Chapter Seven

Systems of Particles and Rotational Motion

7.1 INTRODUCTION

In the earlier chapters we primarily considered the motion of a single particle. (A particle is represented as a point mass. It has practically no size.) We applied the results of our study even to the motion of bodies of finite size, assuming that motion of such bodies can be described in terms of the motion of a particle.

Any real body which we encounter in daily life has a finite size. In dealing with the motion of extended bodies (bodies of finite size) often the idealised model of a particle is inadequate. In this chapter we shall try to go beyond this inadequacy. We shall attempt to build an understanding of the motion of extended bodies. An extended body, in the first place, is a system of particles. We shall begin with the consideration of motion of the system as a whole. The centre of mass of a system of particles will be a key concept here. We shall discuss the motion of the centre of mass of a system of particles and usefulness of this concept in understanding the motion of extended bodies.

A large class of problems with extended bodies can be solved by considering them to be rigid bodies. **Ideally a rigid body is a body with a perfectly definite and unchanging shape. The distances between all pairs of particles of such a body do not change.** It is evident from this definition of a rigid body that no real body is truly rigid, since real bodies deform under the influence of forces. But in many situations the deformations are negligible. In a number of situations involving bodies such as wheels, tops, steel beams, molecules and planets on the other hand, we can ignore that they warp, bend or vibrate and treat them as rigid.

7.1.1 What kind of motion can a rigid body have?

Let us try to explore this question by taking some examples of the motion of rigid bodies. Let us begin with a rectangular block sliding down an inclined plane without any sidewise

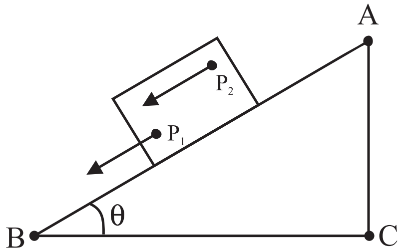


Fig 7.1 Translational (sliding) motion of a block down an inclined plane.
(Any point like P_1 or P_2 of the block moves with the same velocity at any instant of time.)

movement. The block is a rigid body. Its motion down the plane is such that all the particles of the body are moving together, i.e. they have the same velocity at any instant of time. The rigid body here is in pure translational motion (Fig. 7.1).

In pure translational motion at any instant of time all particles of the body have the same velocity.

Consider now the rolling motion of a solid metallic or wooden cylinder down the same inclined plane (Fig. 7.2). The rigid body in this problem, namely the cylinder, shifts from the top to the bottom of the inclined plane, and thus, has translational motion. But as Fig. 7.2 shows, all its particles are not moving with the same velocity at any instant. The body therefore, is not in pure translation. Its motion is translation plus 'something else.'

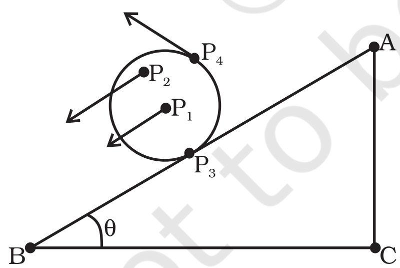
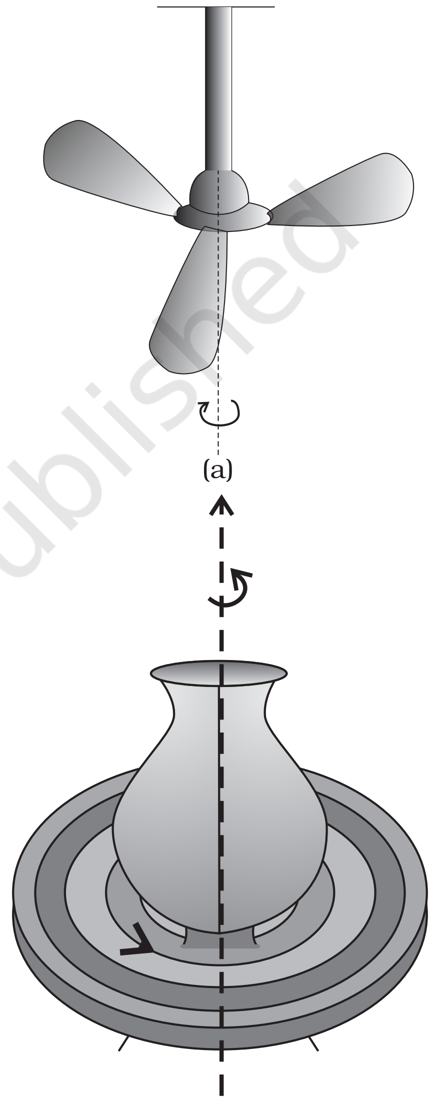


Fig. 7.2 Rolling motion of a cylinder It is not pure translational motion. Points P_1, P_2, P_3 and P_4 have different velocities (shown by arrows) at any instant of time. In fact, the velocity of the point of contact P_3 is zero at any instant, if the cylinder rolls without slipping.

In order to understand what this 'something else' is, let us take a rigid body so constrained that it cannot have translational motion. The most common way to constrain a rigid body so that it does not have translational motion is to fix it along a straight line. The only possible motion of such a rigid body is **rotation**. The line along which the body is fixed is termed as its **axis of rotation**. If you look around, you will come across many examples of rotation about an axis, a ceiling fan, a potter's wheel, a giant wheel in a fair, a merry-go-round and so on (Fig 7.3(a) and (b)).



(b)

Fig. 7.3 Rotation about a fixed axis
(a) A ceiling fan
(b) A potter's wheel.

