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1. Let A = $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ show that $(aI + bA)^n = a^n I + na^{n-1}\, bA$ where I is the identity matrix of order 2 and n ∈ N.

Solution:
Use Mathematical Induction:

Step 1: Result is true for n = 1

$$(aI + bA)^n = a^n I + na^{n-1} bA$$

Step 2: Assume that result is true for n = k
So,

$$(aI + bA)^k = a^k I + ka^{k-1} bA$$

Step 3: Prove that, result is true for n = k + 1
That is,

$$(aI + bA)^{k+1} = a^{k+1} I + (k+1) a^k bA$$

L.H.S.:

$$(aI + bA)^{k+1} = (aI + bA)^k (aI + bA)$$

$$= (a^k I + ka^{k-1} bA)(aI + bA)$$

$$= a^{k+1} I \times I + ka^k bAI + a^k bAI + ka^{k-1} b^2 A.A$$

Here, A.A = $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ = 0
This implies

$$= a^{k+1} I + (k+1) a^k bA$$

= R.H.S.

Thus, result is true.

Therefore, p(n) is true.

2. If A = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, prove that A^n = $\begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$, n ∈ N

Solution:
Let us say, p(n) = A^n

Use Mathematical Induction:

Step 1: Result is true for n = 1

$$p(1) = A = \begin{bmatrix} 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Step 2: Assume that result is true for n = k
So,

$$p(1) = A = \begin{bmatrix} 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$p(k) = A^k = \begin{bmatrix} 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \end{bmatrix}$$

Step 3: Prove that, result is true for n = k + 1
That is,

$$p(k+1) = A^{k+1} = \begin{bmatrix} 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \end{bmatrix}$$

L.H.S.:

$A^{k+1} = A^k A$

$$\begin{bmatrix} 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3^{k} & 3^{k} & 3^{k} \\ 3^{k} & 3^{k} & 3^{k} \\ 3^{k} & 3^{k} & 3^{k} \end{bmatrix}$$

= R.H.S.

Thus, result is true.

Therefore, By Mathematical Induction p(n) is true for all natural numbers.

3. If A = $\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ then prove that A^n = $\begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ where n is any positive integer.

Solution:

Use Mathematical Induction:

Step 1: Result is true for n = 1.

$A^1 = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

Step 2: Assume that result is true for n = k
So,

$$A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$$

Step 3: Prove that, result is true for n = k + 1
That is,

$$A^{k+1} = \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ (k+1) & 1-2(k+1) \end{bmatrix}$$

L.H.S.:

$A^{k+1} = A^k A$

$$= \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ (k+1) & 1-2(k+1) \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

Using result from step 2.

$$= \begin{bmatrix} 3+6k-4k & -4-8k+4k \\ 3k+1-2k & -4k-1+2k \end{bmatrix}$$

$$= \begin{bmatrix} 3+2k & -4-4k \\ 1+k & -1-2k \end{bmatrix}$$

$$= \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ (k+1) & 1-2(k+1) \end{bmatrix}$$

= R.H.S.

Thus, result is true.

Therefore, By Mathematical Induction result is true for all positive integers.

4. If A and B are symmetric matrices, prove that AB – BA is a skew symmetric matrix.

Solution:
Step 1: If A and B are symmetric matrices, then A’ = A and B’ = B

Step 2: (AB – BA)’ = (AB)’ – (BA)’

(AB – BA)’ = B’A’ – A’B’ [Using Reversal law]

(AB – BA)’ = BA – AB [Using eq. (i)]

(AB – BA)’ = – (AB – BA)

Therefore, (AB – BA) is a skew symmetric.

5. Show that the matrix B'AB is symmetric or skew symmetric according as A is symmetric or skew symmetric.

Solution:

We know that, (AB)' = B' A'

(B'AB)' = [B'(AB]' = (AB)' (B')'

This implies, (B'AB)' = B'A'B ..say equation (1)

If A is a symmetric matrix, then A' = A

Using eq. (i) (B'AB)' = B'AB

Therefore, B'AB is a symmetric matrix.

Again,

If A is a skew symmetric matrix then A' = – A

Using equation (i), (B'AB)' = B'(– A)B = – B'AB

So, B'AB is a skew symmetric matrix.

