Government of Karnataka

MATHEMATICS

8

EIGHTH STANDARD

Part-I

KARNATAKA TEXT BOOK SOCIETY (R)
No.4, 100 Feet Ring Road
Banashankari 3rd Stage, Bengaluru - 560 085

PREFACE

The Textbook Society, Karnataka has been engaged in producing new textbooks according to the new syllabi which in turn are designed on NCF -2005 since June 2010. Textbooks are prepared in 12 languages; seven of them serve as the media of instruction. From standard 1 to 4 there is the EVS, mathematics and 5th to 10th there are three core subjects namely mathematics, science and social science.

NCF -2005 has a number of special features and they are :

- Connecting knowledge to life activities
- Learning to shift from rote methods
- Enriching the curriculum beyond textbooks
- Learning experiences for the construction of knowledge
- Making examinations flexible and integrating them with classroom experiences
- Caring concerns within the democratic policy of the country
- Make education relevant to the present and future needs.
- Softening the subject boundaries-integrated knowledge and the joy of learning.
- The child is the constructor of knowledge

The new books are produced based on three fundamental approaches namely :

i. Constructive approach,

ii. Spiral Approach

iii. Integrated approach.

The learner is encouraged to think, engage in activities, master skills and competencies. The materials presented in these books are integrated with values. The new books are not examination oriented in their nature. On the other hand they help the learner in the total development of his/her personality, thus help him/her become a healthy member of a healthy society and a productive citizen of this great country, India.

Mathematics is essential in the study of various subjects and in real life. NCF 2005 proposes moving away from complete calculations, construction of a framework of concepts, relate mathematics to real life experiences and cooperative learning.

Many students have a maths phobia and in order to help them overcome this phobia, jokes, puzzles, riddles, stories and games have been included in textbooks. Each concept is introduced through an activity or an interesting story at the primary level. The contri¬butions of great Indian mathematicians are mentioned at appropriate places.

The Textbook Society expresses grateful thanks to the chairpersons, writers, scrutinisers, artists, staff of DIETs and CTEs and the members of the Editorial Board and printers in helping the Text Book Society in producing these textbooks.

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FOREWORD

The Government of India through NCERT have brought out NCF-2005 to revise the curriculum of schools and suggested all the states to introduce revised textbooks in the schools based on the new curriculum. Accordingly state Governments took up the work and requested respective DSERTs to start introducing new curriculum and texts. Karnataka Government has suggested to its DSERT to take up the challenge to fulfil the vision of NCF-2005. DSERT, Karnataka started the process: constituted committeestorevisethe syllabi,identified the writersand requested these people to write texts books based on the new syllabi incorporating the expectations of NCF-2005. Karnataka Text Book Society, took the initiative and coordinated the whole programme of writing these text books.

The current work, a text book in mathematics for 8th standard, is a step taken in this direction. An effort has been made here to look at the mathematics needed at 8th standard through a different lens. At first glance, this may look a totally unconventional approach. Some may feel that it is hard on the part of 8th standard students. On the other hand that is the correct age for the students to learn new concepts and ideas. Students are receptive to new intellectual challenges. It is the onus of the teachers to teach new things to the students and prepare them to the challenges of the ever changing world. This text book is also an effort to integrate our students with the national mainstream where CBSE has surged forward and parents think that their wards will be better off by learning CBSE texts.

We have tried here to tell something new about numbers and number system. Similarly, some thing new about graphs, postulates of geometry and congruency of triangles are also introduced with more expectations. Quadrilaterals have been introduced now itself. There are optional problems at the end to challenge the students.

It is my earnest request to all my teacher friends to take up the new challenge. Let the parents of our students not feel that their wards are always in the back seats.

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ಪರಿಷ್ಕರಣೆ ಕುರಿತು

ಒಂದರಿಂದ ಹತ್ತನೇ ತರಗತಿಯ ಪಠ್ಯಪುಸ್ತಕಗಳ ಬಗ್ಗೆ ಪ್ರಕಟಗೊಂಡ ಸಾರ್ವಜನಿಕ ಅಭಿಪ್ರಾಯಗಳನ್ನು ಗಮನಿಸಿ ಮಾನ್ಯ ಮುಖ್ಯಮಂತ್ರಿಯವರೂ ಅರ್ಥಸಚಿವರೂ ಆಗಿರುವ ಶ್ರೀ ಸಿದ್ದರಾಮಯ್ಯನವರು ತಮ್ಮ 2014–15 ರ ಬಜೆಟ್‌ನಲ್ಲಿ ತಜ್ಞರ ಸಮಿತಿಯನ್ನು ರಚಿಸುವ ಘೋಷಣೆ ಮಾಡಿದರು. ತಜ್ಞರು ಅನುಸರಿಸಬೇಕಾದ ಮೂಲ ಆಶಯವನ್ನು ಹೀಗೆ ಹೇಳಿದರು: "ಪಠ್ಯಪುಸ್ತಕಗಳನ್ನು ಸಾಮಾಜಿಕ ಸಾಮರಸ್ಯ, ನೈತಿಕಮೌಲ್ಯಗಳು, ವ್ಯಕ್ತಿತ್ವವಿಕಸನ, ವೈಜ್ಞಾನಿಕ ಮತ್ತು ವೈಚಾರಿಕ ಮನೋಭಾವ, ಜಾತ್ಯತೀತತೆ ಮತ್ತು ರಾಷ್ಟ್ರೀಯ ಬದ್ಧತೆಗಳಿಗೆ ಅನುವಾಗುವಂತೆ ತಜ್ಞರ ಸಮಿತಿಯನ್ನು ಪುನರ್ ರಚಿಸಲಾಗುವುದು" – ಇದು ಬಜೆಟ್ ಭಾಷಣದಲ್ಲಿ ಸಾದರಪಡಿಸಿದ ಆಶಯ.

ಆನಂತರ ಶಿಕ್ಷಣ ಇಲಾಖೆಯು ಒಂದರಿಂದ ಹತ್ತನೇ ತರಗತಿಯವರೆಗಿನ ಪಠ್ಯಪುಸ್ತಕಗಳ ಪರಿಷ್ಕರಣೆಗಾಗಿ 27 ಸಮಿತಿಗಳನ್ನು ರಚಿಸಿ ದಿನಾಂಕ: 24.11.2014 ರಂದು ಆದೇಶಹೊರಡಿಸಿತು. ಈ ಸಮಿತಿಗಳು ವಿಷಯವಾರು ಮತ್ತು ತರಗತಿವಾರು ಮಾನದಂಡಕ್ಕನುಗುಣವಾಗಿ ರಚಿತವಾದವು. ವಿವಿಧ ಪಠ್ಯವಿಷಯಗಳಿಗೆ ಸಂಬಂಧಿಸಿದ ತಜ್ಞರು, ಅಧ್ಯಾಪಕರು ಈ ಸಮಿತಿಗಳಲ್ಲಿದ್ದಾರೆ. ಈಗಾಗಲೇ ಲಿಖಿತವಾಗಿ ಬಂದಿರುವ ಅನೇಕ ಆಕ್ಷೇಪಗಳು ಮತ್ತು ವಿಶ್ಲೇಷಣೆಗಳನ್ನು ಪರಿಶೀಲಿಸಿ, ತಪ್ಪು ಒಪ್ಪುಗಳ ಬಗ್ಗೆ ನಿರ್ಣಹಿಸಿ ಪಠ್ಯಪುಸ್ತಕಗಳನ್ನು ಪರಿಷ್ಕರಿಸುವ ಹೊಣೆಹೊತ್ತ ಈ ಸಮಿತಿಗಳಿಗೆ 'ಅಗತ್ಯಹಿದ್ದಲ್ಲಿ ಪಠ್ಯವಸ್ತುವನ್ನು ಪರಿಷ್ಕರಿಸಿ ನಂತರ ಪಠ್ಯಪುಸ್ತಕಗಳನ್ನು ಪರಿಷ್ಕರಿಸುವ' ಸ್ವಾತಂತ್ರ್ಯವನ್ನು 24.11.2014 ರ ಆದೇಶದಲ್ಲೇ ನೀಡಲಾಗಿತ್ತು. ಆನಂತರ 19.09.2015 ರಂದು ಹೊಸ ಆದೇಶ ಹೊರಡಿಸಿ 'ಅಗತ್ಯವಿದ್ದಲ್ಲಿ ಪುನರ್ ರಚಿಸುವ' ಸ್ವಾತಂತ್ರ್ಯವನ್ನು ನೀಡಲಾಯಿತು. ಹೀಗೆ ಸಮಗ್ರ ಪರಿಷ್ಕರಣೆಗೊಂಡ ಪಠ್ಯಪುಸ್ತಕಗಳನ್ನು 2016–17 ರ ಬದಲು 2017–18ನೇ ಶೈಕ್ಷಣಿಕ ವರ್ಷದಲ್ಲಿ ಜಾರಿಗೊಳಿಸಲಾಗುವುದೆಂದು ಇದೇ ಆದೇಶದಲ್ಲಿ ತಿಳಿಸಲಾಯಿತು.

ಅನೇಕ ವ್ಯಕ್ತಿಗಳೂ ಸಂಘಟನೆಗಳೂ ಸ್ವಯಂಪ್ರೇರಿತರಾಗಿ ಪಠ್ಯಪುಸ್ತಕಗಳ ಮಾಹಿತಿದೋಷ, ಆಶಯದೋಷಗಳನ್ನು ಪಟ್ಟಿಮಾಡಿ ಶಿಕ್ಷಣ ಸಚಿವರಿಗೆ, ಪಠ್ಯಪುಸ್ತಕ ಸಂಘಕ್ಕೆ ಕಳುಹಿಸಿದ್ದರು. ಅವುಗಳ ಪರಿಶೀಲನೆಮಾಡಿದ್ದಲ್ಲದೆ, ಸಮಿತಿಗಳಾಚೆಗೆ ಅನೇಕ ಸಂವಾದಗಳನ್ನು ಏರ್ಪಡಿಸಿ ವಿಚಾರ ವಿನಿಮಯ ಮಾಡಿಕೊಂಡಿದ್ದೇವೆ. ಪ್ರಾಥಮಿಕ ಮತ್ತು ಪ್ರೌಢಶಾಲಾ ಅಧ್ಯಾಪಕರ ಸಂಘಗಳ ಜೊತೆ ಚರ್ಚಿಸಿದ್ದಲ್ಲದೆ ಪ್ರಶ್ನಾವಳಿ ಸಿದ್ಧಪಡಿಸಿ ಅಧ್ಯಾಪಕರಿಗೆ ಕಳಿಸಿ ಅಭಿಪ್ರಾಯಗಳನ್ನು ಸಂಗ್ರಹಿಸಲಾಗಿದೆ. ಅಧ್ಯಾಪಕರು, ವಿಷಯಪರಿವೀಕ್ಷಕರು ಮತ್ತು ಡಯಟ್ ಪ್ರಾಂಶುಪಾಲರ ಪ್ರತ್ಯೇಕ ಸಭೆಗಳನ್ನು ನಡೆಸಿ ವಿಶ್ಲೇಷಣಾತ್ಮಕ ಅಭಿಮತಗಳನ್ನು ಪಡೆಯಲಾಗಿದೆ. ವಿಜ್ಞಾನ, ಸಮಾಜವಿಜ್ಞಾನ, ಗಣಿತ, ಭಾಷೆ ಸಾಹಿತ್ಯಗಳಿಗೆ ಸಂಬಂಧಿಸಿದ ತಜ್ಞರಿಗೆ ಮೊದಲೇ ಪಠ್ಯಪುಸ್ತಕಗಳನ್ನು ಕಳುಹಿಸಿ ಆನಂತರ ಸಭೆ ನಡೆಸಿ ಚರ್ಚಿಸಲಾಗಿದೆ. ಮಹಿಳಾ ಸಂಘಟನೆ ಹಾಗೂ ವಿಜ್ಞಾನ ಸಂಬಂಧಿ ಸಂಸ್ಥೆಗಳನ್ನು ಆಹ್ವಾನಿಸಿ ಚಿಂತನೆ ನಡೆಸಲಾಗಿದೆ. ಹೀಗೆ ಎಲ್ಲಾ ಮೂಲಗಳಿಂದ ಪಡೆದ ಅರಿವಿನ ಆಧಾರದ ಮೇಲೆ ಪಠ್ಯಪುಸ್ತಕಗಳನ್ನು ಅಗತ್ಯವಿದ್ದಕಡೆ ಪರಿಷ್ಕರಿಸಲಾಗಿದೆ.

ಬಹುಮುಖ್ಯವಾದ ಇನ್ನೊಂದು ಸಂಗತಿಯನ್ನು ಇಲ್ಲಿ ಹೇಳಬೇಕು. ಕೇಂದ್ರೀಯ ಶಾಲಾ **(N.C.E.R.T)** ಪಠ್ಯಪುಸ್ತಕಗಳ ಜೊತೆ ರಾಜ್ಯದ ಪಠ್ಯಪುಸ್ತಕಗಳನ್ನು ತೌಲನಿಕವಾಗಿ ಅಧ್ಯಯನ ಮಾಡಿ ಸಲಹೆಗಳನ್ನು ನೀಡಲು ವಿಜ್ಞಾನ, ಗಣಿತ ಮತ್ತು ಸಮಾಜವಿಜ್ಞಾನಕ್ಕೆ ಸಂಬಂಧಿಸಿದಂತೆ ತಜ್ಞರ ಮೂರು ಸಮಿತಿಗಳನ್ನು ರಚಿಸಲಾಯಿತು. ಈ ಸಮಿತಿಗಳು ನೀಡಿದ ತೌಲನಿಕ ವಿಶ್ಲೇಷಣೆ ಮತ್ತು ಸಲಹೆಗಳನ್ನು ಆಧರಿಸಿ ರಾಜ್ಯ ಪಠ್ಯಪುಸ್ತಕಗಳ ಗುಣಮಟ್ಟವನ್ನು ಹೆಚ್ಚಿಸಲಾಗಿದೆ. ಕೇಂದ್ರೀಯ ಶಾಲಾ ಪಠ್ಯಪುಸ್ತಕಗಳಿಗಿಂತ ನಮ್ಮ ರಾಜ್ಯದ ಪಠ್ಯಪುಸ್ತಕಗಳ ಗುಣಮಟ್ಟ ಕಡಿಮೆಯಾಗದಂತೆ ಕಾಯ್ದುಕೊಳ್ಳಲಾಗಿದೆ. ಜೊತೆಗೆ ಆಂಧ್ರ, ತಮಿಳುನಾಡು, ಮಹಾರಾಷ್ಟ್ರಗಳ ಪಠ್ಯಪುಸ್ತಕಗಳ ಜೊತೆ ನಮ್ಮ ಪಠ್ಯಪುಸ್ತಕಗಳನ್ನು ಪರಿಶೀಲಿಸಲಾಗಿದೆ.

ಇನ್ನೊಂದು ಸ್ಪಷ್ಟನೆಯನ್ನು ನೀಡಲು ಬಯಸುತ್ತೇನೆ. ನಮ್ಮ ಸಮಿತಿಗಳು ಮಾಡಿರುವುದು ಪರಿಷ್ಕರಣೆಯೇ ಹೊರತು ಪಠ್ಯಪುಸ್ತಕಗಳ ಸಮಗ್ರ ರಚನೆಯಲ್ಲ. ಆದ್ದರಿಂದ ಈಗಾಗಲೇ ರಚಿತವಾಗಿರುವ ಪಠ್ಯಪುಸ್ತಕಗಳ ಸ್ವರೂಪಕ್ಕೆ ಎಲ್ಲಿಯೂ ಧಕ್ಕೆಯುಂಟುಮಾಡಿಲ್ಲ. ಲಿಂಗತ್ವ ಸಮಾನತೆ, ಪ್ರಾದೇಶಿಕ ಪ್ರಾತಿನಿಧ್ಯ, ರಾಷ್ಟ್ರೀಯ ಸಮಗ್ರತೆ, ಸಮಾನತೆ, ಸಾಮಾಜಿಕ ಸಾಮರಸ್ಯಗಳ ಹಿನ್ನೆಲೆಯಲ್ಲಿ ಕೆಲವು ಪರಿಷ್ಕರಣೆಗಳು ನಡೆದಿವೆ. ಹೀಗೆ ಪರಿಷ್ಕರಿಸುವಾಗ ರಾಷ್ಟ್ರೀಯ ಪಠ್ಯಕ್ರಮ ಚೌಕಟ್ಟು ಮತ್ತು ರಾಜ್ಯ ಪಠ್ಯಕ್ರಮ ಚೌಕಟ್ಟುಗಳನ್ನು ಮೀರಿಲ್ಲವೆಂದು ತಿಳಿಸಬಯಸುತ್ತೇನೆ: ಜೊತೆಗೆ ನಮ್ಮ ಸಂವಿಧಾನದ ಆಶಯಗಳನ್ನು ಅನುಸರಿಸಲಾಗಿದೆ. ಸಮಿತಿಗಳು ಮಾಡಿದ ಪರಿಷ್ಕರಣೆಯನ್ನು ಮತ್ತೊಮ್ಮೆ ಪರಿಶೀಲಿಸಲು ವಿಷಯವಾರು ಉನ್ನತ ಪರಿಶೀಲನ ಸಮಿತಿಗಳನ್ನು ರಚಿಸಿ ಅಭಿಪ್ರಾಯಪಡೆದು ಅಳವಡಿಸಲಾಗಿದೆ.

ಹೀಗೆ ಕ್ರಮಬದ್ಧವಾಗಿ ನಡೆದ ಕೆಲಸಕ್ಕೆ ತಮ್ಮನ್ನು ತಾವು ಸಂಪೂರ್ಣ ತೊಡಗಿಸಿಕೊಂಡ 27 ಸಮಿತಿಗಳ ಅಧ್ಯಕ್ಷರು ಮತ್ತು ಸದಸ್ಯರನ್ನು ಹಾಗೂ ಉನ್ನತ ಪರಿಶೀಲನಾ ಸಮಿತಿಯ ಸಮಸ್ತರನ್ನು ಕೃತಜ್ಞತೆಯಿಂದ ನೆನೆಯುತ್ತೇವೆ. ಅಂತೆಯೇ ಸಮಿತಿಗಳ ಕೆಲಸ ಸುಗಮವಾಗಿ ನಡೆಯುವಂತೆ ವ್ಯವಸ್ಥೆಮಾಡಲು ಕಾರ್ಯಕ್ರಮ ಅಧಿಕಾರಿಗಳಾಗಿ ನಿಷ್ಠೆಯಿಂದ ದುಡಿದ ಪಠ್ಯಪುಸ್ತಕ ಸಂಘದ ಎಲ್ಲಾ ಅಧಿಕಾರಿಗಳನ್ನೂ ನೆನೆಯುತ್ತೇವೆ. ಸಹಕರಿಸಿದ ಸಿಬ್ಬಂದಿಗೂ ನಮ್ಮ ವಂದನೆಗಳು. ಅಭಿಪ್ರಾಯ ನೀಡಿ ಸಹಕರಿಸಿದ ಸರ್ವ ಸಂಘಟನೆಗಳು ಮತ್ತು ತಜ್ಞರಿಗೆ ಧನ್ಯವಾದಗಳು.

ನರಸಿಂಹಯ್ಯ
ವ್ಯವಸ್ಥಾಪಕ ನಿರ್ದೇಶಕರು
ಕರ್ನಾಟಕ ಪಠ್ಯಪುಸ್ತಕ ಸಂಘ (ರಿ)
ಬೆಂಗಳೂರು-85

ಪ್ರೊ. ಬರಗೂರು ರಾಮಚಂದ್ರಪ್ಪ
ಸರ್ವಾಧ್ಯಕ್ಷರು
ಪಠ್ಯಪುಸ್ತಕ ಪರಿಷ್ಕರಣ ಸಮಿತಿ
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Part - I

INDIAN MATHEMATICS -A BRIEF INTRODUCTION

Indian Mathematics dates back to the **Vedic times**. The first significant mathematical texts of Vedic times are *Shulva Sutras*. *Shulva* is a sanskrit word for chord. These contain the details of construction of sacrificial altars. These ancient texts introduce **surds** of the type $\sqrt{2}, \sqrt{3}$, etc. (In fact most of the ancient mathematics was developed because of the interest in **Yajna and Yaga** and astrology.) *Baudhayana Sutra* and *Apastamba Sutra* give a very good approximation to $\sqrt{2}$ in the form

$$1 + \frac{1}{3} + \frac{1}{3 \times 4} - \frac{1}{3 \times 4 \times 3 \times 4}$$

which is correct up to 5 decimal places.

The classical Pythagoras' theorem is stated in the above *sutras*, far earlier than the Greeks discovered it. Another ancient unsolved problem known as *squaring a circle* finds its place in *Shulva sutras*. One is required to construct, using only a ruler and acompass, a square whose area is equal to that of the given circle. *Shulva sutras* give approximate methods for constructing such a square. This remained unsolved over two thousand years, and only in 18^{th} century it was proved that such a construction is impossible.

Indian mathematicians are credited with being the first to give an approximate value for π. Aryabhata I (476AD) gave an approximate value for π as 3.1416; he mentions that a circle of diameter 20000 units has circumference approximately equal to 62832 units. It is interesting to note that Aryabhata I clearly mentions, in the fifth century itself, that π is not rational and he is using its approximate value. Only in 1761, Lambart proved that π is an **irrational** number, and in 1882, Lindeman proved that π is, in fact, a **transcendental** number.

The most remarkable contribution for which the entire world still salutes India is the invention of the **decimal system** by introducing **zero** and **infinity**. If you really want to appreciate the simplicity of the decimal system and the concept of place value, you must first study the prevalent **Roman** system of representing numbers. According to Florian Cajori, an eminent historian of great repute " **of all the mathematical discoveries, no one has contributed more to the general progress of intelligence than Zero**." Using base 10, Indians were able to grasp very large numbers (See Unit 8 Exponents for more details).

Ancient Jain contribution to mathematics is another important milestone in the history of Indian Mathematics. Their findings are recorded in famous Jain texts, dating back to 500 BC to 200 BC. Here again, you see an approximate value for π as $\sqrt{10}$ and it is calculated up to 13 decimal places.

Ancient India has vastly contributed in the areas: the methods of *Arithmetic* called **Vyakthaganita**; The method of *Algebra* called **Avyakthaganita**. Ancient Indian Mathematicians had introduced all the four operations: addition, multiplication, subtraction and division. They also knew how to operate with fractions, solving simple equations, finding square and square-root, finding cube and cube-root, and also knew about permutation and combination.

Mahavira (9^{th} century AD), a great Jain mathematician from *Karnataka*, gave the well known formula

$$^{n}C_{r} = \frac{n!}{(n-r)!r!}$$

for the first time in the history of mathematics, in his *Ganita Sara Sangraha*. Aryabhata I is one of our greatest mathematicians and astronomers of all times. He is the one who systematically developed mathematics and is called, justifiably, the **father of Algebra**. He gave tables to trigonometric ratio Sine, called **jya** in Sanskrit, for angles from 0 to 90 degrees at intervals of $3\frac{3}{4}$ degrees. He was the first Indian to declare that Earth is round and that stars appear to move from East to West for a stationary observer on the Earth.

Aryabhata I was followed by Bhaskara I (6^{th} century AD) who provided interesting geometrical treatment for many algebraic formulae and gave a very good rational approximation for $\sin\theta$, even for large values of angle θ. Brahmagupta (628 AD) was the first to give a formula for the area of a cyclic quadrilateral in the form $\sqrt{(s-a)(s-b)(s-c)(s-d)}$, where a,b,c,d are the sides of the quadrilateral and s, the semi-perimeter. He is also the first mathematician to obtain cyclic quadrilaterals with rational sides.

Another important area, where our ancient mathematicians made significant contributions, is the solution of equations of the form $ax + by = c$, and $x^2 - Ny^2 = 1$ in integers, where a,b,c,N are given integers, and N is generally a square-free positive integer. Such equations are called Diophantine equations in modern terminology. The second equation is wrongly called **Pell**'s equation by the great Euler, but the name still continues. These days, many authors call the equation of the type $x^2 - Ny^2 = 1$ as Brahmagupta-Pell's equation. Aryabhata I discusses $ax + by = c$, whereas Brahmagupta made a significant contribution in the understanding of the equation $x^2 - Ny^2 = 1$. However, later, Bhaskara II (popularly called Bhaskaracharya) developed a new method, called **Chakravala method**, for solving $x^2 - Ny^2 = 1$ in integers. Here is an interesting thing about the equation $x - Ny^2 = 1$. In 1657, Fermat (who is famous for his contribution to number theory) proposed the problem of solving $x^2 - 61y^2 = 1$ for integers x, y, to European mathematicians. But none was able to solve this problem. In 1732, Euler gave a complete solution to this equation. But surprisingly, as a matter of divine coincidence, the same equation $x^2 - 61y^2=1$ was solved by Bhaskara II (1150 AD) in his *Bijaganitam* by his **chakravala** method more than 5 centuries before. The smallest solution of $x^2 - 61y^2=1$ given by Bhaskara II is $x = 226153980$ and $y = 1766319049$. Bhaskara II also introduced calculus, the concept of derivative, though not rigorously. He clearly mentions equivalent of $d(\sin x)= \cos x\, dx$.

We also see in the works of Bhaskara II and Mahavira very beautiful and interesting problems, rich in poetic imagination. Incidentally, Bhaskara II was born in the current **Vijayapura** district of **Karnataka** state (the exact place of his birth is still a point of debate among the scholars) and moved to the current **Maharashtra** state. Here we give two problems, originally stated in verses, whose translation runs as follows:

1. (From *Lilavati* of Bhaskaracharya)

   A beautiful maiden asks me which is the number when multiplied by 3, then increased by three-fourths of the product, divided by 7, diminished by one-third of the quotient, multiplied by itself, diminished by 52, the square-root found, addition of 8, division by 10 gives the number 2?

2. (From *Ganita Sara Sangraha* of Mahavira)

   Three merchants find a purse lying on a road. One merchant says, " If I keep the purse, I shall have twice as much money as the two of you together." "Give me the purse and I shall have three times as much" said the second merchant. The third merchant said, "I shall be better off than either of you if I keep the purse; I shall have five times as much as the two of you together." How much money is in the purse? How much money does each merchant have?

The achievements of Indian Mathematicians are remarkable in certain specific areas: arithmetic; theory of equations; spherical trigonometry and astronomy; geometrical treatment of algebraic equations; plane trigonometry; and mensuration. Unfortunately, after Bhaskara II, Indian Mathematics went into hibernation, except for a brief period during which the Kerala school by Nilakanta and Madhava made some important contributions in the area of series approximation to *tan* function. With Ramanujan, at the end of 19th century, it recovered its earlier glory. Ramanujan was a prodigious mathematician. In his brief span of 32 years, he made wonderful contributions to number theory, hyper-geometric series, divergent series, elliptic functions and integrals, and mock-theta function. Even today, world-wide mathematicians are trying to understand the depth of his mathematics and trying to prove the conjectures he made. It is also worthwhile to mention that Chadrashekhara Samantha of Orissa, at the end of 19th century, made some important contributions to astronomy.

(For more details about the Indian mathematics see *"Indian Mathematics and Astronomy: Some landmarks"* by Dr S Bala Chandra Rao, published by Bharatiya Vidya Bhavan, Bengaluru)

UNIT 1

PLAYING WITH NUMBERS

After studying this unit you learn:

- to write a given natural number in its general form (in base10).
- to formulate some games and puzzles involving numbers.
- given two positive integers, how to divide one by the other to get the quotient and the remainder.
- divisibility tests for 4,3,9,5,11.
- construction of a 3×3 magic square.
- about some unsolved problems involving numbers.

Introduction

Numbers have played an important role in the intellectual development of mankind. This still forms a perfect play-ground for activities of children. One can create simple puzzles which are brain-teasers. They can be used to play (mental) games among children. We shall explore some nice properties of numbers which help in engaging children with puzzles and these puzzles will help in arousing the curiosity of children. On the other hand the numbers can help us in placing some of the hitherto unproved conjectures and perhaps those will help you to further explore the wonderful universe of numbers.

You write 76 or 315 and say these are natural numbers. For example, you say that 6 is the digit in the unit's place of 76, and 7 is the digit in ten's place. Similarly, looking at 315, you say that 5 is in the unit's place, 1 is in the ten's place and 3 is in the hundred's place. You understand the place value of each digit, given a natural number. We explore a little bit more about these concepts and create puzzles using them.

Looking at the numbers 2, 24, 46, 88 or 122, you immediately say that these are even numbers and each is divisible by 2. Can you say whether a number is divisible by 3 by just looking at the number? Can you say that a number is divisible by 4,5, 9 or 11 without actually dividing it by these? Can you device simple rules which help us to decide whether a number is divisible by 3,4,5,9,11?

Numbers in general form

Consider the number 45. We write this as

$$45 = 40 + 5 = (4 \times 10) + (5 \times 1).$$

Similarly, $34 = 30 + 4 = (3 \times 10) + (4 \times 1)$. What can we do with 354? Observe

$$354 = 300 + 50 + 4 = (3 \times 100) + (5 \times 10) + (4 \times 1).$$

Activity 1: Write the following numbers in the form described as above: 75, 88, 121, 361, 1024, 2011, 4444, 2345.

Can you see that any natural number can be written in the above form? It is immaterial how many digits are there in the number. Suppose you have 123456789, a 9-digit number. You may write it as

$$\begin{aligned}123456789 &= 100000000 + 20000000 + 3000000 + 400000 \\ &\qquad + 50000 + 6000 + 700 + 80 + 9 \\ &= (1 \times 100000000) + (2 \times 10000000) + (3 \times 1000000) \\ &\qquad + (4 \times 100000) + (5 \times 10000) + (6 \times 1000) \\ &\qquad + (7 \times 100) + (8 \times 10) + (9 \times 1).\end{aligned}$$

You will learn later that this can be written in a compact form as

$$\begin{aligned}123456789 &= (1 \times 10^8) + (2 \times 10^7) + (3 \times 10^6) + (4 \times 10^5) + (5 \times 10^4) \\ &\qquad + (6 \times 10^3) + (7 \times 10^2) + (8 \times 10^1) + (9 \times 10^0).\end{aligned}$$

This is called the base 10 representation of the given natural numbers or the **generalised form** of the number. This system of representing a number using base 10 was invented by early Indian mathematicians.

Consider, for example, 136. You write this in the generalised form as:

$$136 = (1 \times 100) + (3 \times 10) + (6 \times 1).$$

Can you see that 6 is associated with 1; 3 is associated with 10; and 1 is associated with 100? This is the reason, 6 is called the digit in the unit's place; 3 is the digit in the ten's place; and 1 is the digit in the hundred's place.

Suppose you have a number abcd, with the digits in the unit's place, ten's place hundred's place and thousand's place respectively as d, c, b and a. Then its generalised form is

$$abcd = (a \times 1000) + (b \times 100) + (c \times 10) + (d \times 1).$$

To avoid the confusion that abcd may represent the product of a, b, c and d, the number is written in the form $\overline{abcd}$. Thus

$$\overline{abcd} = (a \times 1000) + (b \times 100) + (c \times 10) + (d \times 1).$$

Indian mathematics emerged in the Indian subcontinent from 1200 BC until the end of the 18th century and after that the modern era dawned. In the classical period of Indian mathematics (400AD to 1200 AD), important contributions were made by scholars like Aryabhata, Brahmagupta, and Bhaskara II. The decimal number system in use today and the binary number system were first recorded in Indian mathematics. Indian mathematicians made early contributions to the study of the concept of zero as a number, negative numbers, arithmetic, and algebra. These mathematical concepts were transmitted to the Middle East, China, and Europe and led to further developments that now form the foundations of many areas of mathematics.

All mathematical works were orally transmitted until approximately 500 BC; thereafter, they were transmitted both orally and in manuscript form. The oldest mathematical document produced on the Indian subcontinent is the birch bark Bakhshali Manuscript, discovered in 1881 in the village of Bakhshali, near Peshawar (modern day Pakistan).

The representation using base 10 is only a convenient thing. One can use different bases and represent numbers. For example computers use base 2 representation (called binary codes) and base 16 representation (hexadecimal codes). However, in daily life the use of decimal system (base 10 representation) is the most useful thing and the Indian contribution is forever remembered.

Exercise 1.1

1. Write the following numbers in generalised form:
   39, 52, 106, 359, 628, 3458, 9502, 7000.
2. Write the following in the decimal form:
   (i) $(5 \times 10) + (6 \times 1)$; **(ii)** $(7 \times 100) + (5 \times 10) + (8 \times 1)$;
   (iii) $(6 \times 1000) + (5 \times 10) + (8 \times 1)$; **(iv)** $(7 \times 1000) + (6 \times 1)$;
   (v) $(1 \times 1000) + (1 \times 10)$.

Some games and puzzles involving digits

Here we describe some properties of numbers which will help you to evolve a game to amuse your friends.

Game 1.

You can play a trick with your friend. You do this in several steps.

Step 1. Ask your friend to choose a 2-digit number in his mind and not to reveal it to you.

Step 2. Tell him to reverse the digits of the number he chose and get another number.

Step 3. Now tell him to add both the numbers and divide the sum by 11.

Step 4. Surprise him by telling him that the remainder is 0.

At no stage he reveals you the number or its reversal or their sum. Still you can conclude that the remainder of the sum when divided by 11 is zero.

For example suppose your friend chooses 41. The number obtained by reversing its digits is 14. Their sum is 41 + 14 = 55. When 55 is divided by 11, the remainder is 0.

Are you not curious how it works?

Suppose the two digit number is $\overline{ab}$. Then you know that $\overline{ab} = (a \times 10) + (b \times 1)$. The reversed number is $\overline{ba} = (b \times 10) + (a \times 1)$. Thus you get the sum of a number and its reversal as:

$$\overline{ab} + \overline{ba} = (a \times 10) + (b \times 1) + (b \times 10) + (a \times 1) = 11\,(a + b).$$

Now you see why the remainder when divided by 11 is zero.

Activity 2: You can create a game of your own. Instead of taking the sum of a 2-digit number and its reversal, suppose you take their positive difference. Take several examples, say, 21, 34, 86, 79, 95. Which divisor is common to all the differences: $21-12$, $43-34$, $86-68$, $97-79$, $95-59$? What is the game you can formulate?

Game 2.

This time, tell your friend to choose a 3-digit number and to keep it in his mind. Let him get the number obtained by reversing the digits of the original number and tell him to find the difference between the original number and the reversed number. Ask him to divide this difference by 99. You may surprise him by telling the remainder is zero, even if you do not know any thing about his choice. For example, if your friend chooses 891, the reversed number is 198 and their difference is $891-198=693=99\times 7$. Hence the remainder is zero after dividing the difference by 99.

Activity 3: Take several 3 digit numbers, say, 263, 394, 512, 765, 681, 898, 926. Find the difference between each number and the number obtained by its reversal. Find the remainder when the difference is divided by 99 in each case.

Why does this work in general? If the number chosen is $\overline{abc}$, then the reversed number is $\overline{cba}$. Thus

$$\begin{aligned}\overline{abc}-\overline{cba} &= (a\times 100)+(b\times 10)+(c\times 1)-(c\times 100)-(b\times 10)-(a\times 1)\\ &= (99\times a)-(99\times c)\\ &= 99\,(a-c).\end{aligned}$$

Hence you see that the difference is always divisible by 99.

Game 3.

Now you start with a 3-digit number, say 132. You can get two more numbers 213 and 321, by cyclically permuting the digits of 132. Add all of them; you get

$$132+213+321=666=18\times 37.$$

Repeat this with numbers 196, 225, 308, 446, 589, 678, 846. Do you observe that in each case the resulting number is divisible by 37? Can you now formulate a game using this property?

Statement 1.

Given a 3-digit number $\overline{abc}$, consider two more numbers obtained by cyclical permutation of its digits, namely, $\overline{bca}$ and $\overline{cab}$. Then 37 divides the sum $\overline{abc} + \overline{bca} + \overline{cab}$.

The proof is not hard. Let us see how it works for 132. We have

$$\begin{aligned} 132 &= (1 \times 100) + (3\times 10) + (2\times 1), \\ 321 &= (3 \times 100) + (2\times 10) + (1\times 1), \\ 213 &= (2 \times 100) + (1\times 10) + (3\times 1). \end{aligned}$$

Thus we get

$$\begin{aligned} 132 + 321 + 213 &= 1 \times (10 +100 + 1) + 3 \times (100 + 10 + 1) + 2 \times (100 + 1 + 10) \\ &= (1 + 3 + 2) \times 111 \\ &= 6 \times 3 \times 37. \end{aligned}$$

The same method works for any number $\overline{abc}$. Using the general form, you get

$$\begin{aligned} \overline{abc} &= (a \times 100) + (b \times 10) + (c \times 1), \\ \overline{bca} &= (b \times 100) + (c \times 10) + (a \times 1), \\ \overline{cab} &= (c \times 100) + (a \times 10) + (b \times 1). \end{aligned}$$

Hence

$$\begin{aligned} \overline{abc} + \overline{bca} + \overline{cab} &= (a \times 100) + (b \times 10) + (c \times 1) + (b \times 100) + (c \times 10) + (a \times 1) \\ &\quad + (c \times 100) + (a \times 10) + (b \times 1) \\ &= 111\,(a + b + c). \end{aligned}$$

But $111 = 37 \times 3$ and hence the number on the right side is divisible by 37. We may conclude that $\overline{abc} + \overline{bca} + \overline{cab}$ is divisible by 37.

Alpha numerals and puzzles

You can create puzzles involving numbers and letters of an alphabet. Look at the following examples.

Example 1. Find the digit represented by P in the following addition.

Solution:

$$\begin{array}{r} 4\ 1\ P \\ +\ Q\ 1\ 5 \\ \hline 5\ 2\ 6 \\ \hline \end{array}$$

You see that P, being a digit, cannot exceed 9. The only way you can arrive to 6 from 5 is adding 1. Hence $P = 1$. Similarly, you get $Q = 1$. You may check that $411 + 115 = 526$.

Example 2. Find the digits A and C in the following multiplication.

Solution:

$$\begin{array}{r} 3\ A \\ \times\ 1\ 2 \\ \hline C\ 8\ 4 \\ \hline \end{array}$$

Here the last digit of $2 \times A$ is 4. Hence either $A = 2$ or $A = 7$. Which one to choose?

Suppose $A = 2$. Then you get the product as $32 \times 12 = 384$. This shows that $C = 3$. On the other hand, if $A = 7$, then the product is $37 \times 12 = 444$. However, the digit in the ten's place of the result must be 8 and not 4. We may therefore reject $A = 7$. We conclude $A = 2$ and $C = 3$.

Example 3. In the following addition, A, B, C represent different digits. Find them and the sum.

Solution:

$$\begin{array}{r} A\ A \\ +\ B\ B \\ +\ C\ C \\ \hline B\ A\ C \\ \hline \end{array}$$

Here you observe that the last digit of $A + B + C$ is C, so that $A + B = 10$. (Why is that $A + B = 0$ not possible?) Since C is a digit, $C \leq 9$. Hence the carry from unit's place to ten's place is 1.

Since we are adding only 3 digits, the carry in ten's place of the sum cannot exceed 2. Hence B cannot be more than 2. Thus you observe that $B = 1$ or 2. The addition of digits in ten's place gives (along with carry 1 from unit's place) $A + B + C + 1 = 10 + C + 1$ and this must leave remainder A when divided by 10.

If $B = 1$, then $A = 9$ and hence $C + 1 = 9$ giving $C = 8$. We get $99 + 11 + 88 = 198$, which is a correct answer. If $B = 2$, you get $A = 8$ and $C + 1 = 8$ giving $C = 7$. But then $88 + 22 + 77 = 187$. This does not fit in as hundred's place in the sum is 1 but not 2.

The correct answer is $99+11+88 = 198$.

Exercise 1.2

1. In the following, find the digits represented by the letters:

(i)
$$\begin{array}{r} 3 \\ +\ B \\ \hline 7 \\ \hline \end{array}$$

(ii)
$$\begin{array}{r} 1\ 6 \\ +\ 2\ A \\ \hline B\ 1 \\ \hline \end{array}$$

(iii)
$$\begin{array}{r} 2\ A \\ \times\ \ A \\ \hline 12A \\ \hline \end{array}$$

(iv)
$$\begin{array}{r} 1\ A\ A \\ +\ 1\ A\ A \\ \hline 2\ A\ A \\ \hline \end{array}$$

(v)
$$\begin{array}{r} 1\ A \\ \times\ 1\ A \\ \hline 1\ B\ A \\ \hline \end{array}$$

(vi)
$$\begin{array}{r} 3\ A \\ \times\ \ A \\ \hline 2\ B\ A \\ \hline \end{array}$$

2. In the adjacent sum, *A, B, C* are consecutive digits. In the third row, *A, B, C* appear in some order. Find *A, B, C*.

$$\begin{array}{rccc} & A & B & C \\ + & C & B & A \\ + & - & - & - \\ \hline 1 & 2 & 4 & 2 \\ \hline \end{array}$$

Divisibility and remainders

One of the important properties related to integers is the concept of **divisibility.** In your earlier class, you have studied how to divide a natural number by another natural number to get a **quotient** and a **remainder.** If you divide 91 by 13, you see that 13 completely divides 91 and you do not get any remainder. On the other hand, dividing 85 by 15, you see that 15 × 5 = 75 and 15 × 6 = 90, so that you cannot divide 85 completely by 15. On actual division, you get 5 as quotient and 10 as remainder.

$$\begin{array}{l} 13\ )\ 91\ (\ 7 \\ \quad\ \ \ 91 \\ \quad\ \ \ \overline{00} \end{array} \qquad\qquad \begin{array}{l} 15\ )\ 85\ (\ 5 \\ \quad\ \ \ 75 \\ \quad\ \ \ \overline{10} \end{array}$$

Suppose you divide 304 by 12.The quotient is 25 and the remainder is 4. If you divide 887 by 17, you obtain the quotient 52 and the remainder 3.

$$\begin{array}{r|l} 12)\ 304\ (25 & \\ \underline{24} & \\ 64 & \\ \underline{60} & \\ 4 & \end{array} \qquad \begin{array}{r|l} 17)\ 887\ (52 & \\ \underline{85} & \\ 37 & \\ \underline{34} & \\ 3 & \end{array}$$

We write these divisions in the following form:

$$91 = (13 \times 7) + 0,$$
$$85 = (15 \times 5) + 10,$$
$$304 = (12 \times 25) + 4,$$
$$887 = (17 \times 52) + 3.$$

Do you observe that $0 < 13, 10 < 15, 4 < 12, 3 < 17$? Can you conclude that the remainder does not exceed the number from which you divide?

Activity 4: Find the quotient and the remainder in each of the following cases:

(i) 100 divided by 2,3,5,7,11,13,17,23,29 and 31.

(ii) 300 divided by 37, 41,43,47,53,59,61,67.

Our observation can be put in a formal way:

> **Given a non-negative integer a and another integer $b > 0$, there exist unique integers q and r such that $a = (b \times q) + r$, where $0 \le r < b$. We say b divides a if the remainder is zero, that is, $r = 0$.**

A similar statement can be made when a number is divided by another non-zero number. You will learn more about these in higher classes. The above statement can be used as a basis for a nice game which you can play with your friends.

Game 4.

Ask your friend to choose a number smaller than 1000. Tell him to divide this number by 7, 11, 13 respectively and ask for the remainders obtained by these divisions. Using these remainders, you can construct the number chosen by your friend.

Suppose your friend has chosen 128. Then the remainder when divided by 7 is 2; the remainder when divided by 11 is 7 and the remainder when divided by 13 is 11. Now form the sum

$$(2 \times 715) + (7 \times 364) + (11 \times 924).$$

If you simplify this, you get 14142. Divide this by 1001. You see that

$$14142 = (1001 \times 14) + 128,$$

so that the remainder is 128. This is the number chosen by your friend. Are you not thrilled?

These are the steps in this game.

Step 1. Tell your friend to choose a number less than 1000, in his mind.

Step 2. Tell him to divide the number by 7, 11, 13 and ask him to give you three remainders.

Step 3. Now you construct the number he thought of using the three remainders as follows. Suppose the remainders he gives you are r_1 (remainder when divided by 7), r_2 (remainder when divided by 11), and r_3 (remainder when divided by 13); multiply r_1 by 715, r_2 by 364 and r_3 by 924; take care that you are doing the correct multiplication. Add all three numbers so obtained and divide the resulting number by 1001. The remainder you obtain is the number chosen by your friend.

Take another example, say 212. Observe that

$$212 = (7 \times 30) + 2; \qquad 212 = (11 \times 19) + 3; \qquad 212 = (13 \times 16) + 4.$$

Thus $r_1 = 2$, $r_2 = 3$ and $r_3 = 4$. We obtain

$$(r_1 \times 715) + (r_2 \times 364) + (r_3 \times 924) = (2\times 715) + (3 \times 364) + (4 \times 924) = 6218.$$

Divide 6218 by 1001. The remainder is 212, the number you started with.

You may be wondering how such a game works. Suppose you start with an arbitrary number $a < 1000$. Let r_1, r_2, r_3 be the remainders when a is divided by 7, 11, 13 respectively. Then you can write

$$a = 7q_1 + r_1, \quad a = 11q_2 + r_2, \quad a = 13q_3 + r_3,$$

for some integers q_1, q_2, q_3. This shows that

$$r_1 = a - 7q_1, \quad r_2 = a - 11q_2, \quad r_3 = a - 13q_3.$$

Hence

$$\begin{aligned} 715r_1 + 364r_2 + 924r_3 &= 715(a - 7q_1) + 364(a - 11q_2) + 924(a - 13q_3) \\ &= a(715 + 364 + 924) - (7 \times 715)q_1 - (11 \times 364)q_2 - (13 \times 924)q_3. \end{aligned}$$

However you may notice that

$$\begin{aligned} 7 \times 715 &= 7 \times 11 \times 13 \times 5, \\ 11 \times 364 &= 11 \times 7 \times 13 \times 4, \\ 13 \times 924 &= 13 \times 7 \times 11 \times 12. \end{aligned}$$

And $1001 = 7 \times 11 \times 13$. Do you now understand why we consider the remainders when divided by 7, 11 and 13? Hence you get

$$715r_1 + 364r_2 + 924r_3 = a \times 2003 - 1001(5q_1 + 4q_2 + 12q_3).$$

You may also observe that $a \times 2003 = (a \times 1001 \times 2) + a$. When you divide $715r_1 + 364r_2 + 924r_3$ by 1001, you are left with a as all other terms are divisible by 1001. Since $a < 1000$, a is indeed the remainder. But that is the number you have started with.