Let us try to understand what rotation is, what characterises rotation. You may notice that **in rotation of a rigid body about a fixed**

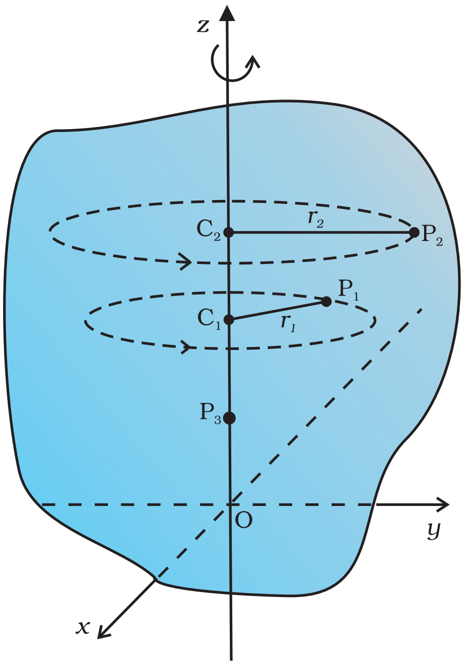


Fig. 7.4 *A rigid body rotation about the z-axis (Each point of the body such as P_1 or P_2 describes a circle with its centre (C_1 or C_2) on the axis. The radius of the circle (r_1 or r_2) is the perpendicular distance of the point (P_1 or P_2) from the axis. A point on the axis like P_3 remains stationary).*

axis, every particle of the body moves in a circle, which lies in a plane perpendicular to the axis and has its centre on the axis. Fig. 7.4 shows the rotational motion of a rigid body about a fixed axis (the *z*-axis of the frame of reference). Let P_1 be a particle of the rigid body, arbitrarily chosen and at a distance r_1 from fixed axis. The particle P_1 describes a circle of radius r_1 with its centre C_1 on the fixed axis. The circle lies in a plane perpendicular to the axis. The figure also shows another particle P_2 of the rigid body, P_2 is at a distance r_2 from the fixed axis. The particle P_2 moves in a circle of radius r_2 and with centre C_2 on the axis. This circle, too, lies in a plane perpendicular to the axis. Note that the circles described by P_1 and P_2 may lie in different planes; both these planes, however, are perpendicular to the fixed axis. For any particle on the axis like P_3, $r = 0$. Any such particle remains stationary while the body rotates. This is expected since the axis is fixed.

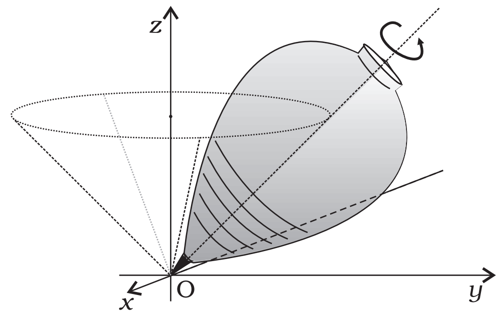


Fig. 7.5 (a) *A spinning top (The point of contact of the top with the ground, its tip O, is fixed.)*

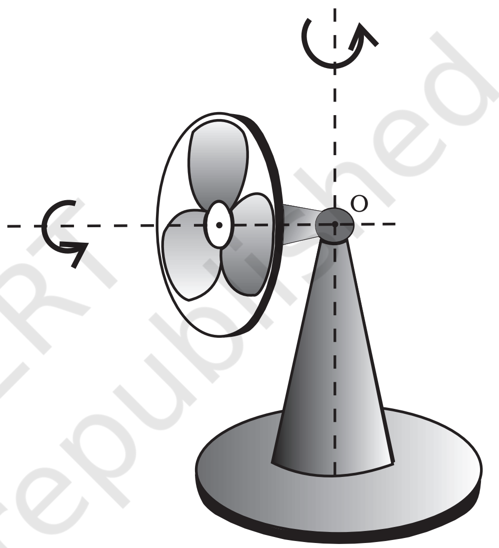


Fig. 7.5 (b) *An oscillating table fan. The pivot of the fan, point O, is fixed.*

In some examples of rotation, however, the axis may not be fixed. A prominent example of this kind of rotation is a top spinning in place [Fig. 7.5(a)]. (We assume that the top does not slip from place to place and so does not have translational motion.) We know from experience that the axis of such a spinning top moves around the vertical through its point of contact with the ground, sweeping out a cone as shown in Fig. 7.5(a). (This movement of the axis of the top around the vertical is termed **precession**.) Note, the **point of contact of the top with ground is fixed**. The axis of rotation of the top at any instant passes through the point of contact. Another simple example of this kind of rotation is the oscillating table fan or a pedestal fan. You may have observed that the axis of rotation of such a fan has an oscillating (sidewise) movement in a horizontal plane about the vertical through the point at which the axis is pivoted (point O in Fig. 7.5(b)).

While the fan rotates and its axis moves sidewise, this point is fixed. Thus, in more general cases of rotation, such as the rotation of a top or a pedestal fan, **one point and not one line,** of the rigid body is fixed. In this case the axis is not fixed, though it always passes through the fixed point. In our study, however, we mostly deal with the simpler and special case of rotation in which one line (i.e. the axis) is fixed. Thus, for us rotation will be about a fixed axis only unless stated otherwise.

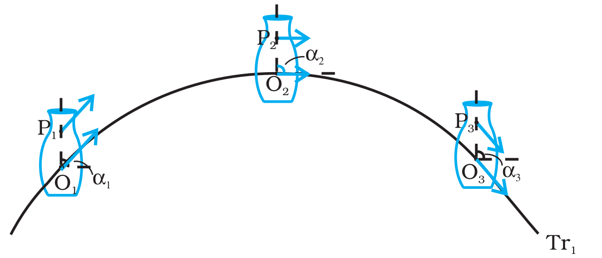


Fig. 7.6(a) Motion of a rigid body which is pure translation.

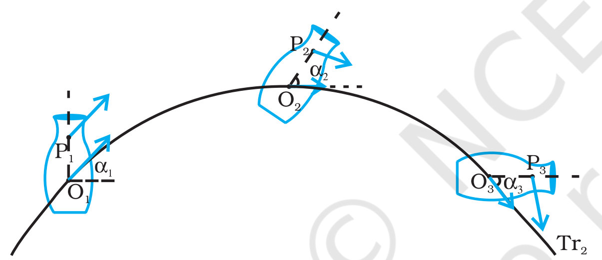


Fig. 7.6(b) Motion of a rigid body which is a combination of translation and rotation.

Fig 7.6 (a) and 7.6 (b) illustrate different motions of the same body. Note P is an arbitrary point of the body; O is the centre of mass of the body, which is defined in the next section. Suffice to say here that the trajectories of O are the translational trajectories Tr_1 and Tr_2 of the body. The positions O and P at three different instants of time are shown by O_1, O_2, and O_3, and P_1, P_2 and P_3, respectively, in both Figs. 7.6 (a) and (b) . As seen from Fig. 7.6(a), at any instant the velocities of any particles like O and P of the body are the same in pure translation. Notice, in this case the orientation of OP, i.e. the angle OP makes with a fixed direction, say the horizontal, remains the same, i.e. $\alpha_1 = \alpha_2 = \alpha_3$. Fig. 7.6 (b) illustrates a case of combination of translation and rotation. In this case, at any instants the velocities of O and P differ. Also, α_1, α_2 and α_3 may all be different.