6. Find the values of x, y, z if the matrix A = $\begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfies the equation A'A = I.

Solution:

Given matrix is A = $\begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$

Transpose of A = A' = $\begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}$

Now, A'A = I (Given)

$$\begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}\begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0+x^2+x^2 & 0+xy-xy & 0-xz+xz \\ 0+xy-xy & 4y^2+y^2+y^2 & 2yz-yz-yz \\ 0-zx+zx & 2yz-yz-yz & z^2+z^2+z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This implies:

$$\begin{bmatrix} 2x^2 & 0 & 0 \\ 0 & 6y^2 & 0 \\ 0 & 0 & 3z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To find the values of unknowns, equate corresponding matrix entries:
We have,

$2x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

$6y^2 = 1 \Rightarrow y = \pm\frac{1}{\sqrt{6}}$ and

$3z^2 = 1 \Rightarrow z = \pm\frac{1}{\sqrt{3}}$

7. For what value of x.

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$$

Solution:

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$$

$$\begin{bmatrix} 1+4+1 & 2+0+0 & 1+2+2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$$

$$\begin{bmatrix} 6 & 2 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = 0$$

$$\begin{bmatrix} 0+4+4x \end{bmatrix} = 0$$

Therefore, 4 + 4x = 0 => x = -1.

8. If A = $\begin{bmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{-1} & \mathbf{2} \end{bmatrix}$, show that A^2 – 5A + 7I = 0.

Solution:

$A^2 = A\,A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$

$5A = 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}$

$7I = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

Now, **A^2 – 5A + 7I = 0**

L.H.S.

$A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

R.H.S.

Hence Proved.

9.

Find x, if $\begin{bmatrix} x & -5 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$

Solution:

$$\begin{bmatrix} x-0-2 & 0-10-0 & 2x-5-3 \end{bmatrix}\begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} x-2 & -10 & 2x-8 \end{bmatrix}\begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = 0$$

$$\left[(x-2)x-10(4)+(2x-8)1\right] = 0$$

$$[x^2 - 2x - 40x + 2x - 8] = 0$$

$$[x^2 - 48] = [0]$$

or x^2 – 48 = 0 => x = $\pm 4\sqrt{3}$

10. A manufacturer produces three products x, y, z, which he sells in two markets. Annual sales are indicated below:

Market	**Products**	**Products**	**Products**
I	**10,000**	**2,000**	**18,000**
II	**6,000**	**20,000**	**8,000**

(a) If unit sales prices of x, y, and z are Rs. 2.50, Rs. 1.50 and Rs. 1.00 respectively, find the total revenue in each market with the help of matrix algebra.

(b) If the unit costs of the above three commodities are Rs. 2.00, Rs. 1.00 and 50 paise respectively. Find the gross profit.

Solution:

(a)
If unit sales prices of x, y, and z are Rs. 2.50, Rs. 1.50 and Rs. 1.00 respectively.
Total revenue in market I and II can be shown with the help of matrix as: Basically Revenue Matrix

$$\begin{bmatrix} 10,000 & 2,000 & 18,000 \\ 6,000 & 20,000 & 8,000 \end{bmatrix}\begin{bmatrix} 2.5 \\ 1.5 \\ 1 \end{bmatrix}$$

Solving above matrix, we have,

$$= \begin{bmatrix} 25,000+3,000+18,000 \\ 15,000+30,000+8,000 \end{bmatrix}$$

$$= \begin{bmatrix} 46,000 \\ 53,000 \end{bmatrix}$$

Therefore, the total revenue in Market I = Rs. 46,000 and in Market II = Rs. 53,000.

