right)$$

$$=\tan^{-1}\left(\dfrac{a\sin\theta}{a\sqrt{1-\sin^2\theta}}\right)$$

$$=\tan^{-1}\left(\dfrac{a\sin\theta}{a\cos\theta}\right)$$

= tan^{-1}(tan θ)

= θ

= sin^{-1}(x/a)

10. $\tan^{-1}\left(\dfrac{3a^2x-x^3}{a^3-3ax^2}\right)$, a > 0; $\dfrac{-a}{\sqrt{3}}<x<\dfrac{a}{\sqrt{3}}$

Solution:
After dividing numerator and denominator by a^3 we have

$$\tan^{-1}\left(\frac{3\left(\dfrac{x}{a}\right)-\left(\dfrac{x}{a}\right)^3}{1-3\left(\dfrac{x}{a}\right)^2}\right)$$

Put x/a = tan θ and θ = tan⁻¹(x/a)

$$= \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right)$$

= tan⁻¹ (tan 3 θ)

= 3 θ

= 3 tan⁻¹(x/a)

Find the values of each of the following:

11.
$$\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$$

Solution:

$$= \tan^{-1}\left[2\cos\left(2\sin^{-1}\sin\frac{\pi}{6}\right)\right]$$

$$= \tan^{-1}\left[2\cos\left(2\times\frac{\pi}{6}\right)\right]$$

= tan⁻¹ (2 cos π/3)

= tan⁻¹(2 x ½)

= tan⁻¹ (1)

= tan⁻¹ (tan (π/4))

= π/4

12. cot (tan⁻¹a + cot⁻¹a)

Solution:

cot (tan⁻¹a + cot⁻¹a) = cot π/2 = 0

Using identity: tan⁻¹a + cot⁻¹a = π/2

13.

$$\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right], \; |x|<1, y>0 \text{ and } xy<1$$

Solution:

Put x = tan θ and y = tan Φ, we have

$$\tan\frac{1}{2}\left[\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}+\cos^{-1}\frac{1-\tan^2\phi}{1+\tan^2\phi}\right]$$

= tan1/2[sin⁻¹ sin 2 θ + cos⁻¹ cos 2 Φ]

= tan (1/2) [2 θ + 2 Φ]

= tan (θ + Φ)

$$= \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$$

= (x+y) / (1-xy)

14. If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, **then find the value of x.**

Solution:
We know that, sin 90 degrees = sin π/2 = 1

So, given equation turned as,

$$\sin^{-1}\frac{1}{5}+\cos^{-1}x=\frac{\pi}{2}$$

$$\cos^{-1}x=\frac{\pi}{2}-\sin^{-1}\frac{1}{5}$$

Using identity: $\sin^{-1} t + \cos^{-1} t = \pi/2$

We have, $\cos^{-1} x = \cos^{-1} \dfrac{1}{5}$

Which implies, the value of x is 1/5.

15. If $\tan^{-1} \dfrac{x-1}{x-2} + \tan^{-1} \dfrac{x+1}{x+2} = \dfrac{\pi}{4}$ **, then find the value of x.**

Solution:

We have reduced the given equation using below identity:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \dfrac{x+y}{1-xy}$$

$$\tan^{-1} \dfrac{\dfrac{x-1}{x-2} + \dfrac{x+1}{x+2}}{1 - \left(\dfrac{x-1}{x-2}\right)\left(\dfrac{x+1}{x+2}\right)} = \dfrac{\pi}{4}$$

or

$$\tan^{-1} \dfrac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)-(x-1)(x+1)} = \dfrac{\pi}{4}$$

or

$$\tan^{-1} \dfrac{x^2+2x-x-2+x^2-2x+x-2}{x^2-4-(x^2-1)} = \dfrac{\pi}{4}$$

or $\dfrac{2x^2-4}{x^2-4-x^2+1} = \tan\left(\dfrac{\pi}{4}\right)$

or (2x^2 - 4)/-3 = 1

or 2x^2 = 1

or x = $\pm \dfrac{1}{\sqrt{2}}$

The value of x is either $\dfrac{1}{\sqrt{2}}$ or $-\dfrac{1}{\sqrt{2}}$

Find the values of each of the expressions in Exercises 16 to 18.