The rolling motion of a cylinder down an inclined plane is a combination of rotation about a fixed axis and translation. Thus, the 'something else' in the case of rolling motion which we referred to earlier is rotational motion. You will find Fig. 7.6(a) and (b) instructive from this point of view. Both these figures show motion of the same body along identical translational trajectory. In one case, Fig. 7.6(a), the motion is a pure translation; in the other case [Fig. 7.6(b)] it is a combination of translation and rotation. (You may try to reproduce the two types of motion shown using a rigid object like a heavy book.)

We now recapitulate the most important observations of the present section: **The motion of a rigid body which is not pivoted or fixed in some way is either a pure translation or a combination of translation and rotation. The motion of a rigid body which is pivoted or fixed in some way is rotation.** The rotation may be about an axis that is fixed (e.g. a ceiling fan) or moving (e.g. an oscillating table fan). We shall, in the present chapter, consider rotational motion about a fixed axis only.

7.2 CENTRE OF MASS

We shall first see what the centre of mass of a system of particles is and then discuss its significance. For simplicity we shall start with a two particle system. We shall take the line joining the two particles to be the x- axis.

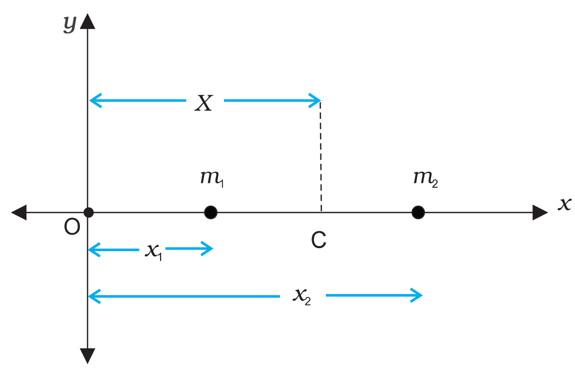


Fig. 7.7

Let the distances of the two particles be x_1 and x_2 respectively from some origin O. Let m_1 and m_2 be respectively the masses of the two particles. The centre of mass of the system is that point C which is at a distance X from O, where X is given by

$$X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \tag{7.1}$$

In Eq. (7.1), X can be regarded as the mass-weighted mean of x_1 and x_2. If the two particles have the same mass $m_1 = m_2 = m$ then

$$X = \frac{m x_1 + m x_2}{2m} = \frac{x_1 + x_2}{2}$$

Thus, for two particles of equal mass the centre of mass lies exactly midway between them.

If we have n particles of masses m_1, m_2, ...m_n respectively, along a straight line taken as the x- axis, then by definition the position of the centre of the mass of the system of particles is given by

$$X = \frac{m_1 x_1 + m_2 x_2 + ... + m_n x_n}{m_1 + m_2 + ... + m_n} = \frac{\sum m_i x_i}{\sum m_i} \tag{7.2}$$

where x_1, x_2,...x_n are the distances of the particles from the origin; X is also measured from the same origin. The symbol $\sum$ (the Greek letter sigma) denotes summation, in this case over n particles. The sum

$$\sum m_i = M$$

is the total mass of the system.

Suppose that we have three particles, not lying in a straight line. We may define x and y-axes in the plane in which the particles lie and represent the positions of the three particles by coordinates (x_1,y_1), (x_2,y_2) and (x_3,y_3) respectively. Let the masses of the three particles be m_1, m_2 and m_3 respectively. The centre of mass C of the system of the three particles is defined and located by the coordinates (X, Y) given by

$$X = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} \tag{7.3a}$$

$$Y = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} \tag{7.3b}$$

For the particles of equal mass $m = m_1 = m_2 = m_3$,

$$X = \frac{m(x_1 + x_2 + x_3)}{3m} = \frac{x_1 + x_2 + x_3}{3}$$

$$Y = \frac{m(y_1 + y_2 + y_3)}{3m} = \frac{y_1 + y_2 + y_3}{3}$$

Thus, for three particles of equal mass, the centre of mass coincides with the centroid of the triangle formed by the particles.

Results of Eqs. (7.3a) and (7.3b) are generalised easily to a system of n particles, not necessarily lying in a plane, but distributed in space. The centre of mass of such a system is at (X, Y, Z), where

$$X = \frac{\sum m_i x_i}{M} \tag{7.4a}$$

$$Y = \frac{\sum m_i y_i}{M} \tag{7.4b}$$

and $$Z = \frac{\sum m_i z_i}{M} \tag{7.4c}$$

Here $M = \sum m_i$ is the total mass of the system. The index i runs from 1 to n; m_i is the mass of the i^{th} particle and the position of the i^{th} particle is given by (x_i, y_i, z_i).

Eqs. (7.4a), (7.4b) and (7.4c) can be combined into one equation using the notation of position vectors. Let $\mathbf{r}_i$ be the position vector of the i^{th} particle and $\mathbf{R}$ be the position vector of the centre of mass:

$$\mathbf{r}_i = x_i \hat{\mathbf{i}} + y_i \hat{\mathbf{j}} + z_i \hat{\mathbf{k}}$$

and $$\mathbf{R} = X \hat{\mathbf{i}} + Y \hat{\mathbf{j}} + Z \hat{\mathbf{k}}$$

Then $$\mathbf{R} = \frac{\sum m_i \mathbf{r}_i}{M} \tag{7.4d}$$

The sum on the right hand side is a vector sum.

Note the economy of expressions we achieve by use of vectors. If the origin of the frame of reference (the coordinate system) is chosen to be the centre of mass then $\sum m_i \mathbf{r}_i = 0$ for the given system of particles.