(b) If the unit costs of the above three commodities are Rs. 2.00, Rs. 1.00 and 50 paise respectively.
Total cost prices of all the products in market I and II can be shown with the help of matrix as: Basically Cost Matrix

$$\begin{bmatrix} 10,000 & 2,000 & 18,000 \\ 6,000 & 20,000 & 8,000 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 0.5 \end{bmatrix}$$

Solving above matrix, we have,

$$\begin{bmatrix} 20,000+2,000+9,000 \\ 12,000+20,000+4,000 \end{bmatrix}$$

$$= \begin{bmatrix} 31,000 \\ 36,000 \end{bmatrix}$$

From (a) and (b),

The profit collected in two markets is given in matrix form as

Profit matrix = Revenue matrix – Cost matrix

$$\begin{bmatrix} 46,000 \\ 53,000 \end{bmatrix} - \begin{bmatrix} 31,000 \\ 36,000 \end{bmatrix} = \begin{bmatrix} 15,000 \\ 17,000 \end{bmatrix}$$

Therefore, the gross profit in market I and market II = Rs. 15000 + Rs. 17000 = Rs. 32,000.

11. Find the matrix X so that X

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$$

Solution:

Let X = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

We have,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} a+4b & 2a+5b & 3a+6b \\ c+4d & 2c+5d & 3c+6d \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$$

Equate all the corresponding elements:

a + 4b = -7 ….(1)

2a + 5b = -8 …(2)

3a + 6b = - 9 ..(3)

c + 4d = 2 …(4)

2c +5d = 4 …(5)

3c + 6d =6 ….(6)

Solving (1) and (2), we have a = 1 and b = -2
Solving (4) and (5), we have c = 2 and d = 0

So X = $\begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$

12. If A and B are square matrices of the same order such that AB = BA, then prove by induction that $AB^n = B^nA$. Further, prove that $(AB)^n = A^nB^n$ for all n ∈ N.

Solution:

Use Mathematical Induction, to prove $AB^n = B^nA$

Step 1: Result is true for n = 1

AB = BA

Step 2: Assume that result is true for n = k
So,
$AB^k = B^kA$

Step 3: Prove that, result is true for n = k + 1
That is,

$AB^{k+1} = B^{k+1} A$

L.H.S.:

$AB^{k+1} = AB^k B$

Using result of Step 2, we have

$= B^kA B$

$= B^{k+1} A$

= R.H.S.

Thus, by Mathematical Induction the result is true.

Again, prove that $(AB)^n = A^nB^n$

Use Mathematical Induction:

Step 3: Result is true for n = 1

(AB) = AB

Step 4: Assume that result is true for n = k
So, $(AB)^k = A^kB^k$

Step 3: Prove that, result is true for n = k + 1
That is, $(AB)^{k+1} = A^{k+1}B^{k+1}$

L.H.S.: $(AB)^{k+1} = (AB)^k\ (AB)$

$= A^kB^k\ (AB)$ (using step 2 result)

$= A^k\ (B^k A)\ B$

$= A^k\ (A\ B^k)\ B$

$= (A^k\ A)\ (B^k\ B)$

$= A^{k+1}B^{k+1}$

= R.H.S.
Thus, result is true for n = k+1.
Therefore, by Mathematical Induction we have $(AB)^n = A^nB^n$ for all n ∈ N.

13. If A = $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ is such that A^2 = I, then:

(A) $1+\alpha^2+\beta\gamma=0$ (B) $1-\alpha^2+\beta\gamma=0$

(C) $1-\alpha^2-\beta\gamma=0$ (D) $1+\alpha^2-\beta\gamma=0$

Solution:
Option (C) is correct.

$A^2 = I$

$$\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \alpha^2+\beta\gamma & 0 \\ 0 & \beta\gamma+\alpha^2 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

or $\alpha^2+\beta\gamma=1$

or $1-\alpha^2-\beta\gamma=0$

14. If the matrix A is both symmetric and skew symmetric, then:

(A) A is a diagonal matrix

(B) A is a zero matrix

(C) A is a square matrix

(D) None of these

Solution:
Option (B) is correct.

15. If A is a square matrix such that $A^2 = A$, then $(I + A)^3 - 7A$ is equal to:

(A) A **(B) I – A**

(C) I **(D) 3A**

Solution:

Option (C) is correct.

Explanation:

As $A^2 = A$

Multiplying both sides by A,

$A^3 = A^2 A = A A = A^2 = A$

Again,
$(I + A)^3 - 7A = I^3 + A^3 + 3I^2A + 3IA^2 - 7A$

Using $A^2 = A$ and $A^3 = A$, we have

$= I + 7A - 7A = I$