16. $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$

Solution:

Given expression is $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$

First split $\frac{2\pi}{3}$ as $\frac{(3\pi-\pi)}{3}$ or $\pi - \frac{\pi}{3}$

After substituting in given we get,

$\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \sin^{-1}\left(\sin(\pi - \frac{\pi}{3})\right) = \frac{\pi}{3}$

Therefore, the value of $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ is $\frac{\pi}{3}$

17. $tan^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$

Solution:

Given expression is $tan^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$

First split $\frac{3\pi}{4}$ as $\frac{(4\pi-\pi)}{4}$ or $\pi - \frac{\pi}{4}$

After substituting in given we get,

$tan^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right) = \tan^{-1}\left(\tan(\pi - \frac{\pi}{4})\right) = -\frac{\pi}{4}$

The value of $tan^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$ is $\frac{-\pi}{4}$.

18. $\tan\left(sin^{-1}\left(\frac{3}{5}\right) + \cot^{-1}\frac{3}{2}\right)$

Solution:

Given expression is $\tan\left(sin^{-1}\left(\frac{3}{5}\right) + \cot^{-1}\frac{3}{2}\right)$

Putting, $sin^{-1}\left(\frac{3}{5}\right) = x$ and $\cot^{-1}(\frac{3}{2}) = y$

Or sin(x) = 3/5 and cot y = 3/2

Now, sin(x) = 3/5 => cos x = $\sqrt{1 - \sin^2 x}$ = 4/5 and sec x = 5/4

(using identities: cos x = $\sqrt{1 - \sin^2 x}$ and sec x = 1/cos x)

Again, tan x = $\sqrt{\sec^2 x - 1}$ = $\sqrt{\frac{25}{16} - 1}$ = ¾ and tan y = 1/cot(y) = 2/3

Now, we can write given expression as,

$\tan(\sin^{-1}\left(\frac{3}{5}\right) + \cot^{-1}\frac{3}{2})$ = tan(x + y)

= $\frac{\tan x + \tan y}{1 - \tan x \tan y}$ = $\frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{3}{4} \times \frac{2}{3}}$

= 17/6

19. $\cos^{-1}(\cos\frac{7\pi}{6})$ **is equal to**

(A) 7π/6   (B) 5 π/6   (C) π/3   (D) π/6

Solution:

Option (B) is correct.

Explanation:

$\cos^{-1}(\cos\frac{7\pi}{6})$ = $\cos^{-1}(\cos(2\pi - \frac{7\pi}{6})$

(As cos (2 π − A) = cos A)

Now $2\pi - \frac{7\pi}{6} = \frac{12\pi - 7\pi}{6} = \frac{5\pi}{6}$

20. $\sin\left[\dfrac{\pi}{3} - \sin^{-1}\left(-\dfrac{1}{2}\right)\right]$ **is equal to**

(A) ½ (B) 1/3 (C) ¼ (D) 1

Solution:

Option (D) is correct

Explanation:

First solve for: $\sin^{-1}\left(-\dfrac{1}{2}\right)$

$$\sin^{-1}\left(-\dfrac{1}{2}\right) = \sin^{-1}\left(-\sin\dfrac{\pi}{6}\right) = \sin^{-1}\left[\sin\left(-\dfrac{\pi}{6}\right)\right]$$

$= -\pi/6$

Again,

$$\sin\left[\dfrac{\pi}{3} - \sin^{-1}\left(-\dfrac{1}{2}\right)\right]$$

$$= \sin\left[\dfrac{\pi}{3} - \left(-\dfrac{\pi}{6}\right)\right]$$

$$= \sin\left[\dfrac{\pi}{3} + \dfrac{\pi}{6}\right]$$

$= \sin(\pi/2)$

$= 1$

21. tan⁻¹ √3 – cot⁻¹ (-√3) is equal to

(A) π   **(B) -π/2**   **(C) 0**   **(D) 2√3**

Solution:

Option (B) is correct.

Explanation:

tan⁻¹ √3 – cot⁻¹ (-√3) can be written as

$$= \tan^{-1} \tan \frac{\pi}{3} - \cot^{-1}\left(-\cot\frac{\pi}{6}\right)$$

$$= \frac{\pi}{3} - \cot^{-1}\left[\cot\left(\pi - \frac{\pi}{6}\right)\right]$$

$$= \frac{\pi}{3} - (\pi - \frac{\pi}{6})$$

$$= \frac{\pi}{3} - \frac{5\pi}{6}$$

$$= \frac{-3\pi}{6}$$

$$= -\pi/2$$

Miscellaneous Exercise   Page No: 51

Find the value of the following:

1. $\cos^{-1}(\cos\frac{13\pi}{6})$

Solution:

First solve for, $\cos\frac{13\pi}{6} = \cos(2\pi + \frac{\pi}{6}) = \cos\frac{\pi}{6}$

Now: $\cos^{-1}(\cos\frac{13\pi}{6}) = \cos^{-1}(\cos\frac{\pi}{6}) = \frac{\pi}{6} \in [0, \pi]$

[As $\cos^{-1}\cos(x) = x$ if $x \in [0, \pi]$]

So the value of $\cos^{-1}(\cos\frac{13\pi}{6})$ is $\frac{\pi}{6}$.