Activity 5:

Check the game 4 with some more numbers: 804, 515, 676, 938, 97, 181.

Exercise 1.3

1. Find the quotient and the remainder when each of the following number is divided by 13

   8, 31, 44, 85, 1220.

2. Find the quotient and the remainder when each of the following number is divided by 304

   128, 636, 785, 1038, 2236, 8858.

3. Find the least natural number larger than 100 which leaves the remainder 12 when divided by 19.

4. What is the least natural number you have to add to 1024 to get a multiple of 181?

Divisibility tests

If a number ends with any of the digits 0, 2,4,6 or 8, you immediately say the number is divisible by 2. What is your reasoning? You write any such number a as $a = 10k+r$, where r is the remainder when divided by10. Hence r is one of the numbers 0,2,4,6,8. You now see that 10 is divisible by 2 and r is also divisible by 2. You conclude that 2 divides a.

It is natural to think whether such simple tests are available for divisibility by other numbers. We explore some of them here.

1. Divisibility by 4

If a number is divisible by 4, it has to be divisible by 2(why?). Hence the digit in the unit's place must be one of 0,2,4,6,8. But look at the following numbers: 10,22,34,46,58. You see the last digit in each of these numbers are as required, yet none of them is divisible by 4? Thus you may conclude that it is not possible to decide the divisibility on just reading the last digit. Perhaps, the last two digits may help.

If a number has two digits, you may decide the divisibility by actually dividing it by 4. All you need is to remember the multiplication table for 4. Suppose the given number is large, say it has more then 2 digits. Consider the numbers, for example, 112 and 122. You see that $112 = 100 + 12$, and both 100 and12 are divisible by 4. You may conclude that 112 is divisible by 4. But $122 = 100 + 22$; here 100 is divisible by 4, but 22 is not. Hence 122 is not divisible by 4. We invoke the following fundamental principle on divisibility:

Statement 1

If a and b are integers which are divisible by an integer $m \neq 0$, then m divides $a + b$, $a - b$ and ab.

How does this help us to decide the divisibility of a large number by 4? Suppose you have a number a with more than 2 digits. Divide this by 100 to get a quotient q and remainder r: $a = 100q + r$, where $0 \leq r < 100$. Since 4 divides 100, you will immediately see that a is divisible by 4 if and only if r is divisible by 4. But r is the number formed by the last two digits of a. Thus you may arrive at the following test:

> **Statement 2**
> A number a (having more than one digit) is divisible by 4 if and only if the 2-digit number formed by the last two digits of a is divisible by 4.

Example 4. Check whether 12456 is divisible by 4

Solution: Here, the number formed by the last two digits is 56. This is divisible by 4 and hence so is 12456.

Example 5. Is the number 12345678 divisible by 4?

Solution: The number formed by the last 2 digits is 78, which is not divisible by 4. Hence the given number is not divisible by 4.

Activity 8:

Ask your friend sitting adjacent to you to give several 4, 5 and 6 digit numbers. Test divisibility by 4 for them.

Activity 9:

By dividing several 4 and 5 digit numbers by 8, formulate a divisibility test by 8.

2. Divisibility by 3 and 9

Consider the numbers 2,23,234,2345, 23456, 234567. We observe that among these 6 numbers, only 234 and 234567 are divisible by 3. Here, we cannot think of the number formed by the last two digits, or for that matter even three digits. Note that 3 divides 234, but it does not divide 34. Similarly, 3 divides 456, but 3 does not divide 23456.

Activity 10: Write down numbers 1, 11, 21, 31, 41, . . . , 141, 151. (All numbers from 1 to 151 which ends in 1.) Form the sum of the digits of each number and tabulate them. Check which numbers are divisible by 3 and whether the digital sum of that number is also divisible by 3? What do you observe?

Consider the numbers 234 and 234567. The sum of the digits of the first number is 9 and that of the second is 27. You see that both 9 and 27 are divisible by 3. (In fact, divisible by 9.) Let us explore this in a general case for a 2-digit, 3-digit and 4-digit numbers. If $n = \overline{ab}$ is 2-digit number, then

$$n = \overline{ab} = (10 \times a) + b = 9a + (a + b).$$

This shows that n is divisible by 3 if and only if $a + b$ is divisible by 3. Similarly for $m = \overline{pqr}$, we have

$$m = \overline{pqr} = 100p + 10q + r = (99p + 9q) + (p + q + r),$$

and you may observe that m is divisible by 3 if and only if $p + q + r$ is divisible by 3. Do you observe that you can deduce divisibility test for 9 as well: 9 divides m if and only if 9 divides $p + q + r$? Now you do not have any problem to extend the test for a 4-digit number or a number with more digits. Observe that the sum of the digits of 234567 is 27. You can check that 9 divides 234567.

> **Statement 3**
> An integer a is divisible by 3 if and only if the sum of the digits of a is divisible by 3. An integer b is divisible by 9 if and only if the sum of the digits of b is divisible by 9.

Example 6. Check whether the number 12345321 is divisible by 3. Is it divisible by 9?

Solution: The sum of the digits is $1 + 2 + 3 + 4 + 5 + 3 + 2 + 1 = 21$. Hence the number is divisible by 3, but not by 9. In fact $12345321 = (9 \times 1371702) + 3$.

Example 7. Is 444445 divisible by 3?

Solution: The sum of the digits is 25, which is not divisible by 3. Hence 444445 is not divisible by 3. Here the remainder is 1.

3. Divisibility by 5 and 10

Activity 11: Take all multiples of 5 from 51 to 100. Tabulate the last digits of each multiple of 5.

Do you see that 0 or 5 appears as the digit in the unit's place for every multiple of 5? Does this observation help you to formulate a divisibility test for 5 and also for 10?

> **Statement 4**
> An integer a is divisible by 5 if and only if it ends with 0 or 5. A number is divisible by 10 if and only if ends with 0.

Example 8. How many numbers from 101 to 200 are divisible by 5?

Solution: Write down all numbers from 101 to 200 which end with 0 or 5: 105, 110, 115, 120, 125, 130, 135, 140, 145, 150, 155, 160, 165, 170, 175, 180, 185, 190, 195, 200. There are 20 such numbers.

Example 9. Is the number 12345 divisible by 15?

Solution: Note that $15 = 3 \times 5$. Hence the given number must be divisible by both 3 and 5. (This is also sufficient to prove that the given number is divisible by 15. However, a general rule is false. For example, 4 divides 12 and 6 divides 12, but their product 24 does not divide 12. Can you formulate some rule?) It is obviously divisible by 5, as its last digit is 5. The sum of the digits is $1+2+3+4+5 = 15$ and it is divisible by 3. Hence 3 also divides 12345. We conclude that 15 divides 12345.

Example 10. How many numbers from 201 to 250 are divisible by 5, but not by 3?

Solution: Here again, the numbers divisible by 5 are 205, 210, 215, 220, 225, 230, 235, 240, 245, 250. Now you compute the digital sum of these numbers: you get 7,3,8,4,9,5,10,6,11,7. Among these, the only numbers divisible by 3 are 3,6,9. Thus among the 10 numbers divisible by 5, only three numbers are also divisible by 3. The remaining 7 numbers are not divisible by 3.

3. Divisibility by 11

Consider the number 4587. You may check that it is divisible by 11. (In fact, $4587 = 11 \times 417$.) We may also write

$$\begin{aligned} 4587 &= (4 \times 1000) + (5 \times 100) + (8 \times 10) + 7 \\ &= (4 \times 1001) + (5 \times 99) + (8 \times 11) + (-4 + 5 - 8 + 7) \\ &= (11 \times 91 \times 4) + (11 \times 9 \times 5) + (11 \times 8) - (4 - 5 + 8 - 7). \end{aligned}$$

Each of the numbers in the first three brackets is divisible by 11. Hence the divisibility of 4587 by 11 is now related to the divisibility of $4 - 5 + 8 - 7$, which involves only the digits of the given number. The important point to note here is that the sign alternates with + and −. We also observe that $4 - 5 + 8 - 7 = 0$, which is divisible by 11.

Consider a 3 digit number 429. You may easily check that 429 is divisible by 11: $429 = 11 \times 39$. On the other hand.

$$\begin{aligned} 429 &= (4 \times 100) + (2 \times 10) + 9 \\ &= (4 \times 99) + (2 \times 11) + (4 - 2 + 9). \end{aligned}$$

Since $4 - 2 + 9 = 11$ is divisible by 11, you may conclude that 11 divides 429, without actually dividing it by 11.

How do you test a general 3-digit or 4-digit number? Suppose $n = \overline{abc}$ is a 3-digit number. Then

$$\begin{aligned} \text{n} &= 100a + 10b + \text{c} \\ &= 99a + 11b + (a - b + c). \end{aligned}$$

Thus n is divisible by 11 if and only if 11 divides $a - b + c$. If $m = \overline{pqrs}$ is a 4-digit number, then

$$\begin{aligned} \text{n} &= 1000p + 100q + 10r + s \\ &= 1001p + 99q + 11r - (p - q + r - s). \end{aligned}$$

Hence 11 divides n if and only if 11 divides $p - q + r - s$. This may be extended to numbers with any number of digits, with due care.

Statement 5

Given a number n in decimal form, put alternatively − and + signs between the digits and compute the sum. The number is divisible by 11 if and only if this sum is divisible by 11. Thus a number is divisible by 11 if and only if the difference between the sum of the digits in odd places and the sum of the digits in even places is divisible by 11.

Example 11. Is the number 23456 divisible by 11?

Solution: Observe that $2-3+4-5+6 = 4$ and hence not divisible by 11. The test indicates that 23456 is not divisible by 11. In fact $23456 = (11 \times 2123) + 4$.

A **palindrome** is a number which reads the same from left to right or right to left. Thus a palindrome is a number n such that by reversing the digits of n, you get back n. For example 232 is a 3-digit palindrome; 5445 is a 4-digit palindrome.

Example 12. Find all 3-digit palindromes which are divisible by 11.

Solution: A 3-digit palindrome must be of the form $\overline{aba}$, where $a \neq 0$

and b are digits. This is divisible by 11 if and only if $2a - b$ is divisible by 11. This is possible only if $2a - b = 0$ or $2a - b = 11$ or $2a - b = -11$. Since $a \geq 1$ and $b \leq 9$, we see that $2a - b \geq 2 - 9 = -7 > -11$. Hence $2a - b = -11$ is not possible. Suppose $2a - b = 0$. Then $2a = b$; thus $a = 1, b = 2$; $a = 2, b = 4$; $a = 3, b = 6$; and $a = 4, b = 8$ are possible. We get the numbers 121, 242, 363, 484. For $a = 6, b = 1$, we see that $2a - b = 12 - 1 = 11$, and hence divisible by 11. Similarly $a = 7, b = 3$; $a = 8, b = 5$; and $a = 9, b = 7$ give the combinations for which $2a - b$ is divisible by 11.We get four more numbers: 616, 737, 858, and 979.

Thus the required numbers are: 121, 242, 363, 484, 616, 737, 858, 979.

Example 13. Prove that 12456 is divisible by 36 without actually carrying out the division.

Solution: First notice that $36 = 4 \times 9$. Hence it is enough to prove that the given number is separately divisible by 4 and 9 (then it will be divisible by their LCM which is 36). Consider the number formed by the last two digits: 56. It is divisible by 4. Hence 12456 is divisible by 4. On the other hand the sum of the digits is 18 and divisible by 9. Hence 12456 is divisible by 9 as well. Combining, we get the result.

Exercise 1.4

1. Without actual division, using divisibility rules, classify the following numbers as divisible by 3, 4, 5, 11.

   803, 875, 474, 583, 1067, 350, 657, 684, 2187, 4334, 1905, 2548.

2. How many numbers from 1001 to 2000 are divisible by 4?

3. Suppose a 3-digit number $\overline{abc}$ is divisible by 3. Prove that $\overline{abc} + \overline{bca} + \overline{cab}$ is divisible by 9.

4. If $\overline{4a3b}$ is divisible by 11, find all possible values of $a + b$.

5. Prove that a 4-digit palindrome is always divisible by 11.

Magic squares

Can you arrange the numbers from 1 to 9 in 3 rows and 3 columns such that the sum of the numbers in each row, each column and each diagonal are all identical? Look at the following arrangement. (Fig. 1)

8	**1**	**6**
3	**5**	**7**
4	**9**	**2**

Fig. 1

6	**1**	**8**
7	**5**	**3**
2	**9**	**4**

Fig. 2

You observe that the sum of the numbers in each row is 15, the column sum is 15 and the diagonal sum is 15. You can also arrange the numbers as in Fig. 2. Can you see that there is a resemblance between these two magic squares? The middle column is (1, 5, 9) in both the squares. The right most column (8, 3, 4) in the first square is the left most column in the second. Similarly, the left most column (6, 7, 2) in the first square is the right most column in the second. Thus the second magic square is obtained by flipping the right most column and the left most column. The sum 15 is called the **magic sum.**

Is there a way of constructing such a magic square? Start with the middle cell of the topmost row and put 1 there. Thus we begin with the following:

	1	

Now we follow the following rules:

Rule 1. If there is a vacant cell along the diagonal from left to right, you fill it with the next number. (Here there is a vacant cell after 4 along the diagonal. Hence we fill it with 5.

4		

→

	5	
4		

Rule 2. If there is no vacant cell along the diagonal and if there are columns further, you fill the bottom cell of the next column with the next number. Follow Rule 1. (Here there is no vacant cell along the diagonal after 1. Hence we go to the bottom most cell in the next column and fill with 2.)

	1	

→

	1	
		2

Rule 3. If there is no vacant cell along the diagonal and if there are no columns further, go to the row above the cell you have reached and fill the left most cell of this row with the next number and follow Rule 1. (Here there is no vacant cell along the diagonal to move from 2 and we are already in the last column. Hence we move to the row above and fill the left most cell with 3.)

	1	
		2

→

	1	
3		
		2

Rule 4. If at any stage you encounter a cell which has already been filled, go to the cell below the cell you have reached and then continue with Rule 1. (Here you cannot move from 3 to the next cell along the diagonal since you have 1 already there. Hence we go to the cell below 3 and fill with this 4.)

	1	
3		

→

	1	6
3	5	
4		

Rule 5. If you are at the end of the main diagonal, fill the number below the last cell in the diagonal with the next number. And follow the appropriate rule further. (Here you have 6 at the end of the main diagonal. Hence we go to the cell below it and fill it with 7.)

	1	6
3	5	
4		

→

	1	6
3	5	7
4		

Let us see how it works for a 3×3 magic square. We start with the central cell of the first row and put 1 there. Now apply Rule 2, as we cannot move diagonally. We fill the bottom cell of the next column, that is column 3, with number 2. Again, we cannot move diagonally, nor do we have column further. We apply rule 3 and move to the row above and fill the left most cell with 3. Now you see that you cannot move along the diagonal as the cell there is already filled. Hence we use Rule 4 and go to the cell below the cell we are in. We fill this with 4 and move diagonally to fill 5 and 6. Again, we cannot move further and we are on the main diagonal. We use Rule 5 and fill the cell below the last cell on the diagonal with 7. Now we use Rule 3 and fill the left most cell of the row above with 8. Using Rule 2, we now fill the bottom most cell of the next column with 9. And you have the magic square!

The sequence of operations are shown below

	1	

→ Rule 2 →

	1	
		2

→ Rule 3 →

	1	
3		
		2

→ Rule 4 →

	1	
3		
4		2

→ Rule 1 →

	1	
3	5	
4		2

→ Rule 1 →

	1	6
3	5	
4		2

→ Rule 5 →

	1	6
3	5	7
4		2

→ Rule 3 →

8	1	6
3	5	7
4		2

→ Rule 2 →

8	1	6
3	5	7
4	9	2

Activity 6: Using the central cell of the first column as the starting point, construct a 3×3 magic square with numbers from 1 to 9. How does this compare with the magic square in Fig.1 ? What relation is there between the magic sum and the number in the central cell of the magic square?

Using the numbers from 3 to 11, construct a magic square.

Solution: We use the same sequence of operations as we have used earlier, but the starting number is 3 instead of 1.

	3	

→ Rule 2 →

	3	
		4

→ Rule 3 →

	3	
5		
		4

→ Rule 4 →

	3	
5		
6		4

→ Rule 1 →

	3	
5	7	
6		4

→ Rule 1 →

	3	8
5	7	
6		4

→ Rule 5 →

	3	8
5	7	9
6		4

→ Rule 3 →

10	3	8
5	7	9
6		4

→ Rule 2 →

10	3	8
5	7	9
6	11	4

Here the magic sum is 21.

Activity 7: Construct a 5 × 5 magic square using the above rules and with numbers from 1 to 25. What relation is there between the magic sum and the number in the central cell of the magic square?

> **Using the five rules, it is possible to construct an** $m \times m$ **magic square with the numbers from 1 to** m^2**, for any odd natural number** $m > 1$.

Exercise 1.5

1. Using the numbers from 5 to 13, construct a 3×3 magic square. What is the magic sum here? What relation is there between the magic sum and the number in the central cell?
2. Using the numbers from 9 to 17, construct a 3×3 magic square. What is the magic sum here? What relation is there between the magic sum and the number in the central cell?
3. Starting with the middle cell in the bottom row of the square and using numbers from 1 to 9, construct a 3 × 3 magic square.
4. Construct a 3 × 3 magic square using all odd numbers from 1 to 17.
5. Construct a 5 × 5 magic square using all even numbers from 1 to 50.

You have seen the first few primes are:

2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73....

There are infinitely many primes. Among these look at the pairs:

(3,5), (5,7), (11,13), (17,19), (29,31), (41,43), (59,61), (71,73).

These are primes which differ by 2. That is: the difference between them is 2. Such a pair of primes is called a *twin prime.* It is an outstanding unsolved problem: Are there infinitely many twin primes?

Another classical unsolved problem involving numbers is *Goldbach's conjecture.* It dates back to 1742. Consider even numbers greater than 2: observe $4 = 2+2$, $6 = 3 + 3$, $8 = 3 + 5$, $10 = 3 + 7$, $12 = 5 + 7$, $14 = 3 + 11$, $16 = 5 + 11$, and so on. It was conjectured by a German mathematician Goldbach that every even number greater than 2 is a sum of two primes. It has been extensively verified using computers and there is a strong suspicion that the conjecture is true. But no mathematical proof is available till today.

A natural number n is called a *perfect number* if it is equal to the sum of all its proper positive divisors.(That is the sum of all its positive divisors excluding the number itself.) The first perfect number is 6. It has three proper positive divisors, 1,2,3; and $1 + 2 + 3 = 6$. Similarly, 28 has proper divisors, 1,2,4,7,14; and $1 + 2 + 4 + 7 + 14 = 28$. The next numbers are 496 and 8128. All these were discovered by Euclid. Euclid also proved that $2^{p-1}(2^p - 1)$, with p a prime, is a perfect number whenever $2^p - 1$ is a prime. Prime numbers of the form $2^p - 1$ are called Mersenne primes (after a seventh-century mathematician called Marin Mersenne). It is easy to prove that $2^n - 1$ is prime only if n is prime. But not all numbers of the form $2^p - 1$, where p a prime, are prime numbers. For example, $2^{11} - 1 = 2047 = 23 \times 89$ and hence not a prime. Using this idea and computers, one can generate some perfect numbers. The numbers $2^{p-1}(2^p - 1)$, where p is equal to

2,3,5,7,13,17,19,31,61,89,107,127,521,607,1279,2203,2281

are the first few perfect numbers. There are some unsolved problems related to perfect numbers.

1. Are there infinitely many perfect numbers? (Equivalently, are there infinitely many Mersenne primes?)
2. All the perfect numbers so far known are even perfect numbers. This naturally raises the question: Is there any odd perfect number?

Glossary

Divisibility: an integer a is said to be divisible by another non-zero integer b, if $a = qb$ for some integer q.

Quotient: if $a = qb$, for some integers a, $b \neq 0$ and q, then q is the quotient upon division of a by b.

Remainder: if $a = bq + r$, where $0 \leq r < b$ for some integer a and natural number b, then r is the remainder upon division of a by b.

Palindrome: a number which reads the same from left to right and right to left.

Puzzle: any mind twister.

Alpha numeral: a letter appearing in an equation and taking some numerical value.

Conjecture: a statement which is believed to be true, but without any substantiating mathematical proof.

Magic square: a square consisting of smaller squares and each smaller square filled with numbers such that the row sum, the column sum and the diagonal sum are all equal.

Perfect number: a natural number whose all positive divisors smaller than the number add up to the number.

Twin-primes: pairs of prime numbers such that the difference of numbers in each pair is 2.

Mersenne primes: prime numbers of the form $2^p - 1$.

(If $2^p - 1$ is a prime, it forces p to be a prime.)

Points to remember

- Any natural number can be written in the generalised form using base 10.
- Given any two integers a and $b > 0$, there exist unique integers q and r such that $a = bq + r$, where $0 \leq r < b$.
- A number is divisible by 4 if and only if the number formed by the last two digits is divisible by 4.

- A number is divisible by 3 or 9 if and only if the sum of the digits is divisible by 3 or 9 respectively.
- A number is divisible by 5 if and only if it ends in 0 or 5.
- A number is divisible by 11 if and only if the difference between the sum of oddly placed digits and the sum of evenly placed digits is divisible by 11.

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Answers

Exercise 1.1

1. (i) $(3 \times 10) + (9 \times 1)$ (ii) $(5 \times 10) + (2 \times 1)$ (iii) $(1 \times 100) + (6 \times 1)$ (iv) $(3 \times 100) + (5 \times 10) + (9 \times 1)$ (v) $(6 \times 100) + (2 \times 10) + (8 \times 1)$ (vi) $(3 \times 1000) + (4 \times 100) + (5\times 10) + (8 \times 1)$ (vii) $(9 \times 1000) + (5\times 100) + (2\times 1)$ (viii) (7×1000). **2.** (i) 56 (ii) 758 (iii) 6058 (iv) 7006 (v) 1010.

Exercise 1.2

1. (i) $B = 4$ (ii) $A = 5$, $B = 4$ (iii) $A = 5$ (iv) $A = 0$ (v) two solutions. $A = 0$, $B = 0$ and $A = 1$, $B = 2$ (vi) $A = 6$, $B = 1$. **2.** $A = 3$, $B = 4$, $C = 5$.

Exercise 1.3

1. If $s \rightarrow (q, r)$ denotes the quotient q and remainder r, when s is divided by 13, then:

$8 \rightarrow (0,8)$; $31 \rightarrow (2,5)$; $44 \rightarrow (3,5)$; $85 \rightarrow (6,7)$; $1220 \rightarrow (93,11)$.

2. If $s \rightarrow (q,r)$ denotes the quotient q and remainder r, when s is divided by 304, then

$128 \rightarrow (0,128)$; $636 \rightarrow (2,28)$; $785 \rightarrow (2,177)$; $1038 \rightarrow (3,126)$; $2236 \rightarrow (7, 108)$; $8858 \rightarrow (29, 42)$.

3. 107. **4.** 62.

Exercise 1.4

2. 250 numbers. **4.** The only numbers in the required form and divisible by 11 are 4939, 4037, 4136, 4235, 4334, 4433, 4532, 4631, 4730. Hence $a + b = 18$ (in the case of 4939), or $a + b = 7$.

Exercise 1.5

1.

12	5	10
7	9	11
8	13	6

Magic sum is 27. Central number is 9. We have $27 = 3 \times 9$.

2.

16	9	14
11	13	15
12	17	10

Magic sum is 39. Central number is 13. We have $39 = 3 \times 13$.

3.

2	9	4
7	5	3
6	1	8

This is only one magic square. You can construct different magic squares for different positions of 1, as you have seen earlier.

4.

15	1	11
5	9	13
7	17	3

5.

34	48	2	16	30
46	10	14	28	32
8	12	26	40	44
20	24	38	42	6
22	36	50	4	18

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UNIT 2

ALGEBRAIC EXPRESSIONS

After studying this unit you learn:

- the meaning and types of polynomials.
- addition and subtraction of polynomials.
- multiplication of polynomials: monomials by monomials, binomial by monomials, binomial by binomial $(x + a)(x + b)$, $(a + b)^2$, $(a - b)^2$ and $(a + b)(a - b)$ etc.

Introduction

Let us first review some of the things you have learn earlier. A symbol which has a fixed value is called a **constant**.

Examples: 5, -7, $2\frac{3}{5}$, $\sqrt{5}$, $2+\sqrt{3}$, π, etc.

A symbol which does not have any fixed value, but may be assigned value (values) according to the requirement is called a **variable** or a **literal**.

Examples: p, q, x, y, z etc.

Note:

1. Combination of a constant and a variable is a variable.

Examples: $3x$, $(4+p)$, $\frac{6}{x}$, $\frac{x}{7}$, $x - 4$, $9x$ etc.

2. Combination of two or more variables is either a variable or a constant.

Examples: xy, $\frac{x}{y}$, $(x - y)$, $(y - x)$, $-x$, $(x + y)$, xyz, $\frac{xy}{z}$, $13 + x - y$, $14x - y$, $10 - xy$, $\frac{7x}{y}$, $\frac{8}{xy}$ etc. Observe $(4 + x) + (4 - x) = 8$ which is a constant.

A **term** is a number (constant), a variable, or a combination of (product or quotient of) numbers and variables.

Examples: $9, x, 3x, 4xy, \frac{7x}{15y}, \frac{21}{xy}, \frac{yz}{x}$ etc.

A single term or a combination of two or more terms connected by additive (both addition and subtraction) and multiplicative (both multiplication and division) symbols form an **algebraic expression**.

Examples: $7 - y$, $3x^2 - 4y$, $6xy$, $6+x^2 - 3x$, $(\frac{7x}{x})+4y - 6z$ etc.

Note : The signs of multiplication and division do not separate terms. For example: $9x^2 4y$ or $\frac{4x^2}{7y}$ are single terms.

An algebraic expression in which each term contains only the variable(s) with non negative integral exponent(s) is called a **polynomial**.

Examples: $x^2 - 4x$, $x - 4xy + y$, $6 - 5y + xy + xy$, 4.

A polynomial which contains *only one term* is called a **monomial**.

Examples: 4, $\frac{5}{11}$, x, $6x$, $8xy$, $7x^2y$, yzx, $5x^2yz$ etc.

A polynomial which contains two terms is called a **binomial**.

Examples: $7 + x$, $xy - 7$, $5xy - 3x$, $3x^2 - 6xy$, $yz^2 + 2z$.

A polynomial which contains three terms is called a **trinomial**.

Examples: $4 + x + y$, $6x + 15 - y$, $ax^2 + bx + c$, $ax + by + 2$.

Note: $\frac{x}{y} + 2$ is not a polynomial, it is only an algebraic expression.

Exercise 2.1

1. Separate the constants and variables from the following:

$12 + z$, 15, $\frac{-x}{5}$, $\frac{-3}{7}$, x, 3, $\frac{2}{3}xy$, $\frac{5xy}{2}$, 7, $7 - x$, $6x + 4y$, $-7z$, $\frac{8yz}{3x}$, $y + 4$, $\frac{y}{4}$ and $\frac{2x}{8yz}$.

2. Separate the monomials, binomials and trinomials from the following:

$7xyz$, $9 - 4y$, $4y^2 - xz$, $x - 2y + 3z$, $7x + z^2$, $8xy$, $\frac{8}{5}x^2y^2$, $4 + 5y - 6z$.

Algebraic expressions

Consider $9x$. It contains two factors: 9 and x. We say 9 is the **numerical coefficient** (or **constant** or **arithmetical coefficient**); x is called as **variable** (or **literal factor** or **literal coefficient**). Or consider $9xy$. Here there are two variables x and y. We say $9x$ is the coefficient of y and $9y$ is the coefficient of x. Observe the following tabular columns:

Product	Coefficient	Numerical Coefficient (constant)	Literal Coefficient (variable)
$-8xy$	$-8x$ is the coefficient of y	-8	x
	$-8y$ is the coefficient of x	-8	y
	xy is the coefficient of	–	xy
	-8 is the coefficient of xy	-8	–

Note:
1. If the literal factor (variable) has no sign, it is taken as positive.
2. If the variable has no power, it is taken as 1.
3. If the variable has no numerical coefficient, it is taken as 1. For example, x means $+1x$.
4. In a polynominal with single variable, the highest power of the variable is called the power of the polynomial.

Like and unlike terms

Terms having the same variable with same exponents are called **like terms**.

Examples:

$$5x,\ 2x,\ 7x,\ -9x,\ \frac{1}{3}x \text{ etc.;}$$
$$x^2,\ 2x^2,\ 6x^2,\ 9x^2,\ \frac{1}{7}x^2 \text{ etc.;}$$
$$x^3,\ 3x^3,\ 7x^3,\ -9x^3,\ \frac{1}{9}x^3 \text{ etc.}$$

Terms having the same variable with different exponents or different variables with same/different exponents are called **unlike terms**.

Examples: x, x^2, x^3, x^4, x^5 etc.; x, m, n, p etc.; $-x$, xy, xy^2 etc.

Addition and Subtraction of polynomials

Let us first review properties of addition and multiplication in the set of all integers:

1. Sum of two positive integers is a positive integer; $(+7) + (+5) = +7 + 5 = +12$.
2. Sum of two negative integers is a negative integer; $(-7) + (-5) = -7 - 5 = -12$.
3. Sum of a positive integer and a negative integer is positive if the absolute value of the negative integer is smaller than the positive integer; $(+7) + (-5) = +7 - 5 = +2$.
4. Sum of a positive integer and a negative integer is negative if the absolute value of the negative integer is larger than the positive integer; $(-7) + (+5) = -7 + 5 = -2$
5. Product of two positive integers is also a positive integer; $(+7) \times (+5) = +35$.

6. Product of two negative integers is a positive integer; $(-7)\times(-5) = +35$
7. Product of a positive integer and a negative integer is a negative integer; $(+7) \times (-5) = -35$.
8. Product of a negative integer and a positive integer is a negative integer; $(-7) \times (+5) = -35$.

Make the following rules for addition and subtraction of two polynomials:

1. like terms can be added or subtracted;
2. unlike terms cannot be added or subtracted;
3. while adding or subtracting like terms, their numerical coefficients are added or subtracted;

Example 1. Add $5x^2y$, $-7x^2y$ and $9x^2y$.

Solution:
$$\begin{aligned}(5x^2y) + (-7x^2y) + (9x^2y) &= (5 + (-7) + 9)\, x^2y \\ &= (5 - 7 + 9)\, x^2y \\ &= 7x^2y.\end{aligned}$$

This is **horizontal addition**. We can also have **vertical addition**:

$$\begin{array}{r} +5\,x^2y \\ -7\,x^2y \\ +9\,x^2y \\ \hline 7\,x^2y \\ \hline \end{array}$$

You may observe that we are adding the coefficients and retaining the variable term as it is.

Example 2. Add: $7x^2 - 4x + 5$ and $9x - 10$.

Solution: Here there are unlike terms. We can add only like terms. We write like terms one below the other to facilitate easy addition.

$$\begin{array}{rrr} 7x^2 & -4x & +5 \\ & +9x & -10 \\ \hline 7x^2 & +5x & -5 \\ \hline \end{array}$$

Example 3. Add $8xy + 4yz - 7zx$, $6yz + 11zx - 6y$ and $-5xz + 6x - 2yx$.

Solution: Here the again there are many unlike terms. We write like terms one below the other to facilitate easy addition. We are also using the commutative property: $xy = yx$ and $xz = zx$.

8xy	+4yz	$-7zx$		
	+6yz	$+11x$.		$-6y$
−2xy		−5zx	+ 6x	
$+6xy$	+10yz	$-xy$	+ 6x	$-6y$

Example 4. Subtract $2x^3 - x^2 + 4x - 6$ from $x^3 + 5x^2 - 4x + 6$.

Solution: We write like terms one below the other to facilitate easy subtraction. Note that we are subtracting relevant coefficients. Subtracting a negative number is equivalent to adding the negative of that number.

$+1x^3$	$+5x^2$	$-4x$	$+6$	→	**Minuend**
$+2x^3$	$-x^2$	$+4x$	-6	→	**Subtrahend**
(-2)	$(+1)$	(-4)	$(+6)$		
$-1x^3$	$+6x^2$	$-8x$	$+12$		

Hence we have changed the signs of coefficients in the subtrahend and added the coefficients. You can also do this in a quick way, once you understand the meaning of subtraction, as follows:

$$\begin{aligned}(x^3 + 5x^2 - 4x + 6) - (2x^3 - x^2 + 4x - 6) &= x^3 + 5x^2 - 4x + 6 - 2x^3 + x^2 - 4x + 6 \\ &= (1-2)\,x^3 + (5+1)\,x^2 + (-4-4)\,x + (6+6) \\ &= -1x^3 + 6x^2 - 8x + 12 \\ &= -x^3 + 6x^2 - 8x + 12.\end{aligned}$$

Exercise 2.2

1. **Classify into like terms:**

   $4x^2,\ \frac{1}{3}x,\ -8x^3,\ xy,\ 6x^3,\ 4y,\ -74x^3,\ 8xy,\ 7xyz,\ 3x^2$.

2. **Simplify:**

   (i) $7x - 9y + 3 - 3x - 5y + 8$; (ii) $3x^2 + 5xy - 4y^2 + x^2 - 8xy - 5y^2$.

3. **Add:**

   (i) $5a + 3b$, $a - 2b$ and $3a + 5b$; (ii) $x^3 - x^2y + 5xy^2 + y$, $-x^3 - 9xy^2 + y$, and $3x^2y + 9xy^2$.

4. **Subtract:**

   (i) $-2xy + 3xy^2$ from $8xy$; (ii) $a - b - 2c$ from $4a + 6b - 2c$.

Multiplication of Polynomials

Observe the following products: (i) $5x \times 6x^2 = (5 \times 6) \times (x \times x^2) = 30x^3$;

(ii) $2x \times 6y \times 8z = (2x \times 6y) \times (8z)$

$= ((2 \times 6) \times (x \times y)) \times (8z) = (12xy) \times (8z)$

$= (12 \times 8) \times (xy \times z) = 96xyz.$

We can also write this in one step: $2x \times 6y \times 8z = (2\times6\times8) \times (x\times y\times z) = 96xyz$.

> **Note:** coefficient of the product = the product of the coefficients of expressions;
>
> algebraic factor of the product = the product of all the algebraic factors.

Example 5. Find the product of $6x$ and $-7x^2y$.

Solution: We have $(6x)\times(-7x^2y) = (6\times(-7)) \times (x \times x^2y) = (-42)\,x^3y$.

> **Note:** We are using $x \times x^2y = (x \times x^2)\,y = x^3y$. In other words, we multiply similar variables and use the law of indices for simplifying the expression: $x^m \times x^n = x^{m+n}$ for all integers m,n, which you will study later.

Multiplying a monomial by a monomial

Example 6. Find the product of $4x \times 5y \times 7z$.

Solution: We have $4x \times 5y\times 7z = (4 \times 5 \times 7)\times(x \times y \times z) = 140xyz$.

Example 7. What is the product of $2l^2m \times 3lm^2$?

Solution: We have $2l^2m\times 3lm^2 = (2 \times 3)\times(l^2 \times l)\times(m \times m^2) = 6l^3m^3$. We are using the law of indices: $x^m \times x^n = x^{m+n}$.

Multiplying a monomial by a binomial

Consider the product $9\times 103 = 927$. We may also write this in the form

$$9 \times 103 = 9 \times (100 + 3) = (9 \times 100) + (9 \times 3) = 900 + 27 = 927.$$

Do you see that we have used the distributive property of multiplication over addition? We adopt the same strategy when a binomial is involved. Thus

$$2x\,(3x + 5xy) = ((2x)\times(3x)) + ((2x)\times(5xy)) = 6x^2 + 10x^2y.$$

Example 8. Determine the product $(8y+3) \times 4x$.

Solution: We have

$$\begin{aligned}(8y+3) \times (4x) &= (4x)\times(8y+3)\\ &= (4x \times 8y) + (4x)\times 3\\ &= 32xy + 12x.\end{aligned}$$

Here we have used the commutativity of the product and right distributivity. We can also use left distributive law and get

$$(8y + 3) \times (4x) = \big((8y) \times (4x)\big) + \big((3 \times (4x)\big)$$
$$= 32yx + 12x$$
$$= 32xy + 12x,$$

because $xy = yx$, the commutative property of the product.

Important points: We are using all the properties of numbers to the algebraic variables; associativity, commutativity, law of indices, distributive property. This is because, basically variables represent numbers when we substitute numbers for these variables, all these properties are true. As an extrapolation, we expect them to hold for variables as well.

Multiplying a binomial by a binomial

Consider the product of two binomials $(4a + 6b)$ and $(5a + 7b)$. We have

$$(4a + 6b)(5a + 7b) = 4a(5a + 7b) + 6b\,(5a + 7b)$$
$$= \big((4a)(5a)\big) + \big((4a)(7b)\big) + \big((6b)(5a)\big) + (6b)(7b)$$
$$= 20a^2 + 28ab + 30ab + 42b^2$$
$$= 20a^2 + 42b^2 + 58ab.$$

Exercise 2.3

1. Complete the following table of products of two monomials:

First → Second ↓	$3x$	$-6y$	$4x^2$	$-8xy$	$9x^2y$	$-11x^3y^2$
$3x$						
$-6y$						
$4x^2$						
$-8xy$						
$9x^2y$						
$-11x^3y^2$						

2. Find the products:
   (i) $(5x+8)3x$ (ii) $(-3pq)(-15p^3q^2-q^3)$ (iii) $\frac{2x}{5}(3a^3-3b^3)$ (iv) $-x^2(x-15)$.
3. Simplify the following:
   (i) $(2xy-xy)(3xy-5)$ (ii) $(3xy^2+1)(4xy-6xy^2)$
   (iii) $(3x^2+2x)(2x^2+3)$ (iv) $(2m^3+3m)(5m-1)$.

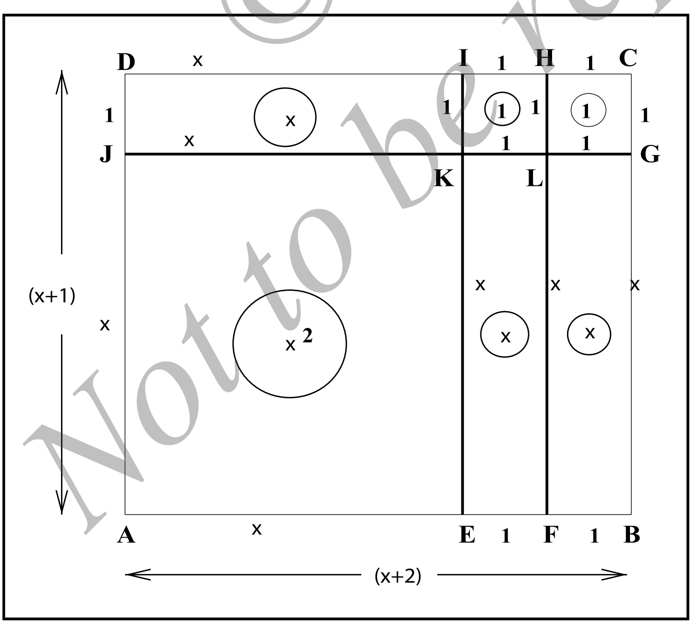
Special product

Now we study a special product, a product of two binomials. Consider the product:

$$
\begin{aligned}
(x+a)(x+b) = x(x+b) + a(x+b) &= x^2 + xb + ax + ab \\
&= x^2 + ax + bx + ab \\
&= x^2 + (a+b)x + ab.
\end{aligned}
$$

We have used commutative property and the distributive property: $xb = bx$, $(ax + bx) = (a + b)x$. We say $(x + a)(x + b) = x^2 + (a + b)x + ab$ is an **identity**. Thus

$$(x+2)(x+1) = x^2 + (2+1)x + (2 \times 1) = x^2 + 3x + 2.$$



We can also represent the identity $(x+2)(x+1) = x^2+3x+2$, pictorially. You observe that the area of the rectangle is $(x+2)(x+1)$. The rectangle is split in to smaller squares and rectangles: squares $AEKJ$, $KLHI$ and $LGCH$ having areas x^2, 1 and 1 respectively; rectangles $EFLK$, $FBGL$, $JKID$ having areas x, x, x respectively.

Thus the area of $ABCD$ is $x^2 + 1 + 1 + x + x + x = x^2 + 3x + 2$.

We thus obtain $(x+2)(x+1) = x^2 + 3x + 2$.

What is the value of $(x+a)(x+b)$? If we replace x by y what happens to the identity: $(x+a)(x+b) = x^2 + (a+b)x + ab$? Can you see that you again get an identity?

Example 9. Find the product of $(x+6)(x+7)$.

Solution: We observe that $(x+6)(x+7) = x^2+(6+7)x+(6\times7)$

$$= x^2+13x+42.$$

Example 10. Determine the product of $(x+8)(x-4)$

Solution: Using the identity $(x+a)(x+b) = x^2+(a+b)x+ab$, we get

$$(x+8)(x-4) = x^2+(8-4)x+8\times(-4)$$

$$= x^2+4x-32.$$

Example 11. Compute $(2x+5)(2x+3)$.

Solution: We know that $(x+a)(x+b) = x^2+(a+b)x+ab$. Thus

$$(2x+5)(2x+3) = (2x)^2+(5+3)(2x)+(5\times3)$$

$$= 4x^2+16x+15,$$

Example 12. Find the product of 103×98 using the above identity.

Solution: We observe that

$$\begin{aligned} 103 \times 98 &= (100+3)(100-2) \\ &= (100)^2 + ((3)+(-2))\,100 + ((3\times(-2))) \\ &= 10000 + (1\times100) + (-6) \\ &= 10094. \end{aligned}$$

where we have taken $x = 100$, $a = 3$ and $b = -2$ in the expansion:

$(x+a)(x+b) = x^2 + (a+b)x + ab$.

Example 13. Find the product of $(p^2 - 5)(p^2 - 3)$.

Solution: We have

$$(p^2 - 5)(p^2 - 3) = (p^4 + \left((-5)+(-3)\right)(p^2) + (-5) \times (-3)$$

$$= p^4 - 8p^2 + 15.$$

Identities

An **identity** is an equality which is true for every value of the variable in it.

For example $(x+3)(x+2) = x^2 + 5x + 6$ is an identity. If you give any value for the variable x, you see that the left side and right side coincide.

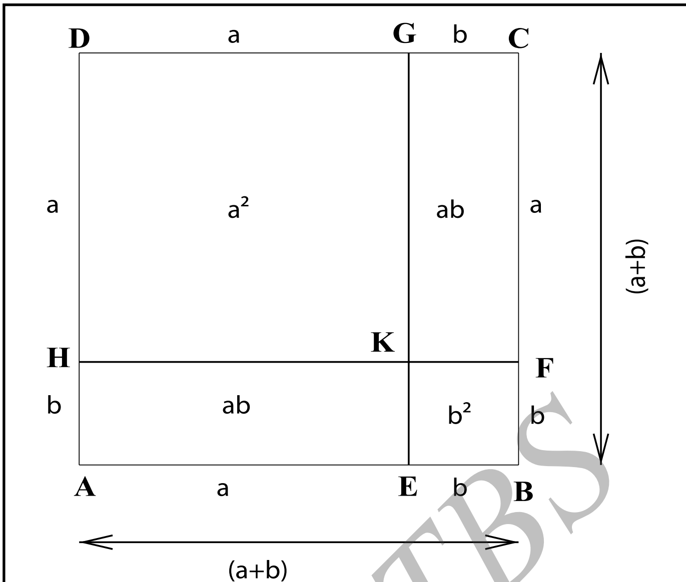
We have some special identities, which are helpful in solving problems. Consider

$$(a+b)^2 = (a+b)(a+b) = a(a+b) + b(a+b) = a^2 + ab + ba + b^2$$

$$= a^2 + ab + ab + b^2$$

$$= a^2 + b^2 + 2ab.$$

Observe that we have used the commutative property: $ab = ba$. We can have a pictorial proof of this identity using geometrical squares

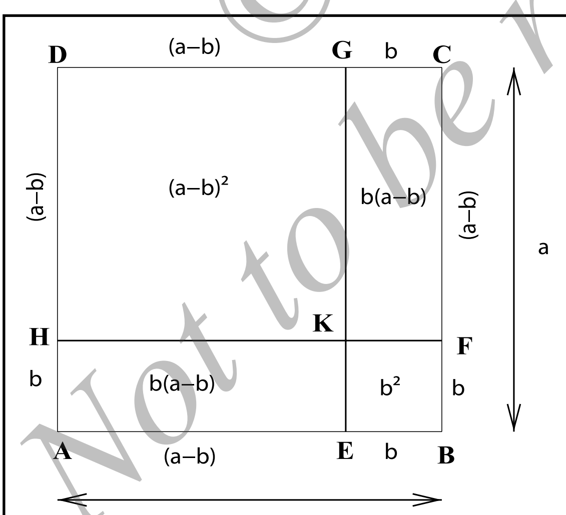


Observe that the square $ABCD$ has area equal to $(a+b)^2$, since its side-length is $a+b$. Now we divide the square in to two smaller squares and two rectangles; squares $HKGD$ having area a^2 and the square $EBFK$ having area b^2; the rectangle $KFCG$ having area ab and the rectangle $AEKH$ having area ab. Thus the area of $ABCD$ is $a^2 + b^2 + ab + ab = a^2 + b^2 + 2ab$.