A rigid body, such as a metre stick or a flywheel, is a system of closely packed particles; Eqs. (7.4a), (7.4b), (7.4c) and (7.4d) are therefore, applicable to a rigid body. The number of particles (atoms or molecules) in such a body is so large that it is impossible to carry out the summations over individual particles in these equations. Since the spacing of the particles is small, we can treat the body as a continuous distribution of mass. We subdivide the body into n small elements of mass; $\Delta m_1, \Delta m_2 \ldots \Delta m_n$; the i^{th} element Δm_i is taken to be located about the point (x_i, y_i, z_i). The coordinates of the centre of mass are then approximately given by

$$X = \frac{\sum (\Delta m_i) x_i}{\sum \Delta m_i}, Y = \frac{\sum (\Delta m_i) y_i}{\sum \Delta m_i}, Z = \frac{\sum (\Delta m_i) z_i}{\sum \Delta m_i}$$

As we make n bigger and bigger and each Δm_i smaller and smaller, these expressions become exact. In that case, we denote the sums over i by integrals. Thus,

$$\sum \Delta m_i \to \int \mathrm{d}m = M,$$

$$\sum (\Delta m_i) x_i \to \int x \, \mathrm{d}m,$$

$$\sum (\Delta m_i) y_i \to \int y \, \mathrm{d}m,$$

and $$\sum (\Delta m_i) z_i \to \int z \, \mathrm{d}m$$

Here M is the total mass of the body. The coordinates of the centre of mass now are

$$X = \frac{1}{M}\int x \, \mathrm{d}m, Y = \frac{1}{M}\int y \, \mathrm{d}m \text{ and } Z = \frac{1}{M}\int z \, \mathrm{d}m \quad (7.5a)$$

The vector expression equivalent to these three scalar expressions is

$$\mathbf{R} = \frac{1}{M}\int \mathbf{r} \, \mathrm{d}m \quad (7.5b)$$

If we choose, the centre of mass as the origin of our coordinate system,

$$\mathbf{R} = \mathbf{0}$$

i.e., $\int \mathbf{r} \, \mathrm{d}m = \mathbf{0}$

or $$\int x \, \mathrm{d}m = \int y \, \mathrm{d}m = \int z \, \mathrm{d}m = 0 \quad (7.6)$$

Often we have to calculate the centre of mass of homogeneous bodies of regular shapes like rings, discs, spheres, rods etc. (By a homogeneous body we mean a body with uniformly distributed mass.) By using symmetry consideration, we can easily show that the centres of mass of these bodies lie at their geometric centres.

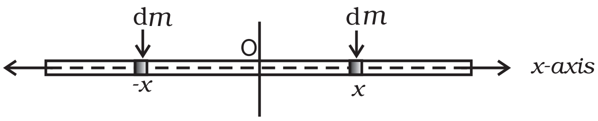


Fig. 7.8 *Determining the CM of a thin rod.*

Let us consider a thin rod, whose width and breath (in case the cross section of the rod is rectangular) or radius (in case the cross section of the rod is cylindrical) is much smaller than its length. Taking the origin to be at the geometric centre of the rod and x-axis to be along the length of the rod, we can say that on account of reflection symmetry, for every element dm of the rod at x, there is an element of the same mass dm located at $-x$ (Fig. 7.8).

The net contribution of every such pair to the integral and hence the integral $\int x \, \mathrm{d}m$ itself is zero. From Eq. (7.6), the point for which the integral itself is zero, is the centre of mass. Thus, the centre of mass of a homogenous thin rod coincides with its geometric centre. This can be understood on the basis of reflection symmetry.

The same symmetry argument will apply to homogeneous rings, discs, spheres, or even thick rods of circular or rectangular cross section. For all such bodies you will realise that for every element dm at a point (x, y, z) one can always take an element of the same mass at the point $(-x, -y, -z)$. (In other words, the origin is a point of reflection symmetry for these bodies.) As a result, the integrals in Eq. (7.5 a) all are zero. This means that for all the above bodies, their centre of mass coincides with their geometric centre.

▶ ***Example 7.1*** Find the centre of mass of three particles at the vertices of an equilateral triangle. The masses of the particles are 100g, 150g, and 200g respectively. Each side of the equilateral triangle is 0.5m long.

Answer

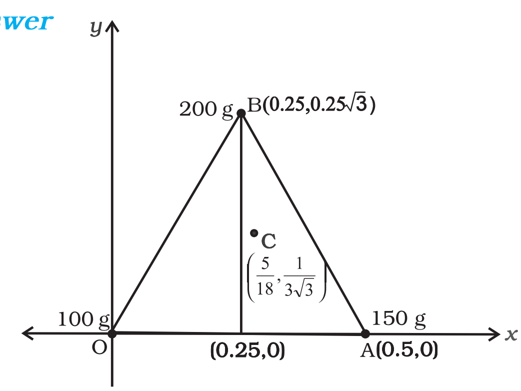


Fig. 7.9

With the x–and y–axes chosen as shown in Fig. 7.9, the coordinates of points O, A and B forming the equilateral triangle are respectively (0,0), (0.5,0), $(0.25, 0.25\sqrt{3})$. Let the masses 100 g, 150g and 200g be located at O, A and B be respectively. Then,

$$X = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

$$= \frac{[100(0) + 150(0.5) + 200(0.25)] \text{ g m}}{(100 + 150 + 200) \text{ g}}$$

$$= \frac{75 + 50}{450} \text{ m} = \frac{125}{450} \text{ m} = \frac{5}{18} \text{ m}$$

$$Y = \frac{[100(0) + 150(0) + 200(0.25\sqrt{3})] \text{ g m}}{450 \text{ g}}$$

$$= \frac{50\sqrt{3}}{450} \text{ m} = \frac{\sqrt{3}}{9} \text{ m} = \frac{1}{3\sqrt{3}} \text{ m}$$

The centre of mass C is shown in the figure. Note that it is not the geometric centre of the triangle OAB. Why? ◀

▶ ***Example 7.2*** Find the centre of mass of a triangular lamina.

Answer The lamina (ΔLMN) may be subdivided into narrow strips each parallel to the base (MN) as shown in Fig. 7.10

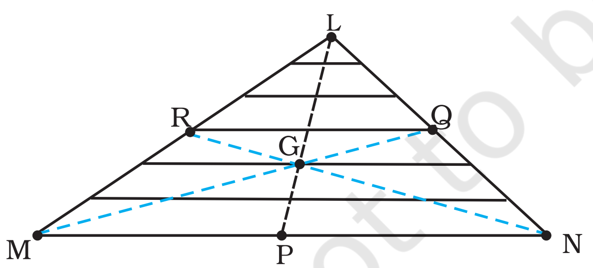


Fig. 7.10

By symmetry each strip has its centre of mass at its midpoint. If we join the midpoint of all the strips we get the median LP. The centre of mass of the triangle as a whole therefore, has to lie on the median LP. Similarly, we can argue that it lies on the median MQ and NR. This means the centre of mass lies on the point of concurrence of the medians, i.e. on the centroid G of the triangle. ◀

▶ ***Example 7.3*** Find the centre of mass of a uniform L-shaped lamina (a thin flat plate) with dimensions as shown. The mass of the lamina is 3 kg.