2. $tan^{-1}(\tan\frac{7\pi}{6})$

Solution:

First solve for, $\tan\frac{7\pi}{6} = \tan(\pi + \frac{\pi}{6}) = \tan\frac{\pi}{6}$

Now: $tan^{-1}(\tan\frac{7\pi}{6}) = \tan^{-1}(\tan\frac{\pi}{6}) = \frac{\pi}{6} \in (-\pi/2, \pi/2)$

[As $\tan^{-1}\tan(x) = x$ if $x \in (-\pi/2, \pi/2)$]

So the value of $tan^{-1}(\tan\frac{7\pi}{6})$ is $\frac{\pi}{6}$.

3. Prove that $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$

Solution:
Step 1: Find the value of cos x and tan x

Let us consider $\sin^{-1}\frac{3}{5} = x$, then sin x = 3/5

So, cos x = $\sqrt{1 - \sin^2 x} = \sqrt{1 - \left(\frac{3}{5}\right)^2}$ = 4/5

tan x = sin x/ cos x = ¾

Therefore, x = tan⁻¹ (3/4), substitute the value of x,

$$\Rightarrow \quad \sin^{-1}\frac{3}{5} = \tan^{-1}\left(\frac{3}{4}\right) \quad \text{.....(1)}$$

Step 2: Solve LHS

$$2\sin^{-1}\frac{3}{5} = 2\tan^{-1}\frac{3}{4}$$

Using identity: 2tan⁻¹ x = $\tan^{-1}$ = $\tan^{-1}(\frac{2x}{1-x^2})$, we get

$$= \tan^{-1}\left(\frac{2\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^2}\right)$$

= tan⁻¹(24/7)

= RHS

Hence Proved.

4. Prove that $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$

Solution:

Let $\sin^{-1}\left(\frac{8}{17}\right) = x$ then sin x = 8/17

Again, cos x = $\sqrt{1 - \sin^2 x}$ = $\sqrt{1 - \frac{64}{289}}$ = 15/17

And tan x = sin x / cos x = 8/ 15

Again,

Let $\sin^{-1}\left(\frac{3}{5}\right) = y$ then sin y = 3/5

Again, cos y = $\sqrt{1 - \sin^2 y}$ = $\sqrt{1 - \frac{9}{25}}$ = 4/5

And tan y = sin y / cos y = ¾

Solve for tan(x + y), using below identity,

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \, \tan y}$$

$$= \frac{\frac{8}{15} + \frac{3}{4}}{1 - \frac{8}{15} \times \frac{3}{4}}$$

$$= \frac{32 + 45}{60 - 24}$$

= 77/36

This implies x + y = tan⁻¹(77/36)

Substituting the values back, we have

$\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$ (Proved)

5. Prove that $\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$

Solution:

Let $\cos^{-1}\frac{4}{5} = \theta$ Let $\cos^{-1}\frac{12}{13} = \phi$

$\cos\theta = \frac{4}{5}$ $\cos\phi = \frac{12}{13}$

$\sin\theta = \sqrt{1 - \cos^2\theta}$ $\sin\phi = \sqrt{1 - \cos^2\phi}$

$= \sqrt{1 - \frac{16}{25}}$ $= \sqrt{1 - \frac{144}{169}}$

$= \frac{3}{5}$ $= \frac{5}{13}$

Solve the expression, Using identity: cos (θ + φ) = cosθ cos φ - sinθ sin φ

= 4/5 x 12/13 – 3/5 x 5/13

= (48-15)/65

= 33/65

This implies cos (θ + φ) = 33/65

or θ + φ = cos⁻¹ (33/65)

Putting back the value of θ and φ, we get

$$\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$$

Hence Proved.