Similarly we obtain

$$(a-b)^2 = a^2 + b^2 - 2ab.$$

And its pictorial proof is as follows.



Consider a square $ABCD$ with side-length a so that its area is equal to a^2 . Now we divide the square into two smaller squares and two rectangles; squares $HKGD$ having area $(a-b)^2$ and the square $EBFK$ having area b^2; the rectangle $KFCG$ having area $b(a-b)$ and the rectangle $AEKH$ having area $b(a-b)$.(We assume a is greater than b.) Now the area of $HKGD$ is obtained by subtracting the areas of $EBKF$, $KFCG$ and $AEKH$ from that of $ABCD$.

Hence we get

$$(a-b)^2 = a^2 - b^2 - b(a-b) - b(a-b) = a^2 - b^2 - ba + b^2 - ba + b^2$$
$$= a^2 - 2ab + b^2.$$

We also have

$$(a+b)(a-b) = a^2 - ab + ba - b^2 = a^2 - b^2.$$

These are called **standard identities**.

Example 14. Find $(2x+3y)^2$.

Solution: We use the identity: $(a+b)^2 = a^2 + 2ab + b^2$. Taking $a = 2x$ and $b = 3y$, we get

$$\begin{aligned}(2x+3y)^2 &= (2x)^2 + 2\,(2x)\,(3y) + (3y)^2 \\ &= 4x^2 + 12xy + 9y^2.\end{aligned}$$

Example 15. What is the expansion of $(4p-3q)^2$?

Solution: Here we use $(a-b)^2 = a^2 - 2ab + b^2$. Taking $a = 4p$ and $b = 3q$, we obtain

$$\begin{aligned}(4p-3q)^2 &= (4p)^2 - 2(4p)(3q) + (3q)^2 \\ &= 16p^2 - 24pq + 9q^2.\end{aligned}$$

Example 16. Compute $(4.9)^2$.

Solution: We can use identities for this problem. Observe

$$\begin{aligned}(4.9)^2 = (5-0.1)^2 &= 5^2 - 2(5)(0.1) + (0.1)^2 \\ &= 25 - 1 + 0.01 \\ &= 24.01.\end{aligned}$$

You may verify this by directly computing $(4.9)^2$.

Example 17. Compute 54×46.

Solution: Here again, identities are useful. We make use of

$(a+b)(a-b) = a^2 - b^2$. Taking $a=50$ and $b=4$,

$$\begin{aligned}54 \times 46 = (50+4)(50-4) &= (50)^2 - (4)^2 \\ &= 2500 - 16 \\ &= 2484.\end{aligned}$$

Activity 1:

Represent the identity $(a+b)(a-b) = a^2 - b^2$ pictorially as has been done in the case of other identities.

Exercise 2.4

1. Find the product:

   (i) $(a+3)(a+5)$ (ii) $(3t+1)(3t+4)$

   (iii) $(a-8)(a+2)$ (iv) $(a-6)\ (a-2)$.

2. Evaluate using suitable identities :

   (i) 53×55 (ii) 102×106 (iii) 34×36 (iv) 103×96.

3. Find the expression for the product $(x + a)\ (x + b)\ (x + c)$ using the identity $(x + a)\ (x + b) = x^2 + (a + b)\ x + ab$

4. Using the identity $(a + b)^2 = a^2 + 2ab + b^2$, simplify the following:

   (i) $(a + 6)^2$ (ii) $(3x + 2y)^2$ (iii) $(2p+3q)^2$ (iv) $(x^2+5)^2$.

5. Evaluate using the identity $(a + b)^2 = a^2 + 2ab + b^2$

   (i) $(34)^2$ (ii) $(10.2)^2$ (iii) $(53)^2$ (iv) $(41)^2$

6. Use the identity $(a - b)^2 = a^2 - 2ab + b^2$ to compute:

   (i) $(x - 6)^2$ (ii) $(3x - 5y)^2$ (iii) $(5a - 4b)^2$ (iv)$(p^2 - q^2)^2$

7. Evaluate using the identity $(a - b)^2 = a^2 - 2ab + b^2$

   (i) $(49)^2$ (ii) $(9.8)^2$ (iii) $(59)^2$ (iv) $(198)^2$

8. Use the identity $(a + b)(a - b) = a^2 - b^2$ to find the products:

   (i) $(x-6)\ (x+6)$ (ii) $(3x + 5)\ (3x + 5)$

   (iii) $(2a + 4b)\ (2a - 4b)$ (iv) $\left(\frac{2x}{3} + 1\right)\left(\frac{2x}{3} - 1\right)$.

9. Evaluate these using identity:

   (i) 55×45 (ii) 33×27 (iii) 8.5×9.5 (iv) 102×98

10. Find the product:

    (i) $(x-3)(x+3)(x2+9)$ (ii) $(2a+3)\ (2a-3)\ (4a^2+9)$ (iii) $(p+2)\ (p-2)\ (p^2+4)$

    (vi) $\left(\frac{1}{2}m - \frac{1}{3}\right)\left(\frac{1}{2}m + \frac{1}{3}\right)\left(\frac{1}{4}m^2 + \frac{1}{9}\right)$ (v) $(2x-y)\ (2x+y)\ (4x^2+y^2)$

    (vi) $(2x - 3y)(2x + 3y)(4x^2 + 9y^2)$

Glossary

Constant: any symbol which has a fixed value.

Variable: a symbol which can be given any value as desired.

Algebraic expression: a combination of constants and variables connected by algebraic operations.

Polynomial: an algebraic expression in which the variables have non-negative integral powers.

Term: a part of an algebraic expression which does not involve addition and subtraction, but may be connected by multiplication and division.

Coefficient: the companion term of a variable.

Monomial: a polynomial containing only one term.

Binomial: a polynomial containing two terms.

Trinomial: a polynomial containing three terms.

Degree: the largest power of a variable in a polynomial (if it has more than one variable, then one has to take the sum of the powers of variables in each term and take the maximum of all these sums).

Identity: equality of two algebraic expressions which is valid for all the values of the variables in it.

Points to remember

- An expression involving variables and constants combined using the algebraic operations, namely addition, multiplication, subtraction and division is an **algebraic expression**.
- While adding two algebraic expressions, we add only like terms.
- While multiplying two expressions, we multiply term-by-term and use the laws of exponents to simplify it.
- In a polynomial, the powers of variable(s) in it are non-negative integers.
- An identity is an equality of two algebraic expressions which is valid for all values of the variable(s) in it.

✸✸✸✸✸

Answers

Exercise 2.1

1. Constants: $15, \frac{-3}{7}, \sqrt{3}, 7$; **Variables:** $12+z, \frac{-x}{5}, \sqrt{x}, \frac{2}{3}xy; \frac{5xy}{2}, 7-x,$ $6x+4y, -7z, \frac{8yz}{4x}, y+4, \frac{y}{4}, \frac{2x}{8yz},$

2. Monomials: $7xyz$, $8xy$, $\frac{8}{5}x^2y^2$; **Binomials:** $9-4y$, $4y^2-xz$, $7x+z$;
Trinomials: $x-2y+3z$, $4+5y-6z$.

Exercise 2.2

1. $\{4x, 3x^2\}$, $\{xy, 8xy\}$, $\{-8x, 6x, -74x^3\}$, $\left\{\frac{1}{3}x\right\}$, $\{7xyz\}$.

2. (i) $4x-14y+11$; (ii) $4x^2-3xy-9y$. **3.** (i) $9a+6b$; (ii) $2x^2y+5xy^2+2y^3$.

4. (i) $10x^2y-3xy^2$; (ii) $3a+7b$.

Exercise 2.3

1.

First → Second ↓	$3x$	$-6y$	$4x^2$	$-8xy$	$9x^2y$	$-11x^3y^2$
$3x$	$9x^2$	$-18xy$	$12x^3$	$-24x^2y$	$27x^3y$	$-33x^4y^2$
$-6y$	$-18xy$	$36y^2$	$-24x^2y$	$48xy^2$	$-54x^2y^2$	$66x^3y^3$
$4x^2$	$12x^3$	$-24x^2y$	$16x^4$	$-32x^3y$	$36x^4y$	$-44x^6y^2$
$-8xy$	$-24x^2y$	$48xy^2$	$-32x^3y$	$64x^2y^2$	$-72x^3y^2$	$88x^4y^3$
$9x^2y$	$27x^3y$	$-54x^2y^2$	$36x^4y$	$-72x^3y^2$	$8x^4y^2$	$-99x^5y^3$
$-11x^3y^2$	$-33x^4y^2$	$66x^3y^3$	$-44x^5y^2$	$88x^4y^3$	$-99x^5y^3$	$121x^6y^4$

2. (i) $15x^2+24x$; (ii) $45p^4q^3+3pq^4$; (iii) $\frac{6}{5}a^3x \;\; \frac{6}{5}b^3x$; (iv) $-x^3+15x$.

3. (i) $6x^3y^2-10x^2y-3x^2y^2+5xy$; (ii) $12x^3y^3-18x^3y^4+4xy-6xy^2$;

(iii) $6x^4+4x^3+9x^2+6x$; (iv) $10m^3+12m^2-3m$.

Exercise 2.4

1. (i) $a^2+8a+15$; (ii) $9t^2+15t+4$; (iii) $a^2-6a-16$; (iv) $a^2-8a+12$. **2.** (i) 2915; (ii) 10812; (iii) 1224; (iv) 9888. **3.** $x^3+(a+b+c)x^2+(ab+bc+ca)x+abc$.
4. (i) $a^2+12a+36$; (ii) $9x^2+12xy+4y^2$; (iii) $4p^2+12pq+9q^2$; (iv) x^4+10x^2+25.

5. (i) 1156; (ii) 104.04; (iii) 2809; (iv) 1681. **6.** $x^2-12x+36$; (ii) $9x^2-30xy+25y^2$; (iii) $25a^2-40ab+16b^2$; (iv) p^4-2pq^2+q.

7. (i) 2401; (ii) 96.04; (iii) 3481; (iv) 39204. **8.** x^2-36; (ii) $9x^2-25$;
(iii) $4a^2-16b^2$; (iv) $\left(\frac{4x^2}{9}-1\right)$. **9.** (i) 2475; (ii) 851; (iii) 80.75;

(iv) 9996. **10.** (i) x^4-81; (ii) $16a^4-81$; (iii) p^4-16; (iv) $\left(\frac{1}{16}m^4-\frac{1}{81}\right)$
(v) $16x^4-y^4$; (vi) $16x^4-81y^4$

✡✡✡✡✡

UNIT 3

AXIOMS, POSTULATES AND THEOREMS

After studying this unit you learn:

- the meaning of undefined terms, axioms, postulates and hypothesis.
- that the lines, points, plane, space are undefined terms in Euclidean geometry.
- various types of angles and relation among these angles.
- the properties of parallel lines and about Euclid's fifth postulate.

Introduction

In earlier classes, you have studied many geometrical objects: straight lines, triangles, quadrilaterals and circles. You have also studied some geometrical properties of these objects: angles and different types of angles; triangle inequality (the sum of two sides of a triangle is greater than the third side); medians; altitudes; area of a triangle and a circle. Most of these are taught to you through observations.You may wonder that these were developed by our ancestors more than 2000 years ago.

Indeed, the concept of geometry is very old. Egyptian civilisation developed the early geometrical methods and measurements. In fact, geometry is derived from two Greek words: **Geo** to mean Earth and **metron** meaning measurement. When the Nile river flooded the whole region, the cultivated land used to submerge in water erasing all the boundaries. Hence Egyptians developed certain geometrical methods to demarcate the boundaries afresh. They also introduced area of plane figures and volume of some three dimensional objects which were used as granaries. Perhaps **Pyramids**, which still occupy a place among *seven wonders* of the world and whose construction is certainly one of the greatest human achievements, will give you an idea how much Egyptians were advanced in the use of geometry.

Thus the ancient geometry developed through the practical requirement of measuring land. However, a systematic treatment of geometry started with ancient Greek around 2500 years ago. They are the first one to realise the need to conceptualise the geometrical ideas.

The practical geometry used point, line and plane without bothering much what they mean. But, the Greek Philosophers and Mathematicians were more interested in proving statements by **deductive reasoning**. It was, perhaps, Thales (640BC-546BC) who first introduced the concept of **proof**. He realised the need for proving a statement by logical reasoning. Many more Greeks, like Appolonius, Plato, Pythagoras, Diophantus and Ptolemy made enormous contributions to the systematic development of Geometry and other areas of Mathematics and laid the firm foundation to make Mathematics a science of logical reasoning.

However, it was **Euclid** who collected all these contributions to Geometry and other branches of mathematics into **thirteen volumes** of a book called **the Elements** along with his own original ideas.

Euclid (around 300 BC) was a Greek mathematician often called the **Father of Geometry**. He was a contemporary of Ptolemy (323 BC-283 BC), another famous Greek mathematician of antiquity. Euclid's work **Elements** is one of the most influential work in the whole history of mathematics which changed the face of Mathematics laying the foundation for the future development.

Euclid deduced his results, what is now known as **Euclidean Geometry**, from a small number of principles called **Axioms and Geometrical postulates**. Euclid has also contributed to other branches of Mathematics. His proof that there are infinitely many prime numbers is a classic example of the deductive reasoning that Euclid employed in his works. Nothing much is known about Euclid's life.

The date and place of his birth are unknown. Like-wise, the date and circumstances of his death are also unknown. All we know about him is through the references made by other people in their work. The picture of Euclid we have today is also through the imagination of an artist.

In ancient India, **Sulva Sutras** are perhaps the first record of Mathematical lore, especially geometry (600 BC to 300 BC). These are records of the mathematical principles developed during Vedic period and subsequent time. The **Sulva Sutras** contain several geometrical principles.

The Indian geometry developed out of religious needs of constructing **sacrificial altars** to propitiate Gods and later for the study of **Eclipses**. The **Baudhayana Sutra**, the most ancient among the **Sulva Sutras** says *the diagonals of a rectangle bisect each other*. This also contained the idea of the celebrated **Pythagoras's theorem**, but unfortunately no proof was given.

The **Sulva Sutras** also gave methods of constructing a square whose area equals the area of a given circle. The construction involved approximating π and the approximation used was 3.088, which is quite close to the present day approximations to π.

With the passage of time, others also made significant contributions: Aryabhata I, Bhaskara I, Varahamihara Brahmagupta, Mahaviracharya. Bhaskaracharya II, Madhava, Nilakantha Somayaji, and many others made innumerable contribution to the advancement of Mathematics.

Axioms and Postulates

You have seen that a straight angle is defined as the angle measuring 180 degrees. You use a protractor to measure angle. However, the calibration of protractor is such that when you place it on a straightline, you read 180 degrees. Thus the protractor is designed to measure 180 degrees when a straight angle is given. Can you see that you have to go around from protractor to straight angle to protractor?

This was the major difficulty faced by Greek mathematicians while developing Geometry as a pure deductive science. They had to depend on certain primitive notions like points, straight lines and planes and space. But this was not enough to deduce everything. They had to set up certain statements, whose validity was accepted unquestionably, applicable to geometry alone. They had to depend on some more statements applicable to all of mathematics and science in general and Geometry in particular.

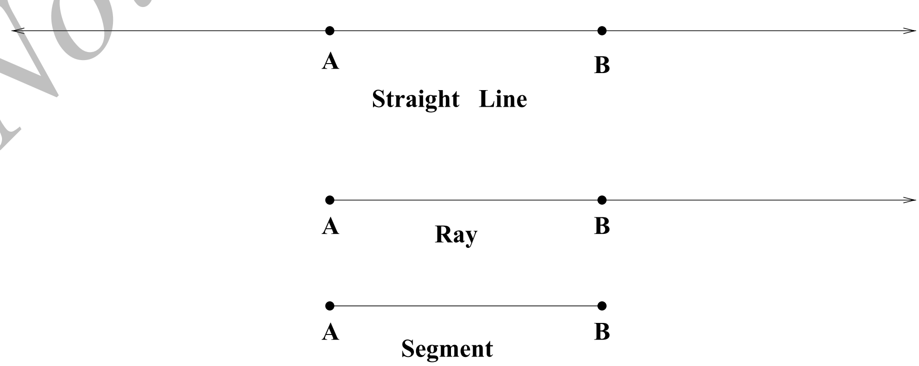
The general statements which are accepted without question and which are applicable to all branches of science are commonly referred as **Axioms**.

The statements which are particular to Geometry and accepted without question are called **Postulates**. Any result you further prove depends on these axioms and postulates.

Another problem with the deductive method is how to define some geometrical terms. For example, you all have an intuitive idea what a point is. But can you define a point? When you define some thing, you must do it so using what you know already. Hence you have to depend on some undefined terms.

In Euclid's geometry, the undefined terms are *point, line, plane*. They are only certain **abstract ideas**. Thus you cannot see a *point*. If you take a sharp pencil and make a dot on a paper, that approximately resembles a point. Similarly, you cannot see a *line*. When a point moves in both the *directions*, it produces a *straight line*. A *line* is *endless*.

If A and B are points on a line, we denote the straight line by $\overleftrightarrow{AB}$. If a straight line is cut, you get two pieces: each piece is called a *ray*. Thus a ray has an *initial point* and extends *indefinitely* in one direction. If A is the initial point of a ray and B is any other point on a ray, we denote the ray by $\overrightarrow{AB}$. Take a line and choose any two points A and B on it. The part of the line between A and B is called a *line segment* and is denoted by $\overline{AB}$.

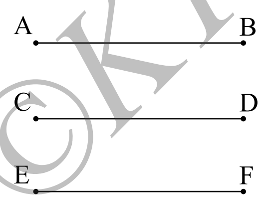


Likewise, you cannot define a plane. Intuitively, a *plane* is flat infinite surface without any *thickness*. A black board or the surface of still water in a big tank resemble finite part of a plane. Thus, there are *undefined* terms in Euclid's geometry. With suitable axioms and geometrical postulates, Euclid's geometry tells you what edifice can be built. Let us study these axioms and postulates.

I. Axioms

There are certain elementary statements, which are self evident and which are accepted without any questions. These are called **axioms**. These statements are also applicable to other areas of mathematics and science. Euclid used the following statements which he called **Common Notions**.

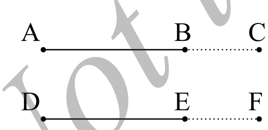
Axiom 1: Things which are equal to the same thing are equal to one another.



AB = CD, EF = CD
implies ($\Rightarrow$) AB = EF

Suppose you have three baskets A, B and C having mangoes, oranges and bananas. Suppose A and B have equal number of fruits and B and C also have equal number of fruits. Can you conclude that A and C have equal number of fruits?

Axiom 2: If equals are added to equals, the wholes are equal.

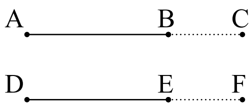


AB=DE and BC=EF
$\Rightarrow$ AC=DF

Suppose you have two line segments $\overline{AB}$ and $\overline{DE}$ of equal length. Add $\overline{BC}$ to $\overline{AB}$ and add $\overline{EF}$ to $\overline{DE}$. If BC = EF, then AC = DF.

Take a basket A of 10 mangoes and a basket B of 10 oranges. Add 5 apples to both of these baskets. Do you see that the number of fruits in both the baskets are equal? (equal to 15)

Axiom 3: If equals are subtracted from equals, then the remainders are equal.



$AC = DF$ and $BC = EF$

$\Rightarrow AB = DE$

Suppose you have two line segments $\overline{AC}$ and $\overline{DF}$ of equal length. Remove $\overline{BC}$ from $\overline{AC}$ and $\overline{EF}$ from $\overline{DF}$ respectively. If $BC = EF$, then $AB = DE$.

Take a basket A of 10 mangoes and B of 10 oranges. Remove 2 fruits from each basket. Then A and B again have equal number of fruits.

Axiom 4: Things which coincide with one another must be equal to one another.

This means that if two geometric figures can fit completely one into other, then they are *essentially* the same.

Axiom 5: The whole is greater than the part.

If AC is a part of AB, then $AB > AC$

Take a container of water. Remove some water from it. Will the remaining volume of water the same as the original volume?

Euclid's common notions are these first five axioms. The first three concern "equals" or "equal things"; the fourth is interpreted now to mean that if two figures, such as line segments, angles, triangles, or circles, are such that one can be moved to coincide with the other, the figures are equal. In modern language they are called congruent. Two ideas run through these common notions: (1) that geometrical figures can be treated as magnitudes, and (2) that, if one figure is visibly part of another (perhaps after a motion), then the magnitude of the part is less than the magnitude of the whole. For adding and comparing, you should have magnitudes of the same kind. For example, you cannot add area to length. Or you cannot compare a triangle with a point. Axiom 5 can be used to define *greater than*.

If b is a part of a, then a is greater than b. Again this comparison is between magnitudes of the same kind. (For example, you cannot take out a point from a line and say *line is greater than a point* which is clearly meaningless). Since *point* has no magnitude of any kind, you cannot compare two points. But you can say some line segment is larger than some other line segment as you can compare their lengths.

II. Postulates

Apart from these **common notions**, Euclid also made the following **postulates** to deduce new **propositions**.

Postulate 1. A straight line segment can be drawn joining any two points.

Postulate 2. Any straight line segment can be extended indefinitely in a straight line.

Postulate 3. Given any straight line segment, a circle can be drawn having the segment as radius and one end point as center.

Postulate 4. All right angles are congruent.

Postulate 5. If a straight line meets two other lines, so as to make the two interior angles on one side of it together less than two right angles, the other straight lines will meet if produced on that side on which the angles are less than two right angles.

The **Postulate 5** is the famous **Euclid's parallel postulate**. What it asserts is that two distinct straight lines in a plane are either *parallel* or meet exactly in one point. This postulate cannot be proved as a theorem, although this was attempted by many people. Euclid himself used only the first four postulates ("absolute geometry") for the first 28 propositions of the **Elements**, but was forced to invoke the parallel postulate on the 29th.

In 1823, **Janos Bolyai** and **Nicolai Lobachevsky** independently realized that entirely self-consistent "non-Euclidean geometries" could be created in which the parallel postulate did not hold. The parallel postulate is actually equivalent to: any two straight lines in the plane either do not meet at all or meet in one point.

While proving his propositions, Euclid made several tacit assumptions. For example, Postulate 1 says that there is a line passing through any two given points. But what Euclid had in his mind seems to be given any two distinct point in the plane, there is a *unique* line passing through these two points. On the other hand, you can draw *infinitely many* lines through a given point in the plane.

Postulate 2 says that given a line segment in the plane, this can be extended to a unique straight line. Postulate 4 is concerned about *right angles*. But this is not defined any where by Euclid. What he seems to have thought was that *angle by a straight line* is made up of two right angles.

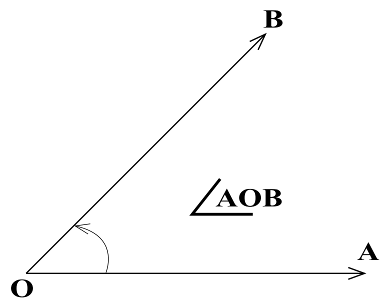
By present day standards, there are several inconsistencies in Euclid's *Elements*. Nevertheless, it is definitely **the first book**, based on rigorous mathematical principles. In recent years, many attempts have been made to introduce new set of undefined objects, axioms and postulates, so that the terms hitherto undefined could be defined using these new objects, axioms and postulates. The advancement of set theory and axioms on number systems make it possible to define a *point*,a *line* or a *plane* using **coordinate systems**.

Exercise 3.1

1. What are undefined objects in Euclid's geometry?
2. What is the difference between an axiom and a postulate?
3. Give an example for the following axioms from your experience:

   (a) If equals are added to equals, the wholes are equal.
   (b) The whole is greater than the part.
4. What is the need of introducing axioms?
5. You have seen earlier that the set of all natural numbers is closed under addition (closure property). Is this an axiom or something you can prove?

Lines and Angles



Suppose you have a ray $\overrightarrow{OA}$ on a *plane* with end point O. With the same end point O, consider another ray $\overrightarrow{OB}$ in the same plane. You observe that $\overrightarrow{OB}$ is obtained from $\overrightarrow{OA}$ through suitable rotation around the point O. We say $\overrightarrow{OB}$ subtends an *angle* with $\overrightarrow{OA}$. The amount of rotation is the **measure** of this angle.

We use a numerical measurement called **degree** to measure angles. We use the notation a° to denote a degrees. The rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are called the **sides** of the angle and O is called the **vertex** of the angle. The angle subtended by the rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is denoted by $\angle AOB$ or $A\hat{O}B$.

Note: Consider two rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$. If X is any point on $\overrightarrow{OA}$, then the rays $\overrightarrow{OA}$ and $\overrightarrow{OX}$ are the same. Similarly, for any point Y on the ray $\overrightarrow{OB}$, the rays $\overrightarrow{OY}$ and $\overrightarrow{OB}$ are the same. Thus $\angle AOB = \angle XOY$.

Activity 1:

Construct an angle which measures 40° using protractor.

Activity 2:

Take two rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$ and measure the angle between them using a protractor.

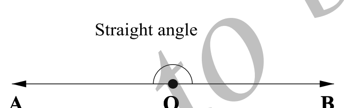
Warning! Even with the best protractor, scale and pencil, you may not be able to produce an angle which measures exactly 40°. Your eyes also play an important role, as parallax error normally creep in. Nevertheless your construction is good enough for all practical purposes.

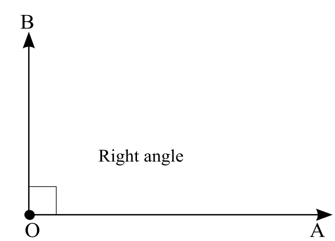
Recall what you have studied about different types of angles: **straight angle**, **right angle**, **acute angle**, **obtuse angle**, **reflex angle**, **complete angle**, **adjacent angles**, **complementary angles** and **supplementary angles**.

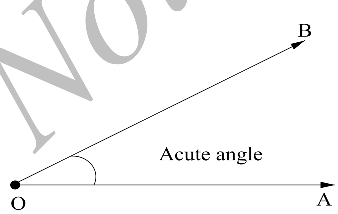
Consider a straight line and let O be point on the straight line. Then O divides the straight line in to two rays. If B is to the *left* of O on the line and A to the *right* of O, then there are two rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$. The angle between these two rays is called a **straight angle**.

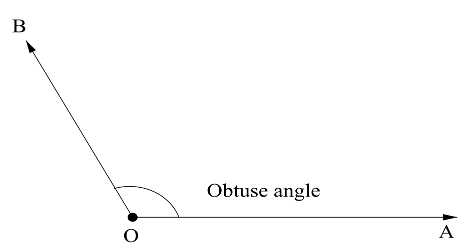
If you set your protractor such that its centre coincides with O, then you see that the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is $180°$. However, your protractor is so calibrated that it measures any straight angle exactly $180°$. Thus you cannot define a **straight angle** using a measuring device.

We will see that in Euclid's geometry, this is taken as one of the **postulates**. But once you know a straight angle, you can define all other types of angles. For example, a **right angle** is that angle which measures $90°$ or you need half of a protractor. Similarly, an **acute angle** is one having measure less than $90°$ and an **obtuse angle** is that angle which measures more than $90°$ but less than a straight angle. A **reflex angle** is an angle measuring more than $180°$ but less than $360°$. Finally, a **complete angle** is an angle measuring $360°$. This corresponds to a complete revolution of a ray $\overrightarrow{OA}$ around the initial point O.

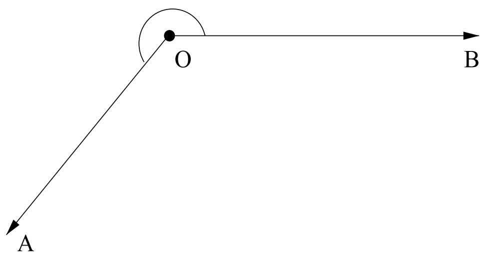




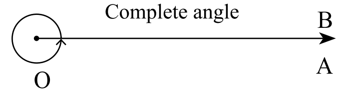




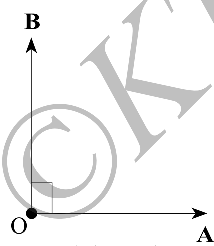
Reflex angle



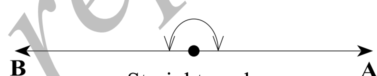
Complete angle



We say two rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are **perpendicular** to each other if the angle between them is $90°$ and we write $\overrightarrow{OA} \perp \overrightarrow{OB}$. We say two rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are **supplementary rays** if the angle between them is $180°$. Observe that in this case $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are in **opposite** directions.



Right angle



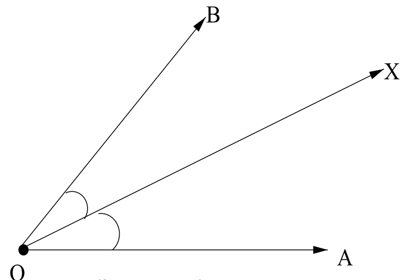
Straight angle

Suppose you have a straight line and O is a point on this line. Then O divides the straight line into two rays: if B is to the left of O and A to the right of O, then the straight line is made up of two rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$, the angle between them being one straight angle or $180°$.

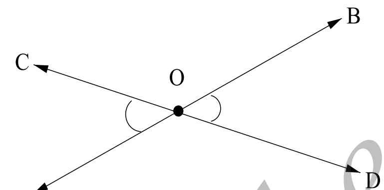
Two angles are said to be **supplementary angles** if their sum is $180°$. Similarly, two angles are said to be **complementary** if they add up to $90°$.

Two angles are said to be **adjacent angles**, if both the angles have a common vertex and a common side. If two straight lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at a point O, then you see that *four* angles are formed at O: if the first line is divided by O in to two rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$ and if the second line is divided in to two rays $\overrightarrow{OC}$ and $\overrightarrow{OD}$, then you get four angles $\angle AOC$, $\angle COB$, $\angle BOD$ and $\angle DOA$.

The pair of angles $\angle$AOC and $\angle$BOD are called **vertically opposite angles**. Observe the other pair $\angle$COB and $\angle$DOA is also a pair of vertically opposite angles.



Adjacent angles



Vertically opposite angles

While measuring lengths of line segments and angles, we observe the following rules. These were not stated separately by Euclid, but tacitly assumed by him in the derivation of new propositions. We take them as additional postulates.

Rule 1. **Every line segment has a positive length.** (The length of the line segment $\overline{AB}$ is denoted by AB or |AB|.)

Rule 2. **If a point** C **lies on a line segment** $\overline{AB}$, **then the length of** $\overline{AB}$ **is equal to the sum of the lengths of** $\overline{AC}$ **and** $\overline{CB}$; **that is** AB = AC + CB.

Rule 3. **Every angle has a certain magnitude. A straight angle measures** $180°$.

Rule 4. **If** $\overrightarrow{OA}$, $\overrightarrow{OB}$ **and** $\overrightarrow{OC}$ **are such that** $\overrightarrow{OC}$ **lies between** $\overrightarrow{OA}$ **and** $\overrightarrow{OB}$, **then** $\angle AOB = \angle AOC + \angle COB$.

Rule 5. **If the angle between two rays is zero then they coincide. Conversely, if two rays coincide, the angle between them is either** zero **or an integral multiple of** $360°$.

> **Note:** While measuring angles, we use the following convention: if the angle is measured anti-clock-wise, it is positive. If it is measured clock-wise, then it is negative.

Note: What is the minimum number of axioms needed to develop a self-consistent geometry is a question in Mathematical Philosophy. However, here we do not bother about the minimum number required nor about retaining *original* axioms and postulates made by Euclid. In fact there were several gaps in the set of axioms used by Euclid and later mathematicians added some more to those used by Euclid.

Activity 3:

On a sheet of paper draw a straight line $\overleftrightarrow{AB}$. Choose a point O on it. Draw a ray $\overrightarrow{OC}$ on it. Measure angles $\angle BOC$ and $\angle COA$ using a protractor. What is the sum $\angle BOC + \angle COA$? Repeat this with different positions of $\overrightarrow{OC}$. What do you find?

You will always find that the sum of these two angles is 180°. Can you prove this using axioms and postulates.?

Note: Perhaps,you may realise here the need for a logical proof. No matter which configuration you take with a line and a ray standing on it, you see that the sum of the two adjacent angles always add up to 180°. However, this does not deny that there is a case of a line and a ray on it such that the sum of the two adjacent angles is different from 180°. This is inherent in the structure itself as there are infinitely many possibilities of a line and a ray on it, and you cannot verify your finding with all of them. This is the reason why Mathematicians look for logical proofs based on axioms and postulates or on the propositions which have already been proved.

Let us see what is the statement we need to prove. We have to take an arbitrary line and an arbitrary ray standing on it. Then there are two adjacent angles formed by the line and the ray. We have to show that the sum of these adjacent angles is 180°. We put this as a **proposition**. A **Proposition** is a statement which is to be proved using the axioms and postulates. It also depends on what is given in the proposition called **hypotheses** which are used in the proof of the statement.

Proposition 1. Let $\overleftrightarrow{AB}$ be a straight line and $\overrightarrow{OC}$ be a ray standing on the line $\overleftrightarrow{AB}$. Then $\angle BOC + \angle COA = 180°$.

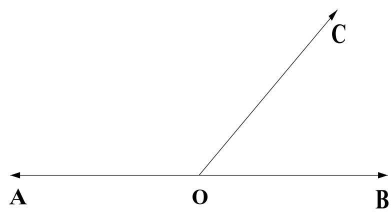


Fig. 1

Before getting on to the proof, let us see what are given and what we need to prove.

Given: A ray $\overrightarrow{OC}$ stands on a straight-line $\overleftrightarrow{AB}$ forming two adjacent angles $\angle BOC$ and $\angle COA$.

To prove: $\angle BOC + \angle COA = 180°$.

Proof: We have $\angle BOC + \angle COA = \angle BOA$ (by rule 4).

But $\angle BOA$ is a straight angle determined by the line $\overleftrightarrow{AB}$.
By rule 3, $\angle BOA = 180°$.
Now we can invoke Axiom 1. $\angle BOC + \angle COA$ and $180°$ are both equal to the same thing, $\angle BOA$.
We conclude that $\angle BOC + \angle COA = 180°$.

Look at the proposition once again. It says: if a ray stands on a straight line, then the sum of two adjacent angles formed is $180°$. That is, the two adjacent angles are *supplementary*. This is the general nature of all our new propositions: given that a certain statement S is true, some other statement R is true. We say S is the **hypothesis** and R is the **conclusion**.(Here *a ray stands on a straight line* is the statement S which is the hypothesis and *the sum of adjacent angles is* $180°$ is the statement R, which is the conclusion.)

What is the converse of a proposition? Naturally, hypothesis and conclusion must change their places. If our original proposition has S as hypothesis and R as conclusion, the **converse** proposition must have R as hypothesis and S as conclusion. In the present context the converse is: *If there are three rays* $\overrightarrow{OA}$, $\overrightarrow{OB}$ *and* $\overrightarrow{OC}$ *such that* $\angle BOC$ *and* $\angle COA$ *are adjacent angles (that is* $\overrightarrow{OC}$ *is between* $\overrightarrow{OA}$ *and* $\overrightarrow{OB}$*) and if* $\angle BOC + \angle COA = 180°$, *then* A,O,B *all lie on the same straight line.* We say A,O,B are **collinear** if all of them lie on the same straight line. We put forth this as another proposition.

Proposition 2. Let $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ be three rays such that $\overrightarrow{OC}$ is between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Suppose $\angle BOC + \angle COA = 180°$. Then A,O,B are collinear, that is, they lie on the same straight-line.

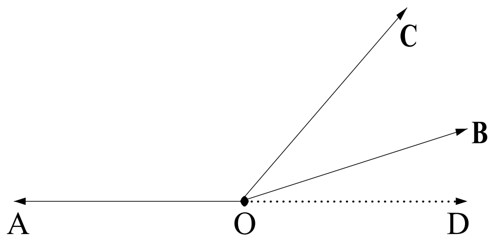


Fig. 2

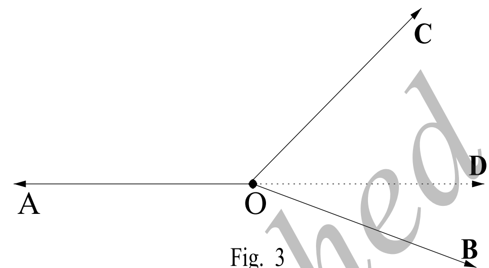


Fig. 3

Given: Three rays $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are such that $\angle BOC$ and $\angle COA$ are adjacent angles which add up to $180°$.

To prove: A,O,B all lie on the same line..

Construction: Extend $\overrightarrow{AO}$ to D such that A,O,D all lie on the same line $\overleftrightarrow{AD}$.

Proof: Using proposition 1, we have $\angle DOC + \angle COA = 180°$. But we are given that $\angle BOC + \angle COA = 180°$. Using Axiom1, it follows that

$$\angle DOC + \angle COA = \angle BOC + \angle COA.$$

Now use Axiom 3 to get $\angle DOC = \angle BOC$. There are two possibilities: $\overrightarrow{OB}$ lies between $\overrightarrow{OD}$ and $\overrightarrow{OC}$ (see Fig.2) or $\overrightarrow{OD}$ lies between $\overrightarrow{OB}$ and $\overrightarrow{OC}$ (see Fig. 3). In the first case, using rule 4, we get

$$\angle BOC = \angle DOC = \angle DOB + \angle BOC,$$

and Axiom 3 implies that $\angle DOB = 0$. In the second case, rule 4 gives

$$\angle DOC = \angle BOC = \angle BOD + \angle DOC,$$

and Axiom 3 implies that $\angle BOD = 0$. Thus the angle between the rays $\overrightarrow{OB}$ and $\overrightarrow{OD}$ is *zero*. Using rule 5, we conclude that the rays $\overrightarrow{OB}$ and $\overrightarrow{OD}$ coincide. This means B and O are on the line $\overleftrightarrow{AD}$. Thus A,O,B are collinear.

> **Note:** Proposition 1 and proposition 2 are two geometrical statements which are **converses** of each other. In geometry, if some statement is true, then many times its converse is also true. However, this is not universally valid. There may be statements which are true, but whose converses may fail. Later you will see that an equilateral triangle is isosceles, but an isosceles triangle need not be equilateral.

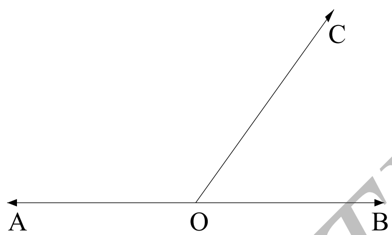


Fig. 4

Example 1. In the adjoining figure, if $\angle COA - \angle BOC = 50^\circ$, find these angles.

Solution: We know by proposition 1,

$$\angle BOC + \angle COA = 180^\circ .$$

Adding, two relations, we get

$$2\angle COA = 230^\circ .$$

(Which axiom is used here?)

This implies that $\angle COA = 115^\circ$. (Which axiom is needed here?) Now

$$\angle BOC = 180^\circ - \angle COA = 180^\circ - 115^\circ$$

$$\angle BOC = 65^\circ$$

Example 2. In the adjoining figure,if the angles $\angle AOB$, $\angle BOC$, $\angle COD$ are in the ratio 1:2:3 and AD is a straight line, find the measures of all the angles.

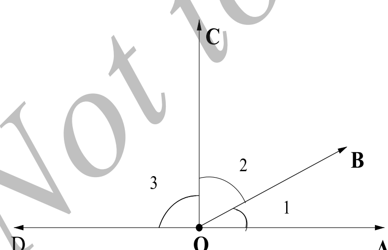


Fig. 5

Solution: Using the proposition 1, we see that

$\angle AOB + \angle BOC + \angle COD = 180^\circ$.

But the given hypothesis is $\angle BOC = 2\angle AOB$ and $\angle COD = 3\angle AOB$.

Thus we get $6\angle AOB = 180^\circ$.

$\angle AOB = 30^\circ$.

$\angle BOC = 2 \times 30^\circ = 60^\circ$ and

$\angle COD = 3 \times 30^\circ = 90^\circ$.

Definition: Suppose $\angle AOB$ is an angle formed by two rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$. If $\overrightarrow{OP}$ is another ray between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ such that $\angle AOP = \angle POB$, we say $\overrightarrow{OP}$ bisects $\angle AOB$ or $\overrightarrow{OP}$ is the **angle bisector** of $\angle AOB$. We observe that in this case $\angle AOP = \angle POB = \frac{1}{2}\angle AOB$.

Activity 4:

Draw a straight line $\overleftrightarrow{AB}$ and a ray $\overrightarrow{OC}$ on $\overleftrightarrow{AB}$. Measure $\angle BOC$ and $\angle COA$. Construct ray $\overrightarrow{OP}$ such that it bisects $\angle BOC$. Similarly, construct ray $\overrightarrow{OQ}$ such that, it bisects $\angle COA$. Measure $\angle POQ$. Do you see that $\angle POQ = 90°$?. Repeat this taking different rays $\overrightarrow{OC}$ on $\overleftrightarrow{AB}$. Do you always find that $\angle POQ = 90°$? Can you formulate this as a proposition?

Proposition 3. Let $\overleftrightarrow{AB}$ be a straight line and let $\overrightarrow{OC}$ be a ray standing on it. Let $\overrightarrow{OP}$ be the bisector of $\angle BOC$, and let $\overrightarrow{OQ}$ be the bisector of $\angle COA$. Then $\angle POQ = 90°$.

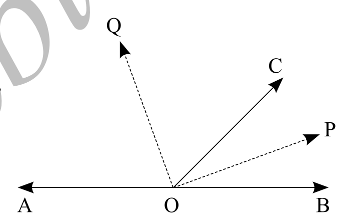


Fig. 6

Given: $\overrightarrow{OP}$ bisects $\angle BOC$ and $\overrightarrow{OQ}$ bisects $\angle COA$.

To prove: $\angle POQ = 90°$.

Proof:

Since $\overrightarrow{OP}$ bisects $\angle BOC$, we have

$$\angle POC = \frac{1}{2}\angle BOC. \qquad (1)$$

Since $\overrightarrow{OQ}$ bisects $\angle COA$, we also have

$$\angle COQ = \frac{1}{2}\angle COA. \qquad (2)$$

Adding these two and using rule 4, we obtain

$$\angle POQ = \frac{1}{2}(\angle BOC + \angle COA).$$

By proposition 1, $\angle BOC + \angle COA = 180°$. Thus we obtain

$$\angle POQ = \frac{1}{2} \times 180° = 90°.$$

Activity 5: Take two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersecting at a point O. Measure $\angle BOD$, $\angle DOA$, $\angle AOC$, $\angle COB$. Compare $\angle BOD$ and $\angle AOC$. Similarly compare $\angle DOA$ and $\angle COB$. Do you observe something pertinent? Repeat this with different positions of $\overleftrightarrow{CD}$ with respect to $\overleftrightarrow{AB}$. Can you formulate a new geometrical proposition?

Proposition 4. If two straight lines intersect at a point, then the vertically opposite angles are equal.

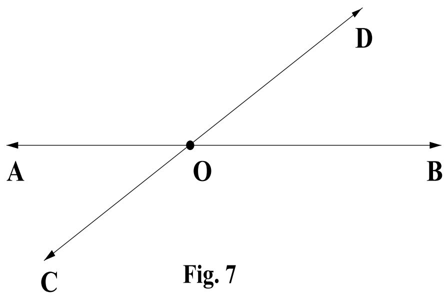


Fig. 7

Given: $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersecting at a point O.

To prove: $\angle BOD = \angle AOC$ and $\angle DOA = \angle COB$.

Proof: Consider the straight line $\overleftrightarrow{AB}$ and the ray $\overrightarrow{OD}$ standing on it. Then $\angle BOD$ and $\angle DOA$ are adjacent angles.

$$\angle BOD + \angle DOA = 180^\circ . \qquad (1)$$

Similarly, considering the straight line $\overleftrightarrow{CD}$ and the ray $\overrightarrow{OA}$, we see that $\angle DOA$ and $\angle AOC$ are adjacent angles. Thus proposition 1 again gives

$$\angle DOA + \angle AOC = 180^\circ. \qquad (2)$$

Using Axiom1, we can compare (1) and (2)and get

$$\angle BOD + \angle DOA = \angle DOA + \angle AOC. \qquad (3)$$

Since we can remove $\angle DOA$ using Axiom 3, we obtain

$$\angle BOD = \angle AOC.$$

A similar argument gives $\angle DOA = \angle COB$.

Example 3. Let $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ be straight lines intersecting at O. Let $\overrightarrow{OP}$ be the bisector of $\angle BOD$ and $\overrightarrow{OQ}$ be the bisector of $\angle AOC$. Prove that Q,O,P are collinear.