Answer Choosing the X and Y axes as shown in Fig. 7.11 we have the coordinates of the vertices of the L-shaped lamina as given in the figure. We can think of the L-shape to consist of 3 squares each of length 1m. The mass of each square is 1kg, since the lamina is uniform. The centres of mass C_1, C_2 and C_3 of the squares are, by symmetry, their geometric centres and have coordinates (1/2,1/2), (3/2,1/2), (1/2,3/2) respectively. We take the masses of the squares to be concentrated at these points. The centre of mass of the whole L shape (X, Y) is the centre of mass of these mass points.

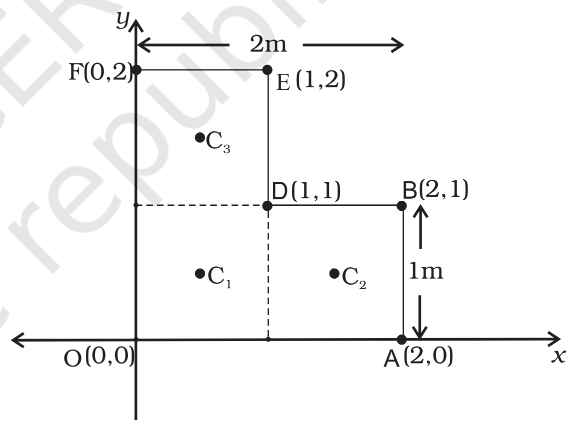


Fig. 7.11

Hence

$$X = \frac{[1(1/2) + 1(3/2) + 1(1/2)] \text{ kg m}}{(1+1+1) \text{ kg}} = \frac{5}{6} \text{m}$$

$$Y = \frac{[[1(1/2) + 1(1/2) + 1(3/2)]] \text{ kg m}}{(1+1+1) \text{ kg}} = \frac{5}{6} \text{m}$$

The centre of mass of the L-shape lies on the line OD. We could have guessed this without calculations. Can you tell why? Suppose, the three squares that make up the L shaped lamina of Fig. 7.11 had different masses. How will you then determine the centre of mass of the lamina?

7.3 MOTION OF CENTRE OF MASS

Equipped with the definition of the centre of mass, we are now in a position to discuss its physical importance for a system of particles. We may rewrite Eq.(7.4d) as

$$M\mathbf{R} = \sum m_i \mathbf{r}_i = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots + m_n\mathbf{r}_n \qquad (7.7)$$

Differentiating the two sides of the equation with respect to time we get

$$M\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = m_1\frac{\mathrm{d}\mathbf{r}_1}{\mathrm{d}t} + m_2\frac{\mathrm{d}\mathbf{r}_2}{\mathrm{d}t} + \ldots + m_n\frac{\mathrm{d}\mathbf{r}_n}{\mathrm{d}t}$$

or

$$M\mathbf{V} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \ldots + m_n\mathbf{v}_n \qquad (7.8)$$

where $\mathbf{v}_1 (= \mathrm{d}\mathbf{r}_1/\mathrm{d}t)$ is the velocity of the first particle $\mathbf{v}_2 (= d\mathbf{r}_2/dt)$ is the velocity of the second particle etc. and $\mathbf{V} = \mathrm{d}\mathbf{R}/\mathrm{d}t$ is the velocity of the centre of mass. Note that we assumed the masses m_1, m_2, ... etc. do not change in time. We have therefore, treated them as constants in differentiating the equations with respect to time.

Differentiating Eq.(7.8) with respect to time, we obtain

$$M\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = m_1\frac{\mathrm{d}\mathbf{v}_1}{\mathrm{d}t} + m_2\frac{\mathrm{d}\mathbf{v}_2}{\mathrm{d}t} + \ldots + m_n\frac{\mathrm{d}\mathbf{v}_n}{\mathrm{d}t}$$

or

$$M\mathbf{A} = m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + \ldots + m_n\mathbf{a}_n \qquad (7.9)$$

where $\mathbf{a}_1 (= \mathrm{d}\mathbf{v}_1/\mathrm{d}t)$ is the acceleration of the first particle, $\mathbf{a}_2 (= \mathrm{d}\mathbf{v}_2/\mathrm{d}t)$ is the acceleration of the second particle etc. and $\mathbf{A} (= \mathrm{d}\mathbf{V}/\mathrm{d}t)$ is the acceleration of the centre of mass of the system of particles.

Now, from Newton's second law, the force acting on the first particle is given by $\mathbf{F}_1 = m_1\mathbf{a}_1$. The force acting on the second particle is given by $\mathbf{F}_2 = m_2\mathbf{a}_2$ and so on. Eq. (7.9) may be written as

$$M\mathbf{A} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n \qquad (7.10)$$

Thus, the total mass of a system of particles times the acceleration of its centre of mass is the vector sum of all the forces acting on the system of particles.

Note when we talk of the force $\mathbf{F}_1$ on the first particle, it is not a single force, but the vector sum of all the forces on the first particle; likewise for the second particle etc. Among these forces on each particle there will be **external** forces exerted by bodies outside the system and also **internal** forces exerted by the particles on one another. We know from Newton's third law that these internal forces occur in equal and opposite pairs and in the sum of forces of Eq. (7.10), their contribution is zero. Only the external forces contribute to the equation. We can then rewrite Eq. (7.10) as

$$M\mathbf{A} = \mathbf{F}_{ext} \qquad (7.11)$$

where $\mathbf{F}_{ext}$ represents the sum of all external forces acting on the particles of the system.

Eq. (7.11) states that **the centre of mass of a system of particles moves as if all the mass of the system was concentrated at the centre of mass and all the external forces were applied at that point.**

Notice, to determine the motion of the centre of mass no knowledge of internal forces of the system of particles is required; for this purpose we need to know only the external forces.

To obtain Eq. (7.11) we did not need to specify the nature of the system of particles. The system may be a collection of particles in which there may be all kinds of internal motions, or it may be a rigid body which has either pure translational motion or a combination of translational and rotational motion. Whatever is the system and the motion of its individual particles, the centre of mass moves according to Eq. (7.11).