6. Prove that $\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$

Solution:

Let $\cos^{-1}\frac{12}{13} = \theta$ | Let $\sin^{-1}\frac{3}{5} = \phi$

So $\cos\theta = \frac{12}{13}$ | So $\sin\phi = \frac{3}{5}$

$\sin\theta = \sqrt{1-\cos^2\theta}$ | $\cos\phi = \sqrt{1-\sin^2\phi}$

$= \sqrt{1-\frac{144}{169}}$ | $= \sqrt{1-\frac{9}{25}}$

$= \frac{5}{13}$ | $= \frac{4}{5}$

Solve the expression, Using identity: sin (θ + φ) = sin θ cos φ + cos θ sin φ

= 12/13 x 3/5 + 12/13 x 3/5

= (20+36)/65

= 56/65

or sin (θ + φ) = 56/65

or θ + φ) = sin $^{-1}$ 56/65

Putting back the value of θ and φ, we get

$$\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$$

Hence Proved.

7. Prove that $\tan^{-1}\left(\frac{63}{16}\right) = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$

Solution:

Let $\sin^{-1}\frac{5}{13} = \theta$ Let $\cos^{-1}\frac{3}{5} = \phi$

so $\sin\theta = \frac{5}{13}$ so $\cos\phi = \frac{3}{5}$

$\cos\theta = \sqrt{1-\sin^2\theta}$ $\sin\phi = \sqrt{1-\cos^2\phi}$

$\quad = \sqrt{1-\dfrac{25}{169}}$ $\quad = \sqrt{1-\dfrac{9}{25}}$

$\quad = \dfrac{12}{13}$ $\quad = \dfrac{4}{5}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{5}{12}$ $\tan\phi = \dfrac{\sin\phi}{\cos\phi} = \dfrac{4}{3}$

Solve the expression, Using identity:

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi}$$

$$= \frac{\dfrac{5}{12}+\dfrac{4}{3}}{1-\dfrac{5}{12}\times\dfrac{4}{3}}$$

= 63/16

(θ + φ) = tan^{-1} (63/16)

Putting back the value of θ and φ, we get

$$\tan^{-1}\left(\frac{63}{16}\right) = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$$

Hence Proved.

8. Prove that $\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$
Solution:

LHS = $\left(\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right)\right) + \left(\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right)\right)$

Solve above expressions, using below identity:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right)$$

After simplifying, we have

= tan^{-1} (6/17) + tan^{-1} (11/23)

Again, applying the formula, we get

$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right)$$

After simplifying,

= tan^{-1}(325/325)

= tan^{-1} (1)

= π/4

9. Prove that $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\frac{1-x}{1+x}$ **, x ∈ (0, 1)**

Solution:

Let $\tan^{-1}\sqrt{x} = \theta$, then $\sqrt{x}$ = tan θ

Squaring both the sides

tan² θ = x

Now, substitute the value of x in $\frac{1}{2}\cos^{-1}\frac{1-x}{1+x}$, we get

$$= \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

= ½ cos-1 (cos 2 θ)

= ½ (2 θ)

= θ

$= \tan^{-1}\sqrt{x}$

10. Prove that $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}$ **, x ∈ (0, π/4)**

Solution:
We can write 1+ sin x as,

$$1+\sin x = \cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\cos\frac{x}{2}\sin\frac{x}{2} = \left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2$$

And

$$1-\sin x = \cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\cos\frac{x}{2}\sin\frac{x}{2} = \left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2$$

LHS:

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$$

$$= \cot^{-1}\left[\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right]$$

$$= \cot^{-1}\left(\frac{2\cos(\frac{x}{2})}{2\sin(\frac{x}{2})}\right)$$

$$= \cot^{-1}(\cot(x/2)$$

$$= x/2$$

11. Prove that $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x, \ -\frac{1}{\sqrt{2}} \le x \le 1$

[Hint: Put x = cos 2 θ]

Solution:

Put $x = \cos 2\theta$ so, $\theta = \frac{1}{2}\cos^{-1}x$

$$\text{LHS} = \tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)$$

Divide each term by $\sqrt{2}\cos\theta$

$$= \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\tan\dfrac{\pi}{4}-\tan\theta}{1+\tan\dfrac{\pi}{4}\tan\theta}\right)$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4}-\theta\right)$$

$$= \frac{\pi}{4}-\theta$$

$$= \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$$

= RHS

Hence proved

12. Prove that $\dfrac{9\pi}{8}-\dfrac{9}{4}\sin^{-1}\dfrac{1}{3}=\dfrac{9}{4}\sin^{-1}\dfrac{2\sqrt{2}}{3}$

Solution:

LHS $= \dfrac{9\pi}{8}-\dfrac{9}{4}\sin^{-1}\dfrac{1}{3}$

$$= \frac{9}{4}\left(\frac{\pi}{2}-\sin^{-1}\frac{1}{3}\right)$$

$$= \frac{9}{4}\cos^{-1}\frac{1}{3}$$

.......(1)

(Using identity: $\sin^{-1}\theta+\cos^{-1}\theta=\dfrac{\pi}{2}$)

Let $\theta = \cos^{-1}(1/3)$, so $\cos\theta = 1/3$

As

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{1}{9}} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$

Using equation (1), $\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

Which is right hand side of the expression.