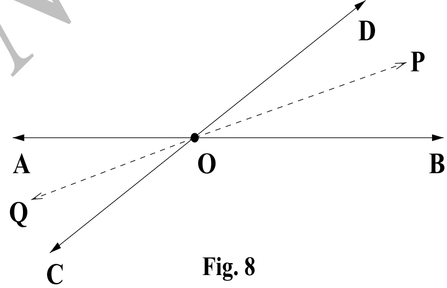


Fig. 8

Solution: We have to show that $\angle POQ = 180^\circ$. Since OP bisects $\angle BOD$, we have

$$\angle POD = \frac{1}{2}\angle BOD \qquad (1)$$

Similarly, using the given hypothesis that $\overrightarrow{OQ}$ bisects $\angle AOC$, we also obtain

$$\angle AOQ = \frac{1}{2}\angle AOC. \qquad (2)$$

However, we have

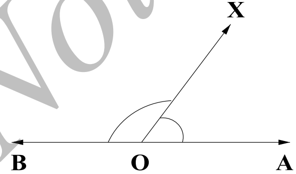
$$\begin{aligned}
\angle POQ &= \angle POD + \angle DOA + \angle AOQ \quad \text{(using rule 4)} \\
&= \angle DOA + \frac{1}{2}(\angle BOD + \angle AOC) \text{ (from (1) and (2))} \\
&= \angle DOA + \frac{1}{2} \times 2\angle AOC \text{ } (\angle BOD = \angle AOC \text{ as vertical opposite angles)} \\
&= \angle DOA + \angle AOC \\
&= 180^\circ \text{ (using proposition 1).}
\end{aligned}$$

It follows that P,O,Q are collinear.

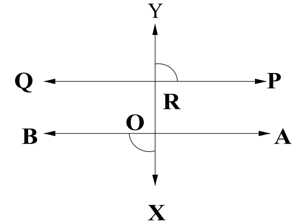
Exercise 3.2

1. Draw diagrams illustrating each of the following situation:
   (a) Three straight lines which do not pass through a fixed point.
   (b) A point and rays emanating from that point such that the angle between any two adjacent rays is an acute angle.
   (c) Two angles which are not adjacent angles, but still supplementary.
   (d) Three points in the plane which are equidistant from each other.
2. Recognise the type of angles in the following figures:

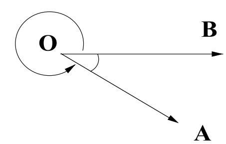
(i)



(ii)

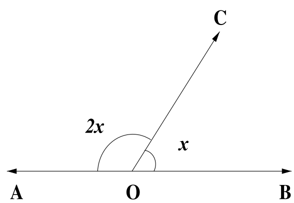


(iii)

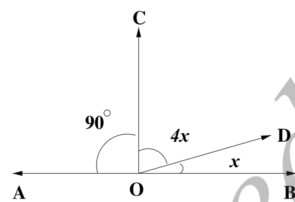


3. Find the value of x in each of the following diagrams:

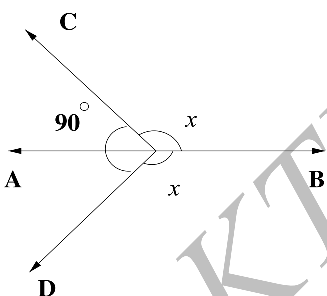
(i)



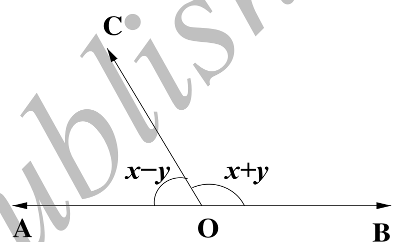
(ii)



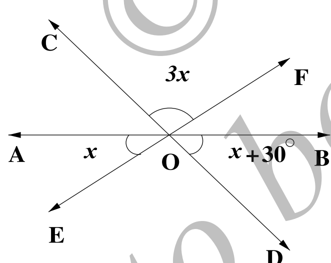
(iii)



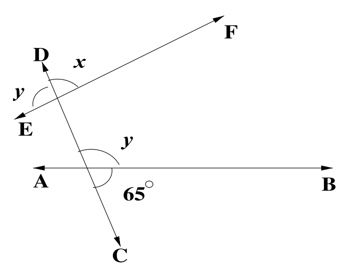
(iv)



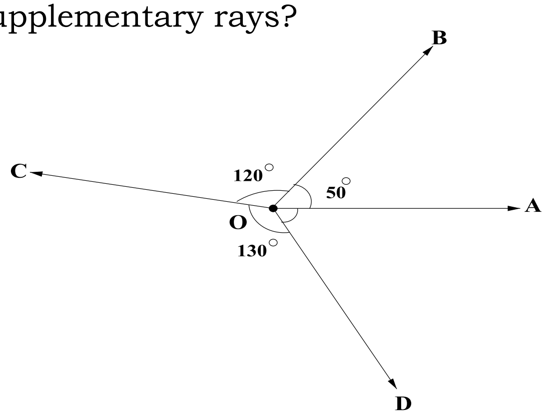
(v)



(vi)



4. Which pair of angles are supplementary in the following diagram? Are they supplementary rays?



5. Suppose two adjacent angles are supplementary. Show that if one of them is an obtuse angle, then the other angle must be acute.

Parallel lines and Euclid's fifth postulate

Take any two distinct straight lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. We first show that there is at most one point common to them. Suppose the contrary and assume that P and Q are two distinct points which lie on both the lines. But we know that by postulate 1 that there is a *unique* line $\overleftrightarrow{PQ}$ passing through P and Q. Since P and Q are on $\overleftrightarrow{AB}$, we must have $\overleftrightarrow{AB} = \overleftrightarrow{PQ}$. A similar argument shows that $\overleftrightarrow{CD}=\overleftrightarrow{PQ}$. Using Axiom 1, we obtain $\overleftrightarrow{AB} = \overleftrightarrow{CD}$, which contradicts the assumption that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are distinct. Thus given two distinct lines, either they do not have any point in common or there is one point common to them. In the latter case, the two straight line **intersect** at this common point.

We say two straight lines are **parallel** to each other, if either they are identical or they do not intersect. Thus two distinct lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel to each other if and only if they do not share any common point.

Let us get back to Euclid's Postulate 5:

Postulate 5. If a straight line meets two other lines, so as to make the two interior angles on one side of it together less than two right angles, the other straight lines will meet if produced on that side on which the angles are less than two right angles.

This is one of the most complicated postulate made by Euclid. In later years, many attempts have been made to arrive at a simpler equivalent versions of Euclid's fifth postulate.

Suppose $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are two straight lines and let $\overleftrightarrow{PQ}$ be a line which meets $\overleftrightarrow{AB}$ in L and $\overleftrightarrow{CD}$ in M.(see Fig. 9) If a line intersects two or more lines, it is called a *transversal* to those set of lines. Here $\overleftrightarrow{PQ}$ is a transversal to $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. There are *eight* angles formed by these three lines: $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$, $\angle 5$, $\angle 6$, $\angle 7$ and $\angle 8$ as shown in figure. In this $\angle 1$, $\angle 4$, $\angle 7$ and $\angle 6$ are called **exterior angles**; $\angle 3$, $\angle 2$, $\angle 5$ and $\angle 8$ are called **interior angles**. The angles $\angle 3$ and $\angle 5$ are called a pair of **interior alternate angles**. Observe that $\angle 2$ and $\angle 8$ are also a pair of interior alternate angles. Similarly, the pairs $\angle 1$, $\angle 7$ and $\angle 4$, $\angle 6$ are called pairs of **exterior alternate angles**. Angles $\angle 1$ and $\angle 5$ are called a pair **corresponding angles**. There are three more pairs of corresponding angles: $\angle 2$, $\angle 6$; $\angle 3$, $\angle 7$; and $\angle 4$, $\angle 8$.

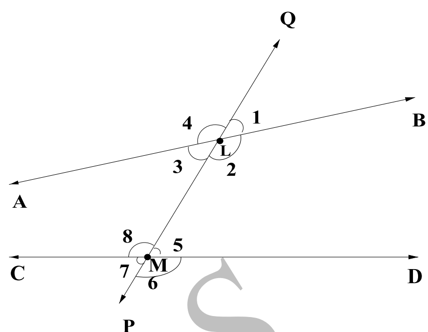


Fig. 9

Look at $\angle 3$ and $\angle 8$. They are the interior angles on the same side of the line $\overleftrightarrow{PQ}$. According to Postulate 5, if $\angle 3 + \angle 8 < 180°$, then $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ meet on the *left* side of $\overleftrightarrow{PQ}$. (If it happens that $\angle 2 + \angle 5 < 180°$, then they must meet on the right side of $\overleftrightarrow{PQ}$.) Let us explore more on the condition $\angle 3 + \angle 8 \neq 180°$. We have the following proposition.

Proposition 5. If a transversal cuts two parallel lines, then the sum of two interior angles on the same side of the transversal is equal to 180°.

Given: a transversal intersecting two parallel lines.

To prove: the sum of interior angles on the same side of transversal is equal to 180°.

Proof: Suppose the result is not true. (see fig 9.) If $\angle 3 + \angle 8 \neq 180°$, then either you must have $\angle 3 + \angle 8 < 180°$ or $\angle 3 + \angle 8 > 180°$. In the first case $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ meet on the *left* side of $\overleftrightarrow{PQ}$. Suppose $\angle 3 + \angle 8 > 180°$ We observe that

$$\begin{aligned}\angle 3 + \angle 8 + \angle 2 + \angle 5 &= (\angle 3 + \angle 2) + (\angle 8 + \angle 5)\\ &= \angle ALB + \angle CMD\\ &= 180° + 180° = 360°.\end{aligned}$$

Thus

$$\angle 2 + \angle 5 = 360° - (\angle 3 + \angle 8) < 360° - 180° = 180°.$$

$\Rightarrow\ \angle 2+\angle 5 < 180^\circ$. Hence Postulate 5 tells us that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ meet on the *right* side of $\overleftrightarrow{PQ}$.

We conclude that: if the sum of the interior angles on the same side of $\overleftrightarrow{PQ}$ is not equal to 180°, then $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ meet at some point (either to the left or to the right of $\overleftrightarrow{PQ}$). Thus if $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel, then the sum of the interior angles on the same side of any transversal $\overleftrightarrow{PQ}$ is equal to 180°. This completes the proof of the proposition.

This was one of the methods frequently employed by Euclid and also by later mathematicians to prove new propositions. If you want to prove that "S **implies** R", it is enough to prove "**not** R **implies not** S." This is called method of **reductio ad absurdum**. This is a latin word with meaning "reduction to the absurdity." Formally we say "S implies R" is equivalent to "not R implies not S." Hence if R is not true, then S cannot be true. This is also known as proof by contradiction.

Thus Postulate 5 implies that given a pair of parallel lines and a transversal, the sum of the interior angles on the same side of the transversal is equal to 180°.

Is the converse true? Given two straight lines and a transversal such that the sum of two internal angles on the same side of the transversal is equal to 180°, does it follow that the two lines are parallel?. Here we make use of a fairly simple equivalent version of **parallel postulate** of Euclid. This was first given by a Scottish mathematician called Playfair.

Playfair's postulate: Given a line in a plane and a point outside the line in the same plane, there is a unique line passing through the given point and parallel to the given line.

We have the following statement.

Proposition 6. If a transversal cuts two distinct straight lines in such a way that the sum of two interior angles on the same side of the transversal is equal to 180^o, then the two lines are parallel to each other.

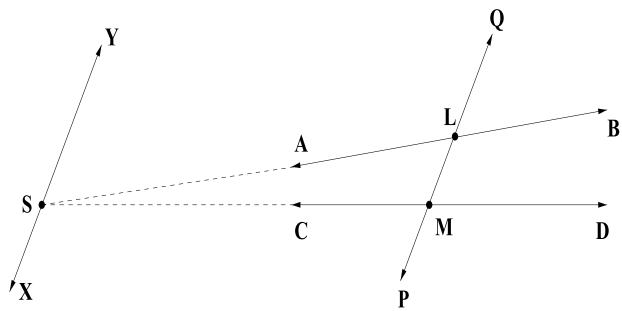


Fig. 10

Given: $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are two straight lines. And a transversal $\overleftrightarrow{PQ}$ intersecting $\overleftrightarrow{AB}$ at L and $\overleftrightarrow{CD}$ at M respectively; and $\angle ALM + \angle LMC = 180^\circ$.

To prove: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

Proof: We prove this using the method of contradiction. Assume that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are not parallel. Then they must meet at some point, say S. (see Fig.10) By Playfair's postulate, there is a unique line $\overleftrightarrow{XY}$ passing through S and parallel to $\overleftrightarrow{PQ}$.

Since $\overleftrightarrow{XY} \parallel \overleftrightarrow{PQ}$, we have $\angle QLS + \angle LSY = 180^\circ$ (they are internal angles on the same side of the transversal $\overleftrightarrow{SB}$ to the parallel lines $\overleftrightarrow{XY}$ and $\overleftrightarrow{PQ}$). But $\angle QLS + \angle ALM = 180^\circ$ (they are adjacent angles formed by the ray LA standing on the line $\overleftrightarrow{PQ}$). Hence it follows that $\angle LSY = \angle ALM$. (Which axioms are needed here?) But $\angle ALM + \angle LMC = 180^\circ$ (given data). We also have $\angle LMC + \angle MSY = 180^\circ$ (since they are the sum of the internal angles on the same side of the transversal $\overleftrightarrow{SD}$ cutting the parallel lines $\overleftrightarrow{XY}$ and $\overleftrightarrow{PQ}$). Thus we get $\angle ALM = \angle MSY$. It now follows that $\angle LSY = \angle MSY$. But $\angle MSY = \angle MSL + \angle LSY$. We obtain $\angle MSL = 0$. Hence $\overleftrightarrow{SB}$ and $\overleftrightarrow{SD}$ coincide. This forces that the straight lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are the same, contradicting that they are distinct lines. We conclude that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

Activity 6:

Draw two parallel lines. Draw a transversal and measure different angles formed by intersections. You will see that:

1. Any pair of alternate angles are equal.
2. Any pair of corresponding angles are equal.

Repeat the same with different positions of the transversal. You will see that the same results repeat. We formulate this as a theorem.

A theorem is a proposition that has been proved logically on the basis of previously established statements.

Theorem 1. If two parallel lines are cut by a transversal, then

(i) each pair of alternate angles are equal;

(ii) each pair of corresponding angles are equal.

Given: $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ two distinct parallel lines and $\overleftrightarrow{PQ}$ a transversal intersecting $\overleftrightarrow{AB}$ in L and $\overleftrightarrow{CD}$ in M. (see Fig. 11)

To prove: $\angle 3 = \angle 5$ and $\angle 1 = \angle 5$.

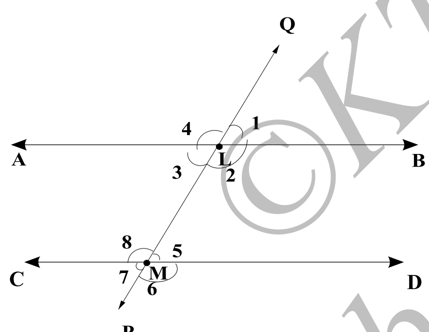


Fig. 11

Proof: Since $\angle 3$ and $\angle 8$ are two internal angles on the same side of the transversal $\overleftrightarrow{PQ}$ cutting the parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, we know that

$$\angle 3 + \angle 8 = 180^\circ .$$

But $\angle 8$ and $\angle 5$ are the adjacent angles formed by the ray MP standing on the line $\overleftrightarrow{CD}$. Hence we also know that

$$\angle 8 + \angle 5 = 180^\circ .$$

Comparing, we see that $\angle 3 = \angle 5$.

Again observe that $\angle 2 + \angle 3 = 180^\circ = \angle 8 + \angle 5$. Using $\angle 3 = \angle 5$, we get $\angle 2 = \angle 8$.

We also observe that $\angle 1 = \angle 3$, since they are vertically opposite angles. Using this with $\angle 3 = \angle 5$, we conclude that $\angle 1 = \angle 5$.

Thus the pair $\angle 1, \angle 5$ of corresponding angles are equal.

Similarly we can prove $\angle 2 = \angle 6$, $\angle 4 = \angle 8$ and $\angle 3 = \angle 7$, $\angle 1 = \angle 7$, $\angle 4 = \angle 6$.

Think it over! There are two statements about parallel lines and a transversal: (i) if a transversal cuts two distinct parallel lines, then any pair of alternate angles are equal; (ii) if a transversal cuts two distinct parallel lines, then any pair of corresponding angles are equal. But these two are not independent statements. You can easily prove any of them assuming the other and using propositions 1 and 4.

What is the converse of theorem 1? If there are two distinct straight lines and a transversal such that any pair of alternate angles are equal, can we prove that the two lines are parallel to each other? Note that if we are able to prove this result, you can also prove that: given two distinct lines and a transversal such that any pair of corresponding angles are equal, then the lines are parallel(use the previous observation). Thus we have the following theorem.

Theorem 2. Suppose a transversal cuts two distinct straight lines such that a pair of alternate angles are equal, then the two lines are parallel to each other.

Given: Two straight lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ and a transversal $\overleftrightarrow{PQ}$ (see Fig. 11) and $\angle 3 = \angle 5$.

To prove: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

Proof: We know that $\angle 8 + \angle 5 = 180°$(since they are supplementary angles). By the given hypothesis, we know that $\angle 3 = \angle 5$. We thus obtain $\angle 3 + \angle 8 = 180°$. However, $\angle 3$ and $\angle 8$ are the internal angles on the same side of the transversal to the line $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. By proposition 6, we conclude that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

Corollary: If a transversal cuts a pair of straight lines in such a way that a pair of corresponding angles are equal, then the two lines are parallel to each other.

Proof: Referring to Fig. 11, we are given, say, $\angle 1 = \angle 5$. But $\angle 1 = \angle 3$, since they are vertically opposite angles. We obtain $\angle 3 = \angle 5$. Hence by theorem 2, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

Activity 7:

Draw two parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. Draw a line $\overleftrightarrow{XY}$ parallel to $\overleftrightarrow{CD}$. Draw a transversal $\overleftrightarrow{PQ}$. Let it cut $\overleftrightarrow{AB}$ at L, $\overleftrightarrow{CD}$ at M and $\overleftrightarrow{XY}$ at N. Measure $\angle BLQ$ and $\angle YNQ$. Do you see that they measure the same. Repeat this with different positions of $\overleftrightarrow{PQ}$.

Example 4. Two lines which are parallel to a common line are parallel to each other.

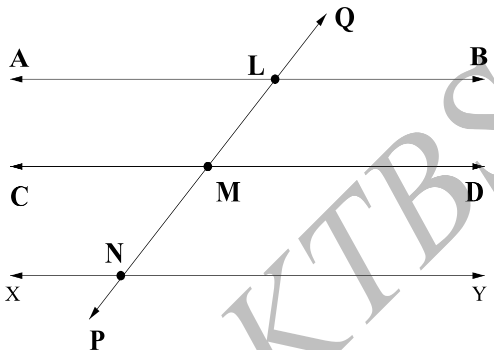


Fig. 12

Solution: Suppose $\overleftrightarrow{AB}$ and $\overleftrightarrow{XY}$ are two lines which are parallel to a common line $\overleftrightarrow{CD}$. We show that $\overleftrightarrow{AB} \parallel \overleftrightarrow{XY}$ Draw a transversal $\overleftrightarrow{PQ}$, cutting $\overleftrightarrow{AB}$ at L, $\overleftrightarrow{CD}$ at M and $\overleftrightarrow{XY}$ at N, respectively. We observe that

$$\angle BLP = \angle DMP,$$

as they are corresponding angles made by transversal $\overleftrightarrow{PQ}$ with the parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. Similarly,

$$\angle D\,M\,P = \angle Y\,N\,P \text{ (Why?)}.$$

Using Axiom 1, we obtain $\angle BLP = \angle YNP$. Now we can use corollary to theorem 2 and conclude that $\overleftrightarrow{AB} \parallel \overleftrightarrow{XY}$.

Example 5. Let $\overleftrightarrow{AB}$ be a straight line. Let $\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ be two straight lines such that each of them is **perpendicular** to $\overleftrightarrow{AB}$. Prove that $\overleftrightarrow{CD} \parallel \overleftrightarrow{EF}$.

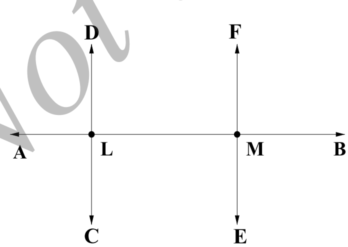


Fig. 13

Solution: Let $\overleftrightarrow{AB}$ intersect $\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ at L and M respectively. Since $\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$, we have $\angle DLA = 90°$. Using $\overleftrightarrow{EF} \perp \overleftrightarrow{AB}$, we also get $\angle FMA = 90°$. Thus $\angle DLA = \angle FMA$. But these are corresponding angles made by the transversal $\overleftrightarrow{AB}$ with the lines $\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$. Hence by corollary to theorem 2, we conclude that $\overleftrightarrow{CD} \parallel \overleftrightarrow{EF}$.

Example 6. Show that the angle bisectors of a pair of alternate angles made by the transversal with two parallel lines are parallel to each other.

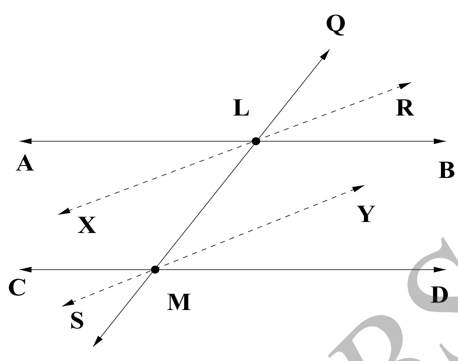


Fig. 14

Solution: We are given two parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, and transversal $\overleftrightarrow{PQ}$. Consider the pair of alternate angles $\angle ALM$ and $\angle LMD$. Let $\overrightarrow{LX}$ be the bisector of $\angle ALM$; let $\overrightarrow{MY}$ be the bisector of $\angle LMD$. Extend the ray $\overrightarrow{LX}$ to the straight line $\overleftrightarrow{XR}$ and the ray $\overrightarrow{MY}$ to the straight line $\overrightarrow{SY}$ as shown in the figure (see Fig. 14). We have to show that $\overrightarrow{XR} \parallel \overrightarrow{SY}$. Consider the lines $\overleftrightarrow{XR}$ and $\overleftrightarrow{SY}$ with transversal $\overleftrightarrow{PQ}$.

We have

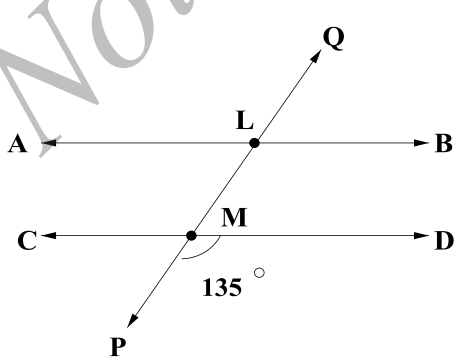
$$\angle XLM = \frac{1}{2}\angle ALM, \quad \angle LMY = \frac{1}{2}\angle LMD.$$

However, $\angle ALM = \angle LMD$ (why?).

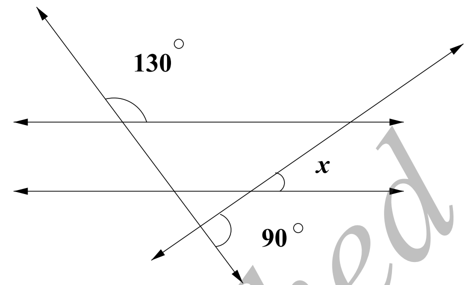
We hence obtain $\angle XLM = \angle LMY$. But $\angle XLM$ and $\angle LMY$ are a pair of alternate angles made by the
transversal $\overleftrightarrow{PQ}$ with the lines $\overleftrightarrow{XR}$ and $\overleftrightarrow{SY}$. It follows, by theorem 2, that $\overleftrightarrow{XR} \parallel \overleftrightarrow{SY}$.

Exercise 3.3

1. Find all the angles in the following figure



2. Find the value of x in the diagram below.



3. Show that if a straight line is perpendicular to one of the two or more parallel lines, then it is also perpendicular to the remaining lines.
4. Let $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ be two parallel lines and $\overleftrightarrow{PQ}$ be a transversal. Show that the angle bisectors of a pair of two internal angles on the same side of the transversal are perpendicular to each other.

Glossary

Undefined objects: those objects in mathematics which cannot be defined using the terms already known.

Axioms: certain statements which are valid in all branches of mathematics whose validity is taken for granted without seeking mathematical proofs.

Postulates: some statements which are taken for granted in a particular branch of mathematics.

Hypothesis: certain conditions assumed while proving a proposition

Adjacent angles: a pair of angles made by a ray standing on a line.

Complementary angles: a pair of angles which add up to $90°$.

Supplementary angles: a pair of angles which add up to $180°$.

Straight angle: an angle formed by a straight line; equal to $180°$.

Complete angle: an angle which measures $360°$.

Reflex angle: an angle which measures more than $180°$, but less than $360°$.

Linear pair: a pair of angles which make a straight line.

Vertically opposite angles: when two straight lines intersect each other, a pair of angles which do not form a linear pair are vertically opposite angles.

Collinear: points all lying on the same straight line.

Parallel lines: A pair of lines which do not intersect in a plane.

Alternate angles: when a transversal cuts a pair of lines, the angles formed by the transversal which is not a linear pair and lying on both sides of the transversal.

Corresponding angles: when a transversal cuts a pair of lines, the angles formed by the transversal which lie on the same side of the transversal and also on the similar side of the two lines.

Points to remember

- Mathematics is a game in which certain objects are given and you have to play the game according to certain pre-laid rules.
- In geometry, the objects are points, lines plane; the rules are axioms and postulates of geometry.
- Euclid's fifth postulate– equivalent formulation is that given any line and a point out side that line, there is a unique line passing through the given point and parallel to the given line– is the one which gives Euclidean geometry. Changing this postulate will lead to different geometry.

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Answers

Exercise 3.2

3. (i) 60° (ii) 18° (iii) 135° (iv) 90° (v) 30° (vi) 65° .

Exercise 3.3

1. $\angle DML = 45^\circ$ $\angle BLQ = 45^\circ$ $\angle MLB = 135^\circ$ $\angle CMP = 45^\circ$ $\angle CML = 135^\circ$ $MLA = 45^\circ$ $\angle QLA = 135^\circ$. **2.** 40° .

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UNIT 4

FACTORISATION

After studying this unit you learn to:

- factorise an algebraic expression by taking out common factors.
- factorise an expression by grouping appropriate terms.
- factorise an expression which is a difference of two square expressions.
- factorise a trinomial expression.
- factorise a square trinomial expression by using known identities.

Introduction

The process of writing a given algebraic expression as a product of two or more expressions is called **factorisation**. Each of the expression (constant or variable) which form the product is called a **factor** of the given algebraic expression.

For example: (i) Consider $7xy$. It has factors: 7, x, y, $7x$, $7y$, xy and $7xy$; (ii) $(x+3)$ and $(x+2)$ are factors of x^2+5x+6.

Recall the process of factorisation of a number. Given an integer, we write this as a product of other integers. This ultimately leads to prime factorisation of a number. In some sense, algebraic expressions also follow similar way. Given an expression, you will be able to write the expression as a product of its factors.

Note: (1) Factorisation and multiplication are reverse processes. In multiplication, we multiply different expressions and get a new expression. In factorisation, we split the given expression into simpler expressions whose product turns out to be the given expression. (2) You can always use 1 as a factor: $(x+5) = 1 \times (x+5)$. But this does not give any thing new. This is called the trivial factorisation. Many times one has to consider such factorisations as well.

Different methods of factorisation

There are many ways of factorising a given expression. We study them here.

1. Factorisation taking common factors

Consider the following examples:

Example 1: $5x^2 - 10x$

Here you see that $5x$ is the HCF of $5x^2$ and $10x$. We can remove them from both the terms. Thus we get

$$5x^2 - 10x = (5x)(x) - (5x)2$$
$$= (5x)(x - 2).$$

Example 2: $4a + 12b = 4(a + 3b)$.

Example 3: $3x^2y - 6xy^2 + 9xy = 3xy\,(x - 2y + 3)$.

Example 4: $a^3 - a^2 + a = a(a^2 - a + 1)$.

You see that we look at the HCF of all the terms in the expression and take out the HCF to get a factorisation.

2. Factorisation by grouping

We follow several steps in this process:

Step I: Arrange the terms of the given expression in suitable groups such that each group has a common factor;

Step II: Factorise each group;

Step III: Take out the factor which is common to each group.

Example 5: Factorise $ax - bx + ay \ \ by$.

Solution: We group them as

$$(ax - bx) + (ay - by) = (a - b)\,x + (a - b)\,y = (a - b)\,(x + y).$$

Do you see that the distributive law is used here? We can also do this as follows:

$$ax - bx + ay - by = (ax + ay) - (bx + by)$$
$$= a\,(x + y) - b\,(x + y)$$
$$= (a - b)(x + y).$$

Example 6: Get the factors of $y^3 - 3y^2 + 2y - 6 - xy + 3x$.

Solution: Here again we group the terms in the expression and factorise:

$$y^3 - 3y^2 + 2y - 6 - xy + 3x = (y^3 - 3y^2) + (2y - 6) - (xy - 3x)$$
$$= y^2(y - 3) + 2\,(y - 3) - x\,(y - 3$$
$$= (y^2 + 2 - x)\,(y - 3).$$

3. Factorisation of difference of two squares

We know from earlier unit that $(a+b)(a-b) = a^2 - b^2$ for all a,b. This leads to a nice factorisation when the given expression can be written as difference of two squares.

Example 7: Factorise $36a^2 - 49b^2$.

Solution: Observe that $36a^2 = (6a)^2$ and $49b^2 = (7b)^2$. Thus we get

$$36a^2 - 49b^2 = (6a)^2 - (7b)^2 = (6a + 7b)(6a - 7b).$$

Example 8: Factorise $\frac{x^2}{y^2} - \frac{9}{16}$.

Solution: Here again, we write

$$\frac{x^2}{y^2} - \frac{9}{16} = \left(\frac{x}{y}\right)^2 - \left(\frac{3}{4}\right)^2 = \left(\frac{x}{y} + \frac{3}{4}\right)\left(\frac{x}{y} - \frac{3}{4}\right).$$

Example 9: Compute $(4.5)^2 - (1.5)^2$.

Solution: We have

$$\begin{aligned}(4.5)^2 - (1.5)^2 &= (4.5+1.5)(4.5-1.5)\\ &= 6 \times 3 = 18.\end{aligned}$$

Exercise 4.1

1. Resolve in to factors:

   (i) $x^2 + xy$ (ii) $3x^2 - 6x$ (iii) $(1.6)a^2 - (0.8)\,a$ (iv) $5 - 10m - 20n$

2. Factorise:

   (i) $a^2 + ax + ab + bx$ (ii) $3ac + 7bc - 3ad - 7bd$ (iii) $3xy - 6zy - 3xt + 6zt$

   (iv) $y^3 - 3y^2 + 2y - 6 - xy + 3x$

3. Factorise:

   (i) $4a^2 - 25$ (ii) $x^9 - \frac{9}{16}$ (iii) $x^4 - y^4$ (iv) $\left(7\frac{3}{10}\right)^2 - \left(2\frac{1}{10}\right)^2$ (v) $(0.7)^2 - (0.3)^2$

   (vi) $(5a - 2b)^2 - (2a - b)^2$

Factorisation of trinomials

We have seen earlier how to multiply two binomials of the form $(x+a)$ and $(x+b)$: $(x+a)(x+b) = x^2 + (a+b)x + ab$.

We can also proceed in the reverse direction. Given the trinomial of the form $x^2 + (a+b)\,x + ab$, we can factorise this to get $x^2 + (a+b)\,x + ab = (x+a)(x+b)$. But generally, the trinomial is not given in this form. You may be given in the form $x^2 + mx + n$, where m,n are some numbers. You must be able to write $m = (a+b)$ and $n = a\times b$ to bring the given trinomial in to a factorisable form. This needs some properties of numbers. We study them here.

The sum and product of two numbers are positive if and only if both the numbers are positive.

This says that if $a + b$ and ab are positive then so are a,b. The converse is also true. Thus 6 and 5 are positive; $5 = 3+2$ and $6 = 3\times 2$; both 3 and 2 are positive.

The sum of two numbers is negative and their product positive if and only if both the numbers are negative.

Thus $a+b$ negative and ab positive if and only if both a and b are negative. If we are given numbers 21 and −10, we see that $-10 = (-7) + (-3)$ and $21 = (-7)(-3)$.

We say 7 is the **absolute value** of both 7 and −7.Thus given an integer a, we define its **absolute value** by $|a| = a$ if $a > 0$; $|a| = -a$ if $a < 0$; and $|a| = 0$ if $a = 0$. Observe $-8 < -6$ but $|-8| = 8 > 6 = |-6|$.

The sum of two numbers is positive and their product negative if and only if one of the numbers is positive and the other negative, and the positive number has larger absolute value than the negative number.

This means $a + b$ is positive and ab negative only if one of a,b is positive and the other negative; and if a is positive and b is negative, then $|a| > |b|$; if a is negative and b positive, then $|a| < |b|$. For example, we see that if $a + b = 7$ and $ab = -18$, then $a = 9$ and $b = -2$ or $a = -2$ and $b = 9$.

The sum of two numbers is negative and their product negative if and only if one of the numbers is positive and the other negative, and the positive number has smaller absolute value than the negative number.

Thus $a + b$ is negative and ab negative if and only if one of a,b is positive and the other negative; if a is positive and b is negative, then

$|a| < |b|$; if a is negative and b positive, then $|a| > |b|$. For example, if $a + b = -12$ and $ab = -28$, we can write $a = 2$ and $b = -14$ or $a = -14$ and $b = 2$.

Here we have not mentioned any thing about the nature of numbers. They can be integers, rational numbers or even **real numbers**, which you will study in your next class.

Example 10. Factorise $6x^2 + 11x + 3$.

Solution: Here you can adopt the standard method of splitting and grouping:

$$6x^2 + 11x + 3 = 6x^2 + 9x + 2x + 3 = (6x^2 + 9x) + (2x + 3)$$
$$= 3x(2x + 3) + 1(2x + 3)$$
$$= (3x + 1)(2x + 3).$$

The genuine problem is how to arrive at this splitting. Suppose we are looking for a factorisation of the form $6x^2 + 11x + 3 = (ax + b)(cx + d)$. This gives, after expansion,

$$6x^2 + 11x + 3 = acx^2 + (ad + bc)x + bd.$$

After comparing terms of different degrees,

$$ac = 6,\ ad + bc = 11,\ bd = 3.$$

Thus $acbd = 18$ or $(ad)(bc) = 18$; and $ad + bc = 11$. You have two numbers whose product is 18 and their sum is 11. You will immediately conclude that $ad = 9, bc = 2$ or $ad = 2, bc = 9$. Thus you may write

$$6x^2 + 11x + 3 = acx^2 + (ad + bc)x + bd = 6x^2 + 9x + 2x + 3,$$

which is the splitting we have used. Observe we can also use the other pair of values for (ad, bc); $ad = 2$, $bc = 9$. We get

$$6x^2 + 11x + 3 = (6x^2 + 2x) + (9x + 3) = 2x(3x + 1) + 3(3x + 1)$$
$$= (2x + 3)(3x + 1),$$

which is the same as the original factorisation.

Rule: If you want to factorise a trinomial of the form $x^2 + px + q$, you must be able to find numbers a and b such that $a \cdot b = q$ and $a + b = p$. Then $x^2 + px + q = (x + a)(x + b)$.

> You may notice here that a polynomial of degree 2 is written as a product of two polynomials, each of degree 1, in the process of factorisation. This helps us in understanding the structure of a polynomial of degree 2. You will see in later classes that this is an easy way of solving a polynomial equation, if you are able to factorise the polynomial.

Example 11. Factorise $x^2 - 9x+20$.

Solution: Again you must be able to find two numbers a and b such that $ab = 1\times 20$ (the product of the coefficients of x^2 term and the constant term) and $a+b = -9$. Here the product is positive and the sum is negative. Hence both the numbers must be negative. This can be achieved by taking $a = -5$ and $b = -4$. Thus

$$\begin{aligned} x^2 - 9x + 20 &= (x^2 - 5x) + (-4x + 20) = x(x-5) - 4(x-5) \\ &= (x-4)(x-5). \end{aligned}$$

Factorising a square trinomial

Any algebraic expression, which can be written either in the form $a^2 + 2ab + b^2$ or in the form $a^2 - 2ab + b^2$, is called a **square trinomial**. For example, $x^2 + 2x + 1$ is a square trinomial. We can have an immediate factorisation for such a trinomial using $a^2 + 2ab + b^2 = (a+b)(a+b)$ or $a^2 - 2ab + b^2 = (a-b)(a-b)$.

Example 12. Factorise $4x^2 + 12xy + 9y^2$.

Solution: We observe that

$$\begin{aligned} 4x^2 + 12xy + 9y &= (2x)^2 + 2(2x)(3y) + (3y)^2 \\ &= (2x+3y)^2. \end{aligned}$$

Thus its factors are two equal expressions: $2x + 3y$ and $2x + 3y$.

Example 13. Is $x^2 - 6xy + 36y^2$ a square trinomial?

We observe that $x^2 - 6xy + 36y = (x)^2 - (x)(6y) + (6y)^2$,

which is not in the standard form. Thus $x^2 - 6xy + 36y^2$ is not a square trinomial.

Think it over! Is it possible to factorise x^2+1? Alternatively, is it possible to find two numbers whose sum is *zero* and whose product is *one*?

Exercise 4.2

1. In the following, you are given the product pq and the sum $p + q$. Determine p and q:

   (i) $pq = 18$ and $p + q = 11$ (ii) $pq = 32$ and $p + q = -12$

   (iii) $pq = -24$ and $p+q = 2$ (iv) $pq = -12$ and $p+q = 11$

   (v) $pq = -6$ and $p+q = -5$ (vi) $pq = -44$ and $p + q = -7$.

2. Factorise:

   (i) $x^2 + 6x + 8$ (ii) $x^2 + 4x + 3$ (iii) $a^2 + 5a + 6$

   (iv) $a^2 - 5a + 6$ (v) $a^2 - 3a - 40$ (vi) $x^2 - x - 72$.

3. Factorise:

   (i) $x^2 + 14x + 49$ (ii) $4x^2 + 4x + 1$ (iii) $a^2 - 10a + 25$

   (iv) $2x^2 - 24x + 72$ (v) $p^2 - 24p + 144$ (vi) $x^3 - 12x^2 + 36x$.

<u>Glossary</u>

Common factor: given two or more expressions, the factor of each expression which is common to all the expressions.

Factorisation: the process of writing an algebraic expression as a product of more than one algebraic expressions.

Points to remember:

- Factorisation is the reverse process to the formation of the products;
- One can factorise some expressions using proper grouping and splitting of its terms.

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Answers

Exercise 4.1

1. (i) $x(x+y)$ (ii) $3x(x-2)$

(iii) $(0.8)a(2a-1)$ (iv) $5(1-2m-4n)$

2. (i) $(a+x)(a+b)$ (ii) $(3a+7b)(c-d)$

(iii) $(x-2z)(3y-3t)$ (iv) $(y-3)(y^2+2-x)$

3. (i) $(2a+5)(2a-5)$ (ii) $\left(x+\frac{3}{4}\right)\left(x-\frac{3}{4}\right)$

(iii) $(x^2+y^2)(x+y)(x-y)$ (iv) $\frac{1222}{25}$

(v) 0.4 (vi) $(7a-3b)(3a-b)$

Exercise 4.2

1. (i) $p=9$, $q=2$ (ii) $p=-8$, $q=-4$ (iii) $p=6$, $q=-4$

(iv) $p=12$, $q=-1$ (v) $p=-6$, $q=1$ (vi) $p=-11$’ $q=4$.

2. (i) $(x+4)(x+2)$ (ii) $(x+3)(x+1)$ (iii) $(a+3)(a+2)$

(iv) $(a-3)(a-2)$ (v) $(a-8)(a+5)$ (vi) $(x-9)(x+8)$

3. (i) $(x+7)(x+7)$ (ii) $(2x+1)(2x+1)$ (iii) $(a-5)(a-5)$

(iv) $2(x-6)(x-6)$ (v) $(p-12)(p-12)$ (vi) $x(x-6)(x-6)$.

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UNIT 5

SQUARES, SQUARE ROOTS, CUBES, CUBE ROOTS

After studying this unit, you learn:

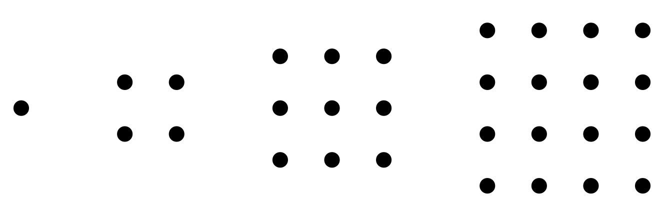
- perfect squares and square root of perfect squares.
- to recognise the digits in unit's place of a perfect square.
- to obtain the remainders when a perfect square is divided by 3 and 4.
- different occasions leading to perfect squares.
- some methods of finding perfect squares and square-roots of perfect squares.
- perfect cubes and cube-roots of perfect cubes.

Introduction

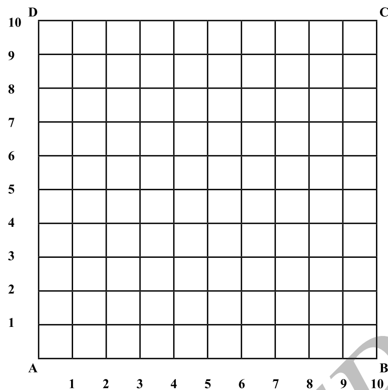
Look at the numbers of the form 1, 4, 9, 16, 25 and so on. What do you recognise? For example, $4 = 2 \times 2$, $25 = 5 \times 5$. You see that each such number is the product of two equal numbers. Similarly, you see that $8 = 2 \times 2 \times 2$, $64 = 4 \times 4 \times 4$. Each number is the product of three equal numbers. These numbers are given special names. In this chapter we study some properties of such numbers. We also study the reverse process: whenever a number is the product of two equal numbers or three equal numbers, can we find these equal numbers?

Perfect squares

Look at the following diagrams.



How many dots do you find in each figure ? You recognise them as 1, 4, 9, 16.



Suppose you have a square $ABCD$ of side-length 10 units. Divide the square into smaller unit squares (as in the adjoining figure) using lines parallel to the sides. Can you count that there are 100 unit squares?

Activity 1: Repeat this with squares of side-length 8, 12, 15 units and tabulate your findings.

Observe that $1 = 1 \times 1$, $4 = 2 \times 2$, $9 = 3 \times 3$, $16 = 4 \times 4$, $100 = 10 \times 10$.

> If a is an integer and $b = a \times a$, we say b is a **perfect square.**

Hence $1, 4, 9, 16, 100$ are all perfect squares. Since $0 = 0 \times 0$, we see that 0 is also a perfect square.

If a is an integer, we denote $a \times a$ by a^2 (we read this as **Square of** a or simply a-**square**). Thus $36 = 6^2$, $81 = 9^2$. Thus a perfect square is of the form m^2, where m is an integer.

Here you may be observing something more. For example $4 = 2 \times 2$ and $4 = (-2) \times (-2)$; in the second representation, you again have equal integers, but negative this time. There is nothing strange about this. The property is inherent in the number system: if you multiply two negative integers, you get a positive integer. Thus for any natural number m, we get $m^2 = m \times m = (-m) \times (-m)$. This also tells something about the nature of perfect squares. If m is a natural number, $m^2 = m \times m$ is also a natural number and hence m^2 is positive. If $m = 0$, then $m^2 = 0 \times 0 = 0$. If m is a negative integer, then $m = -n$ for some natural number n. Hence $m^2 = (-n) \times (-n) = n^2$, which again is positive.

> **Thus a perfect square is either equal to 0 or must be a positive integer. It can never be a negative integer.**

We have seen that $1 = 1^2$ and $4 = 2^2$ are perfect squares. Can 2 and 3 also be written as a product of two equal integers? Your intuition tells that this cannot happen. Can we see this mathematically? Suppose m and n are two natural numbers such that $m < n$. Then it is easy to see that $m^2 < n^2$ (why?).

If at all 2 is a perfect square, $2 = n^2$ for some natural number. Then $1 < 2 < 4$ gives $1 = 1^2 < n^2 < 2^2 = 4$. This forces $1 < n < 2$ (why?). Thus n is a natural number strictly between 1 and 2. But we know that given a natural number k, there is no natural number between k and its *successor* $k + 1$. Thus no natural number exists between 1 and 2. We conclude that 2 is not a perfect square.

Similarly, you can conclude that 3 is not a perfect square. This may be extended to prove that any natural number n, such that $m^2 < n < (m+1)^2$, cannot be a perfect square.

Look at the following table:

a	1	2	3	8	−7	−12	20	−15
a^2	1	4	9	64	49	144	400	225

Do you see that squares of 2,8,-12,20 are even numbers, where as the squares of 1,3, −7, −15 are all odd numbers. What do you infer?

Statement 1. The square of an even integer is even and the square of an odd integer is odd.

This is not hard to prove. If m is even, then $m = 2n$ for some integer n and $m^2 = (2n) \times (2n) = 4n^2$ is an even integer. If m is odd, then $m = 2k + 1$ for some integer k, so that

$$\begin{aligned} m^2 &= (2k+1)(2k+1) \\ &= [(2k+1) \times (2k)] + (2k+1) \times 1 \\ &= (2k) \times (2k) + (1 \times 2k) + (2k \times 1) + (1 \times 1) \\ &= 4k^2 + 2k + 2k + 1 = 4k^2 + 4k + 1, \end{aligned}$$

which is an odd number.

Consider the first ten perfect squares as in the table.

1^2	2^2	3^2	4^2	5^2	6^2	7^2	8^2	9^2	10^2
1	4	9	16	25	36	49	64	81	100

If you observe the unit's place in these squares, you see that they are 1,4,9,6,5,6,9,4,1,0 in that order. Thus the only digits which can occupy the unit's place in a perfect square are 0,1,4,5,6,9. If you take any number, its unit's digit is one of 0,1,2,3,4,5,6,7,8,9. Hence the square of that number (multiply the number with itself in your mind to get the digit in unit's place) ends with one of the numbers 0,1,4,5,6,9. Can you see that the digits 2,3,7,8 can never occur as the last digit of a perfect square? We can make a formal statement:

Statement 2. A perfect square always ends in one of the digits 0,1,4,5,6,9. If the last digit of a number is 2,3,7 or 8, it *cannot* be a perfect square.