Instead of treating extended bodies as single particles as we have done in earlier chapters, we can now treat them as systems of particles. We can obtain the translational component of their motion, i.e. the motion centre of mass of the system, by taking the mass of the whole system to be concentrated at the centre of mass and all the external forces on the system to be acting at the centre of mass.

This is the procedure that we followed earlier in analysing forces on bodies and solving problems without explicitly outlining and justifying the procedure. We now realise that in earlier studies we assumed, without saying so, that rotational motion and/or internal motion of the particles were either absent or negligible. We no longer need to do this. We have not only found the justification of the procedure we followed earlier; but we also have found how to describe and separate the translational motion of (1) a rigid body which may be rotating as well, or (2) a system of particles with all kinds of internal motion.

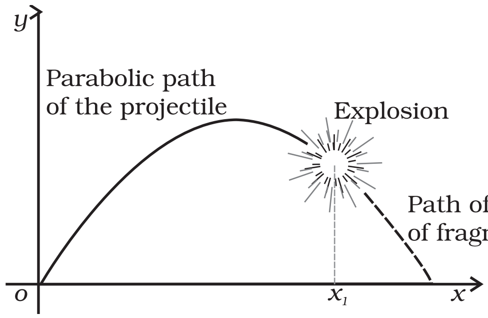


Fig. 7.12 The centre of mass of the fragments of the projectile continues along the same parabolic path which it would have followed if there were no explosion.

Figure 7.12 is a good illustration of Eq. (7.11). A projectile, following the usual parabolic trajectory, explodes into fragments midway in air. The forces leading to the explosion are internal forces. They contribute nothing to the motion of the centre of mass. The total external force, namely, the force of gravity acting on the body, is the same before and after the explosion. The centre of mass under the influence of the external force continues, therefore, along the same parabolic trajectory as it would have followed if there were no explosion.

7.4 LINEAR MOMENTUM OF A SYSTEM OF PARTICLES

Let us recall that the linear momentum of a particle is defined as

$$\mathbf{p} = m\,\mathbf{v} \tag{7.12}$$

Let us also recall that Newton's second law written in symbolic form for a single particle is

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \tag{7.13}$$

where **F** is the force on the particle. Let us consider a system of n particles with masses $m_1, m_2, \ldots m_n$ respectively and velocities $\mathbf{v}_1, \mathbf{v}_2, \ldots \ldots \mathbf{v}_n$ respectively. The particles may be interacting and have external forces acting on them. The linear momentum of the first particle is $m_1\mathbf{v}_1$, of the second particle is $m_2\mathbf{v}_2$ and so on.

For the system of n particles, the linear momentum of the system is defined to be the vector sum of all individual particles of the system,

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \ldots + \mathbf{p}_n$$

$$= m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \ldots + m_n\mathbf{v}_n \tag{7.14}$$

Comparing this with Eq. (7.8)

$$\mathbf{P} = M\,\mathbf{V} \tag{7.15}$$

Thus, **the total momentum of a system of particles is equal to the product of the total mass of the system and the velocity of its centre of mass**. Differentiating Eq. (7.15) with respect to time,

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = M\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = M\mathbf{A} \tag{7.16}$$

Comparing Eq.(7.16) and Eq. (7.11),

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \mathbf{F}_{ext} \tag{7.17}$$

This is the statement of **Newton's second law extended to a system of particles.**

Suppose now, that the sum of external forces acting on a system of particles is zero. Then from Eq.(7.17)

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = 0 \quad \text{or} \quad \mathbf{P} = \text{Constant} \tag{7.18a}$$

Thus, when the total external force acting on a system of particles is zero, the total linear momentum of the system is constant. This is the law of conservation of the total linear momentum of a system of particles. Because of Eq. (7.15), this also means that when the total external force on the system is zero the velocity of the centre of mass remains constant. (We assume throughout the discussion on systems of particles in this chapter that the total mass of the system remains constant.)

Note that on account of the internal forces, i.e. the forces exerted by the particles on one another, the individual particles may have complicated trajectories. Yet, if the total external force acting on the system is zero, the centre of mass moves with *a* constant velocity, i.e., moves uniformly in a straight line like a free particle.

The vector Eq. (7.18a) is equivalent to three scalar equations,

$$P_x = c_1,\ P_y = c_2 \text{ and } P_z = c_3 \qquad (7.18\text{ b})$$

Here P_x, P_y and P_z are the components of the total linear momentum vector **P** along the *x*, *y* and *z* axes respectively; c_1, c_2 and c_3 are constants.

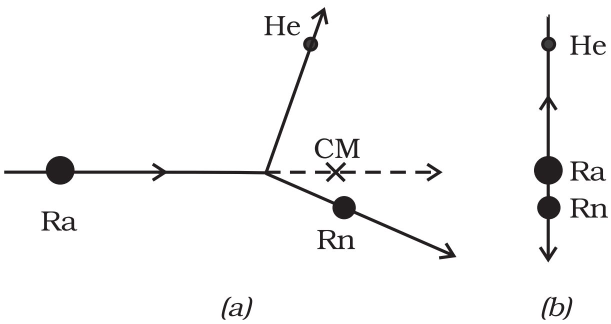


Fig. 7.13 (a) A heavy nucleus (Ra) splits into a lighter nucleus (Rn) and an alpha particle (He). The CM of the system is in uniform motion.
(b) The same spliting of the heavy nucleus (Ra) with the centre of mass at rest. The two product particles fly back to back.

As an example, let us consider the radioactive decay of a moving unstable particle, like the nucleus of radium. A radium nucleus disintegrates into a nucleus of radon and an alpha particle. The forces leading to the decay are internal to the system and the external forces on the system are negligible. So the total linear momentum of the system is the same before and after decay. The two particles produced in the decay, the radon nucleus and the alpha particle, move in different directions in such a way that their centre of mass moves along the same path along which the original decaying radium nucleus was moving [Fig. 7.13(a)].

If we observe the decay from the frame of reference in which the centre of mass is at rest, the motion of the particles involved in the decay looks particularly simple; the product particles

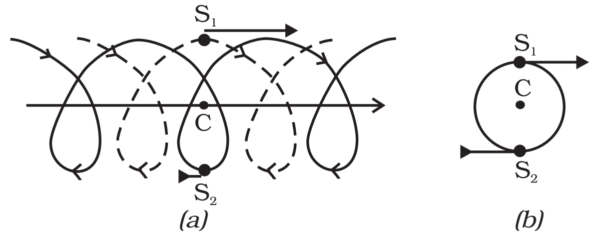


Fig. 7.14 (a) Trajectories of two stars, S_1 (dotted line) and S_2 (solid line) forming a binary system with their centre of mass C in uniform motion.
(b) The same binary system, with the centre of mass C at rest.

move back to back with their centre of mass remaining at rest as shown in Fig.7.13 (b).