Solve the following equations:

13. 2tan⁻¹ (cos x) = tan⁻¹ (2 cosec x)

Solution:

$$2\tan^{-1}(\cos x) = \tan^{-1}(2\cos ec\ x)$$

$$\tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}\left(\frac{2}{\sin x}\right)$$

$$\frac{2\cos x}{1-\cos^2 x} = \frac{2}{\sin x}$$

$$\frac{\cos x}{\sin x} = 1$$

Cot x = 1

x = π/4

14. Solve $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x, (x>0)$

Solution:

Put x = tan θ

$$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x$$

This implies

$$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x$$

$$\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right) = \frac{1}{2}\tan^{-1}\tan\theta$$

$$\tan^{-1}\left(\frac{\tan\dfrac{\pi}{4}-\tan\theta}{\tan\dfrac{\pi}{4}+\tan\theta}\right) = \frac{1}{2}\theta$$

$$\tan^{-1}\tan\left(\frac{\pi}{4}-\theta\right) = \frac{\theta}{2}$$

π/4 − θ = θ /2

or 3θ /2 = π/4

θ = π/6

Therefore, x = tan θ = tan π/6 = 1/√3

15. $\sin\left(\tan^{-1}x\right), |x| < 1$ **is equal to**

(A) $\dfrac{x}{\sqrt{1-x^2}}$ (B) $\dfrac{1}{\sqrt{1-x^2}}$

(C) $\dfrac{1}{\sqrt{1+x^2}}$ (D) $\dfrac{x}{\sqrt{1+x^2}}$

Solution:

Option (D) is correct.

Explanation:

Let $\theta = \tan^{-1}x$ so, $x = \tan\theta$

Again, Let's say

$$\sin\left(\tan^{-1} x\right) = \sin\theta$$

This implies,

$$\sin\left(\tan^{-1} x\right) = \frac{1}{\cos ec\theta} = \frac{1}{\sqrt{1+\cot^2\theta}}$$

Put $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{1}{x}$

Which shows,

$$\sin\left(\tan^{-1} x\right) = \frac{1}{\sqrt{1+\dfrac{1}{x^2}}} = \frac{x}{\sqrt{x^2+1}}$$

16. $\sin^{-1}(1-x) - 2\sin^{-1} x = \dfrac{\pi}{2}$ **then x is equal to**

(A) 0, ½ (B) 1, ½   (C) 0 (D) ½

Solution:

Option (C) is correct.

Explanation:

Put $\sin^{-1} x = \theta$ So, $x = \sin\theta$

Now,

$$\sin^{-1}(1-x) - 2\sin^{-1} x = \frac{\pi}{2}$$

$$\sin^{-1}(1-x) - 2\theta = \frac{\pi}{2}$$

$$\sin^{-1}(1-x) = \frac{\pi}{2} + 2\theta$$

$$1 - x = \sin\left(\frac{\pi}{2} + 2\theta\right)$$

$$1 - x = \cos 2\theta$$

$$1 - x = 1 - 2x^2$$

(As x = sin θ)

After simplifying, we get

x(2x – 1) = 0

x = 0 or 2x – 1 = 0

x = 0 or x = ½

Equation is not true for x = ½. So the answer is x = 0.

17. $\tan^{-1}\left(\dfrac{x}{y}\right) - \tan^{-1}\left(\dfrac{x-y}{x+y}\right)$ **is equal to**

(A) π/2   **(B)** π/3   **(C)** π/4   **(D)** -3 π/4

Solution:

Option (C) is correct.

Explanation:

Given expression can be written as,

$$= \tan^{-1}\left[\frac{\dfrac{x}{y} - \left(\dfrac{x-y}{x+y}\right)}{1 + \dfrac{x}{y}\left(\dfrac{x-y}{x+y}\right)}\right]$$

$$= \tan^{-1}\left[\frac{x(x+y) - y(x-y)}{y(x+y) + x(x-y)}\right]$$

$$= \tan^{-1}\left(\frac{x^2 + xy - xy + y^2}{xy + y^2 + x^2 - xy}\right)$$

$$= \tan^{-1}\left(\frac{x^2 + y^2}{x^2 + y^2}\right)$$

$= \tan^{-1}(1)$

$= \pi/4$