Think it over ! If a number ends in 0,1,4,5,6 or 9, then it is *not necessary* that the number is a perfect square.

In precise mathematical language, we say that a necessary condition for the given number to be a perfect square is that it should end with one of the digits 0,1,4,5,6 or 9, but this condition is not a sufficient condition to ensure that the given number is a perfect square. This helps us to recognise perfect squares.

Exercise 5.1.

1. Express the following statements mathematically:
   (i) square of 4 is 16; (ii) square of 8 is 64; (iii) square of 15 is 225.
2. Identify the perfect squares among the following numbers: 1,2,3,8,36,49,65,67,71,81,169,625,125,900,100,1000,100000.
3. Make a list of all perfect squares from 1 to 500.
4. Write 3-digit numbers ending with 0,1,4,5,6,9, one for each digit, but none of them is a perfect square.
5. Find numbers from 100 to 400 that end with 0,1,4,5,6 or 9, which are perfect squares.

Some facts related to perfect squares

There are some nice properties about perfect squares. We study them here.

(a) Look at the following table:

a	4	10	20	25	100	300	1000
a^2	16	100	400	625	10000	90000	1000000
The number of zeros at the end of a^2	0	2	2	0	4	4	6

What do you observe? The number of zeros at the end of a square is always an even number (it may be equal to 0, but still an even number). Moreover the number of zeros at the end of each square is twice the number of zeros at the end of the number whose square is considered. Can you now formulate this as a statement?

Statement 3. If a number has k zeros at the end, then its square ends in $2k$ zeros.

Thus, if a number ends in odd number of zeros, it cannot be a perfect square. This helps us to rule out certain numbers from the list of perfect squares.

(b) Look at the adjoining table:

a	a^2	The remainder of a^2 when divided by 3	The remainder of a^2 when divided by 4
1	1	1	1
2	4	1	0
3	9	0	1
5	25	1	1
8	64	1	0
11	121	1	1
−6	36	0	0

Do you see that the remainder of a perfect square when divided by 3 is either 0 or 1? Similarly, the remainder of a perfect square when divided by 4 is either 0 or 1. When you divide a number by 3, the possible remainders are 0, 1 and 2. But when you divide a perfect square by 3, the remainder is either 0 or 1, but never 2. Similarly, when you divide a number by 4, the possible remainders are 0,1,2 or 3. However, when you divide a perfect square by 4, the remainder is either 0 or 1, but it can never be 2 and 3.

Statement 4. The remainder of a perfect square, when divided by 3, is either 0 or 1, but never 2. The remainder of a perfect square, when divided by 4, is either 0 or 1, but never 2 and 3.

Think it over !

The remainder of a perfect square, when divided by 8, is either 0 or 1 or 4. It can never be equal to 2,3,5,6,7.

Activity 2:

Take any four consecutive natural numbers and form their product. Add 1 to this product. Check whether it is a perfect square. Repeat this with some more sets of four consecutive natural numbers. Do the same with some sets of consecutive negative integers. What is your observation if one of the four consecutive integers is 0 ?

For example:

$$(1 \times 2 \times 3 \times 4) + 1 = 24 + 1 = 25 = 5^2$$

$$(8 \times 9 \times 10 \times 11) + 1 = 7920 + 1 = 7921 = 89^2.$$

Statement 5. When the product of four consecutive integers is added to 1, the resulting number is a perfect square.

(c) Read the following table:

$$\begin{aligned} 1 &= 1 = 1^2, \\ 1+3 &= 4 = 2^2, \\ 1+3+5 &= 9 = 3^2, \\ 1+3+5+7 &= 16 = 4^2, \\ 1+3+5+7+9 &= 25 = 5^2. \end{aligned}$$

Continue this process for some more rounds, adding the next odd number to the previous sum. You see that you go on getting perfect squares. A careful observation also reveals something more. The sum of the first 4 odd natural numbers is 4^2; the sum of the first 5 odd natural numbers is 5^2. Check this with the sum of the first 8 and 12 odd natural numbers. Can you formulate this as a new statement?

> **Statement 6.** The sum of the first n odd natural numbers is equal to n^2, for every natural number n.

Activity 3:

Consider the numbers 11, 101, 1001, 10001 and compute their squares. Do the same inserting some more zeros. Do you see some pattern?

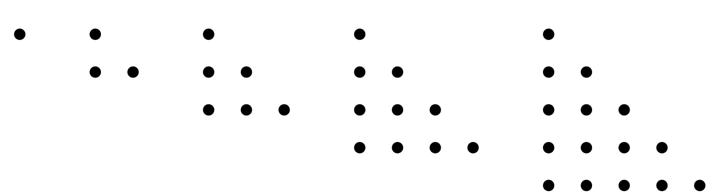
For example:

$$11^2 = 121,$$
$$101^2 = 10201,$$
$$1001^2 = 1002001,$$

and so on. You see that the middle number of each square is always 2; on both sides of 2, zeros appear as digits; and the end digits are equal to 1. The number of zeros on both sides of 2 are equal and equal to the number of zeros in the original number. Thus we can formulate a statement as follows.

> **Statement 7.** Consider the number $N = 1000\cdots01$, where zeros appear k times. (For example, for $k = 6$, you get $N = 10000001$; there are 6 zeros in the middle.) Then $N^2 = 1000\cdots\ 02000\cdots\ 01$, where the number of zeros on both sides of 2 is k.

(d) Look at the following patterns:



To generate such a pattern, put a dot in row 1, put 2 dots in row 2, put 3 dots in row 3, and so on, as shown in the figure. The dots are now arranged in the shape of a triangle. Count the number of dots in each of the triangle. (Single dot is considered as a degenerated triangle.) They are 1,3,6, 10, 15, 21, 28, 36 and so on. These numbers are called **triangular numbers** (you know the reason). You can see how the triangular numbers are formed. For n^{th} triangular number, you form a triangle of dots with n rows and each row contains as many points as the index of that row. If you want to find the 8^{th} triangular number, the number of points in the 8^{th} triangle is

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36.$$

Here are the first few triangular numbers:

$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91.$$

Take any two consecutive triangular numbers and find their sum. For example: $10 + 15 = 25 = 5^2$; $28 + 36 = 64 = 8^2$; $55 + 66 = 121 = 11^2$; $36 + 45 = 81 = 9^2$. You see that the sum of any two consecutive triangular numbers is a perfect square. You can also observe something more. Note that 28 is 7^{th} triangular number and 36 is 8^{th} one; their sum is 8^2. Similarly 66 is 11^{th} triangular number and 78 is 12^{th} one; their sum is $144 = 12^2$. Verify this property for some more pairs.

Statement 8. The sum of n^{th} and $(n + 1)^{th}$ triangular numbers is $(n + 1)^2$.

Exercise 5.2

1. Find the sum $1 + 3 + 5 + \cdots + 51$ (the sum of all odd numbers from 1 to 51) without actually adding them.
2. Express 144 as a sum of 12 odd numbers.
3. Find the 14^{th} and 15^{th} triangular numbers, and find their sum. Verify the **Statement 8** for this sum.
4. What are the remainders of a perfect square when divided by 5?

Methods for squaring a number

Many times, it is easy to find the square of a number without actually multiplying the number with itself. Consider 42^2. We may write $42^2 = (40 + 2)^2$. Thus

$$\begin{aligned} 42^2 &= (40+2)(40+2) \\ &= 40^2 + (40\times 2) + (2\times 40) + 2^2 \\ &= 40^2 + (2\times 40\times 2) + 2^2. \end{aligned}$$

Here we have used the distributive property of integers. Now it is easy to recognise $40^2 = 1600$; $2 \times 40 \times 2 = 160$; and $2^2 = 4$. Hence $42^2 = 1600 + 160 + 4 = 1764$. (You can compute 40^2, $2 \times 40 \times 2$ and 2^2 in mind and add.)

Note. The basis for this method is the identity $(a+b)^2 = a^2 + 2ab + b^2$, which you will study later.

Activity 4:

Find $89^2, 68^2, 96^2$ using the above method.

There is an easy way of computing the square of a number ending with 5. For example, consider 35^2. Take the digit in unit's place, namely 5. Put 25 (= 5^2) first. Remove the unit's digit from the given number and consider the number formed by the remaining digits, which is 3. Consider the product of 3 and its next number 4; $3\times 4 = 12$. Prefix 12 to 25 to get 1225. Check that $35^2 = 1225$.

Take another example, say 105^2. Here the number formed by the remaining digits after the removal of the digit in the unit's place is 10 and its successor is 11. Their product is $10 \times 11 = 110$. Now you may check that $105^2 = 11025$. You may formulate this as a statement.

Statement 9. If $n = \overline{a_1 a_2 \cdots a_k 5}$ (represented in base 10), then n^2 is equal to 25 prefixed by

$$(\overline{a_1 a_2 \cdots a_k}) \times (\overline{a_1 a_2 \cdots a_k} + 1)$$

Exercise 5.3

1. Find the squares of:
   (i) 31 (ii) 72 (iii) 37 (iv) 166.

2. Find the squares of : (i) 85 (ii) 115 (iii) 165.

3. Find the square of 1468 by writing this as $1465+3$.

Square roots

As you know, if the side-length of a square *ABCD* is l, then its area is l^2. Can you reverse the process? Given the area of a square, can we find its side-length?

Suppose the area of a square is 16 cm^2. To find its side-length, we write $l^2 = 16 = 4^2$ and conclude $l = 4$ cm. Here the square of a number is given and we have to find the number. Do you see that we are moving in the opposite direction?

Again consider the following perfect squares:

$$1 = 1^2,\ 4 = 2^2,\ 9 = 3^2,\ 16 = 4^2,\ 49 = 7^2,\ 81 = 9^2,\ 196 = 14^2.$$

In each case the number is obtained by product of two equal numbers. Here we say 1 is square root of 1; 2 is square root of 4; 7 is square root of 49 and so on.

Suppose N is a natural number such that $N = m^2$. The number m is called a square root of N.

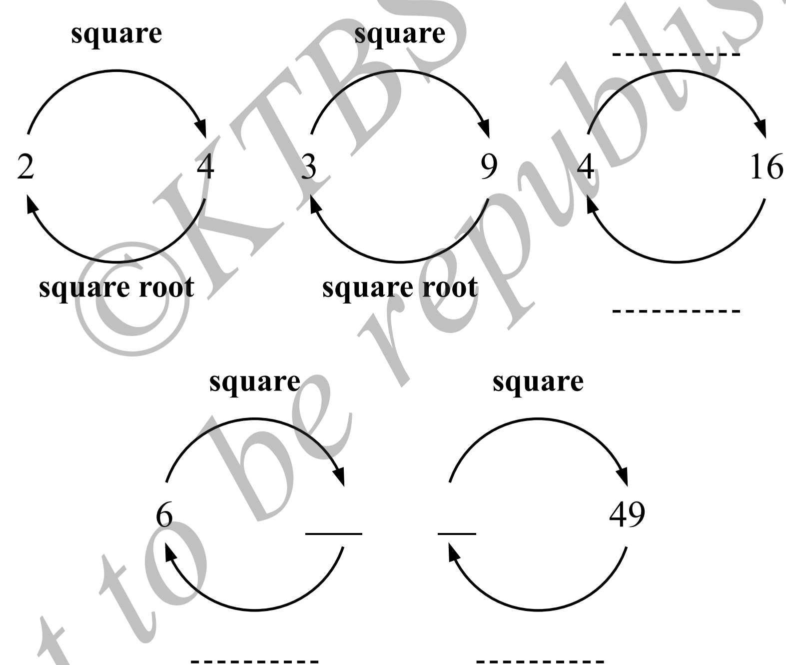
We have seen earlier $m^2 = m \times m = (-m) \times (-m) = (-m)^2$. Thus m^2 has two square roots: m and $-m$. Which one should be taken? For example $16 = 4^2 = (-4)^2$. Thus both 4 and -4 are square roots of 16. It is not clear, which one of these should be taken. Many times, physical context clarify the matter. As in the above example, if the area of a square is 16 units, then its side-length is necessarily 4 units (-4 is not admissible as it cannot be length). However, mathematically both 4 and -4 are acceptable as a square root of 16. We make the following convention.

Whenever the word square root is used, it is always meant to be the positive square root. The square root of N is denoted by $\sqrt{N}$.

Activity 5: Fill in the following blanks looking at the similarity of the statements:

$$1^2 = 1 \quad \Rightarrow \quad \sqrt{1} = 1;$$
$$2^2 = 4 \quad \Rightarrow \quad \sqrt{4} = 2;$$
$$5^2 = 25 \quad \Rightarrow \quad \sqrt{25} = ----;$$
$$11^2 = 121 \quad \Rightarrow \quad ----- = 11;$$
$$----- = 225 \quad \Rightarrow \quad ------ = 15.$$

Activity 6: Fill in the blanks with appropriate word or number looking at the similarity of the statements:



We have learnt earlier that the square of a non-zero integer is always a positive integer. Hence square root is meaningful only for positive integers or possibly 0 (whose square root is 0).

Square root of a perfect square by factorisation

We know that $3 = \sqrt{9}$ and $4 = \sqrt{16}$.

However $9 = 3 \times 3$ and $16 = 2 \times 2 \times 2 \times 2 = 4 \times 4$. Thus we can factorise the given perfect square in terms of their prime factors, combine these prime factors appropriately to write the given perfect square as a product of two equal integers. This will help us to read off the square root of the given perfect square.

Example 1. Find the square root of 5929.

Solution: We do this in several steps.

Step 1. We express 5929 as a product of prime numbers:

$$\begin{array}{r|l} 7 & 5929 \\ \hline 7 & 847 \\ \hline 11 & 121 \\ \hline & 11 \end{array}$$

Thus $5929 = 7 \times 7 \times 11 \times 11$.

Step 2. We arrange these prime factors suitably to write $5929 = (7 \times 11) \times (7 \times 11) = 77 \times 77$.

Step 3. Since $5929 = 77 \times 77$, a product of two equal integers, we conclude $\sqrt{5929} = 77$

Example 2. Find the square root of 6724.

Solution: We observe that 6724 is even, so 2 is its prime factor. Thus $6724 = 2 \times 3362$. Again 3362 is even so that $3362 = 2 \times 1681$. Thus $6724 = 2 \times 2 \times 1681$. Now there is no easy way of finding prime factors of 1681. We must go on checking whether 1681 is divisible by the primes in increasing order, starting from 3. We see that it is not divisible by 3, 5,7,11,13,17,19,23,29,31,37,but it is divisible by 41 and $1681 = 41 \times 41$. Thus we obtain $6724 = 2 \times 2 \times 41 \times 41 = (2 \times 41) \times (2 \times 41) = 82 \times 82$. We conclude: $\sqrt{6724} = 82$.

There is no easy way of finding the prime factors of a given number. There are some algorithms which can be used on computers to find the prime factors of a given large number. However, these algorithms also use up lot of computer time. The fact that there is no easy way of factorising a large number is the basis for modern day security systems used in banks and other financial institutions.

Why some numbers are not perfect squares?

You might have noticed, while finding the square root of a perfect square by factorising it, that each prime factor of the perfect square occurs even number of times. For example, if you take 1296, you factorise it as $1296 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 3$. It has two distinct prime factors, 2 and 3. Now 2 occurs four times and 3 also occurs 4 times. This helps you to write $1296 = (2 \times 2 \times 3 \times 3) \times (2 \times 2 \times 3 \times 3) = 36 \times 36$, as a product of two equal integers. You conclude that $\sqrt{1296} = 36$. The success of this method depends on the fact that the prime factors can be properly paired to get the given number as a product of two equal integers. This is possible because each prime factor occurs even number of times.

Activity 7:

Write all perfect squares between 1000 and 1500. In each case factorise it as a product of prime numbers. Check that in every case, each of the prime factor occurs even times in the product.

Now you can see why a number fails to be a perfect square. In its prime factorisation, some primes may not occur even number of times. Then there is no way of pairing the factors such that the given number is equal to a product of two equal integers. However, you can make it a perfect square on multiplication by a suitable factor or on division by a suitable factor.

Suppose a number is not a perfect square. Take for example, 48. We see that $48 = 2 \times 2 \times 2 \times 2 \times 3$. Here 2 occurs four times, where as 3 occurs only once. Hence we cannot properly pair the prime factors. However, if we multiply 48 by 3, we see that

$$48 \times 3 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 = (2 \times 2 \times 3) \times (2 \times 2 \times 3) = 12 \times 12,$$

and we get a perfect square. Of course you may as well multiply 48 with $3 \times 2 \times 2$ and get

$$48 \times 12 = (2 \times 2 \times 3 \times 2) \times (2 \times 2 \times 3 \times 2) = 24 \times 24,$$

leading to a perfect square. In fact you can multiply 48 by $3k^2$ where k is any positive integer and get as earlier $48 \times 3k^2 = (12k) \times (12k)$, which is a perfect square. However 3 is the least number with which you have to multiply 48 to get a perfect square.

Example 3. Find the least positive integer whose product with 9408 gives a perfect square.

Solution: You can start with 2 and go on dividing by 2 till you get an odd number: $9408 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 147$. Now $147 = 3 \times 49 = 3 \times 7 \times 7$. Thus we get

$$9408 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 7 \times 7.$$

Here both 2 and 7 occur even number of times, where as 3 occurs only once. Hence we have to multiply 9408 by 3 to get a perfect square (you get $9408 \times 3 = (168)^2$).

Let us go back to $48 = 2 \times 2 \times 2 \times 2 \times 3$. Instead of multiplying by 3, we can as well divide by 3:

$$\frac{48}{3} = \frac{2 \times 2 \times 2 \times 2 \times 3}{3} = 4 \times 4.$$

Still we get a perfect square. Here again you ay divide by $3 \times 2 \times 2$ and get

$$\frac{48}{3 \times 2 \times 2} = \frac{2 \times 2 \times 2 \times 2 \times 3}{3 \times 2 \times 2} = 2 \times 2.$$

a perfect square. However 3 is the least number with which you have to divide 48 to get a perfect square.

Example 4. Find the smallest positive integer with which one has to divide 336 to get a perfect square.

Solution: We observe that $336 = 2 \times 2 \times 2 \times 2 \times 3 \times 7$. Here both 3 and 7 occur only once. Hence we have to remove them to get a perfect square. We divide 336 by $3 \times 7 = 21$ and get

$$\frac{336}{21} = 16 = 4^2.$$

The least number required is 21.

Exercise 5.4

1. Find the square root of the following numbers by factorisation:

   (i) 196 (ii) 256 (iii) 10404 (iv) 1156 (v) 13225.

2. Simplify:

   (i) $\sqrt{100} + \sqrt{36}$ (ii) $\sqrt{1360+9}$ (iii) $\sqrt{2704} + \sqrt{144} + \sqrt{289}$
   (iv) $\sqrt{225} - \sqrt{25}$ (v) $\sqrt{1764} - \sqrt{1444}$ (vi) $\sqrt{169} \times \sqrt{361}$.

3. A square yard has area 1764 m^2. From a corner of this yard, another square part of area 784 m^2 is taken out for public utility. The remaining portion is divided in to 5 equal square parts. What is the perimeter of each of these equal parts?

4. Find the smallest positive integer with which one has to multiply each of the following numbers to get a perfect square:

   (i) 847 (ii) 450 (iii) 1445 (iv) 1352.

5. Find the largest perfect square factor of each of the following numbers:

   (i) 48 (ii) 11280 (iii) 729 (iv) 1352.

Perfect squares near to a given number

Let us start with a non-perfect square, say, 72. Observe that $72 = 2 \times 2 \times 2 \times 3 \times 3$, so that 2 appears only an odd number of times. We can multiply 72 by 2 to get $72 \times 2 = 144 = 12^2$. Or we may divide 72 by 2 to get $\frac{72}{2} = 36 = 6^2$. But there are more perfect squares between 6^2 and 12^2, namely $7^2 = 49$, $8^2 = 64$, $9^2 = 81$, $10^2 = 100$ and $11^2 = 121$. Which one is the nearest to 72 among these perfect squares? You see that $8^2 = 64 < 72 < 81 = 9^2$ and $72 - 64 = 8 < 9 = 81 - 72$. Thus we see that 64 is nearer to 72 than 81. Hence 64 is the nearest perfect square to 72.

In fact, given a non-perfect square, there is a unique perfect square nearest to it. Suppose N is the given non-perfect square. You can put it between two consecutive squares; there is a unique n such that $n^2 < N < (n+1)^2$. (Can you say why?). Since n and $n+1$ are two consecutive numbers, one of them is even and the other odd. Hence N cannot be exactly in the middle of n^2 and $(n+1)^2$; if $N - n^2 = (n+1)^2 - N$, then $2N = n^2 + (n+1)^2 = n^2 + n^2 + 2n + 1 = 2n^2 + 2n + 1$ which is impossible since $2N$ is even and $2n^2 + 2n + 1$ is odd. Hence either n^2 is the nearest perfect square to N or $(n+1)^2$ is the nearest perfect square. If n^2 is the nearest perfect square to N, we say n approximates $\sqrt{N}$; if $(n+1)^2$ happens to be the nearest square to N, we say $n+1$ approximates $\sqrt{N}$. Thus, even if N is not a perfect square (so that $\sqrt{N}$ is no more an integer), we can find the nearest integer to $\sqrt{N}$ and get an integer approximation to $\sqrt{N}$.

Example 5. If the area of a square is 90 cm^2, what is its side-length, rounded to the nearest integer?

Solution: Since $A = l^2$, we have $l^2 = 90$. But $81 < 90 < 100$ and 81 is nearer to 90 than 100. Hence the nearest integer to $\sqrt{90}$ is $\sqrt{81} = 9$.

Example 6. A square piece of land has area 112 m^2. What is the closest integer which approximates the perimeter of the land?

Solution: If l is the side-length of a square, its perimeter is $4l$. We know that $l^2 = 112$. Hence

$$(4l)^2 = 16l^2 = (16) \times (112) = 1792.$$

But $42^2 = 1764 < 1792 < 1849 = 43^2$ and 1764 is nearer to 1792 than 1849. Hence the integer approximation for $\sqrt{1792}$ is 42. The approximate value of the perimeter is 42 m.

Caution!

If you take $\sqrt{112}$ to the nearest integer you see that it is 11 and you may be tempted to write the approximate value of the perimeter as $44\ (= 4 \times 11)$ cm. By replacing $\sqrt{112}$ with 11, you have already committed an error and when you multiply by 4, the error increases by 4 times. That is the reason we have multiplied 112 by 16 first and then took the square root to the nearest integer. There is nothing strange in this. The nearest integer to $r = \frac{1}{4}$ is 0. But the nearest integer to $3r = \frac{3}{4}$ is 1, not $3 \times 0 = 0$.

Now you may see the limitations of integers. If you want to find $\sqrt{90}$, you have to take it as 9; if you want $\sqrt{94}$, then you have to take it as 10. But neither of them gives you a true picture of what is the square root of a non-perfect square. The limitation is because: there is no integer between n and $n + 1$. However, this property ceases to be true in the system of rational numbers. You can find a rational number between any two rational numbers. This will help in moving further close to the square root of a non-perfect square. You will learn more about this in your higher classes.

Exercise 5.5

1. Find the nearest integer to the square root of the following numbers:

   (i) 232 (ii) 600 (iii) 728 (iv) 824 (v) 1729.

2. A piece of land is in the shape of a square and its area is 1000 m^2. This has to be fenced using barbed wire. The barbed wire is available only in integral lengths. What is the minimum length of the barbed wire that has to be bought for this purpose?

3. A student was asked to find $\sqrt{961}$. He read it wrongly and found $\sqrt{691}$ to the nearest integer. How much small was his number from the correct answer?

Perfect cubes

Read the following table:

$$\begin{aligned} 1 &= 1 \times 1 \times 1; \\ 8 &= 2 \times 2 \times 2; \\ 27 &= 3 \times 3 \times 3; \\ 125 &= 5 \times 5 \times 5 \end{aligned}$$

You observe that each number is written as a product of 3 equal integers.

We say that an integer N is a perfect cube if N can be written as a product of three equal integers. If $N = m \times m \times m$, we say N is the cube of m and write $N = m^3$(read as cube of m or simply m-cube).

Consider a few more examples:

$$(-4) \times (-4) \times (-4) = -64 = (-4)^3,$$

$$(-5) \times (-5) \times (-5) = -125 = (-5)^3$$

$$(-8) \times (-8) \times (-8) = -512 = (-8)^3$$

Do you see that the negative numbers are also perfect cubes? Contrast this with perfect squares. A non-zero perfect square is necessarily a positive integer. However, perfect cubes can as well be negative.

Example 7. Find the cube of 6.

Solution : We have $6^3 = 6 \times 6 \times 6 = 36 \times 6 = 216$.

Example 8. What is the cube of 20?

Solution: Again $(20)^3 = 20 \times 20 \times 20 = (400) \times 20 = 8000$.

You have studied about a solid called *cube*. It is a solid having equal length, breadth and height. If l is the side-length of a *cube*, then its volume $V = l^3$ cubic units.

Example 9. If a cube has side-length 10 cm, what is its volume?

Solution: We have $V = 10 \times 10 \times 10 = 1000$ cm^3.

Example 10. Find the smallest integer larger than 1 which is a perfect square as well as a perfect cube.

Solution: Start with any number n and multiply it with itself 6 times to get a number N. We observe that

$$N = n \times n \times n \times n \times n \times n = (n \times n) \times (n \times n) \times (n \times n)$$
$$= (n^2) \times (n^2) \times (n^2) = (n^2)^3.$$

Thus N is the cube of n^2. On the other hand you may also observe that

$$N = n \times n \times n \times n \times n \times n = (n \times n \times n) \times (n \times n \times n)$$
$$= (n^3) \times (n^3) = (n^3)^2.$$

Hence N is also the square of n^3. Thus N is both a perfect cube and a perfect square. Taking $n = 2$, we get the least number: $N = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$. You may verify that $64 = 4^3$ and $64 = 8^2$.

Example 11. Show that 6 is not a perfect cube.

Solution: We observe that $1 < 6 < 8$ so that $1^3 < 6 < 2^3$. Since no integer exists between 1 and 2, 6 cannot be written as a product of three equal integers. Hence 6 is not a perfect cube.

Exercise 5.6

1. Looking at the pattern, fill in the gaps in the following:

2	3	4	-5	—	8	—
$2^3 = 8$	$3^3 =$ —	— $= 64$	— = —	$6^3 =$ —	— = —	— $= -729$

2. Find the cubes of the first five odd natural numbers and the cubes of the first five even natural numbers. What can you say about the parity of the odd cubes and even cubes?
3. How many perfect cubes you can find from1 to100? How many from −100 to 100?
4. How many perfect cubes are there from 1 to 500? How many are perfect squares among these cubes?
5. Find the cubes of 10, 30, 100, 1000. What can you say about the zeros at the end?
6. What are the digits in the unit's place of the cubes of 1,2,3,4,5,6,7,8,9,10? Is it possible to say that a number is not a perfect cube by looking at the digit in unit's place of the given number, just like you did for squares?

Srinivasa Ramanujan (1887-1920) is undoubtedly the greatest Indian mathematician of all times. He was self-taught and had an uncanny mathematical manipulative ability. He was not able to pass his school examinations in India, and had to be content with a clerical position in the Port Trust of Madras. However, he continued to create his own mathematics, obtained lot of hitherto unknown results. He sent these results G. H. Hardy who at once recognized Ramanujan's intrinsic mathematical ability and arranged for him to travel to Cambridge.

Because of his lack of formal training, Ramanujan sometimes did not differentiate between formal proof and apparent truth based on intuitive or numerical evidence. His intuition and computational ability allowed him to determine and state highly original and unconventional results which continued to defy formal proof until recently.

Ramanujan had an intimate familiarity with numbers, and excelled especially in number theory. J. Littelewood (a collaborator of G.H.Hardy) exclaimed that every integer was a personal friend of Ramanujan. His familiarity with numbers may be demonstrated by the following incident. During an illness in England, Hardy visited Ramanujan in the hospital. When Hardy remarked that he had taken taxi number 1729, a singularly dull number, Ramanujan immediately responded that this number was actually quite remarkable: it is the smallest integer that can be represented in two ways by the sum of two cubes: $1729 = 1^3 + 12^3 = 9^3 + 10^3$ (**Hardy-Ramanujan number**).

Unfortunately, Ramanujan's health deteriorated rapidly in England, perhaps due to the unfamiliar climate, food, and to the isolation which Ramanujan felt as the sole Indian in a culture which was largely foreign to him. Ramanujan was sent home to recuperate in 1919, but tragically died the next year at the very young age of 32.

Activity 8: (More on **Hardy-Ramanujan numbers**) Express 4104 and 13832 as a sum of two perfect cubes in two different ways. Find some numbers which can be expressed as a sum of two different perfect squares in two or more ways. Explore more on this topic.

Cube root

You know how to find the volume of a cube, given its side-length. Can you reverse the process? Given the volume of a cube, is it possible to find its side-length?

Suppose the volume of a cube is 125 cm^3. If l is its side-length, you write $l^3 = 125$ and conclude $l = 5$ cm. Here we say 5 is the cube root of 125 and write $5 = \sqrt[3]{125}$.

If N is number and m is another number such that $N = m^3$, we say m is the cube root of N and write $m = \sqrt[3]{N}$.

Remark: The definition of square root and cube root makes sense even for non-integers (for square root, the number must be non-negative). At this stage we confine only to integers and do not get involved in generality.

Contrast the definition of cube root with that of square root. Given a perfect square, there are two possible square roots; positive and negative. This is because, the square of a non-zero integer is always positive and $(-n)^2 = n^2$ for any integer n. Such a thing cannot happen for cubes. If n is positive, then n^3 is positive; if n is negative, n^3 is also negative. Hence cube root of a perfect cube is negative or positive depending on the negativity or positivity of the given perfect cube. This shows that we can, unambiguously, talk of the cube root of a perfect cube. There is no need to follow a convention as in the case of square roots.

As in the case of square roots, we can find the cube root of a perfect cube by prime factorisation.

Example 12. Find the cube root of 216 by factorisation.

Solution: Observe

$$\begin{aligned} 216 = 2 \times (108) = 2 \times 2 \times (54) = 2 \times 2 \times 2 \times 27 &= 2 \times 2 \times 2 \times 3 \times 3 \times 3 \\ &= (2 \times 3) \times (2 \times 3) \times (2 \times 3) \\ &= 6 \times 6 \times 6. \end{aligned}$$

Hence $\sqrt[3]{216} = 6$.

Example 13. Find the cube root of -17576 using factorisation.

Solution: Let us first find the cube root of 17576. As earlier, we have

$$\begin{aligned} 17576 = 2 \times (8788) = 2 \times 2 \times (4394) &= 2 \times 2 \times 2 \times (2197) \\ &= 2 \times 2 \times 2 \times 13 \times (169) \\ &= 2 \times 2 \times 2 \times 13 \times 13 \times 13 \\ &= (2 \times 13) \times (2 \times 13) \times (2 \times 13) \\ &= 26 \times 26 \times 26. \end{aligned}$$

This shows that $-17576 = (-26) \times (-26) \times (-26)$. Thus - $\sqrt[3]{17576} = -26$.

Example 14. What is the least positive integer with which you have to multiply 243 to get a perfect cube?

Solution: Let us factorise 243. We observe that

$$243 = 3 \times (81) = 3 \times 3 \times 3 \times 3 \times 3.$$

If we multiply 243 by 3, we see that

$$243 \times 3 = 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 9 \times 9 \times 9,$$

and we get a perfect cube. The answer is therefore 3.

If you factorise a positive perfect cube, you may observe that the number of times each prime factor occurs is always a multiple of 3 (just like it is a multiple of 2 for perfect squares). Hence to get the least positive integer whose product with the given integer makes a perfect cube, you have to see how much each prime factor is deficient in the prime factorisation of the given number to be away from a perfect cube.

Often, finding the cube root of a given perfect cube may be time consuming. We seek to find easier methods. We can use the behaviour of the unit's digit of a cube to fix the cube root. We see that the unit's digits of the cubes of numbers ending with 1,2,3,4,5,6,7,8,9,0 are uniquely determined. See the following table:

units' digit of n	1	2	3	4	5	6	7	8	9	0
unit's digit of n^3	1	8	7	4	5	6	3	2	9	0

We also tabulate the cubes of first nine numbers:

n	1	2	3	4	5	6	7	8	9
n^3	1	8	27	64	125	216	343	512	729

Let us see how these help us.

Example 15. Find the cube root of 103823.

Solution: Here the units digit of 103823 is 3. If $n^3 = 103823$, then the units digit of n must be 7. Let us split 103823 as 103 and 823. We observe that $4^3 = 64 < 103 < 125 = 5^3$. Hence

$$40^3 = 64000 < 103823 < 125000 = 50^3.$$

Hence n must lie between 40 and 50. Since the units digit of n is 7, the only such number is 47. You may check that $47^3 = 103823$.

Note: This method works only if you know that the given number is a perfect cube.

Nevertheless, this helps us in estimating the cube root of a non-perfect cube. We can squeeze the given number between two perfect cubes and see which is the nearest one.

Example 16. Find the nearest integer to the cube root of 12345.

Solution: We observe that

$$20^3 = 8000 < 12345 < 27000 = 30^3.$$

Hence $\sqrt[3]{12345}$ must lie between 20 and 30. We do not know whether 12345 is a perfect cube or not. However, we may sharpen the bound: $23^3 = 12167$ and $24^3 = 13824$ and hence $\sqrt[3]{12345}$ must be between 23 and 24. More over 12167 is nearer to 12345 than 13824. Hence the closest integer to $\sqrt[3]{12345}$ is 23.

Exercise 5.7

1. Find the cube root by prime factorisation:
   i)1728 (ii) 3375 (iii)10648 (iv) 46656 (v)15625.
2. Find the cube root of the following by looking at the last digit and using estimation
   (i) 91125 (ii)166375 (iii)704969.
3. Find the nearest integer to the cube root of each of the following:
   (i) 331776 (ii) 46656 (iii) 373248.

Glossary

Perfect square: an integer which is the product of two equal integers.

Triangular numbers: the sum of the first n natural number is called n^{th} triangular number.

Square-root: a number a is square root of b if $b = a^2$.

Perfect cube: an integer which is the product of three equal integers.

Cube-root: a number c whose cube is d is called a cube-root of c.

Prime factor: a prime number which divides an integer a is a prime factor of a.

Irrational number: any real number which is not a rational number.

Points to remember

- A perfect square is the product of two equal integers; a perfect cube is the product of three equal integers.

- A perfect square is always non-negative (0 is also perfect square); a perfect cube may be negative, equal to 0 or may be positive.
- Given a positive number, there are two square-roots, positive and negative. But for any number, there is only one cube-root.
- Given any positive number which is not a square, you can always squeeze it between two consecutive perfect squares.

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Answers

Exercise 5.1

1. (i) $4^2 = 16$; (ii) $8^2 = 64$; (iii) $15^2 = 225$. **2.** 1, 36, 49, 81, 16, 625, 900, 100.

3. 1,4,9,16,25,36,49,64,81,100,121,144,169,196,225,256,289,324, 361,400,441,484. **4.** You can take 200,201,204,205,206,209. None of these is a square as they lie between 196 and 225.
5. 100,121,144,169,196,225,256,289,324,361,400.

Exercise 5.2

1. $1+3+5+\cdots+51 = 26^2 = 676$. **2.** $144 = 12^2 = 1+3+5+\cdots+23$.

3. 105 and 120. Their sum is $225 = 15^2$. **4.** 0,1 or 4.

Exercise 5.3

1. (i) 961; (ii) 5184; (iii)1369; (iv) 27556. **2.** (i) 7225; (ii)13225; (iii) 27225. **3.** 2155024.

Exercise 5.4

1. (i) 14 (ii) 16 (iii) 102 (iv) 34 (v) 115. **2.** (i) 16 (ii) 37 (iii) 81 (iv) 10 (v)4 (vi) 247. **3.** 56 m. **4.** (i) 7 (ii) 2 (iii) 5 (iv) 2. **5.** (i) 16 (ii) 16 (iii) 729 (iv) 676.

Exercise 5.5

1. (i)15 (ii)24 (iii)27 (iv)29 (v)42. **2.** 127 m. **3.** 5.

Exercise 5.6

1.

2	3	4	−5	6	8	−9
$2^3 = 8$	$3^3 = 27$	$4^3 = 64$	$(-5)^3 = -125$	$6^3 = 216$	$8^3 = 512$	$(-9)^3 = -729$

2.

1^3	3^3	5^3	7^3	9^3	2^3	4^3	6^3	8^3	10^3
1	27	125	343	729	8	64	216	512	1000

The cube of an odd number is odd and the cube of an even number is even.

3. There are 4 perfect cubes from 1 to 100; there are 9 perfect cubes from −100 to 100(recall 0 is also a perfect cube). **4.** There are 7 perfect cubes from 1 to 500. Of these 64 is the only perfect square; $64 = 4^3 = 8^2$.

5. The number of zeros at the end is always a multiple of 3. **6.** Each digit occurs at the end of some cube. Hence one cannot conclude that some number is not a cube by looking at the last digit(compare this with perfect squares).

Exercise 5.7

1. (i) 12 (ii) 15 (iii) 22 (iv) 36 (v) 25. **2.** (i) 45 (ii) 55 (iii) 89.
3. (i) 69 (ii) 36 (iii) 72.

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UNIT 6

THEOREMS ON TRIANGLES

After studying this unit, you learn to:

- identify a triangle in a collection of figures.
- classify different types of triangles based on sides and angles.
- recognise the angle sum property of a triangle.
- identify the interior and exterior angles of a triangle.
- establish the relationship between the exterior angle and interior opposite angles.
- prove logically angle sum property of the triangle.
- solve problems based on the angles of a triangle.

Introduction

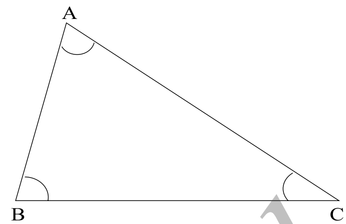
In the previous chapter you have studied the properties lines and angles. You have seen how the axioms of Euclidean geometry helps you to build some nice relations between angles and lines. In this chapter, you shall study about a closed plane figure formed by three non parallel lines, a **triangle**.

A plane figure bound by three non concurrent line segments in plane is called a triangle.

This needs an explanation. When we say a plane figure, we actually mean the linear figure, not the two-dimensional figure. Let A,B,C be three points such that they are not on the same line; we say A,B,C are **non collinear**. Join AB, BC and CA. You get a linear figure which consists of three line segments which meet only at their end points. Such a linear figure is called a triangle. We say A, B and C are the **vertices** of the triangle ABC. The segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ are called the **sides** of the triangle; and the angles $\angle BAC$, $\angle ABC$ and $\angle ACB$ are called the angles of the triangle ABC (or the interior angles of ABC).

A triangle consists 9 elements:

vertices	sides	Angles
A	AB	$\angle BAC$ or $\angle A$
B	BC	$\angle ABC$ or $\angle B$
C	AC	$\angle ACB$ or $\angle C$



Note: When there is no confusion, the sides of a triangle ABC are also denoted by AB, BC and AC. These symbols are also used to denote their respective lengths.The context tells whether the side or the length to be taken.

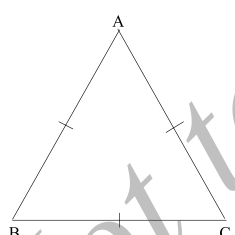
Point to ponder:

(i) A triangle cut in a plane sheet of paper is a triangular sheet, but not a triangle. Only the three line segments constitute a triangle.

Triangles are classified based on the measure of sides and angles.

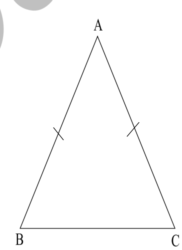
Classification based on sides:

(i) Equilateral triangle; (ii) Isosceles triangle; (iii) Scalene triangle.



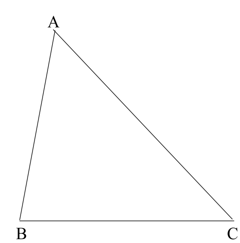
Equilateral triangle: A triangle in which all sides are of equal length is called an equilateral triangle. In triangle ABC,

$$AB = BC = CA.$$



Isosceles triangle: A triangle in which two sides are of equal-length is called an isosceles triangle. In the above triangle ABC,

$$AB = BC.$$



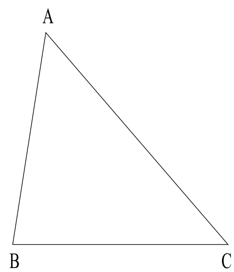
Scalene triangle: A triangle in which all sides are of different lengths is called a scalene triangle In the above triangle ABC,

$$AB \neq BC \neq CA \neq AB.$$

Do you see that an equilateral triangle is also isosceles? But an isosceles triangle need not be equilateral.

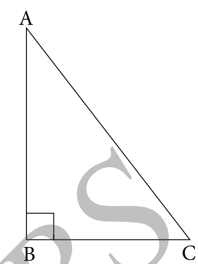
classification based on angles :

(i) Acute angled triangle;



Acute angled triangle: A triangle in which all the angles are less than 90° is called an acute angled triangle. In the above triangle ABC, $\angle ABC < 90°$, $\angle BCA < 90°$, $\angle CAB < 90°$

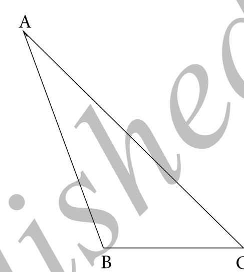
(ii) Right angled triangle;



Right angled triangle: A triangle having an angle equal to 90° is called a right angled triangle. In the above triangle ABC,

$$\angle ABC = 90°.$$

(iii) Obtuse angled triangle.

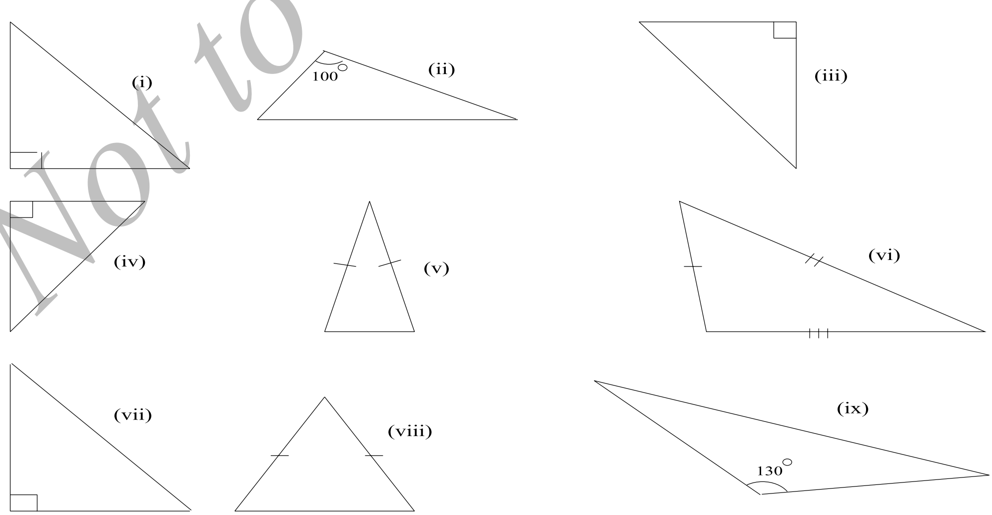


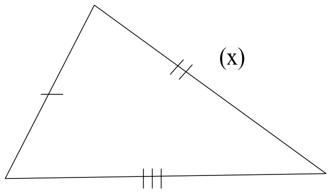
Obtuse angled triangle: If a triangle has an angle greater than 90°, it is called an obtuse angled triangle. In the above triangle ABC

$$\angle ABC > 90°.$$

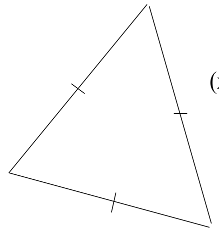
Activity 1:

Name the types of triangles given below:

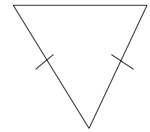




(x)



(xi)

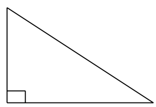


(xii)

Exercise 6.1

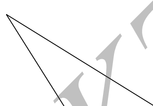
1. Match the following:

(1)



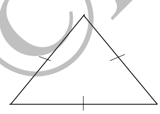
(a) Equilateral triangle

(2)



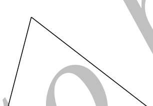
(b) Acute angled triangle

(3)



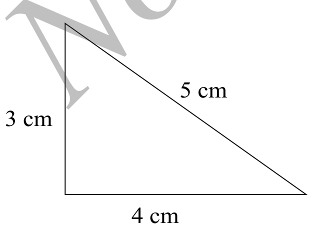
(c) Right angled triangle

(4)

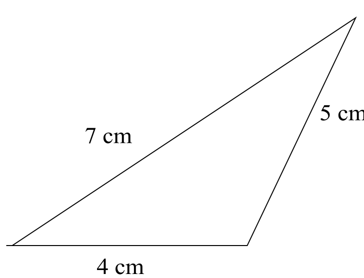


(d) Obtuse angled triangle

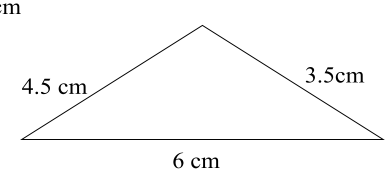
2. Based on the sides, classify the following triangles (figures not drawn to the scales):



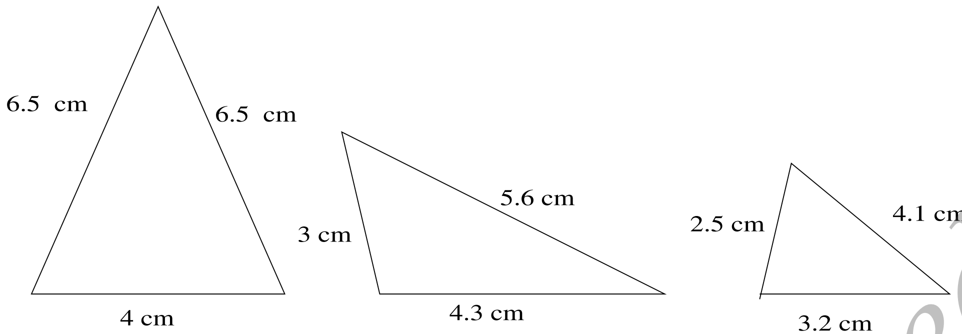
(i)



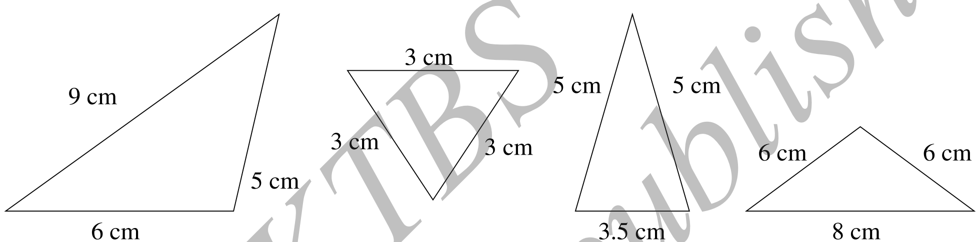
(ii)



(iii)



(iv) (v) (vi)



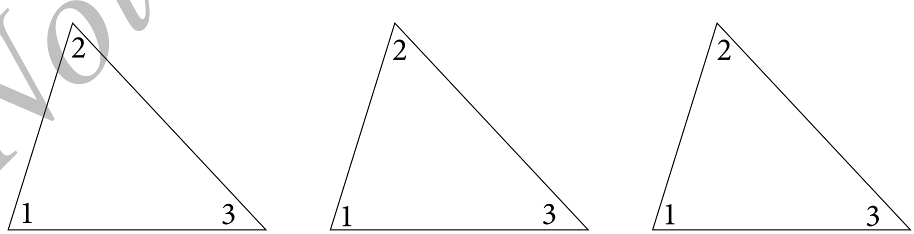
(vii) (viii) (ix) (x)

Sum of interior angles

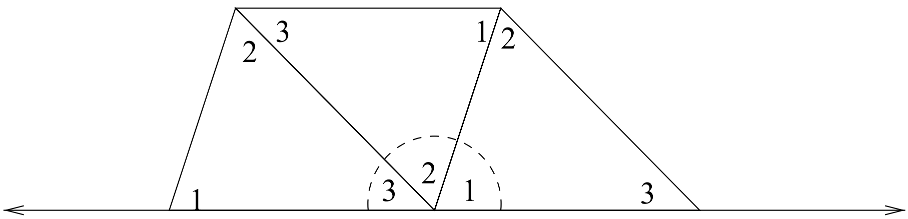
Let us do some paper activity before guessing a geometrical result.