In many problems on the system of particles as in the above radioactive decay problem, it is convenient to work in the centre of mass frame rather than in the laboratory frame of reference.

In astronomy, binary (double) stars is a common occurrence. If there are no external forces, the centre of mass of a double star moves like a free particle, as shown in Fig.7.14 (a). The trajectories of the two stars of equal mass are also shown in the figure; they look complicated. If we go to the centre of mass frame, then we find that there the two stars are moving in a circle, about the centre of mass, which is at rest. Note that the position of the stars have to be diametrically opposite to each other [Fig. 7.14(b)]. Thus in our frame of reference, the trajectories of the stars are a combination of (i) uniform motion in a straight line of the centre of mass and (ii) circular orbits of the stars about the centre of mass.

As can be seen from the two examples, **separating** the motion of different parts of a system into motion **of the centre of mass and motion about the centre of mass** is a very useful technique that helps in understanding the motion of the system.

7.5 VECTOR PRODUCT OF TWO VECTORS

We are already familiar with vectors and their use in physics. In chapter 6 (Work, Energy, Power) we defined the scalar product of two vectors. An important physical quantity, work, is defined as a scalar product of two vector quantities, force and displacement.

We shall now define another product of two vectors. This product is a vector. Two important quantities in the study of rotational motion, namely, moment of a force and angular momentum, are defined as vector products.

Definition of Vector Product

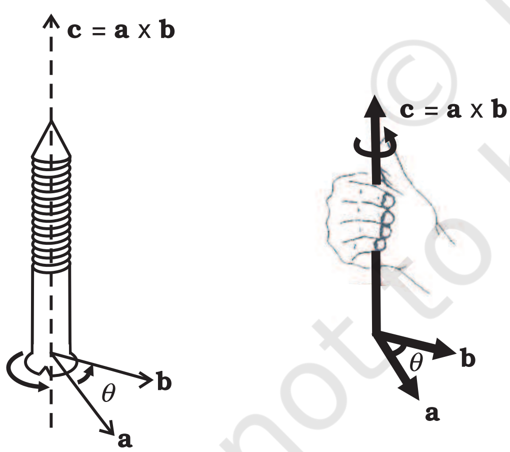
A vector product of two vectors **a** and **b** is a vector **c** such that

(i) magnitude of **c** = $c = ab\sin\theta$ where a and b are magnitudes of **a** and **b** and θ is the angle between the two vectors.

(ii) **c** is perpendicular to the plane containing **a** and **b**.

(iii) if we take a right handed screw with its head lying in the plane of **a** and **b** and the screw perpendicular to this plane, and if we turn the head in the direction from **a** to **b**, then the tip of the screw advances in the direction of **c**. This right handed screw rule is illustrated in Fig. 7.15a.

Alternately, if one curls up the fingers of right hand around a line perpendicular to the plane of the vectors **a** and **b** and if the fingers are curled up in the direction from **a** to **b**, then the stretched thumb points in the direction of **c**, as shown in Fig. 7.15b.



(a) (b)

Fig. 7.15 (a) Rule of the right handed screw for defining the direction of the vector product of two vectors.
(b) Rule of the right hand for defining the direction of the vector product.

A simpler version of the right hand rule is the following : Open up your right hand palm and curl the fingers pointing from **a** to **b**. Your stretched thumb points in the direction of **c**.

It should be remembered that there are two angles between any two vectors **a** and **b** . In Fig. 7.15 (a) or (b) they correspond to θ (as shown) and $(360^0 - \theta)$. While applying either of the above rules, the rotation should be taken through the smaller angle ($<180^0$) between **a** and **b**. It is θ here.

Because of the cross used to denote the vector product, it is also referred to as cross product.

- Note that scalar product of two vectors is commutative as said earlier, **a**.**b** = **b**.**a**

The vector product, however, is not commutative, i.e. **a**   **b** ≠ **b**   **a**

The magnitude of both **a**   **b** and **b**   **a** is the same ($ab\sin\theta$); also, both of them are perpendicular to the plane of **a** and **b**. But the rotation of the right-handed screw in case of **a**   **b** is from **a** to **b**, whereas in case of **b**   **a** it is from **b** to **a**. This means the two vectors are in opposite directions. We have

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

- Another interesting property of a vector product is its behaviour under reflection. Under reflection (i.e. on taking the mirror image) we have $x \to -x, y \to -y$ and $z \to -z$. As a result all the components of a vector change sign and thus $a \to -a$, $b \to -b$. What happens to **a**   **b** under reflection?

$$\mathbf{a} \quad \mathbf{b} \to (-\mathbf{a}) \times (-\mathbf{b}) = \mathbf{a} \times \mathbf{b}$$

Thus, **a**   **b** does not change sign under reflection.

- Both scalar and vector products are distributive with respect to vector addition. Thus,

$$\mathbf{a}.(\mathbf{b}+\mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$$

$$\mathbf{a} \times (\mathbf{b}+\mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

- We may write **c** = **a**   **b** in the component form. For this we first need to obtain some elementary cross products:

(i) **a**   **a** = **0** (**0** is a null vector, i.e. a vector with zero magnitude)

This follows since magnitude of **a**   **a** is $a^2 \sin 0° = 0$.

From this follow the results

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \mathbf{0},\ \ \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \mathbf{0},\ \ \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \mathbf{0}$$

(ii) $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$

Note that the magnitude of $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$ is $\sin 90^0$ or 1, since $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ both have unit magnitude and the angle between them is 90^0. Thus, $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$ is a unit vector. A unit vector perpendicular to the plane of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ and related to them by the right hand screw rule is $\hat{\mathbf{k}}$. Hence, the above result. You may verify similarly,

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}} \ \text{ and } \ \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$$

From the rule for commutation of the cross product, it follows:

$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}},\ \ \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}},\ \ \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$$

Note if $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ occur cyclically in the above vector product relation, the vector product is positive. If $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ do not occur in cyclic order, the vector product is negative.