Activity 2:

Take a sheet of paper, fold it into four folds. On one of its folds, draw a triangle using a scale and a pencil. Then cut the triangle from a pair of scissors. Now you have four identical triangular sheets. Select three of them and mark identical angles as 1, 2, and 3 on each sheet of the paper as shown below.



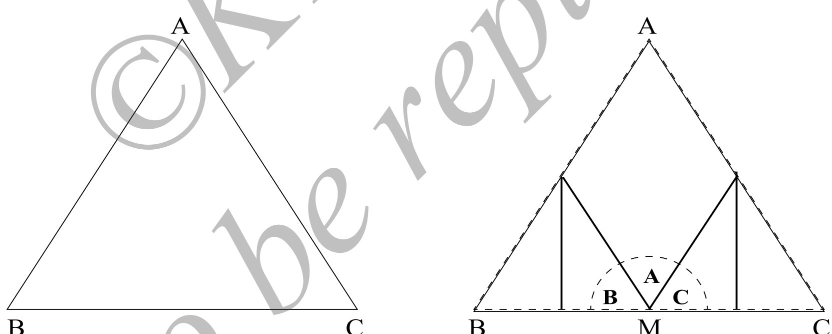
Draw a straight line on a sheet of your note book. Arrange the triangles such that angle 1 of the first triangle, 2 of the second and 3 of the third as shown in the figure.



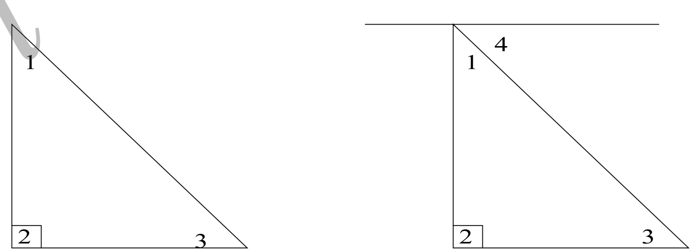
You can find that all the three angles together form a straight angle. But you see that these three angles are precisely the angles of a triangle. Thus you can guess that **the sum of three angles of a triangle is** $180°$**.**

Activity 3:

Draw a triangle ABC on a sheet as shown in the figure. Cut the remaining part of the paper. Fold the triangular sheet such that the vertex A touches the base line of the paper at M. Now fold the vertex B and C to meet the point M. You will find that they make a straight angle. (See the figure below.)



Draw right angled triangle such that angle $\angle 2 = 90°$.(See the figure below.)

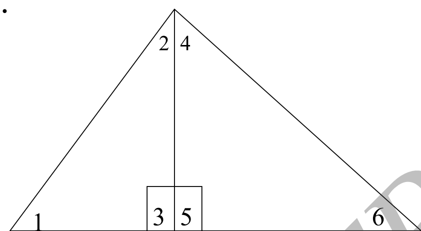


We want to find the sum of the three angles: $\angle 1 + \angle 2 + \angle 3$. Draw the parallel line to base of the triangle and passing through the vertex at the top.

Now we find another angle $\angle 4$ equal to $\angle 3$, because they are alternate interior angles between two parallel lines.
Therefore $\angle 1 + \angle 3 = \angle 1 + \angle 4 = 90^\circ$. This implies that

$$\angle 1 + \angle 2 + \angle 3 = \angle 2 + (\angle 1 + \angle 3) = 90^\circ + 90^\circ = 180^\circ .$$

Thus the sum of interior angles of a right triangle is 180 . Now we make use of this to show that the sum of three interior angles of a triangle is 180° .



Take an arbitrary triangle. This can be split into two right triangles, by drawing a perpendicular to the base. We know that,

$$\angle 1 + \angle 2 + \angle 3 = 180^\circ , \angle 4 + \angle 5 + 6 = 180^\circ .$$

Adding these we obtain

$$\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5 + \angle 6 = 360^\circ .$$

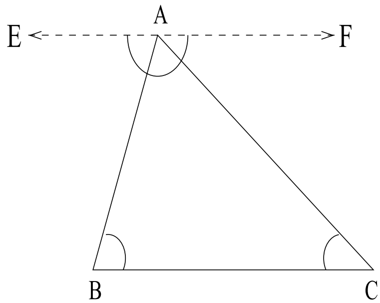
But angles $\angle 3$ and $\angle 5$ are supplementary angles and they make a straight line. Therefore $\angle 3 + \angle 5 = 180^\circ$. Thus we get

$$\angle 1 + \angle 2 + \angle 4 + \angle 6 = 360^\circ - (\angle 3 + \angle 5) = 360^\circ - 180^\circ = 180^\circ .$$

But can you see that $\angle 2 + \angle 4$ is the angle at one of the vertices of the triangle? Hence the sum of the three angles of triangle is 180°

If you observe the proof given above, it consists of two parts. In the first part you prove that the sum of three interior angles of a right triangle is 180°, using a construction; drawing a line parallel to the base line through the top vertex. A general triangle is split into two right angled triangles and we use the result for right triangle for getting on to a general triangle. Can't we construct a parallel to the base through the top vertex for a general triangle and proceed with the proof? We take up this approach below.

Theorem 1. In any triangle, the sum of the three interior angles is 180^o . (Interior angle theorem)



Given: ABC is a triangle.

To prove: $\angle ABC + \angle BCA + \angle CAB = 180^\circ$.

Construction: Through point A, draw the line EF $\parallel$ BC.

Proof: Below we give several statements and the reason for the truth of each statement. Finally we arrive at the desired conclusion.

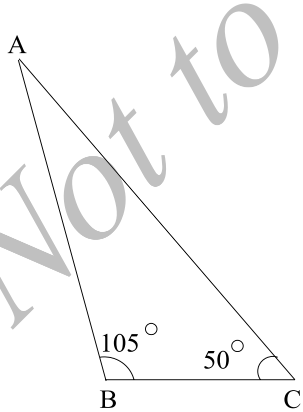
Statement	**Reason**
$\angle ABC = \angle EAB$	alternate angles by the transversal AB with the parallel lines BC and EF
$\angle BCA = \angle FAC$	alternate angles by the transversal AC with the parallel lines
$\angle EAB + \angle BAC + \angle FAC = 180^\circ$	sum of the linear angles at A.

By substituting $\angle EAB = \angle ABC$ and $\angle BCA = \angle FAC$, we finally get

$$\angle ABC + \angle BAC + \angle BCA = 180^\circ.$$

This completes the proof.

Example 1 In a triangle ABC, it is given that $\angle B = 105^\circ$ and $\angle C = 50^\circ$. Find $\angle A$.



Solution: We have, in triangle ABC, (by Theorem 1),

$\angle A + \angle B + \angle C = 180^\circ$

$\Rightarrow \angle A + 105^\circ + 50^\circ = 180^\circ$

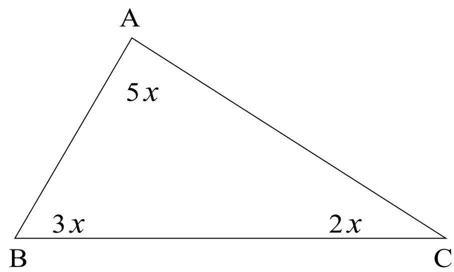
$\Rightarrow \angle A + 155^\circ = 180^\circ$

$\Rightarrow \angle A = 180^\circ - 155^\circ$

$\Rightarrow \angle A = 25^\circ$.

Thus $\angle A$ measures 25°.

Example 2 In the given figure, find all the angles.



Solution: In triangle ABC, if we make use of theorem 1, we get $\angle A + \angle B + \angle C = 180^\circ$. Hence

$5x + 3x + 2x = 180^\circ \Rightarrow 10x = 180^\circ \Rightarrow x = \frac{180^\circ}{10} \Rightarrow x = 18^\circ$.

Hence,

$\angle A = 5x = 90^\circ$;

$\angle B = 3x = 54^\circ$;

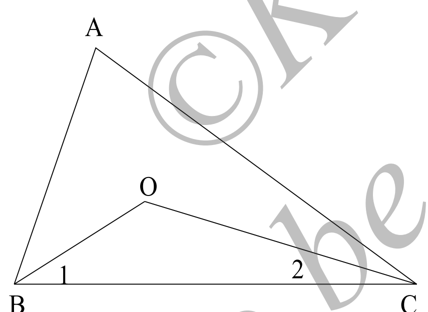
$\angle C = 2x = 36^\circ$.

Example 3 If the bisectors of the angles $\angle ABC$ and $\angle ACB$ of a triangle ABC meet at a point O, then Prove that $\angle BOC = 90^\circ + \frac{1}{2}\angle BAC$.

Solution:

Given: A triangle ABC and the bisectors of $\angle ABC$ and $\angle ACB$ meeting at point O.

To prove: $\angle BOC = 90^\circ + \frac{1}{2}\angle BAC$.



Proof: In triangle BOC we have

$$\angle 1 + \angle 2 + \angle BOC = 180^\circ \qquad \textbf{(1)}$$

In triangle ABC, we have $\angle A + \angle B + \angle C = 180^\circ$. Since BO and CO are bisectors of $\angle ABC$ and $\angle ACB$ respectively, we have

$$\angle B = 2\angle 1 \text{ and } \angle C = 2\angle 2.$$

We therefore get $\angle A + 2(\angle 1) + 2(\angle 2) = 180^\circ$.

Dividing by 2, we get $\frac{\angle A}{2} + \angle 1 + \angle 2 = 90^\circ$. This gives

$$\angle 1 + \angle 2 = 90^\circ - \frac{\angle A}{2} \qquad \textbf{(2)}$$

From (1) and (2), we get

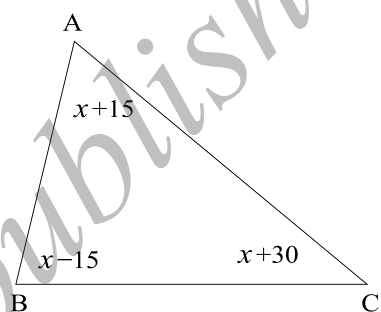
$$90^\circ - \frac{\angle A}{2} + \angle BOC = 180^\circ.$$

Hence

$$\angle BOC = 90^\circ + \frac{1}{2}\angle BAC.$$

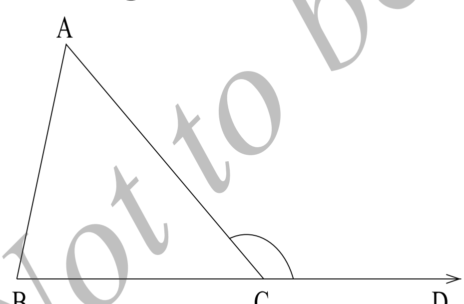
Exercise 6.2

1. In a triangle ABC, if $\angle A = 55°$ and $\angle B = 40°$, find $\angle C$.
2. In a right angled triangle, if one of the other two angles is $35°$, find the remaining angle.
3. If the vertex angle of an isosceles triangle is $50°$, find the other angles.
4. The angles of a triangle are in the ratio 1:2:3. Determine the three angles.
5. In the adjacent triangle ABC, find the value of x and calculate the measure of all the angles of the triangle.



6. The angles of a triangle are arranged in ascending order of their magnitude. If the difference between two consecutive angles is $10°$, find the three angles.

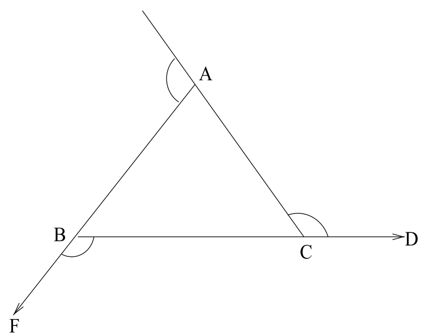
Exterior angles



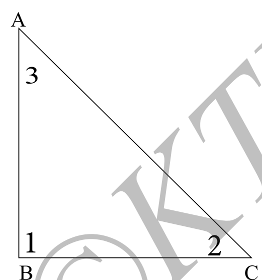
Consider a triangle ABC. If the side BC is produced externally to form a ray $\overrightarrow{BD}$, then $\angle ACD$ is called an **exterior angle** of triangle ABC at C and is denoted by Ext $\angle C$.

Note: If you produce AC to a point E (instead of BC), you get an angle $\angle BCE$. But $\angle ACD = \angle BCE$ as they are vertically opposite angles. Thus Ext$\angle C$ is the same whether you use the side BC or the side AC; it depends only on $\angle C$ of the triangle ABC.

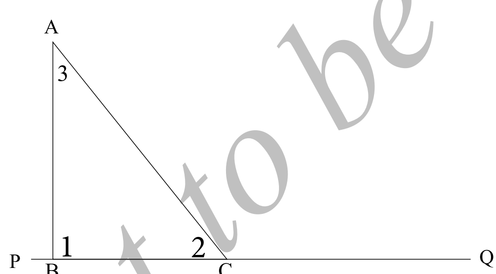
With respect to Ext∠C of triangle ABC, ∠A and ∠B are called **interior opposite angles**.



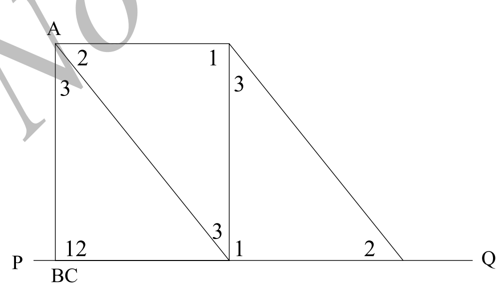
Now in triangle ABC side CA, BC, and AB are produced to form rays $\overrightarrow{CE}$, $\overrightarrow{BD}$ and $\overrightarrow{AF}$. Then ∠BAE, ∠ACD and ∠CBF are exterior angles of the triangle ABC.



Activity 4: Take three sheets of paper of size 8 × 10 cm. Place one above the other and cut three right angled triangles such that one of the corners of each sheet becomes the right angle of the triangle. Now you will have three identical triangles. Mark the angles of each triangle as 1,2,3 as shown in the figure.

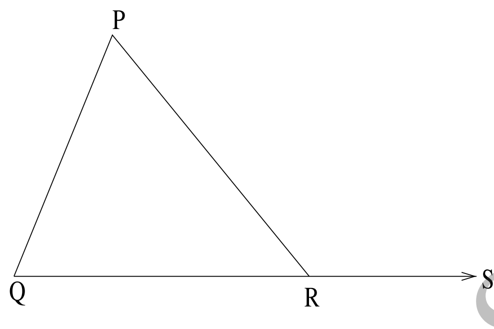


Draw straight line PQ on another sheet of paper and place one of the triangular sheet on the line as shown in figure such that ∠ACQ forms an exterior angle of the triangle ABC.



Now place the remaining two triangular sheets with angles 3 and 1 as shown in the adjacent figure. Can you see that ∠ACQ =∠3 + ∠1? Thus, the measure of an exterior angle of a triangle is equal to the sum of the corresponding two interior opposite angles.

Theorem 2. If a side of triangle is produced, the exterior angle so formed is equal to the sum of the corresponding interior opposite angles. (Exterior angle theorem.)



Given: In triangle PQR, produce QR to S. Then $\angle PRS$ is an exterior angle and the corresponding interior opposite angles are $\angle PQR$ and $\angle QPR$.

To prove: $\angle PRS = \angle QPR + \angle PQR$.

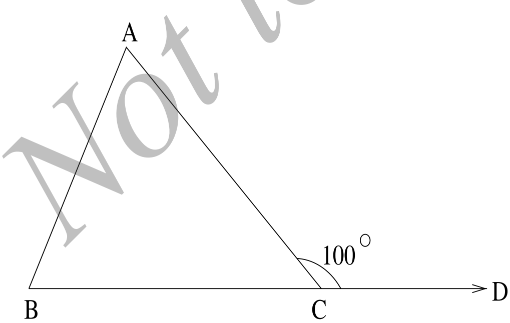
Proof:

Statement	Reason
$\angle QPR + \angle PQR + \angle PRQ = 180°$;	interior angle theorem;
$\angle PRQ + \angle PRS = 180°$;	linear pair
$\angle QPR + \angle PQR + \angle PRQ = \angle PRQ + \angle PRS$;	Axiom 1 (see unit 11);
$\angle QPR + \angle PQR = \angle PRS$;	Axiom 3 (see unit 11).

This completes the proof.

Example 4 An exterior angle of a triangle is 100°and one of the interior opposite angles is 45°. Find the other two angles of the triangle.

Solution: Let ABC be a triangle whose side BC is produced to form an exterior angle $\angle ACD$ such that Ext$\angle C = 100°$. Let $\angle B = 45°$. By exterior angle theorem we have



$\angle ACD = \angle B + \angle A$

$\Rightarrow 100° = 45° + \angle A$

$\Rightarrow \angle A = 100° - 45°$

Hence $\angle A = 55°$. Thus we get

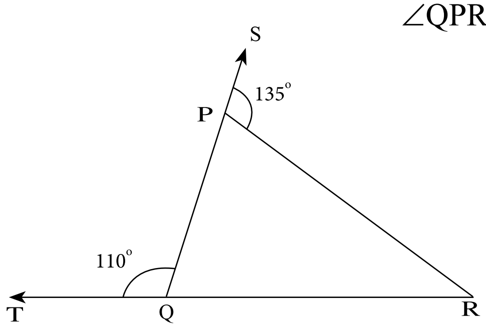
$\angle C = 180° - (\angle A + \angle B)$

$= 180° - (55° + 45°) = 80°$. Hence $\angle C = 80°$.

Example 5 In the given figure, sides QP and RQ of a triangle PQR are produced to the points S and T respectively. If $\angle SPR = 135°$ and $\angle PQT = 110°$, find $\angle PRQ$.

Solution: Since Q, P and S all lie S on the same line,

$$\angle QPR + \angle SPR = 180^\circ.$$



Hence

$$\angle QPR + 135^\circ = 180^\circ$$

or

$$\angle QPR = 180^\circ - 135^\circ = 45^\circ.$$

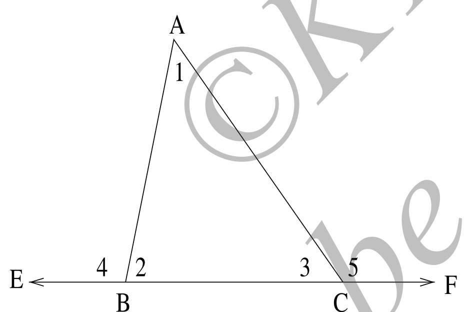
Using exterior angle property in triangle PQR, we have

$$\angle PQT = \angle QPR + \angle PRQ.$$

This gives $110^\circ = 45^\circ + \angle PRQ$.

Solving for $\angle PRQ$, we get $\angle PRQ = 110^\circ - 45^\circ = 65^\circ$.

Example 6 The side BC of a triangle ABC is produced on both sides. Show that the sum of the exterior angles so formed is greater than $\angle A$ by two right angles.



Solution: Draw a triangle ABC and produce BC on both sides to points D and F. Denote the angles as shown in the figure. We have to show that

$\angle 4 + \angle 5 = \angle 1 + 180^\circ$.

By exterior angle theorem, we have

$\angle 4 = \angle 1 + \angle 3$ and $\angle 5 = \angle 1 + \angle 2$.

Adding these two we get

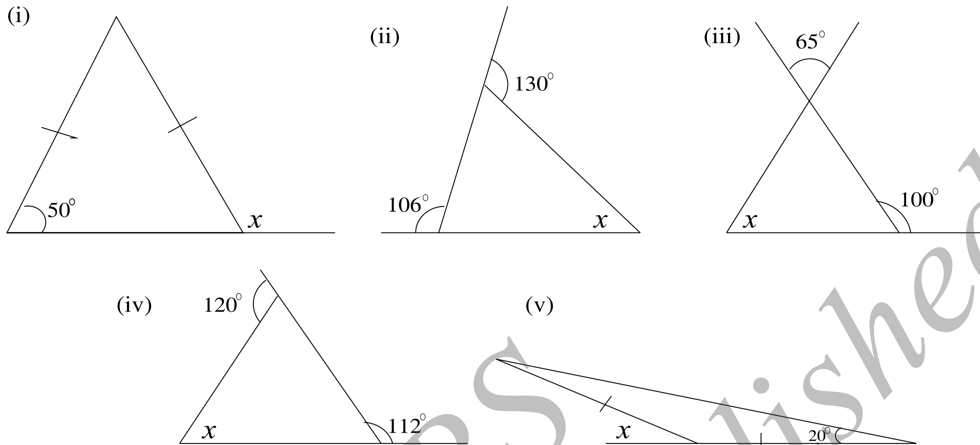
$\angle 4 + \angle 5 = (\angle 1 + \angle 3) + (\angle 1 + \angle 2) = \angle 1 + (\angle 1 + \angle 2 + \angle 3) = \angle 1 + 180^\circ$,

since the sum of all the interior angles of a triangle is 180°.

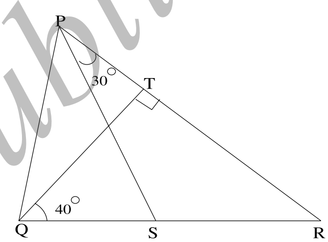
Exercise 6.3

1. The exterior angles obtained on producing the base of a triangle both ways are 104° and 136°. Find the angles of the triangle.
2. Sides BC, CA and AB of a triangle ABC are produced in an order, forming exterior angles $\angle ACD$, $\angle BAE$ and $\angle CBF$. Show that $\angle ACD + \angle BAE + \angle CBF = 360^\circ$.

3. Compute the value of x in each of the following figures:



4. In the figure, QT ⊥ PR, ∠TQR = 40° and ∠SPR = 30° . Find ∠TRS and ∠PSQ.



5. An exterior angle of a triangle is 120°and one of the interior opposite angles is 30° . Find the other angles of the triangle.

Glossary

Exterior angle: the angle subtended externally when a side of a triangle is extended.

Interior opposite angles: angles opposite to the interior angle which is supplementary to the exterior angle.

Points to remember

- Triangles are classified according to their angles and their sides.
- The sum of the three interior angles of a triangle is 180° .
- An exterior angle of a triangle is the sum of two interior opposite angles.

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Answers

Exercise 6.1

1. (i) → (C) (ii) → (D); (iii) → (A) (iv) → (B). **2.** (i) scalene (ii) scalene (iii)scalene (iv) isosceles (v) scalene (vi) scalene (vii) scalene (viii) equilateral (ix) isosceles (x) isosceles.

Exercise 6.2

1. 85°. **2.** 55°. **3.** 65° each. **4** 30°, 60° and 90°. **5** $x = 50^\circ$, $\angle A = 65^\circ$, $\angle B = 35^\circ$, $\angle C = 80^\circ$. **6.** 50°, 60° and 70°.

Exercise 6.3

1. If Ext$\angle B = 136^\circ$ and Ext$\angle C = 104^\circ$, then $\angle A = 60^\circ$, $\angle B = 44^\circ$ and $\angle C = 76^\circ$.

3. (i)130° (ii) 56° (iii) 35° (iv) 52° (v) 40°.

4. $\angle TRS = 50^\circ$, $\angle PSQ = 80^\circ$.

5. The other interior opposite angle is 90° and the third angle is 60°.

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UNIT 7

RATIONAL NUMBERS

After studying this unit you learn:

- the concept of fraction and rational numbers.
- to add and multiply rational numbers.
- the properties of rational numbers with respect to addition and multiplication: closure, associativity, commutativity, distributive properties, and existence of identity and inverse.
- representation of rational numbers on the number-line and density property of rational numbers.
- the gain while moving from integers to rational numbers and the loss.

Introduction

Earlier, you have studied natural numbers and some of their properties; the numbers {1, 2, 3, ...} is called the set of all natural numbers and is denoted by $\mathbb{N}$. You have seen that the sum or product of any two natural numbers is again a natural number: for example $5 + 13 = 18$; $12 \times 15 = 180$. We say that the set of all natural numbers is **closed** under addition and multiplication (or the **closure property** of addition and multiplication of the set of all natural numbers). You have also observed that:

$8 + 12 = 12 + 8$; $13 + (9 + 21) = (13 + 9) + 21$; $15 \times 7 = 7 \times 15$; $3 \times (5 \times 6) = (3 \times 5) \times 6$.

Here you can take any natural number. Thus you have also seen that for all natural numbers m, n, p the following hold:

$m + n = n + m$, (**commutative property** of addition);

$m + (n + p) = (m + n) + p$, (**associative property** of addition);

$m \cdot n = n \cdot m$, (**commutative property** of multiplication);

$m \cdot (n \cdot p) = (m \cdot n) \cdot p$, (**associative property** of multiplication).

You have learnt how to combine addition and multiplication and get yet another property called the **distributive property**. For example

$$5 \times (7 + 8) = 5 \times 15 = 75 = 35 + 40 = (5 \times 7) + (5 \times 8).$$

Thus for all natural numbers m, n, p, you may write

$$m \cdot (n + p) = m \cdot n + m \cdot p.$$

From this you also get

$$(n + p) \cdot m = n \cdot m + p \cdot m,$$

using the commutative property of multiplication and addition. You have also seen that the natural number 1 satisfies $1 \times 8 = 8 \times 1 = 8$; again 8 is irrelevant and the relation $1 \cdot m = m \cdot 1 = m$ holds for all natural numbers m.

You must have wondered why such a natural numbers does not exist for addition; a number u such that $m + u = u + m = m$ for all natural numbers. This is the reason you have added the number 0 to the set of all natural numbers and got the set of all **whole** numbers, W. Thus $W = \{0,1,2,3,...\}$. The number 0 satisfies, for example, $8 + 0 = 0 + 8 = 8$ and $9 \times 0 = 0 \times 9 = 0$. Thus the number 0 obeys certain rules:

$$\begin{aligned} m + 0 &= 0 + m = m, && \text{for all natural numbers } m; \\ m \cdot 0 &= 0 \cdot m = 0, && \text{for all natural numbers } m; \\ 0 + 0 &= 0 \\ 0 \cdot 0 &= 0. \end{aligned}$$

If you multiply two non-zero numbers in W, you get a non-zero number: for example $14 \times 6 \neq 0$. Thus in W, you must have noticed that $m \cdot n = 0$ is possible if and only if either $m = 0$ or $n = 0$(or may be both). Another important property you have studied about the natural number is that it is possible to compare any two natural numbers; for example, if you are asked to compare 12 and 81, you immediately say that 12 is *smaller* than 81; or 81 is *larger* than 12. You write $12 < 81$ or $81 > 12$. Thus given any two natural numbers m, n, you know that either $m < n$ or $m = n$ or $m > n$; and only one of these properties hold. This is called the **ordering** on the set of all natural numbers; and you write $1 < 2 < 3 < 4 < \cdots$. Now we place 0 before 1 and get ordering on W; $0 < 1 < 2 < 3 < 4 < \cdots$. There is one fundamental truth about this ordering.

For example, if you consider the set $E = \{3, 6, 9, ...\}$, the set of all multiples of 3, you see that 3 is the smallest element in it. However this set does not have the largest element. Suppose you take the set of all marks scored by students in your class in a test. If you write the marks in ascending order, you see that one of the marks is the lowest. Any subset of N containing at least one element of $\mathbb{N}$ is called a non-empty subset of N For example, the set of all even natural numbers is a non-empty subset of N The set of all numbers which are both even and odd is an empty subset of N since there is no such number.

Every non-empty subset of natural numbers of $\mathbb{N}$ (or W) has the smallest element.

This is called the **well ordering property** of natural numbers.

Activity 1: Write down five finite non-empty subsets of N and find their smallest elements. Write down two infinite subsets of W and find the least element of these subsets.

There is one distinct disadvantage inherent in N or W. Consider the equation $x + 5 = 3$. You see that there is no natural number m such that $m + 5 = 3$. In fact, for any natural number, you know that $m + 5 > 5 > 3$. This disadvantage is removed in the set of all integers, Z You have seen that you can adjoin to W, another class of numbers called the **negative integers**. For each natural number m, you associate another number $-m$ called the negative of m(or the opposite of m). Thus Z consists of three parts: the set of all natural numbers; 0; and the set of all negative numbers. We usually write this as

$$Z = \{\cdots -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \cdots\}.$$

Here $\mathbb{Z}$ is derived from the German word **Zahlen** (which means number).

You have also seen that you can perform addition and multiplication on $\mathbb{Z}$. If m and n are two natural numbers, then

(i) $(-m) + (-n) = -(m + n)$

(ii) $(-m) + 0 = -m = 0 + (-m)$

(iii) $(-m) + n = \begin{cases} -(m - n), & \text{if } m > n \\ n - m, & \text{if } m < n \\ 0, & \text{if } m = n \end{cases}$

(iv) $(-m)\cdot n = m - (-n) = -(m \cdot n)$

(v) $(-m) \cdot (-n) = m \cdot n$

(vi) $(-m) \cdot 0 = 0 \cdot (-m) = 0.$

Of course, if m and n are two whole numbers, we retain the same old addition and multiplication for them.

With this extension of definition of addition and multiplication, $\mathbb{Z}$ now enjoys several nice properties:

1. **Closure property:** for all integers $a, b,$ both $a + b$ and $a \cdot b$ are also integers
2. **Commutative property:** for all integers a, b
$$a + b = b + a \text{ and } a \cdot b = b \cdot a$$
3. **Associative property:** for all integers a, b, c,
$$a + (b + c) = (a + b) + c \text{ and } a \cdot (b \cdot c) = (a \cdot b) \cdot c$$
4. **Distributive property:** for all integers a, b, c,
$$a \cdot (b + c) = a \cdot b + a \cdot c$$
5. **Cancellation property:** if a, b, c are integers such that $c \neq 0$ and $ac = bc$, then $a = b$ (effectively you can cancel c on both sides).

Note that the cancellation law holds only if $c \neq 0$. For example, you may be tempted to write $3 \cdot 0 = 0 = 5 \cdot 0$ and try to cancel 0 on both sides to end up with an absurd conclusion $3 = 5$. Many of the fallacious conclusions are due to such erroneous cancellations.

What is the advantage in moving from $\mathbb{N}$ to $\mathbb{Z}$? You see that 0 has a special status: $a + 0 = 0 + a = a$ for all integers. We say that 0 is the additive identity of $\mathbb{Z}$ (or 0 is the identity with respect to addition). More over, for each integer a, we have another integer $-a$ such that $a + (-a) = 0 = (-a) + a$; if $a = m$ is a natural number, $-a$ is the negative integer $-m$; if $a = 0$, then $-a = 0$; if a is a negative integer, then $a = -n$ for some natural number n and we take $-a = n$, so that

$$a + (-a) = (-m) + m = 0,$$

by the way we have defined negative numbers. We say $-a$ is the **additive inverse** of a. Every integer has additive inverse. Now you can solve an equation $x + a = b$ in integers, for any two integers a,b. We can take $x = b + (-a)$. Then

$$x + a = (b + (-a)\,) + a = b + ((-a) + a) = b + 0 = b;$$

we have used associativity of addition, the fact that $(-a)$ is the additive inverse of a and that 0 is the additive identity.

We can also order the elements of $\mathbb{Z}$. For any natural number n, we put $-n < 0$. If m and n are two natural numbers such that $m < n$, we put $-n < -m$. You now see that all the elements of $\mathbb{Z}$ can be compared. Every natural number is an integer and we call the natural numbers as **positive integers**.

Here you may be wondering what is the status of subtraction. This is not introduced as a fundamental operation. You have learnt earlier that $12 - 7 = 5$. However, you have learnt now that -7 is the additive inverse of 7. Thus you may think $12 - 7 = 12 + (-7)$ adding 12 and -7. Here you see that the subtraction is a convenient name we have given to the extension of our fundamental operation addition to cover negative integers. This makes sense even when two integers are negative. Suppose you want to find $-8 - 13$. You put this as $(-8) + (-13) = -(8 + 13) = -21$, by the way we have defined addition of two negative integers. Now what is $15 - 21$?. Your answer is clear: adding 15 and (-21). The definition shows that $15 + (-21) = -(21 - 15) = -6$. What is the additive inverse of $-m$? Since $m + (-m) = 0$, we see that m is the additive inverse of $-m$; thus you obtain $-(-m) = m$. The emphasis here is to the fact that $-$ symbol represents additive inverse.

It is a universal law that when you gain something, you must lose something. In $\mathbb{Z}$, the gain is clear: you are able to solve an equation of the form $x + a = b$, whenever a and b are integers. Such an equation cannot be solved in $\mathbb{N}$ unless $b>a$, and a,b natural numbers. On the other hand, you see that in the set of all integers, a non-empty subset need not have the smallest element: if you take $\{\ldots\ -5,-4,-3,-2,-1\}$, the set of all negative integers,this does not have the smallest element (why?).

But the gain we have by going to the integers is much more than the loss incurred.

Consider again $\mathbb{Z}$. Here an equation of the form $ax = b$, where $a \neq 0$ cannot be solved, in general. (You may solve this provided a divides b.) Hence $\mathbb{Z}$ is also inadequate for our purpose. We look forward for a new number system in which we can do better things than we are able to do in $\mathbb{Z}$. We also take care that it has most of the nice properties of $\mathbb{Z}$ and perhaps much more. But remember we have to loose something. We shall see what is the loss and what is the gain?

Exercise 7.1

1. Identify the property in the following statements:

   (i) $2 + (3 + 4) = (2 + 3) + 4$; (ii) $2 \cdot 8 = 8 \cdot 2$; (iii) $8 \cdot (6+5) = (8 \cdot 6) + (8 \cdot 5)$.

2. Find the additive inverses of the following integers:

   6, 9, 123, −76, −85, 1000.

3. Find the integer m in the following:

   (i) $m + 6 = 8$; (ii) $m + 25 = 15$; (iii) $m - 40 = -26$; (iv) $m + 28 = -49$.

4. Write the following in increasing order:

   $$21, -8, -26, 85, 33, -333, -210, 0, 2011.$$

5. Write the following in decreasing order

   $$85, 210, -58, 2011, -1024, 528, 364, -10000, 12.$$

Rational numbers

In your earlier class, you have learnt about fractions; numbers of the form $\frac{p}{q}$, where p and q are natural numbers. For example: $\frac{1}{2}, \frac{1}{3}, \frac{3}{4}, \frac{8}{3}$ and such numbers. You have also learnt how to add and multiply such numbers.

Example 1. Add and multiply $\frac{1}{3}$ and $\frac{8}{5}$

Solution: We have

$$\frac{1}{3} + \frac{8}{5} = \frac{(1 \times 5) + (8 \times 3)}{3 \times 5} = \frac{5 + 24}{15} = \frac{29}{15}$$

Their product is

$$\frac{1}{3} \times \frac{8}{5} = \frac{1 \times 8}{3 \times 5} = \frac{8}{15}$$

If you are given a fraction $\frac{10}{4}$, you write this in the form.

$$\frac{10}{4} = \frac{5 \times 2}{2 \times 2} = \frac{5}{2}$$

by cancelling one 2 in the numerator with another 2 in the denominator. In other words, if there are common factors between the numerator and the denominator, you cancel them for convenience. Thus you do not distinguish between $\frac{10}{4}$ and $\frac{5}{2}$. You also divide one fraction by another: if you want to divide $\frac{1}{3}$ by $\frac{8}{5}$, the result is

$$\frac{1}{3} \div \frac{8}{5} = \frac{1 \times 5}{3 \times 8} = \frac{5}{24}$$

You have learnt a lot about working with fractions. Can all these be put in a formal way? Is it possible to include *negative of fractions* just like we have included negative integers ?

We define a **rational number** as a number of the form $\frac{p}{q}$ where p is an integer and $q > 0$. Here p is called the **numerator** and q is called the **denominator** of the rational number $\frac{p}{q}$. Thus the denominator of a rational number is *always a positive integer*, whereas the numerator could be positive, negative or possibly 0.

Given two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, we say they are **equivalent** if $a \times d = c \times b$. Thus $\frac{10}{4}$ is equivalent to $\frac{5}{2}$ since $10 \times 2 = 20 = 5 \times 4$. We say a rational number $\frac{a}{b}$ is in its **lowest form** or **irreducible form** if a and b do not have any common factors other than 1. Thus $\frac{5}{2}$ is its lowest form, where as $\frac{10}{4}$ is not.

Activity 2: Write ten rational numbers equivalent to $\frac{3}{4}$. How many rational numbers are there which are equivalent to $\frac{3}{4}$?

Thus $\frac{3}{4}$, $\frac{1}{5}$, $\frac{6}{7}$, $\frac{7}{10}$, $\frac{-5}{8}$, $\frac{-6}{11}$ are all rational numbers. You may observe that each integer is also a rational number: given an integer a, you may write it as $\frac{a}{1}$. Thus we do not distinguish between 7 and $\frac{7}{1}$.

Suppose you have a fraction $\frac{3}{4}$. Just like we have defined negative integers using natural numbers (or positive integers), we can also define negative of $\frac{3}{4}$ This is defined as the rational number $\frac{-3}{4}$ This is denoted by $-\frac{3}{4}$

We denote the set of all rational numbers by Q. Thus we can describe Q as: $\quad Q = \left\{ \frac{p}{q} | \, p, q \text{ are integers and } q > 0, \text{ HCF } (p,q) = 1 \right\}$

Here HCF denotes the highest common factor.

At this point, you may wonder why we are not taking negative integers in the denominator. Suppose you take a number of the form $\frac{p}{-q}$, where $q > 0$ is an integer. You may observe that this is equivalent to the rational number $\frac{-p}{q}$, since $p \times q = (-p) \times (-q)$. Thus you are not losing out any rational number by restricting denominators to the set of all positive integers.

We say a rational number is positive if both its numerator and denominator are positive integers.

The term rational number is derived from the word ratio. A rational number is a ratio of two integers where the denominator is not equal to zero.

Exercise 7.2

1. Write down ten rational numbers which are equivalent to $\frac{5}{7}$ and the denominator not exceeding 80.
2. Write down 15 rational numbers which are equivalent to $\frac{11}{5}$ and the numerator not exceeding 180.
3. Write down ten positive rational numbers such that the sum of the numerator and the denominator of each is 11. Write them in decreasing order.
4. Write down ten positive rational numbers such that numerator − denominator for each of them is -2. Write them in increasing order.

5. Is $\frac{3}{-2}$ a rational number? If so, how do you write it in a form conforming to the definition of a rational number (that is, the denominator as a positive integer)?
6. Earlier you have studied decimals 0.9, 0.8 . Can you write these as rational numbers?

Properties of rational numbers

Closure property

You have learnt that the set of all natural numbers and the set of all integers have addition and multiplication satisfying the closure, commutative, associative and distributive properties. Can we similarly define addition and multiplication on the set of all rational numbers having these properties? Our starting point is the addition and multiplication of fractions, which you have learnt in your lower class.

Example 1. Let us find the sum of $\frac{5}{6}$ and $\frac{11}{13}$. It is

$$\frac{5}{6} + \frac{11}{13} = \frac{(5 \times 13) + (11 \times 6)}{6 \times 13}$$
$$= \frac{65 + 66}{78}$$
$$= \frac{131}{78}$$

Similarly, the sum of $\frac{4}{7}$ and $\frac{-3}{5}$ is

$$\frac{4}{7} + \frac{-3}{5} = \frac{(4 \times 5) + ((-3) \times 7)}{7 \times 5}$$
$$= \frac{20 + (-21)}{35}$$
$$= \frac{-1}{35}$$

The sum of $\frac{-7}{4}$ and $\frac{-3}{7}$ is

$$\frac{-7}{4} + \frac{-3}{7} = \frac{(-7) \times 7 + (-3) \times 4}{4 \times 7}$$
$$= \frac{(-49) + (-12)}{28}$$
$$= \frac{-61}{28}$$

Example 2. The product of $\frac{2}{11}$ and $\frac{8}{7}$ is

$$\frac{2}{11} \times \frac{8}{7} = \frac{2 \times 8}{11 \times 7} = \frac{16}{77}$$

Similarly, the product of $\frac{-3}{5}$ and $\frac{-7}{2}$ is:

$$\frac{-3}{5} \times \frac{-7}{2} = \frac{(-3) \times (-7)}{5 \times 2} = \frac{21}{10}.$$

Based on these ideas, we can introduce addition and multiplication of two rational numbers. Given two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, we define

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}; \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Since $b > 0$ and $d > 0$, you see that bd is a natural number. Moreover $ad + cb$ and ac are integers. Thus you may conclude that $\frac{ad + cd}{bd}$ and $\frac{ac}{bd}$ are rational numbers. Thus the sum of two rational numbers is again a rational number. We say that the set of all rationals numbers is closed under addition.

The set of all rational numbers is closed under addition and multiplication.

Activity 3: Take ten pairs of rational numbers. Find the sum of the numbers in each pair. Check that you will always end up with rational numbers. Thus satisfy yourself that the closure with respect to addition holds. Similarly, multiply the two numbers in each pair and satisfy your self that the closure property with respect to multiplication also holds.

<u>**Associative property**</u> You have seen earlier that integers have the properties: $p + (q + r) = (p + q) + r$ and $p \cdot (q \cdot r) = (p \cdot q) \cdot r$. For example: $3 + (5 + 8) = (3 + 5) + 8$ and $3\times (5\times 8) = (3\times 5) \times 8$. Will such things hold for rational numbers as well?

Example 3. Consider three rational numbers $\frac{1}{2}, \frac{4}{5}, \frac{-6}{7}$. Observe that

$$\frac{1}{2}+\left(\frac{4}{5}+\frac{-6}{7}\right)=\frac{1}{2}+\left(\frac{7\times 4+(-6)\times 5}{5\times 7}\right)=\frac{1}{2}+\left(\frac{28-30}{35}\right)$$

$$=\frac{1}{2}+\frac{-2}{35}$$

$$=\frac{35\times 1+(-2)\times 2}{70}$$

$$=\frac{31}{70}.$$

On the other hand, you also have

$$\left(\frac{1}{2}+\frac{4}{5}\right)+\frac{-6}{7}=\left(\frac{5+8}{10}\right)+\frac{-6}{7}=\frac{13}{10}+\frac{-6}{7}$$

$$=\frac{91-60}{70}$$

$$=\frac{31}{70}.$$

Can you conclude

$$\frac{1}{2}+\left(\frac{4}{5}+\frac{-6}{7}\right)=\left(\frac{1}{2}+\frac{4}{5}\right)+\frac{-6}{7}\ ?$$

This is true for any three rational numbers $\frac{a}{b}, \frac{c}{d}$ and $\frac{e}{f}$. You compute both

$$\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right) \quad \text{and} \quad \left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f}.$$

You get

$$\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)=\frac{a}{b}+\frac{cf+ed}{df}=\frac{adf+(cf+de)b}{bdf}$$

$$=\frac{adf+cfb+deb}{bdf}.$$

Similarly,

$$\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f}=\frac{ab+cb}{bd}+\frac{e}{f}=\frac{adf+cbf+ebd}{bdf}.$$

Now can you see that $adf + cfb + deb = adf + cbf + ebd$? What properties of integers have we used here? Thus both the sums are the same.

Similar result is true for multiplication. Consider $\frac{2}{3}, \frac{7}{8}, \frac{11}{13}$ We have

$$\frac{2}{3} \times \left(\frac{7}{8} + \frac{11}{13}\right) = \frac{2}{3} \times \frac{77}{104} = \frac{154}{312};$$

$$\left(\frac{2}{3} \times \frac{7}{8}\right) \times \frac{11}{13} = \frac{14}{24} \times \frac{11}{13} = \frac{154}{312};$$

We conclude that

$$\frac{2}{3} \times \left(\frac{7}{8} + \frac{11}{13}\right) = \left(\frac{2}{3} \times \frac{7}{8}\right) \times \frac{11}{13}.$$

For any three rational numbers $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$ we obtain

$$\frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right) = \frac{a}{b} \cdot \frac{ce}{df} = \frac{ace}{bdf},$$

$$\left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f} = \frac{ac}{bd} \cdot \frac{e}{f} = \frac{ace}{bdf},$$

so that

$$\frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right) = \left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f}.$$

Addition and multiplication are associative on the set of all rational numbers.

Commutative property

You have studied earlier that addition and multiplication satisfy commutative property on the set of all integers. Given any two integers m and n, you have $m + n = n + m$ and $m \cdot n = n \cdot m$; for example $3 + 5 = 5 + 3$ and $3 \times 5 = 5 \times 3$. Do we have similar things for rational numbers?

Example 4. Let us take two rational numbers, say $\frac{8}{11}$ and $\frac{-16}{9}$. Observe that

$$\frac{8}{11} + \frac{-16}{9} = \frac{8 \times 9 + (-16) \times 11}{11 \times 9}$$

$$= \frac{72 - 176}{99} = \frac{-104}{99}.$$

Similarly,

$$\frac{-16}{9}+\frac{8}{11} = \frac{(-16)\times 11+8\times 9}{9\times 11}$$
$$= \frac{-176+72}{99} = \frac{-104}{99}.$$

Thus you will get $$\frac{8}{11}+\frac{-16}{9}=\frac{-16}{9}+\frac{8}{11}.$$

Example 5. Similarly, you may verify that

$$\frac{8}{11}\cdot\frac{-16}{9}=\frac{-16}{9}\cdot\frac{8}{11},.$$

Can we put this in a more general setting? Take any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$. Observe that

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+cb}{bd}, \quad \text{and} \quad \frac{c}{d}+\frac{a}{b}=\frac{cb+ad}{db}.$$

But you know that $ad + cb = cb + ad$ and $bd = db$ (What properties of integers are used here?). Hence you may conclude that

$$\frac{a}{b}+\frac{c}{d}=\frac{c}{d}+\frac{a}{b}.$$

This gives the commutative property of addition. A similar observation can be made for multiplication:

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}=\frac{ca}{db}=\frac{c}{d}\cdot\frac{a}{b}.$$

We obtain the commutative property of multiplication.

Addition and multiplication are commutative on the set of all rational numbers.

Distributive property

Consider the rational numbers $\frac{2}{3}$, $\frac{1}{2}$ and $\frac{1}{9}$. Observe that

$$\frac{2}{3}\cdot\left(\frac{1}{2}+\frac{1}{9}\right)=\frac{2}{3}\cdot\frac{11}{18}=\frac{22}{54}=\frac{11}{27}.$$

Similarly, we have $$\frac{2}{3}\cdot\frac{1}{2}+\frac{2}{3}\cdot\frac{1}{9}=\frac{2}{6}+\frac{2}{27}=\frac{66}{162}=\frac{11}{27}.$$

Note that we have used the property of equivalent fractions. We may conclude that

$$\frac{2}{3}\cdot\left(\frac{1}{2}+\frac{1}{9}\right)=\frac{2}{3}\cdot\frac{1}{2}+\frac{2}{3}\cdot\frac{1}{9}.$$

Activity 4:

Take several triples of rational numbers and verify distributive property. Also prove this as a general statement: if $\frac{p}{q}$, $\frac{r}{s}$ and $\frac{u}{v}$ are rational numbers, then

$$\frac{p}{q}\cdot\left(\frac{r}{s}+\frac{u}{v}\right)=\frac{p}{q}\cdot\frac{r}{s}+\frac{p}{q}\cdot\frac{u}{v},$$

using the definition of addition and multiplication of rational numbers and properties of integers.

In the set of all rational numbers, multiplication is distributive over addition.

Think it over! Given rational numbers $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, can you have

$$\frac{a}{b}+\left(\frac{c}{d}\cdot\frac{e}{f}\right)=\left(\frac{a}{b}+\frac{c}{d}\right)\cdot\left(\frac{a}{b}+\frac{e}{f}\right)?$$

In other words can addition be distributive over multiplication? (That is, can we interchange addition and multiplication in the above distributive property?)

Additive identity

Consider the rational number $\frac{0}{1}$. Observe

$$\frac{7}{8}+\frac{0}{1}=\frac{7\times 1+0\times 8}{8\times 1}=\frac{7}{8}.$$

Similarly,you may verify that

$$\frac{0}{1}+\frac{7}{8}=\frac{7}{8}.$$

For any rational number $\frac{a}{b}$ you see that

$$\frac{a}{b}+\frac{0}{1}=\frac{a\times 1+0\times b}{b\times 1}=\frac{a}{b}.$$

Similarly,you may easily verify that

$$\frac{0}{1} + \frac{a}{b} = \frac{a}{b}.$$

Thus the rational number $\frac{0}{1}$ acts as additive identity. We denote this simply by 0.

The set of all rational numbers has 0 as additive identity; that is $r + 0 = 0 + r = 0$, for all rational numbers r.

Multiplicative identity

Again consider the rational number $\frac{1}{1}$. We have, for example,

$$\frac{11}{12} \times \frac{1}{1} = \frac{11}{12}.$$

For any rational number $\frac{a}{b}$, we observe that

$$\frac{a}{b} \cdot \frac{1}{1} = \frac{a}{b} = \frac{1}{1} \cdot \frac{a}{b}.$$

Hence the rational number $\frac{1}{1}$ (which again is denoted by 1) is identity with respect to multiplication.

The set of all rational numbers has 1 as multiplicative identity; that is $r \cdot 1 = r = 1 \cdot r$, for all rational numbers r.

Additive inverse

Take $\frac{8}{13}$ and $\frac{-8}{13}$. If we add these two, we get

$$\frac{8}{13} + \frac{-8}{13} = \frac{8 \times 13 + (-8 \times 13)}{169} = \frac{0}{169} = 0.$$

This is true for any rational number. For each rational number, $\frac{a}{b}$ consider the rational number $\frac{-a}{b}$. Let us find their sum:

$$\frac{a}{b} + \frac{-a}{b} = \frac{ab + (-a)b}{b^2} = \frac{0}{b^2}.$$

But the rational number $\frac{0}{b^2}$ is the same as $\frac{0}{1}$ as they are equivalent fractions. Thus $\frac{-a}{b}$ is the additive inverse of $\frac{a}{b}$.

For each rational number r, there exists a rational number, denoted by $-r$, such that $r + (-r) = 0 = (-r) + r$.

Multiplicative inverse

You have seen that in the set of integers, an integer may not have multiplicative inverse. For example, 8 has no multiplicative inverse; $8 \times a = 1$ is not possible for any integer a. On the other hand, consider $\frac{7}{5}$ We see that

$$\frac{7}{5} \times \frac{5}{7} = \frac{35}{35} = 1 .$$

This is true for any non zero rational number.

Take any non zero rational number $\frac{a}{b}$. Then $a \neq 0$ and hence $\frac{b}{a}$ is also a rational number, Observe that

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ba} = \frac{1}{1},$$

the multiplicative identity. We have used the fact that $\frac{1}{1}$ and $\frac{ab}{ba}$ are equivalent fractions. You observe here that every non-zero rational number has multiplicative inverse.

For each rational number $r \neq 0$, there exists a rational number, denoted by r^{-1} (or $\frac{1}{r}$), such that $r \cdot r^{-1} = 1 = r^{-1} \cdot r$.

In integers, you have observed another important property, namely **cancellation law**. For example, if $8 \times a = 48$, you write

$$8 \times a = 8 \times 6.$$

You cancel 8 both sides and get $a = 6$. Thus if a,b,c are integers such that $a \neq 0$ and $ab = ac$, then $b = c$. Effectively, you can cancel equal non zero integers on both sides of an equality. This also holds in $\mathbb{Q}$.

For example, we have $\frac{4}{4} = \frac{2}{2}$, as they are equivalent fractions. But

$$\frac{4}{4} = 2 \times \frac{2}{4}, \quad \text{and} \quad \frac{2}{2} = 2 \times \frac{1}{2}.$$

You thus obtain

$$2 \times \frac{2}{4} = 2 \times \frac{1}{2}.$$

Cancelling 2 on both sides, you get $\frac{2}{4} = \frac{1}{2}$, which is true as they are equivalent fractions.

Suppose you have three rational numbers $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$ such that $\frac{a}{b} \neq 0$ and you have

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a}{b} \cdot \frac{e}{f}.$$

Since $\frac{a}{b} \neq 0$, it has multiplicative inverse $\frac{b}{a}$. Multiply both sides by $\frac{b}{a}$:

$$\frac{b}{a} \cdot \left(\frac{a}{b} \cdot \frac{c}{d}\right) = \frac{b}{a} \cdot \left(\frac{a}{b} \cdot \frac{e}{f}\right).$$

Use the associative property to write this as

$$\left(\frac{b}{a} \cdot \frac{a}{b}\right) \cdot \frac{c}{d} = \left(\frac{b}{a} \cdot \frac{a}{b}\right) \cdot \frac{e}{f}.$$

This gives

$$\frac{c}{d} = \frac{e}{f},$$

and we have cancelled $\frac{a}{b}$ on both the sides.

Now we define two more operations on Q : **subtraction** and **division**.

Consider the rational numbers $\frac{4}{13}$ and $\frac{12}{7}$. We want to give meaning for $\frac{4}{13} - \frac{12}{7}$. Recall, you have the additive inverse of $\frac{12}{7}$ which is simply $\frac{-12}{7}$.

We define

$$\frac{4}{13} - \frac{12}{7} = \frac{4}{13} + \frac{-12}{7}.$$

You can simplify this using the definition of addition:

$$\frac{4}{13} + \frac{-12}{7} = \frac{4 \times 7 + (-12) \times 13}{13 \times 7} = \frac{-128}{91}.$$

In other words, subtraction amounts to adding the additive inverse. If $\frac{a}{b}$ and $\frac{c}{d}$ are two rational numbers, then

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{-c}{d} = \frac{ad - bc}{bd}.$$

Similarly, if $\frac{a}{b}$ and $\frac{c}{d}$ are two rational numbers and if $\frac{c}{d}$ is not equal to zero,we define the division of $\frac{a}{b}$ by $\frac{c}{d}$ as follows:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$$

Note that we are multiplying $\frac{a}{b}$ with the multiplicative inverse of $\frac{c}{d}$, which exists as this is a non-zero rational number. If you want to divide $\frac{8}{15}$ by $\frac{-7}{11}$, it is simply

$$\frac{8}{15} \div \frac{-7}{11} = \frac{8}{15} \times \frac{-11}{7} = \frac{-88}{105}.$$

> **The only fundamental operations are addition and multiplication. The subtraction and division are defined in terms of addition and multiplication.**

Let us look at what we have gained from enlarging our number system from $\mathbb{Z}$ to Q. If $a \neq 0$ is an integer, then there is no integer b such that $a \cdot b = b \cdot a = 1$, unless $a = 1$ or $a = -1$. Thus, apart from 1 and −1, no other integer has multiplicative inverse in this set of intergers. On the other hand, every non zero rational number has its multiplicative inverse in Q.

This helps us to solve an equation of the form $rx = s$, where $r \neq 0$ and s are rational numbers. Suppose we have to solve the equation $\frac{3}{8}x = \frac{5}{9}$. You solve this for x by multiplying both sides by $\frac{8}{3}$. Thus.

$$\frac{8}{3} \times \frac{3}{8}x = \frac{8}{3} \times \frac{5}{9} = \frac{40}{27}.$$

You get $x = \frac{40}{27}$. This you can do for any general equation.

Suppose $r = \frac{a}{b}$ and $s = \frac{u}{v}$, where a, u are integers and b, v are natural numbers. Since $r \neq 0$ implies that $a \neq 0$, r has its multiplicative inverse $\frac{b}{a}$. Multiplying both sides by $\frac{b}{a}$, you get

$$\frac{b}{a} \cdot \left(\frac{a}{b} \cdot x\right) = \frac{b}{a} \cdot \frac{u}{v}.$$

This gives $x = \dfrac{bu}{av}$.

Exercise 7.3

1. Name the property indicated in the following:

   (i) $315 + 115 = 430$ (ii) $\frac{3}{4} \cdot \frac{9}{5} = \frac{27}{20}$ (iii) $5 + 0 = 0 + 5 = 5$

   (iv) $\frac{8}{9} \times 1 = \frac{8}{9}$ (v) $\frac{8}{17} + \frac{-8}{17} = 0$ (vi) $\frac{22}{23} \cdot \frac{23}{22} = 1$

2. Check the commutative property of addition for the following pairs:

   (i) $\frac{102}{201}, \frac{3}{4}$ (ii) $\frac{-8}{13}, \frac{23}{27}$ (iii) $\frac{-7}{9}, \frac{-18}{19}$

3. Check the commutative property of multiplication for the following pairs:

   (i) $\frac{22}{45}, \frac{3}{4}$ (ii) $\frac{-7}{13}, \frac{25}{27}$ (iii) $\frac{-8}{9}, \frac{-17}{19}$

4. Check the distributive property for the following triples of rational numbers:

   (i) $\frac{1}{8}, \frac{1}{9}, \frac{1}{10}$ (ii) $\frac{-4}{9}, \frac{6}{5}, \frac{11}{10}$ (iii) $\frac{3}{8}, 0, \frac{13}{7}$

5. Find the additive inverse of each of the following numbers:

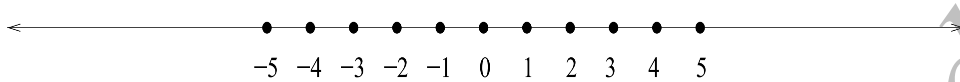
   $\frac{8}{5}, \frac{6}{10}, \frac{-3}{8}, \frac{-16}{3}, \frac{-4}{1}.$

6. Find the multiplicative inverse of each of the following numbers:

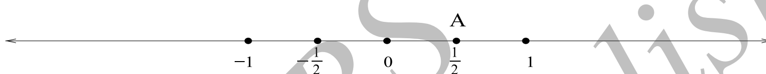
   $2, \frac{6}{11}, \frac{-8}{15}, \frac{19}{18}, \frac{1}{1000}.$

Representation of rational numbers on the Number line

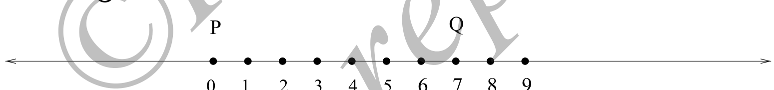
Earlier, you have seen how to represent integers on a line. We choose an infinite line and fix some point on the line. This is denoted by 0. Fix a unit of length and on both sides of 0, go on marking points at equal unit distance. On the right side, at unit distance you get 1. If you move a further unit distance, you get 2 and so on. If you move to the left by unit distance, you get −1. If you further move unit distance to the left, you get −2 and so on. Thus all the integers are represented on the line.



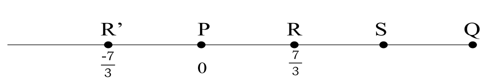
We can also use the same number line to represent rational numbers. For example, we can represent $\frac{1}{2}$ as the mid-point between 0 and 1.



We can obtain *A* by bisecting the line-segment from 0 to 1. Similarly, we can represent $-\frac{1}{2}$ as the mid-point of the line-segment from −1 to 0. How can we get $\frac{7}{3}$?



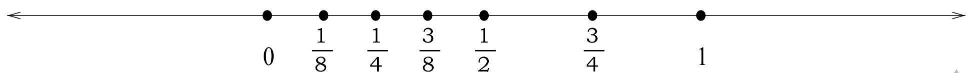
Again consider the line-segment *PQ* from 0 to 7. We divide *PQ* in to 3 equal parts: $PR = RS = SQ$. (Here you need some geometrical constructions, which you will learn later.) Then $PR = \frac{7}{3}$. Hence *R* represents the rational number $\frac{7}{3}$. We can get $-\frac{7}{3}$ by locating a point R' to the left of *P* such that $R'P = PR$.



Activity 5:

Draw a number line and locate the points $\frac{1}{8}, \frac{1}{6}, \frac{1}{3}, \frac{4}{5}, \frac{-3}{8}$, on it.

In this way you can represent each rational number by a unique point on the number line. You may observe that $\frac{2}{4}$ and $\frac{1}{2}$ are represented by the same point on the number line. (Can you see why?) All rational numbers which are equivalent to a given rational number get the same representation on the number line.



Can you now observe what is happening when you represent the rational numbers on the number line? You have seen that you can locate $\frac{1}{2}$ between 0 and 1 as the mid-point of the line segment from 0 to 1. If you take the mid-point of 0 and $\frac{1}{2}$, you get $\frac{1}{4}$; and the mid point of $\frac{1}{2}$ and 1 is $\frac{3}{4}$. The mid point of $\frac{1}{4}$ and $\frac{1}{2}$ is $\frac{3}{8}$. Do you see that the mid point of the line segment joining two points representing two rational numbers on the number line is again a rational number? We may observe here that it is possible to order the rational numbers using the ordering on Z. Suppose $\frac{a}{b}$ and $\frac{c}{d}$ are two rational numbers. We say $\frac{a}{b}<\frac{c}{d}$, if $ad < bc$ Thus

$\frac{6}{7} < \frac{7}{8}$ since $48 < 49$. On the other hand $\frac{-7}{8} < \frac{-6}{7}$, since $-49 < -48$.

Suppose you consider $\frac{2}{7}$ and $\frac{5}{8}$ Then $\frac{2}{7} < \frac{5}{8}$ Then the average of these two is

$$\frac{\frac{2}{7}+\frac{5}{8}}{2} = \frac{51}{112}.$$

Now $\frac{51}{112}$ lies between $\frac{2}{7}$ and $\frac{5}{8}$ In fact

$$\frac{2}{7} < \frac{51}{112} < \frac{5}{8},$$

which may be easily verified.

This is true for any two rational numbers. If we start with two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ such that $\frac{a}{b} < \frac{c}{d}$, then the mid-point of the points representing $\frac{a}{b}$ and $\frac{c}{d}$ represents the number

$$\frac{\frac{a}{b}+\frac{c}{d}}{2}=\frac{ad+bc}{2bd}.$$

This is also a rational number. You may observe that

$$\frac{a}{b}<\frac{ad+bc}{2bd}<\frac{c}{d}.$$

In fact $\quad \frac{a}{b}<\frac{ad+bc}{2bd} \quad \Rightarrow \quad a\times(2bd)<b(ad+bc)$

$\Rightarrow \quad 2ad<ad+bc$ (Cancellation of b)

$\Rightarrow \quad ad<bc \quad \Rightarrow \quad \frac{a}{b}<\frac{c}{d},$

which is given. Similarly, you can prove the other inequality $\frac{ad+bc}{2bd}<\frac{c}{d}$. Thus the rational number $\frac{ad+bc}{2bd}$ is strictly between $\frac{a}{b}$ and $\frac{c}{d}$.

Between any two distinct rational numbers, there is another rational number.

Compare this with the property of integers. Given any integer m, there is no integer between m and its **successor** $m+1$. This is no longer true for rational numbers. We can talk of the **next integer**, but there is nothing like the next rational number. This is precisely the loss we have while moving from integers to rational numbers.

Thus the gain is: we can divide one rational number by another non-zero rational number which helps using solving an equation of the form $rx=s,$ where $r\neq 0$ and s are rational numbers. The loss is: there is no more **the next number** in rationals, which we had in integers.

Exercise 7.4

1. Represent the following rational numbers on the number line:
   $\frac{-8}{5}$; $\frac{3}{8}$; $\frac{2}{7}$; $\frac{12}{5}$; $\frac{45}{13}$.

2. Write the following rational numbers in ascending order:

   $\frac{3}{4}, \frac{7}{12}, \frac{15}{11}, \frac{22}{19}, \frac{101}{100}, \frac{-4}{5}, \frac{-102}{81}, \frac{-13}{7}.$

3. Write 5 rational number between $\frac{2}{5}$ and $\frac{3}{5}$, having the same denominators.

4. How many positive rational numbers less than 1 are there such that the sum of the numerator and denominator does not exceed 10?

5. Suppose $\frac{m}{n}$ and $\frac{p}{q}$ are two positive rational numbers. Where does $\frac{m+p}{n+q}$ lie, with respect to $\frac{m}{n}$ and $\frac{p}{q}$?

6. How many rational numbers are there strictly between 0 and 1 such that the denominator of the rational number is 80?

7. How many rational numbers are there strictly between 0 and 1 with the property that the sum of the numerator and denominator is 70?

Introducing irrational numbers

You have seen that there is no integer whose square is 2. The argument used was: $1^2 = 1 < 2 < 4 = 2^2$ and there is no integer between 1 and 2. However, you have seen now that there are plenty of rational numbers between 1 and 2. In fact there are infinitely many rational numbers which lie between 1 and 2. Hence, it is natural to wonder whether one could get a rational number between 1 and 2 such that its square is 2. And the astounding answer is no! There is no rational number r such that $r^2 = 2$.

The argument is also simple. Suppose, if possible, there is a rational r such that $r^2 = 2$. Write $r = \frac{p}{q}$, such that HCF $(p,q)=1$ You get the relation

$$p^2 = 2q^2.$$

This shows that p^2 is even and hence p itself is even; for, if p is odd, p^2 must be odd. Hence you may write $p = 2a$, for some integer a.

Substitution gives

$$4a^2 = 2q^2 \text{ which implies } q = 2a^2.$$

But then, q is also even. Thus p and q are both even and must have a common factor 2. This contradicts what we know: $\frac{p}{q}$ is in its lowest form. We conclude that no rational number, whose square is 2, exists.

Do you now see that the set of all rational numbers is also inadequate? You cannot solve an equation of the form $x^2=2$ in $\mathbb{Q}$. This is the reason that the mathematicians started enlarging the rational number system to a better number system. One can show that for any natural number n, which is not a perfect square, there is no rational number r such that $r^2=n$. Thus, we need to give a meaning to $\sqrt{n}$, where n is a non-perfect square. This is taken care in the **real number system**.

The number $\sqrt{2}$ or such numbers which are not rational numbers are called **irrational numbers**. One can also prove that there is no rational number whose cube is equal to 2. We denote this by $\sqrt[3]{2}$ called the cube-root of 2. This again is an irrational number. There is a systematic construction of the **real number system** starting from $\mathbb{Q}$, but it involves a better understanding of the structure of $\mathbb{Q}$.

The set of all real numbers consists of two parts: **rational numbers** and **irrational numbers**. Both parts are infinite. However, in some sense, the set of all irrational numbers is much larger than the set of all rational numbers. In this way there are different **infinities** and a hierarchy among infinities. You will enter a fascinating world of **infinities**.

Glossary

Ordering: the comparability of numbers.

Well ordering property: every non-empty subset has the least element.

Positive integers: the integers which can be identified with natural numbers.

Negative integers: those integers which are additive inverses of positive integers.

Cancellation law: the law which enables us to cancel two non-zero equal quantities on both sides of an equality.

Rational number: the numbers which are of the form $\frac{p}{q}$, where p is an integer and q is a natural number.

Lowest form or irreducible form of a rational number: that form $\frac{p}{q}$ of a rational number such that p and q have no common factor.

Additive identity: that number which added to a given number does not change the given number.

Additive inverse: given a number, that number which added to the given number gives the additive identity.

Multiplicative identity: that number which multiplied with the given number does not change the given number.

Multiplicative inverse: given a number, that number which when multiplied with the given number gives the multiplicative identity.

Density property: the inseparable property of numbers; for example rational numbers are dense in number line.

Successor: the next number; integers have this property, where as rationals do not have this property.

Points to remember

- A rational numberis a ratio of the form $\frac{p}{q}$, where p is an integer and q is a natural number.
- The set of all rational numbers is closed under addition and multiplication.
- The set of all rational numbers has associative property and commutative property with respect to both addition and multiplication. Moreover, multiplication is distributive over addition.
- The rational number 0 is the additive identity; 1 is the multiplicative identity.
- Every rational number has its additive inverse; every non-zero rational number has its multiplicative inverse.
- Between any two distinct rational numbers, there are infinitely many rational numbers.
- Every integer has the next integer, but there is no next rational number for any given rational number.

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Answers

Exercise 7.1

1. (i) associative property of addition in $\mathbb{Z}$ (ii) commutative property of multiplication in $\mathbb{Z}$ (iii) distributivity of multiplication over addition in $\mathbb{Z}$

2. −6, −9, −123, 76, 85, −1000. **3.** (i) $m = 2$ (ii) $m = -10$ (iii) m = 14 m = −77. **4.** −333, −210, −26, −8, 0, 21, 33, 85, 2011. **5.** 2011, 528, 364, 210, 85, 12, −58, −1024, −10000.

Exercise 7.2

1. You can take $\frac{10}{14}, \frac{15}{21}, \frac{20}{28}, \frac{25}{35}, \frac{30}{42}, \frac{35}{49}, \frac{40}{56}, \frac{45}{63}, \frac{50}{70}, \frac{55}{77}$.

2. Take $\frac{22}{10}, \frac{33}{15}, \frac{44}{20}, \frac{55}{25}, \frac{66}{30}, \frac{88}{40}, \frac{99}{45}, \frac{110}{50}, \frac{121}{50}, \frac{132}{60}, \frac{143}{65}, \frac{154}{70}, \frac{165}{75}, \frac{176}{80}$,

3. $\frac{10}{1}, \frac{9}{2}, \frac{8}{3}, \frac{7}{4}, \frac{6}{5}, \frac{5}{6}, \frac{4}{7}, \frac{3}{8}, \frac{2}{9}, \frac{1}{10}$.

4. $\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \frac{6}{8}, \frac{7}{9}, \frac{8}{10}, \frac{9}{11}, \frac{10}{12}$.

5. The number $\frac{3}{-2}$ is the same as $\frac{-3}{2}$, as these are equivalent fractions; recall that the denominator is always positive as a convention.

6. $0.9 = \frac{9}{10}$; $0.8 = \frac{8}{10} = \frac{4}{5}$.

Exercise 7.3

1. (i) closure property of addition (ii) closure property of multiplication; (iii) 0 is the additive identity (iv) 1 is the multiplicative identity.

5. $\frac{-8}{5}, \frac{-6}{10}\left(= \frac{-3}{5}\right), \frac{3}{8}, \frac{16}{3}, \frac{4}{1}$. **6.** $\frac{1}{2}, \frac{11}{6}, \frac{-15}{8}, \frac{18}{19}$, 1000

Exercise 7.4

2. $\frac{-13}{7} < \frac{-102}{81} < \frac{-4}{5} < \frac{7}{12} < \frac{3}{4} < \frac{101}{100} < \frac{22}{19} < \frac{15}{11}$.

3. We can take $\frac{13}{30}, \frac{14}{30}, \frac{15}{30}, \frac{16}{30}, \frac{17}{30}$ If you increase both the numerator and denominator and use the property of equivalent fractions, you can get any number of such collections.

4. There are 15 such rational numbers:
$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{3}{4}, \frac{3}{5}, \frac{3}{7}, \frac{4}{5}$.

5. If $\frac{m}{n}$ and $\frac{p}{q}$ are distinct, then $\frac{m+p}{n+q}$ lies between $\frac{m}{n}$ and $\frac{p}{q}$. If $\frac{m}{n} = \frac{p}{q}$, then $\frac{m+p}{n+q} = \frac{m}{n} = \frac{p}{q}$. **6.** 79. **7.** 69.

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UNIT 8

LINEAR EQUATIONS IN ONE VARIABLE

After studying this unit, you learn:

- the meaning of linear equation in one variable.
- to solve a linear equation in one variable.
- to formulate a linear equation from a verbal problem.
- to verify the solution.

Introduction

In this chapter, we study linear equations in one variable with rational co-efficients and solve them in rational number system.

Equality of two different algebraic expression is called an equation. The **standard form** of an **equation** is a statement that an algebraic expression is equal to 0. The statement need not be true for any value of the variables in it; or may be true only for certain values of the variables in it. For example: consider $3x - 5 = 0$. If we are looking for integers x for which this statement is true, you see that no integer satisfies it. On the other hand, if you look for a rational number, then $x = \frac{5}{3}$ makes the statement $3x - 5 = 0$ true. The value of the variable which makes the statement true is called a **solution** of the equation.

Some times you may come across statements like $2x - 5 = x + 6$. This is also an equation, but this is not in the **standard form**. However, this can be brought to standard form; $x - 11 = 0$.

Given an equation, it may not have any solution in some system and may have solution in some other system, as you have noticed earlier. The equation $x^2 - 2 = 0$ has no solution in the set of all rational numbers, but it can be solved in the set of real numbers. In fact one of the motivations to introduce new number systems is that we should be able to solve such equations. You will learn later that the equation $x^2 + 1 = 0$ also cannot be solved in the set of all real numbers, but it can be solved in an enlarged number system called the set of all complex numbers. Thus it is very important to mention where you are seeking the solution.

An equation in one variable is a statement that an algebraic expression in one variable is equal to 0. If the degree of the expression is also 1, we say that the equation is a **linear equation**. The standard form of a linear equation in one variable is $ax + b = 0$, where $a \neq 0$. Here x is the variable and a,b are constants(some numbers). For example: $x - 9 = 0$; $5x - 30 = 0$; $\frac{3}{5}x + \frac{1}{3} = 0$.

What do we mean by solving an equation?. The statement that some expression equals to 0 may not be valid for every value of the variable in it. However, some value(s) of the variable may make the statement true. In this case, that value of the variable which makes the statement true is called a **solution or root** of the given equation.

The process of finding solution or solutions of a given equation is called **solving the equation**.

As you have observed, given a linear equation $ax + b = 0$, the existence of solution depends on the number system where you are seeking solution. If a and b are integers, then there may not be any integer solution unless a divides b. However, if you are in the system of rational numbers, you can always get a rational number as a solution. This is the advantage we had from moving from integers to rational numbers. We are ready to exploit it. With this back-ground we can now make the statement:

> **A linear equation with rational coefficients, $ax + b = 0$, $a \neq 0$, has a unique solution in the rational number system.**

In fact the solution can be written as $x = -ba^{-1}$, where a^{-1} is the multiplicative inverse of a. You have seen earlier, while studying rational numbers, that any non-zero rational number has its multiplicative inverse in the system of rational numbers.

You will see later that this statement is also true in the real number system, provided you look for solution in the real number system. (Infact this is true in any number system in which every non zero number has multiplicative inverse.)

Solving a linear equation in one variable

Consider the equation $5x - 15 = 0$. We want to find the value of x for which $5x - 15 = 0$ holds. Here we make use of important axioms. You will again come across them in Geometry.

1. If equals are added to equals, then the wholes are equal.

If we know $a = b$, then for any c, we get $a+c = b+c$. For example $x- 5 =0$ is given. We add equal number 5 both sides. What the axiom says is that equality remains intact. Thus $(x-5) + 5 = 0 + 5$ which is same as $x =5$. The important point to note is that here a,b,c can be algebraic expressions as well. Thus $3x + 2 = 5- x$ implies that $(3x + 2) + (x - 5) = (5- x)+(x - 5) = 0$ or $4x - 3 = 0$. This axiom and others are important tools in manipulations of expressions.

2. If equals are subtracted from equals, then the remainders are equal.

Thus $a = b$ implies that $a- c = b - c$. Given $x + 5 = 2x - 6$, we can subtract $x - 6$ from both sides and get $(x + 5) - (x - 6) = (2x - 6) - (x - 6)$ or $11 = x$.

Can you see that there is not much difference between **1** and **2** once you consider subtraction as addition of additive inverse. Just like you define additive inverse to numbers, you can do this for polynomials as well and all the properties which are true for integers are true again for polynomials. This holds because in essence each algebraic expression would ultimately represent a number.

Example 1. Solve the equation $x - 15 =0$.

Solution: We add 15 both sides and invoke axiom **1** to get $(x - 15) + 15 = 0 + 15 = 15$. This reduces to $x =15$.

Example 2. Solve the equation $x + 9 =20$.

Solution: Do you see that this is not in the standard form. We can bring this to standard form by subtracting 20 both sides. Thus axiom **2** gives $(x + 9) - 20 = 20 - 20 = 0$ or $x - 11 = 0$. Now this is in standard form. By adding 11 both sides, you will get $x - 11 + 11 = 0 + 11$ or $x = 11$. We can combine both these and reduce the number of steps: subtract 9 both sides. Thus $(x + 9) - 9 = 20 - 9$ or $x =11$.

Example 3. Solve $2x - 3 = x + 8$.

Solution: Here you see that both the sides contain algebraic expression. Suppose you subtract $x - 3$ from both the sides. You get

$$(2x - 3) - (x - 3) = (x +8) - (x - 3) \text{ or equivalently } x =11.$$

3. If equals are multiplied by equal quantities, then the products are equal.

Thus if $a = b$, then $ac = bc$ for any c. For example, suppose $\frac{x}{2} = 1$ is given. You can multiply both the sides from 2 and get $\frac{x}{2} \times 2 = 1 \times 2$ or $x = 2$.

4. If equals are divided by non-zero equal quantities, then the quotients are equal.

This says that if $a = b$ and $c \neq 0$, then $\frac{a}{c} = \frac{b}{c}$ This again is the same as **3** once you think that division is the same as multiplication by multiplicative inverse.

Example 4. Solve $\frac{x}{3} = 9$.

Solution: We multiply both the sides from 3 and get

$$\frac{x}{3} \times 3 = 9 \times 3 = 27.$$

Thus $x = 27$.

Example 5. Solve $\frac{2x}{9} = 5$.

Solution: We multiply both the sides by the same number $\frac{9}{2}$ Note this is same as multiplying first by 9 and then dividing by 2. We get

$$\frac{2x}{9} \times \frac{9}{2} = 5 \times \frac{9}{2} = \frac{45}{2}.$$

We obtain $x = \frac{45}{2}$.

Example 6. Solve $15x = 120$.

Solution: We divide both the sides by 15 and get

$$\frac{15x}{15} = \frac{120}{15} = 8.$$

This gives $x = 8$.

Example 7. Solve $13y = 100$.

We divide both sides by 13 and get

$$\frac{13y}{13} = \frac{100}{13}.$$

This gives $y = \frac{100}{13}$.

Another important component of solving an equation is that of **verifying** the solution. We have to check whether the so called solution we obtain makes the given statement true. This is done by substituting the value of the variable we have got in the given equation and by *verifying the truth of the statement.*

Example 8. Is 2 a solution of the equation $3x - 5 = 19$?

Solution: We substitute $x = 2$, in the given relation. We obtain $3x - 5 = 3(2) - 5 = 6 - 5 = 1$, which is the left hand side or LHS. But the right hand side or RHS is 19. Since $1 \neq 19$, we see that LHS is not equal to RHS for $x = 2$. Hence $x = 2$ is not a solution.

Example 9. Is 7 a solution of $2x - 4 = 10$?

Solution: Substitute $x = 7$ in the given relation. We get LHS $= 2x - 4 = (2 \times 7) - 4 = 14 - 4 = 10$ and RHS $= 10$. Thus we see that LHS = RHS. Therefore, $x = 7$ is the solution of $2x - 4 = 10$.

Example 10. Solve the equation $2x - 3 = 7$.

Solution: Adding 3 both the sides, we get $2x - 3 + 3 = 7 + 3$. This gives $2x = 10$. Now dividing both the sides by 2, we get $\frac{2x}{2} = \frac{10}{2}$ Hence $x = 5$. Let us verify whether we have got a solution. Putting $x = 5$, we get $2x - 3 = 2(5) - 3 = 10 - 3 = 7$. Thus the equation is **satisfied**.

We can reduce the number of steps in solving an equation. Consider the equation $2x - 3 = 7$. We can simply write this as $2x = 7 + 3$. Actually you are adding 3 both sides. But you drop that as a separate step and do the addition mentally to write $2x - 3 + 3 = 2x$. We say, we have **transposed** -3 to the **other side**. In fact this is a convenient way for doing fast calculations. Again you can divide both the sides by 2 in mind and write $x = 5$.

We can also transpose algebraic expressions as well. For example, given $4x - 3 = 3x + 2$, we transpose $3x$ to the left side and -3 to the right side to get $(4x - 3x) = (3+2)$; i.e., $x = 5$.

Example 11. Solve $5x - 12 = 10 - 6x$.

Solution: By transposing $-6x$ to the right side and -12 to the left side, we get $5x + 6x = 10 + 12$. Hence $11x = 22$ or $x = 2$.

> **<u>Rule for transposing:</u> When you transpose an expression from one side to the other side of an equality, you have to change the sign of the expression you are transposing.**

Thus $-6x$ in the previous example becomes $+ 6x$ when transposed from the left side to the right side.

Example 12. Solve $8x - 3 = 9 - 2x$.

Solution: We transpose the variables on one side and constants on the other side and get $8x + 2x = 9 + 3$. Thus $10x = 12$ or $x = \frac{12}{10}$. This reduces 6 to $x = \frac{6}{5}$.

Example 13. Solve $8x + 9 = 3(x - 1) + 7$.

Solution: The equation is $8x + 9 = 3x - 3 + 7 = 3x + 4$. By transposing, we get $8x - 3x = 4 - 9$ or $5x = -5$. We thus get $x = -1$. We can verify this: $8x + 9 = 8(-1) + 9 = -8 + 9 = 1$ and $3(x - 1) + 7 = 3(-1 - 1) + 7 = 3(-2) + 7 = -6 + 7 = 1$. Thus $8x + 9 = 3(x - 1) + 7$ is true for $x = -1$. This checks that $x = -1$ is a solution.

Example 14. Solve $\frac{2}{3}x = \frac{3}{8}x + \frac{7}{12}$.

Solution: Here we multiply by the LCM of the denominators: LCM of 3,8,12 is 24. (Which axiom are you using here?) Thus

$$\left(\frac{2}{3}x\right) 24 = \left(\frac{3}{8}x + \frac{7}{12}\right) 24,$$

and this simplifies to $16x = 9x + 14$.

Hence $7x = 14$ or $x = 2$. Check that $x = 2$ is indeed the solution.

Do you see the advantage of multiplying both the sides by the LCM of all the denominators?. The new expression becomes an equation with integer coefficients. You can easily handle this equation. Thus Axiom 3 helps in transforming an equation with rational coefficients to an equivalent equation with integer coefficients.

Example 15. Solve the equation

$$\frac{2x+7}{5} - \frac{3x+11}{2} = \frac{2x+8}{3} - 5.$$

Solution: Here 2,3,5 appear in the denominators of various fractions. Their LCM is 30. We multiply through out by 30. Thus

$$\frac{2x+7}{5} \times 30 - \frac{3x+11}{2} \times 30 = \frac{2x+8}{3} \times 30 - (5 \times 30).$$

This reduces to

$$6(2x+7) - 15(3x+11) = 10(2x+8) - 150.$$

$$\Rightarrow 12x + 42 - 45x - 165 = 20x + 80 - 150.$$

Transposing appropriate quantities, we get

$$12x - 45x - 20x = -42 + 165 + 80 - 150 \Rightarrow$$
$$-53x = 53.$$

Hence $x = -1$ is the solution. You may verify this by substituting in the equation and checking whether LHS and RHS agree.

Example 16. Solve $(x+4)^2 - (x-5)^2 = 9$.

Solution: This apparently looks like an equation which is not linear. But expanding this using identities, we get

$$x^2 + 8x + 16 - x^2 + 10x - 25 = 9.$$

This reduces to $18x - 9 = 9$. After transposing, we get $18x = 18$. Hence $x = 1$. It is easy to verify that $x = 1$ is a solution.

Think it over!

(1) You have learnt how to solve an equation of the form $ax + b = 0$, **where** $a \neq 0$. **You get** $x = (-b/a)$, **as the unique solution. Suppose you have two variables** x **and** y; **say an equation** $ax + by + c = 0$, **where** $a \neq 0$ **and** $b \neq 0$. **This is again a linear equation, but the number of variables is now 2. Can you solve such an equation? How many solutions** (x, y) **are there for such an equation? If** a, b, c **are integers, is it always possible to find integers** x, y **satisfying the equation? (2) Can you solve** $(x+1)^2 = x^2 + 2x + 1$ **?**

Exercise 8.1

1. Solve the following:

(i) $x + 3 = 11$ (ii) $y - 9 = 21$ (iii) $10 = z + 3$

(iv) $\frac{3}{11} + x = \frac{9}{11}$ (v) $10x = 30$ (vi) $\frac{s}{7} = 4$

(vii) $\frac{3x}{6} = 10$ (viii) $1.6 = \frac{x}{1.5}$ (ix) $8x - 8 = 48$

(x) $\frac{x}{3} + 1 = \frac{7}{15}$ (xi) $\frac{x}{5} = 12$ (xii) $\frac{3x}{5} = 15$

(xiii) $3(x + 6) = 24$ (xiv) $\frac{x}{4} - 8 = 1$ (xv) $3(x + 2) - 2(x - 1) = 7$.

2. Solve the equations:

(i) $5x = 3x + 24$ (ii) $8t + 5 = 2t - 31$ (iii) $7x - 10 = 4x + 11$

(iv) $4z + 3 = 6 + 2z$ (v) $2x - 1 = 14 - x$ (vi) $6x + 1 = 3(x - 1) + 7$

(vii) $\frac{2x}{5} - \frac{3}{2} = \frac{x}{2} + 1$ (viii) $\frac{x - 3}{5} - 2 = \frac{2x}{5}$ (ix) $3(x + 1) = 12 + 4(x - 1)$

(x) $2x - 5 = 3(x - 5)$ (xi) $6(1 - 4x) + 7(2 + 5x) = 53$

(xii) $3(x + 6) + 2(x + 3) = 64$ (xiii) $\frac{2m}{3} + 8 = \frac{m}{2} - 1$

(xiv) $\frac{3}{4}(x - 1) = x - 3$.

Application of linear equations

We shall take up some practical situations leading to linear equations in one variable.

Example 17. Seven times a number, if increased by 11, is 81. Find the number.

Solution : We do this in several steps.

Step 1: First we convert the given data to an appropriate equation. Let the number be x. Apriori, we do not know what this number is. We formulate a linear equation involving the unknown x, using the given data. Now seven times the number means $7x$. Increasing this by 11 leads to $7x + 11$. The problem says that $7x + 11 = 81$. Can you see now that we have a linear equation in the unknown x?.

Step 2: We now have to solve the equation $7x + 11 = 81$. You have already learnt methods for solving such an equation. Transpose 11 to the other side to get $7x = 81 - 11 = 70$. Divide by 7 and you get $x = 10$.

Step 3: We have to check whether the number 10 satisfies the statement of our problem. Now seven times the number gives $7\times 10 = 70$. Adding 11 to this gives 81. And that is precisely the data says. Thus $x = 10$ is indeed the solution.

Example 18. The present age of Siri's mother is three times the present age of Siri. After 5 years, their ages add to 66 years. Find their present ages.

Solution: Again we go through several steps in the solution of this problem. Suppose Siri's present age is x years. Then her mother's age is $3x$ years. After 5 years, Siri's and her mother's respective ages would be $x + 5$ and $3x + 5$ years. The data says that these two numbers would add up to 66. Thus we get the equation

$$(x + 5) + (3x + 5) = 66.$$

Now it is easy to find the value of x. The equation reduces to $4x + 10 = 66$. After transposing 10 to the other side, we obtain $4x = 56$ or $x = 14$. This means Siri's present age is 14 years and her mother's present age is $14\times 3 = 42$ years.

Let us now verify whether these numbers match with the statement of the problem. After 5 years, Siri's age would be $14 + 5 = 19$ years. Her mother's age would be $42 + 5 = 47$ years. Their sum is $19 + 47 = 66$ years which completely matches with the given statement. We conclude that Siri's present age is 14 years and her mother's age is 42 years.

Example 19. The sum of three consecutive even numbers is 252. Find them.

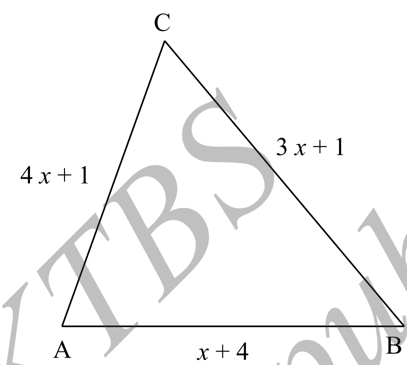
Solution: Let x be the least number among these three consecutive even numbers. Then the other numbers are $x + 2$ and $x + 4$. This is because any two consecutive even numbers differ by 2. The given condition says that

$$x + (x + 2) + (x + 4) = 252.$$

Thus we get $3x + 6 = 252$. This reduces to $3x = 246$. Solving this, we obtain $x = 82$. Hence the numbers are 82, $82 + 2 = 84$ and $82 + 4 = 86$. We check that

$82 + 84 + 86 = 252$, and this verifies the validity of the solution.

Example 20. If the perimeter of a triangle is 14cm and the sides are $x+4$, $3x+1$ and $4x+1$, find x.

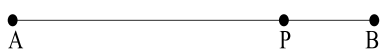


Solution: You know that the perimeter of a triangle is the sum of its three sides. Since the sides are given to be $x+4$, $3x+1$ and $4x+1$, the perimeter is $(x + 4) + (3x + 1) + (4x + 1) = 8x + 6$. The given condition is $8x + 6 = 14$. On solving this, you get $x = 1$. Hence AB the sides are $1+4 = 5$, $(3\times1)+1 = 4$ and $(4\times 1)+1 = 5$ cm. You get a triangle with sides 5,4,5 cm.

> One of the important things to keep in mind is that a mathematical formulation of a problem may not always be physically feasible. Suppose you are asked to find the sides of a triangle, given that they are x, $x+1$ and $x+3$, and its perimeter is 10 units. You write $x + (x+1) + (x+3) = 10$ and solve this to get $x = 2$. You may conclude that the sides are 2,2+1 =3,2+3 =5. But, there is no triangle with sides 2,3,5, since the sides of a triangle must satisfy the *triangle inequality*; the sum of any two sides is greater than the third side. Thus one has to check whether the solution one obtains is a physically valid solution. This is a very important part in the solution of a given problem.

There is nothing strange in this. Mathematics is a game you play using certain rules. As long as you play it safe confining to rules, you will always get an end result. Whether that result is right or wrong, no mathematical law will tell you. You have to go back to the physical situation and see whether the solution you got is a physically correct solution.

Example 21. Let P be a point on a line AB such that P lies between A and B, and $AP = 3PB$. Given that $AB = 10$ cm, find the length of AP.



Solution: Since P lies between A and B, we have $AB = AP + PB$. PB Thus $10 = 3PB + PB = 4PB$. We can solve for PB and get $PB = \frac{10}{4} = \frac{5}{2}$.

Hence $AP = 3PB = 3 \times \frac{5}{2} = \frac{15}{2}$ cm. (Here we are not putting $PB = x$ and get an equation in x. You may directly treat PB as a variable and get an equation involving PB.)

Activity 1:

In the previous problem you have taken that P lies between A and B. It may happen that, on the line AB, P may lie to the left of A or to the right of B. Formulate appropriate equations for both these cases and solve them. One case gives you negative number. This is not a feasible practical situation, as length is non-negative number.

Example 22. The sum of the digits of a two digit number is 12. If the new number formed by reversing the digits is greater than the original number by 54, find the original number.

Solution: Let x be the digit in units place. Then $12 - x$ is the digit in ten's place. Hence the number is

$$(12 - x) \times 10 + x = 120 - 9x.$$

The number obtained by reversing the digits is $10 \times x + 12 - x = 12 + 9x$. The given condition

$$120 - 9x + 54 = 12 + 9x.$$

From this we obtain $18x = 120 + 54 - 12 = 162$. Hence $x = 9$. Therefore the digit in units place 9. The digit is ten's place is $12 - 9 = 3$. The number is hence 39.

Alternate solution: Let the digit in the unit's place be x and the digit in the ten's place be y. Thus the number is $10y + x$. We know that $x + y = 12$. The number obtained by reversing the digits is $10x + y$. The second condition tells that $10x + y = 10y + x + 54$. This reduces to $9(x - y) = 54$ or $x - y = 6$.

Observe that, we have two equations $x + y = 12$ and $x - y = 6$.Adding these, you obtain $(x + y) + (x - y) = 12 + 6 = 18$. Thus $2x = 18$ or $x = 9$. Since $y = 12 - x$, you will also get $y = 12 - 9 = 3$. This shows that the original number is 39.

We verify this. The number obtained by reversing the digits is 93. You may easily check that $93 = 39 + 54$.

Example 23. The sum of two numbers is 75 and they are in the ratio 3:2. Find the numbers.

Solution: The numbers which are in the ratio 3:2 are $3x$ and $2x$. We are given

$$3x + 2x = 75.$$

Thus

$$5x = 75.$$

Solving for x, we get $x = 15$. Hence the numbers are $3x = 45$ and $2x = 30$

We verify: $\frac{45}{30} = \frac{3}{2}$ and $45 + 30 = 75$.

Exercise 8.2

1. If 4 is added to a number and the sum is multiplied by 3,the result is 30. Find the number.
2. Find three consecutive odd numbers whose sum is 219.
3. A number subtracted by 30 gives 14 subtracted by 3 times the number. Find the number.
4. If 5 is subtracted from three times a number, the result is 16. Find the number.
5. Find two numbers such that one of them exceeds the other by 9 and their sum is 81.
6. Prakruthi's age is 6 times Sahil's age. After 15 years, Prakruthi will be 3 times as old as Sahil. Find their age.

7. Ahmed's father is thrice as old as Ahmed. After 12 years, his age will be twice that of his son. Find their present age.
8. Sanju is 6 years older than his brother Nishu. If the sum of their ages is 28 years, what are their present age ?
9. Viji is twice as old as his brother Deepu. If the difference of their ages is 11 years, find their present age.
10. Mrs.Joseph is 27 years older than her daughter Bindu.After 8 years she will be twice as old as Bindu. Find their present age.
11. After 16 years, Leena will be three times as old as she is now. Find her present age.
12. A rectangle has length which is 5 cm less than twice its breadth. If the length is decreased by 5 cm and breadth is increased by 2 cm, the perimeter of the resulting rectangle will be 74cm. Find the length and breadth of the original rectangle.
13. The length of a rectangular field is twice its breadth. If the perimeter of the field is 288 m, find the dimensions of the field.
14. Sristi's salary is same as 4 times Azar's salary. If together they earn Rs 3,750 a month, find their individual salaries.

Glossary

Equation: a statement that a non constant algebraic expression is equal to 0.

Solution: given an equation, any value of the variable which makes the statement true.

Linear equation: an equation of degree one.

Transposition: moving a part of the expression to the other side of the equality; while moving, the sign of the part which has moved changes.

Verification: to check whether the solution obtained satisfies the equation.

Points to remember

- An equation is valid for a certain set of values of the variable(s) in it; an identity is valid for all values of the variable(s) in it.
- Given a problem, setting up an equation conforming to the given data is an important step in the solution of the given problem.
- A mathematical solution may not always be valid physically. One has to check whether the solution obtained by a valid mathematical procedure is also correct for the given physical situation.

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Answers

Exercise 8.1

1. (i) $x = 8$ (ii) $y = 30$ (iii) $z = 7$ (iv) $x = \frac{6}{11}$ (v) $x = 3$ (vi) $s = 28$

(vii) $x = 20$ (viii) $x = 2.4$ (ix) $x = 7$ (x) $x = \frac{-8}{5}$ (xi) $x = 60$ (xii) $x = 25$

(xiii) $x = 2$ (xiv) $x = 36$ (xv) $x = -1$

2. (i) $x = 12$ (ii) $t = -6$ (iii) $x = 7$ (iv) $z = \frac{3}{2}$ (v) $x = 5$

(vi) $x = 1$ (vii) $x = -25$ (viii) $x = -13$ (ix) $x = -5$ (x) $x = 10$

(xi) $x = 3$ (xii) $x = 8$ (xiii) $x = -54$ (xiv) $x = 9$

Exercise 8.2

1 6. **2.** 71, 73, 75. **3.** 11. **4.** 7. **5.** 45 and 36. **6.** 60 and 10. **7.** Ahmed's age 12 and his father's age 36 years. **8.** Nishu's age 11 and Sanju's age 17 years. **9.** Deepu's age 11 and Viji's age 22 years. **10.** Bindu's age19 and Mrs.Joseph's age 46 years. **11.** 8 years. **12.** 25 cm and 15 cm. **13.** 96 m and 48 m. **14.** 3,000 and 750.

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ADDITIONAL PROBLEMS

1. Playing with numbers

1. Choose the correct option:

(a) The general form of 456 is

A. $(4 \times 100) + (5 \times 10) + (6 \times 1)$ **B.** $(4 \times 100) + (6 \times 10) + (5 \times 1)$

C. $(5 \times 100) + (4 \times 10) + (6 \times 1)$ **D.** $(6 \times 100) + (5 \times 10) + (4 \times 1)$

(b) Computers use

A. decimal system **B.** binary system

C. base 5 system **D.** base 6 system

(c) If $\overline{abc}$ is a 3-digit number, then the number

$$n = \overline{abc} + \overline{acb} + \overline{bac} + \overline{bca} + \overline{cab} + \overline{cba}$$

is always divisible by

A. 8 **B.** 7 **C.** 6 **D.** 5

(d) If $\overline{abc}$ is a 3-digit number, then

$$n = \overline{abc} - \overline{acb} + \overline{bac} - \overline{bca} + \overline{cab} - \overline{cba}$$

is always divisible by

A. 12 **B.** 15 **C.** 18 **D.** 21

(e) If $1K \times K1 = K2K$, the letter K stands for the digit

A. 1 **B.** 2 **C.** 3 **D.** 4

(f) The numbers 345111 is divisible by

A. 15 **B.** 12 **C.** 9 **D.** 3

(g) The number of integers of the form 3AB4, where A, B denote some digits, which are divisible by 11 is

A. 0 **B.** 4 **C.** 7 **D.** 9

2. What is the smallest 5-digit number divisible by 11 and containing each of the digits 2,3,4,5,6?

3. How many 5-digit numbers divisible by 11 are there containing each of the digits 2,3,4,5,6?

4. If 49A and A49, where $A > 0$, have a common factor, find all possible values of A.

5. Write 1 to 10 using 3 and 5, each at least once, and using addition and subtraction. (For example, $7 = 5 + 5 - 3$.)

6. Find all 2-digit numbers each of which is divisible by the sum of its digits.

7. The page numbers of a book written in a row gives a 216 digit number. How many pages are there in the book?

8. Look at the following pattern:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
• • • • • •
• • • • • • •

This is called Pascal's triangle. What is the middle number in the 9^{th} row?

9. Complete the adjoining magic square. (Hint: In a 3×3 magic square, the magic sum is three times the central number.)

8		
3	**7**	

10. Find all 3-digit natural numbers which are 12 times as large as the sum of their digits.

11. Find all digits x, y such that $\overline{34x5y}$ is divisible by 36.

12. Can you divide the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 into two groups such that the product of numbers in one group divides the product of numbers in the other group and the quotient is minimum?

13. Find all 8-digit numbers 273A49B5 which are divisible by 11 as well as 25.

14. Suppose a, b are integers such that $2 + a$ and $35 - b$ are divisible by 11. Prove that $a + b$ is divisible by 11.

15. In the multiplication table $A8 \times 3B = 2730$, A and B represent distinct digits different from 0. Find $A + B$.

16. Find the least natural number which leaves the remainders 6 and 8 when divided by 7 and 9 respectively.

17. Prove that the sum of cubes of three consecutive natural numbers is always divisible by 3.

18. What is the smallest number you have to add to 100000 to get a multiple of 1234?

19. Using the digits 4,5,6,7,8, each once, construct a 5-digit number which is divisible by 264.

Answers

1. (a) **A**; (b) **B**; (c) **C**; (d) **C**; (e) **A**; (f) **D**; (g) **D**. **2.** 24365. **3.** 12. **4.** 2, 5, 7, 8. **5.** $1 = 3+3-5$, $2 = 5-3$, $3 = 5+5+5-3-3-3-3-3+3$, $4 = 3+3+3-5$, $5 = 5+5+5+5-3-3-3-3-3$, $6 = 5+5+5+3+3-3-3-3-3-3$, $7 = 5+5-3$, $8 = 5+3$, $9 = 5+5+5+3+3+3-3-3-3-3-3$, $10 = 5+5+5+5+5-3-3-3-3-3$. **6.** 10, 20, 30, 40, 50, 60, 70, 80, 90, 12, 18, 21, 24, 27, 36, 42, 45, 54, 63, 72, 81, 84 **7.** 108 pages. **8.** 70.

9.

8	**9**	**4**
3	**7**	**11**
10	**5**	**6**

10. Only such number is 108. **11.** $x = 4, y = 2$ or $x = 0, y = 6$ or $x = 9, y = 6$. **12.** Take one group as (1, 2, 3, 5, 8, 7) and the other group as (4,6,10). Then $(1\times2\times3\times5\times8\times7)/(4\times6\times10) = 7$ and this is the minimum quotient. **13.** 27314925 and 27364975. **15.** 12. **16.** 62. **18.** 46. **19.** There are 4 numbers: 58476, 48576, 57684, 67584.(Use divisibility by 4 and11 together.)

2. Algebraic expressions

1. Choose the correct answer.

(a) Terms having the same literal factors with same exponents are called

A. exponents **B.** like terms **C.** factors **D.** unlike terms

(b) The coefficient of ab in 2ab is:

A. ab **B.** 2 **C.** 2a **D.** 2b

(c) The exponential form of $a \times a \times a$ is:

A. 3a **B.** 3+a **C.** a^3 **D.** 3− a

(d) Sum of two negative integers is:

A. negative **B.** positive **C.** zero **D.** infinite

(e) What should be added to $a^2 + 2ab$ to make it a complete square?

A. b^2 **B.** 2ab **C.** ab **D.** 2a

(f) What is the product of $(x + 2)(x - 3)$?

A. $2x - 6$ **B.** $3x - 2$ **C.** $x^2 - x - 6$ **D.** $x^2 - 6x$

(g) The value of $(7.2)^2$ is (use an identity to expand):

A. 49.4 **B.** 14.4 **C.** 51.84 **D.** 49.04

(h) The expansion of $(2x - 3y)^2$ is:

A. $2x^2 + 3y^2 + 6xy$ **B.** $4x^2 + 9y^2 - 12xy$

C. $2x^2 + 3y^2 - 6xy$ **D.** $4x^2 + 9y^2 + 12xy$

(i) The product 58×62 is

A. 4596 **B.** 2596 **C.** 3596 **D.** 6596

2. Take away $8x - 7y - 8p + 10q$ from $10x + 10y - 7p + 9q$.

3. Expand:

(i) $(4x + 3)^2$; (ii) $(x + 2y)^2$; (iii) $(x + \frac{1}{x})^2$; (iv) $(x - \frac{1}{x})^2$.

4. Expand:

(i) $(2t + 5)(2t - 5)$; (ii) $(xy + 8)(xy - 8)$; (iii) $(2x + 3y)(2x - 3y)$.

5. Expand:

   (i) $(n-1)(n+1)(n^2+1)$; (ii) $(n-\frac{1}{n})(n+\frac{1}{n})(n^2+\frac{1}{n^2})$

   (iii) $(x-1)(x+1)(x^2+1)(x^4+1)$; (iv) $(2x-y)(2x+y)(4x^2+y^2)$.

6. Use appropriate formulae and compute:

   (i) $(103)^2$; (ii) $(96)^2$; (iii) 107×93; (iv) 1008×992; (v) $185^2 - 115^2$. 2

7. If $x + y = 7$ and $xy = 12$, find $x^2 + y$.

8. If $x + y = 12$ and $xy = 32$, find $x^2 + y$.

9. If $4x^2 + y = 40$ and $xy = 6$, find $2x + y$.

10. If $x - y = 3$ and $xy = 10$, find $x^2 + y$.

11. If $x + \frac{1}{x} = 3$, find $x^2 + \frac{1}{x^2}$ and $x^3 + \frac{1}{x^3}$.

12. If $x + \frac{1}{x} = 6$, find $x^2 + \frac{1}{x^2}$ and $x^4 + \frac{1}{x^4}$.

13. Simplify: (i) $(x+y)^2 + (x-y)^2$; (ii) $(x+y)^2 \times (x-y)^2$.

14. Express the following as difference of two squares:

    (i) $(x+2z)(2x+z)$; (ii) $4(x+2y)(2x+y)$; (iii) $(x+98)(x+102)$;

    (iv) 505×495. (**Hint**. $ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$.)

15. If $a = 3x - 5y$, $b = 6x + 3y$ and $c = 2y - 4x$,

    find (i) $a + b - c$; (ii) $2a - 3b + 4c$.

16. The perimeter of a triangle is $15x^2 - 23x + 9$ and two of its sides are $5x^2 + 8x - 1$ and $6x^2 - 9x + 4$. Find the third side.

17. The two adjacent sides of a rectangle are $2x^2 - 5xy + 3z^2$ and $4xy - x^2 - z$. Find its perimeter.

18. The base and the altitude of a triangle are $(3x - 4y)$ and $(6x + 5y)$ respectively. Find its area.

19. The sides of a rectangle are $2x + 3y$ and $3x + 2y$. From this a square of side length $x+y$ is removed. What is the area of the remaining region?

20. If a, b, c are rational numbers such that $a^2 + b^2 + c^2 - ab - bc - ca = 0$, prove that $a = b = c$.

Answers

1. (a) **B.** (b) **B.** (c) **C.** (d) **A.** (e) **A.** (f) **C.** (g) **C.** (h) **B.** (i) **C.**

2. $2x + 17y + p - q$. **3.** (i) $16x^2 + 24x + 9$ (ii) $x^2 + 4xy + 4y^2$ (iii) $x^2 + (1/x^2) + 2$ (iv) $x^2 + (1/x^2) - 2$. **4.** (i) $4t^2 - 25$ (ii) $xy^2 - 25$ (iii) $4x^2 - 9y$. **5.** (i) $n^4 - 1$ (ii) $n^4 - (1/n^4)$; (iii) $x^8 - 1$; (iv) $16x^4 - y^4$. **6.** (i) 10609; (ii) 9216; (iii) 9951 (iv) 999936 (v) 21000. **7.** 25. **8.** 80. **9.** ±8. **10.** 29 **11.** 7 and 18. **12.** 34 and 1154. **13.** (i) $2(x^2 + y^2)$ $3(x + z)^2 (z - x)^2$ (ii) $x^4 - 2xy^2 + y$.

14. (i) $\left(\frac{3(x+z)}{2}\right)^2 - \left(\frac{(z-x)}{2}\right)^2$ (ii) $(x + 2y)^2 - (2x + y)^2$

(iii) $(x + 100)^2 - 1^2$ (iv) $500^2 - 5^2 . 2$

15. (i) $13x - 4y$ (ii) $-28x - 11y$. **16.** $4x^2 - 22x + 6$. **17.** $2x^2 - 2xy + 4z$.

18. $\frac{(18x^2 - 9xy - 20y^2)}{2}$. **19.** $5x^2 + 11xy + 5y$.

3. Axioms, postulates and theorems

1. Choose the correct option:

(i) If a = 60 and b = a, then b = 60 by ________

A. Axiom 1 **B.** Axiom 2 **C.** Axiom 3 **D.** Axiom 4

(ii) Given a point on the plane, one can draw ________ lines through that point.

A. unique **B.** two **C.** finite number of **D.** infinitely many

(iii) Given two points in a plane, the number of lines which can be drawn to pass through these two points is ________

A. zero **B.** exactly one **C.** at most one **D.** more than one

(iv) If two angles are supplementary, then their sum is ________

A. 90° **B.** 180° **C.** 270° **D.** 360°

(v) The measure of an angle which is 5 times its supplement is ________

A. 30° **B.** 60° **C.** 120° **D.** 150°

2. What is the difference between a pair of supplementary angles and a pair of complementary angles?

3. What is the least number of non-collinear points required to determine a plane?

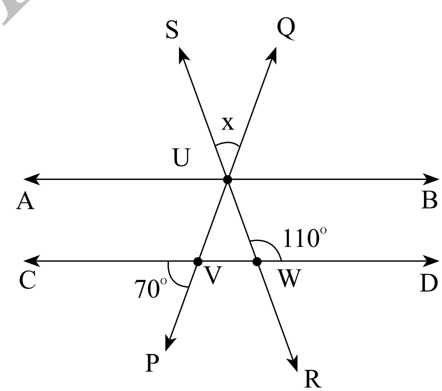
4. When do you say two angles are adjacent?

5. Let $\overline{AB}$ be a segment with C and D between them such that the order of points on the segment is A,C,D,B. Suppose AD = BC. Prove that AC = DB.

6. Let $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ be two straight lines intersecting at O. Let $\overrightarrow{OX}$ be the bisector of $\angle BOD$. Draw $\overrightarrow{OY}$ between $\overrightarrow{OD}$ and $\overrightarrow{OA}$ such that $\overrightarrow{OY} \perp \overrightarrow{OX}$. Prove that $\overrightarrow{OY}$ bisects $\angle DOA$.

7. Let $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ be two parallel lines and $\overleftrightarrow{PQ}$ be a transversal. Let $\overleftrightarrow{PQ}$ intersect $\overleftrightarrow{AB}$ in L. Suppose the bisector of $\angle ALP$ intersect $\overleftrightarrow{CD}$ in R and the bisector of $\angle PLB$ intersect $\overleftrightarrow{CD}$ in S. Prove that $\angle LRS + \angle RSL = 90^\circ$.

8. In the adjoining figure, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel lines. The transversals $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at U on the line $\overleftrightarrow{AB}$. Given that $\angle DWU = 110^\circ$ and $\angle CVP = 70^\circ$, find the measure of $\angle QUS$.



9. What is the angle between the hour's hand and minute's hand of a clock at (i)1.40 hours,(ii) 2.15 hours? (Use 1° =60 minutes.)

10. How much would hour's hand have moved from its position at 12 noon when the time is 4.24 p.m.?

11. Let $\overline{AB}$ be a line segment and let C be the midpoint of $\overline{AB}$. Extend $\overline{AB}$ to D such that B lies between A and D. Prove that AD + BD = 2CD.

12. Let $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ be two lines intersecting at a point O. Let $\overrightarrow{OX}$ be a ray bisecting $\angle BOD$. Prove that the extension of $\overrightarrow{OX}$ to the *left* of O bisects $\angle AOC$.

13. Let $\overrightarrow{OX}$ be a ray and let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be two rays on the same side of $\overrightarrow{OX}$, with $\overrightarrow{OA}$ between $\overleftrightarrow{OX}$ and $\overrightarrow{OB}$. Let $\overrightarrow{OC}$ be the bisector of $\angle AOB$. Prove that

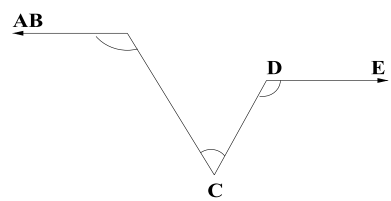
$$\angle XOA + \angle XOB = 2\angle XOC.$$

14. Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be two rays and let $\overrightarrow{OX}$ be a ray between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ such that $\angle AOX > \angle XOB$. Let OC be the bisector of $\angle AOB$.

$$\angle AOX - \angle XOB = 2\angle COX.$$

15. Let $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ be three rays such that $\overrightarrow{OC}$ lies between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Suppose the bisectors of $\angle AOC$ and $\angle COB$ are perpendicular to each other. Prove that B,O,A are collinear.

16. In the adjoining figure, $\overleftrightarrow{AB} \parallel \overleftrightarrow{DE}$.



Prove that

$\angle ABC - \angle DCB + \angle CDE = 180^\circ$.

17. Consider two parallel lines and a transversal. Among the measures of 8 angles formed, how many distinct numbers are there?

Answers

1. (i) **A.** (ii) **D.** (iii) **B.** (iv) **B.** (e) **D.**

3. 3. **8.** 40°. **9.** (i)190°; (ii) $22^\circ 30'$. **10.** 144°.

4. Factorisation

1. Choose the correct answer:

(a) $4a + 12b$ is equal to

A. $4a$ **B.** $12b$ **C.** $4(a+3b)$ **D.** $3a$

(b) The product of two numbers is positive and their sum negative only when

A. both are positive
B. both are negative
C. one positive the other negative
D. one of them equal to zero

(c) Factorising x^2+6x+8, we get

A. $(x+1)(x+8)$ **B.** $(x+6)(x+2)$ **C.** $(x+10)(x-2)$ **D.** $(x+4)(x+2)$

(d) The denominator of an algebraic fraction should not be
A. 1 **B.** 0 **C.** 4 **D.** 7

(e) If the sum of two integers is -2 and their product is -24, the numbers are

A. 6 and 4 **B.** -6 and 4 **C.** -6 and -4 **D.** 6 and -4

(f) The difference $(0.7)^2-(0.3)^2$ simplifies to

A. 0.4 **B.** 0.04 **C.** 0.49 **D.** 0.56

2. Factorise the following:
(i) x^2+6x+9 (ii) $1-8x+16x^2$ (iii) $4x^2-81y^2$ (iv) $4a^2+4ab+b^2$
(v) $a^2b^2+c^2d^2-a^2c^2-b^2d^2$.

3. Factorise the following:
(i) $x^2+7x+12$ (ii) x^2+x-12 (iii) $x^2-3x-18$ (iv) $x^2+4x-21$
(v) $x^2-4x-192$ (vi) x^4-5x^2+4 (vii) x^4-13xy^2+36y.

4. Factorise the following:
(i) $2x^2+7x+6$ (ii) $3x^2-17x+20$ (iii) $6x^2-5x-14$ (iv) $4x^2+12xy+5y$
(v) $4x^4-5x^2+1$.

5. Factorise the following:
(i) x^8-y^8 (ii) ax^4-ax^{12} (iii) xx^2+1 (iv) x^4+5x^2+9.

6. Factorise x^4+4y^4. Use this to prove that 2011^4+64 is a composite number.

Answers

1. (a) **C.** (b) **B.** (c) **D.** (d) **B.** (e) **D.** (f) **A.**

2. (i) $(x+3)^2$ (ii) $(1-4x)^2$ (iii) $(2x+9y)(2x-9y)$ (iv) $(2a+b)^2$ (v) $(a^2-d^2)(b^2-c^2)$.

3. (i) $(x+3)(x+4)$ (ii) $(x+4)(x-3)$ (iii) $(x-6)(x+3)$ (iv) $(x+7)(x-3)$ (v) $(x-16)(x+12)$ (vi) $(x-1)(x+1)(x-2)(x+2)$ (vii) $(x-2y)(x+2y)(x-3y)(x+3y)$.

4. (i) $(2x-3)(x+2)$ (ii) $(x-4)(3x-5)$ (iii) $(x-2)(6x+7)$ (iv) $(2x+y)(2x+5y)$ (v) $(2x-1)(2x+1)(x-1)(x+1)$. **5.** (i) $(x-4)(x+4)(x^2+y^2)(x^4+y^4)$

(ii) $ax^4(a-x)(a+x)(a+x^2)(a+x^4)$ (iii) $(x+x+1)(x^2-x+1)$ (iv) $(x+x+3)(x^2-x+3)$. **6.** $(x^2+2xy+2y^2)(x^2-2xy+2y^2)$.

5. Squares, square roots, cubes and cube roots

1. Match the numbers in the column A with their squares in the column B:

	A		B	Answers
(1)	5	(a)	25	(1) ——
(2)	8	(b)	144	(2) ——
(3)	2	(c)	36	(3) ——
(4)	−6	(d)	484	(4) ——
(5)	−22	(e)	64	(5) ——
(6)	12	(f)	4	(6) ——
		(g)	121	

2. Choose the correct option.

(a) The number of perfect squares from 1 to 500 is:

A. 1 **B.** 16 **C.** 22 **D.** 25

(b) The last digit of a perfect square can never be

A. 1 **B.** 3 **C.** 5 **D.** 9

(c) If a number ends in 5 zeros, its square ends in:

A. 5 zeros **B.** 8 zeros **C.** 10 zeros **D.** 12 zeros

(d) Which could be the remainder among the following when a per fect square is divided by 8?

A. 1 **B.** 3 **C.** 5 **D.** 7

(e) The 6^{th} triangular number is:

A. 6 **B.** 10 **C.** 21 **D.** 28

3. Consider all integers from −10 to 5, and square each of them. How many distinct numbers do you get?

4. Write the digit in unit's place when the following number are squared:

   4,5,9,24,17,76,34,52,33, 2319, 18, 3458, 3453.

5. Write all numbers from 400 to 425 which end in 2,3, 7 or 8. Check if any of these is a perfect square.

6. Find the sum of the digits of $(111111111)^2$.

7. Suppose $x^2 + y^2 = z^2$.

   (i) if $x = 4$ and $y = 3$ find z;
   (ii) if $x = 5$ and $z = 13$, find y;
   (iii) if $y = 15$ and $z = 17$, find x.

8. A sum of ₹ 2304 is equally distributed among several people. Each gets as many rupees as the number of persons. How much does each one get?

9. Define a *new operation* * on the set of all natural numbers by $m*n = m^2 + n^2$.

   (i) Is $\mathbb{N}$ closed under *?
   (ii) Is * commutative on $\mathbb{N}$?
   (iii) Is * associative on $\mathbb{N}$?
   (iv) Is there an identity element in $\mathbb{N}$ with respect to * ?

10. (**Exploration**) Find all perfect squares from 1 to 500, each of which is a sum of two perfect squares.

11. Suppose the area of a square field is 7396 m^2. Find its perimeter.

12. Can 1010 be written as a difference of two perfect squares? [Hint: How many times 2 occurs as a factor of 1010?]

13. What are the remainders when a perfect cube is divided by 7?

14. What is the least perfect square which leaves the remainder 1 when divided by 7 as well as by 11?

15. Find two smallest perfect squares whose product is a perfect cube

16. Find a proper positive factor of 48 and a proper positive multiple of 48 which add up to a perfect square. Can you prove that there are infinitely many such pairs?

Answers

2. (a) **C**; (b) **B**; (c) **C**; (d) **A**; (e) **C**. **3.** 11. **4.** 6,5,1,6,9,6,6,4,9,1,4,4,9. **5.** None of them is a perfect square. **6.** 81. **7.** (i) $z = \pm 5$; (ii) $y = \pm 12$; (iii) $x = \pm 8$. **8.** ₹ 48. **9.** (i)closed; (ii)commutative; (iii)associative; (iv)no, because $m^2 + k^2 = m^2$ implies $k = 0$ and N does not contain 0.

11. 344 m. **12.** If $1010 = a^2 - b^2$ for some integers a and b, then either both a and b are odd or both even. Hence $a^2 - b^2$ is divisible by 4. But 1010 is not divisible by 4. Hence 1010 is not the difference of two perfect squares. **13.** 0,1,6. **14.** $1156 = 34^2$. **15.** 4 and 16.

6. Theorems on triangles

1. Fill up the blanks to make the following statements true:
   (a) Sum of the angles of a triangle is————
   (b) An exterior angle of a triangle is equal to the sum of ———— ———— opposite angles.
   (c) An exterior angle of a triangle is always ———— than either of the interior opposite angles.
   (d) A triangle cannot have more than ————-right angle.
   (e) A triangle cannot have more than———— obtuse angle.

2. Choose the correct answer from the given alternatives:
   (a) In a triangle ABC, $\angle A = 80^\circ$ and $AB = AC$, then $\angle B$ is ———
   A. 50° **B.** 60° **C.** 40° **D.** 70°
   (b) In right angled triangle, $\angle A$ is right angle and $\angle B = 35^\circ$, then $\angle C$ is ———
   A. 65° **B.** 55° **C.** 75° **D.** 45°
   (c) In a triangle ABC, $\angle B = \angle C = 45^\circ$, then the triangle is ———
   A. right triangle **B.** acute angled triangle **C.** obtuse angle triangle **D.** equilateral triangle
   (d) In an equilateral triangle, each exterior angle is ———
   A. 60° **B.** 90° **C.** 120° **D.** 150°

(e) Sum of the three exterior angles of a triangle is ———

A. two right angles **B.** three right angles **C.** one right angle **D.** four right angles

3. In a triangle ABC, $\angle B = 70^\circ$. Find $\angle A + \angle C$.

4. In a triangle ABC, $\angle A = 110^\circ$ and $AB = AC$. Find $\angle B$ and $\angle C$.

5. If three angles of a triangle are in the ratio 2: 3: 5, determine three angles.

6. The angles of triangle are arranged in ascending order of magnitude. If the difference between two consecutive angles is 15°, find the three angles.

7. The sum of two angles of a triangle is equal to its third angle. Determine the measure of the third angle.

8. In a triangle ABC, if $2\angle A = 3\angle B = 6\angle C$, determine $\angle A$, $\angle B$ and $\angle C$.

9. The angles of triangle are $x - 40^\circ$, $x - 20^\circ$ and $^{1}x + 15^\circ$. Find the value of x.

10. In triangle ABC, $\angle A - \angle B = 15^\circ$ and $\angle B - \angle C = 30^\circ$, find $\angle A, \angle B$ and $\angle C$.

11. The sum of two angles of a triangle is 80° and their difference is 20°. Find the angles of the triangle.

12. In a triangle ABC, $\angle B = 60^\circ$ and $\angle C = 80^\circ$. Suppose the bisector of $\angle B$ and $\angle C$ meet at I. Find $\angle BIC$.

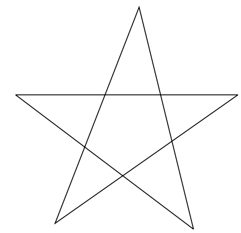
13. In a triangle, each of the smaller angles is half the largest angle. Find the angles.

14. In a triangle, each of the bigger angles is twice the third angle. Find the angles.

15. In a triangle ABC, $\angle B = 50^\circ$ and $\angle A = 60^\circ$. Suppose BC is extended to D. Find $\angle ACD$.

16. In an isosceles triangle, the vertex angle is twice the sum of the base angles. Find the angles of the triangle.

17. Find the sum of all the angles at the five vertices of the adjoining star.



Answers

1. (a)180°; (b) the interior; (c) larger; (d) one; (e) one.

2. (a) **A.** (b) **B.** (c) **A.** (d) **C.** (e) **D.**

3. 110° . **4.** 35° each. **5.** 36° , 54° , 90° . **6.** 45° , 60° , 75° . **7.** 90° .

8. $\angle C = 30°$, $\angle B = 60°$, $\angle A = 90°$. **9.** $x = 90°$. **10.** $\angle A = 80°$, $\angle B = 65°$, $\angle C = 35°$. **11.** 30° , 50° , 100° . **12.** 110° . **13.** 45° , 45° , 90° . **14.** 36° , 72° , 72° . **15.** 110° . **16.** 30° , 30° , 120° . **17.** 180^0

7. Rational numbers

1. Fill in the blanks:
   - (a) The number 0 is not in the set of ——————————-.
   - (b) The least number in the set of all whole numbers is ——————.
   - (c) The least number in the set of all even natural numbers is ——————.
   - (d) The successor of 8 in the set of all natural numbers is ————————.
   - (e) The sum of two odd integers is ———————-.
   - (f) The product of two odd integers is ——————-.
2. State whether the following statements are **true** or **false**:
   - (a) The set of all even natural numbers is a finite set.
   - (b) Every non-empty subset of $\mathbb{Z}$ has the smallest element.
   - (c) Every integer can be identified with a rational number.
   - (d) For each rational number, one can find the next rational number.
   - (e) There is the largest rational number.
   - (f) Every integer is either even or odd.
   - (g) Between any two rational numbers, there is an integer.

3. Simplify:

   (i) $100\,(100-3)-(100\times 100-3)$; (ii) $(20-(2011-201))+(2011-(201-20))$

4. Suppose m is an integer such that $m \neq -1$ and $m \neq -2$. Which is larger $\frac{m}{m+1}$ or $\frac{m+1}{m+2}$? State your reasons.

5. Define an operation * on the set of all rational numbers $\mathbb{Q}$ as follows:

$$r * s = r + s - (r \times s),$$

   for any two rational numbers r,s. Answer the following with justification:

   (i) Is $\mathbb{Q}$ closed under the operation * ?

   (ii) Is * an associative operation on $\mathbb{Q}$?

   (iii) Is * a commutative operation on $\mathbb{Q}$?

   (v) What is a * 1 for any a in $\mathbb{Q}$?

   (vi) Find two integers $a \neq 0$ and $b \neq 0$ such that $a * b = 0$.

6. Find the multiplicative inverses of the following rational numbers:
   $\frac{8}{13}, \frac{12}{17}, \frac{26}{23}, \frac{-13}{11}, \frac{-101}{100}$.

7. Write the following in increasing order:
   $\frac{10}{13}, \frac{20}{23}, \frac{5}{6}, \frac{40}{43}, \frac{25}{28}, \frac{10}{11}$.

8. Write the following in decreasing order:
   $\frac{21}{17}, \frac{31}{27}, \frac{13}{11}, \frac{41}{37}, \frac{51}{47}, \frac{9}{8}$.

9. (a) What is the additive inverse of 0?

   (b) What is the multiplicative inverse of 1?

   (c) Which integers have multiplicative inverses?

10. In the set of all rational numbers, give 2 examples each illustrating the following properties:

    (i) associativity (ii) commutativity (iii) distributivity of multiplication over addition.

11. Simplify the following using distributive property:

(i) $\frac{2}{5} \times \left(\frac{1}{9} + \frac{2}{5}\right)$ (ii) $\frac{5}{12} \times \left(\frac{25}{9} + \frac{32}{5}\right)$ (iii) $\frac{8}{9} \times \left(\frac{11}{2} + \frac{2}{9}\right)$.

12. Simplify the following:

(i) $\left(\frac{25}{9} + \frac{12}{3}\right) + \frac{3}{5}$ (ii) $\left(\frac{22}{7} + \frac{36}{5}\right) \times \frac{6}{7}$

(iii) $\left(\frac{51}{2} + \frac{7}{6}\right) \div \frac{3}{5}$ (iv) $\left(\frac{16}{7} + \frac{21}{8}\right) \times \left(\frac{15}{3} - \frac{2}{9}\right)$.

13. Which is the property that is there in the set of all rationals but not in the set of all integers?

14. What is the value of

$$1 + \cfrac{1}{1 + \cfrac{1}{1+1}}?$$

15. Find the value of $\left(\frac{1}{3} - \frac{1}{4}\right) \div \left(\frac{1}{2} - \frac{1}{3}\right)$.

16. Find all rational numbers each of which is equal to its reciprocal.

17. A bus shuttles between two neigbouring towns every two hours. It starts from 8 AM in the morning and the last trip is at 6 PM. On one day the driver observed that the first trip had 30 passengers and each subsequent trip had one passenger less than the previous trip. How many passengers travelled on that day?

18. How many rational numbers $\frac{p}{q}$ are there between 0 and 1 for which $q < p$?

19. Find all integers such that $\frac{3n+4}{n+2}$ is also an integer.

20. By inserting parenthesis(that is brackets), you can get several values for $2 \times 3 + 4 \times 5$. (For example $((2 \times 3) + 4) \times 5$ is one way of inserting parenthesis.) How many such values are there?

21. Suppose $\frac{p}{q}$ is a positive rational in its lowest form. Prove that $\frac{1}{q} + \frac{1}{p+q}$ is also in its lowest form.

22. Show that for each natural number n, the fraction $\frac{14n+3}{21n+4}$ is in its lowest form.

23. Find all integers n for which the number $(n + 3)(n - 1)$is also an integer.

Answers

1. (a) natural numbers (b) 0 (c) 2 (d) 9 (e) even (f) odd. **2.** (a) false (b) false (c) true (d) false (e) false (f) true (g) false. **3.** (i) 297 (ii) 39. **4.** $m/(m+1) < (m+1)/(m+2)$ do not forget two cases $m < -2$ and $m > -1$. **5.** (i) yes (ii) yes (iii) yes (iv) $a * 1 = 1$ (e) $a = 2$, $b = 2$.

6. $\frac{13}{8}, \frac{17}{12}, \frac{23}{26}, \frac{-11}{13}, \frac{-100}{101}$. **7.** $\frac{10}{13} < \frac{5}{6} < \frac{20}{23} < \frac{25}{28} < \frac{10}{11} < \frac{40}{43}$.

8. $\frac{21}{17} > \frac{13}{11} > \frac{31}{27} > \frac{9}{8} > \frac{41}{37} > \frac{51}{47}$. **9.** (a) 0; (b) 1 (c) 1, −1.

11 (i) $\frac{46}{225}$ (ii) $\frac{413}{108}$ (iii) $\frac{225}{6}$ (iv) $\frac{11825}{504}$ **13.** Every non-zero rational number is invertible, but only ±1 are invertible integers. **14.** $\frac{5}{3}$.

15. $\frac{1}{2}$ **16.** ±1. **17.** 140. **18.** No rational p/q between 0 and 1 for which $q < p$. **19.** $n = 0, -1, -3, -4$. **20.** 4 values: 26, 46, 50, 70.

23. $n = 2, 3, 5, 0, -1, -3$.

8. Linear equations in one variable

1. Choose the correct answer

   (a) The value of x in the equation $5x - 35 = 0$ is:
   A. 2 **B.** 7 **C.** 8 **D.** 11

   (b) If 14 is taken away from one fifth of a number, the result is 20. The equation expressing this statement is:
   A. $(\frac{x}{5}) - 14 = 20$ **B.** $x - (\frac{14}{5}) = (\frac{20}{5})$
   C. $x - 14 = (\frac{20}{5})$ **D.** $x + (\frac{14}{5}) = 20$

   (c) If five times a number increased by 8 is 53, the number is:
   A. 12 **B.** 9 **C.** 11 **D.** 2

   (d) The value of x in the equation $5(x - 2) = 3(x - 3)$ is:
   A. 2 **B.** $\frac{1}{2}$ **C.** $\frac{3}{4}$ **D.** 0

(e) If the sum of two numbers is 84 and their difference is 30, the numbers are:
A. −57 and 27 **B.** 57 and 27 **C.** 57 and −27 **D.** −57 and −27

(f) If the area of a rectangle whose length is twice its breadth is 800 m^2, then the length and breadth of the rectangle are:

A. 60m and 20m **B.** 40m and 20m

C. 80m and 10m **D.** 100m and 8m

(g) If the sum of three consecutive odd numbers is 249, the numbers are

A. 81,83,85 **B.** 79,81,83 **C.** 103,105,107 **D.** 95,97,99

(h) If $\frac{(x+0.7x)}{2} = 0.85$, the value of x is:

A. 2 **B.** 1 **C.** −1 **D.** 0

(i) If $2x - (3x - 4) = 3x - 5$, then x equals:

A. $\frac{4}{9}$ **B.** $\frac{9}{4}$ **C.** $\frac{3}{2}$ **D.** $\frac{2}{3}$

2. Solve: (i) $(3x + 24) \div (2x + 7) = 2$: (ii) $(1 - 9y) \div (11 - 3y) = (\frac{5}{8})$.

3. The sum of two numbers is 45 and their ratio is 7:8. Find the numbers.

4. Shona's mother is four times as old as Shona. After five years, her mother will be three times as old as Shona (at that time). What are their present age?

5. The sum of three consecutive even numbers is 336. Find them.

6. Two friends A and B start a joint business with a capital 60,000. If A's share is twice that of B, how much have each invested?

7. Which is the number when 40 is subtracted gives one-third of the original number?

8. Find the number whose *sixth* part exceeds its *eigth* part by 3.

9. A house and a garden together cost ₹ 8,40,000. The price of the garden is $\frac{5}{7}$ times the price of the house. Find the price of the house and the garden.

10. Two farmers A and B together own a stock of grocery. They agree to divide it by its value. Farmer A takes 72 bags while B takes 92 bags and gives ₹ 8,000 to A. What is the cost of each bag?.

11. A father's age is four times that of his son. After 5 years, it will be three times that of his son.How many more years will take if father's age is to be twice that of his son?

12. Find a number which when multiplied by 7 is as much above 132 as it was originally below it.

13. A person buys 25 pens worth 250, each of equal cost. He wants to keep 5 pens for himself and sell the remaining to recover his money. What should be the price of each pen?

14. The sum of the digits of a two-digit number is 12. If the new number formed by reversing the digits is greater than the original number by 18, find the original number. Check your solution.

15. The distance between two stations is 340 Km. Two trains start simultaneously from these stations on parallel tracks and cross each other. The speed of one of the them is greater than that of the other by 5 Km/hr. If the distance between two trains after 2 hours of their start is 30 Km., find the speed of each train.

16. A steamer goes down stream and covers the distance between two ports in 4 hours while it covers the same distance up stream in 5 hours. If the speed of the steamer upstream is 2 km/hour, find the speed of steamer in still water.

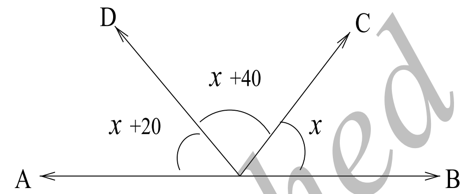
17. The numerator of the rational number is less than its denominator by 3. If the numerator becomes three times and the denominator is increased by 20, the new number becomes $\frac{1}{8}$. Find the original number.

18. The digit at the tens place of a two digit number is three times the digit at the units' place. If the sum of this number and the number formed by reversing its digits is 88, find the number.

19. The altitude of a triangle is five-thirds the length of its corresponding base. If the altitude is increased by 4 cm and the base decreased by 2 cm, the area of the triangle would remain the same. Find the base and altitude of the triangle.

20. One of the angles of a triangle is equal to the sum of the other two angles. If the ratio of the other two angles of the triangle is 4:5, find the angles of the triangle.

21. In the figure, AB is a straight line. Find *x*.



Answers

1 (a) **B.** (b) **A.** (c) **B.** (d) **B.** (e) **B.** (f) **B.** (g) **A.** (h) **B.**

2. (i)10; (ii)$-\frac{47}{57}$. **3.** 21 and 24. **4.** 10 and 40. **5.** 110, 112, 114.

6. A' share ₹ 40,000 and B's share ₹ 20,000. **7.** 60. **8.** 72. **9.** garden ₹ 35,000 and house ₹ 49,000. **10.** ₹ 800. **11.** 15 years. **12.** 33. **13.** ₹ 12.50 **14.** 57. **15.** 90 km and 95 km. **16.** 2.25 km. **17.** $\frac{1}{4}$. **18.** 62. **19.** altitude 20 cm and base 12 cm. **20.** 40°, 50° and 90° **21.** $x = 40$.