Now,

$$\mathbf{a} \times \mathbf{b} = (a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}) \times (b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}})$$

$$= a_x b_y \hat{\mathbf{k}} - a_x b_z \hat{\mathbf{j}} - a_y b_x \hat{\mathbf{k}} + a_y b_z \hat{\mathbf{i}} + a_z b_x \hat{\mathbf{j}} - a_z b_y \hat{\mathbf{i}}$$

$$= (a_y b_z - a_z b_x)\hat{\mathbf{i}} + (a_z b_x - a_x b_z)\hat{\mathbf{j}} + (a_x b_y - a_y b_x)\hat{\mathbf{k}}$$

We have used the elementary cross products in obtaining the above relation. The expression for **a** × **b** can be put in a determinant form which is easy to remember.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

▶ ***Example 7.4*** *Find the scalar and vector products of two vectors.* **a** = $(3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}})$ *and* **b** = $(-2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}})$

Answer

$$\mathbf{a} \cdot \mathbf{b} = (3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) \cdot (-2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}})$$

$$= -6 - 4 - 15$$

$$= -25$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & -4 & 5 \\ -2 & 1 & -3 \end{vmatrix} = 7\hat{\mathbf{i}} - \hat{\mathbf{j}} - 5\hat{\mathbf{k}}$$

Note $\mathbf{b} \times \mathbf{a} = -7\hat{\mathbf{i}} + \hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ ◀

7.6 ANGULAR VELOCITY AND ITS RELATION WITH LINEAR VELOCITY

In this section we shall study what is angular velocity and its role in rotational motion. We have seen that every particle of a rotating body moves in a circle. The linear velocity of the particle is related to the angular velocity. The relation between these two quantities involves a vector product which we learnt about in the last section.

Let us go back to Fig. 7.4. As said above, in rotational motion of a rigid body about a fixed axis, every particle of the body moves in a circle,

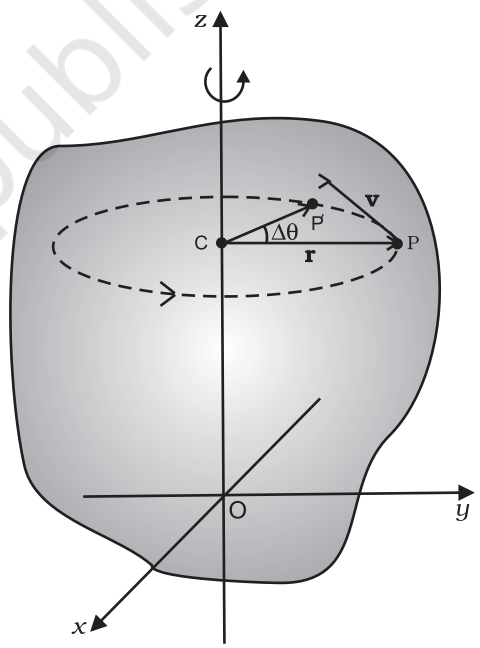


Fig. 7.16 Rotation about a fixed axis. (A particle (P) of the rigid body rotating about the fixed (z-) axis moves in a circle with centre (C) on the axis.)

which lies in a plane perpendicular to the axis and has its centre on the axis. In Fig. 7.16 we redraw Fig. 7.4, showing a typical particle (at a point P) of the rigid body rotating about a fixed axis (taken as the *z*-axis). The particle describes a circle with a centre C on the axis. The radius of the circle is r, the perpendicular distance of the point P from the axis. We also show the linear velocity vector **v** of the particle at P. It is along the tangent at P to the circle.

Let P′ be the position of the particle after an interval of time Δt (Fig. 7.16). The angle PCP′ describes the angular displacement $\Delta\theta$ of the particle in time Δt. The average angular velocity of the particle over the interval Δt is $\Delta\theta/\Delta t$. As Δt tends to zero (i.e. takes smaller and smaller values), the ratio $\Delta\theta/\Delta t$ approaches a limit which is the instantaneous angular velocity $\mathrm{d}\theta/\mathrm{d}t$ of the particle at the position P. We denote the **instantaneous angular velocity** by ω (the Greek letter omega). We know from our study of circular motion that the magnitude of linear velocity v of a particle moving in a circle is related to the angular velocity of the particle ω by the simple relation $\upsilon = \omega r$, where r is the radius of the circle.

We observe that at any given instant the relation $v = \omega r$ applies to all particles of the rigid body. Thus for a particle at a perpendicular distance r_i from the fixed axis, the linear velocity at a given instant v_i is given by

$\omega = d\theta/dt$

$$v_i = \omega r_i \tag{7.19}$$

The index i runs from 1 to n, where n is the total number of particles of the body.

For particles on the axis, $r = 0$, and hence $v = \omega r = 0$. Thus, particles on the axis are stationary. This verifies that the axis is *fixed*.

Note that we use the same angular velocity ω for all the particles. **We therefore, refer to $\boldsymbol{\omega}$ as the angular velocity of the whole body.**

We have characterised pure translation of a body by all parts of the body having the same velocity at any instant of time. Similarly, we may characterise pure rotation by all parts of the body having the same angular velocity at any instant of time. Note that this characterisation of the rotation of a rigid body about a fixed axis is **just another way** of saying as in Sec. 7.1 that each particle of the body moves in a circle, which lies in a plane perpendicular to the axis and has the centre on the axis.

In our discussion so far the angular velocity appears to be a scalar. In fact, it is a vector. We shall not justify this fact, but we shall accept it. For rotation about a fixed axis, the angular velocity vector lies along the axis of rotation, and points out in the direction in which a right handed screw would advance, if the head of the screw is rotated with the body. (See Fig. 7.17a).

The magnitude of this vector is referred as above.

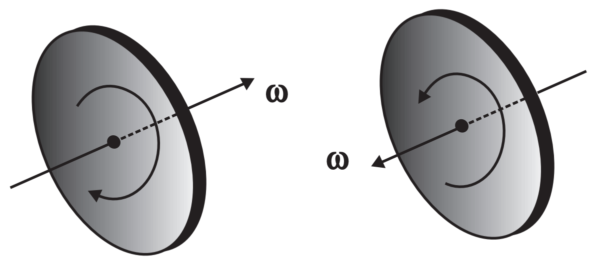


Fig. 7.17 (a) If the head of a right handed screw rotates with the body, the screw advances in the direction of the angular velocity $\boldsymbol{\omega}$*. If the sense (clockwise or anticlockwise) of rotation of the body changes, so does the direction of* $\boldsymbol{\omega}$*.*

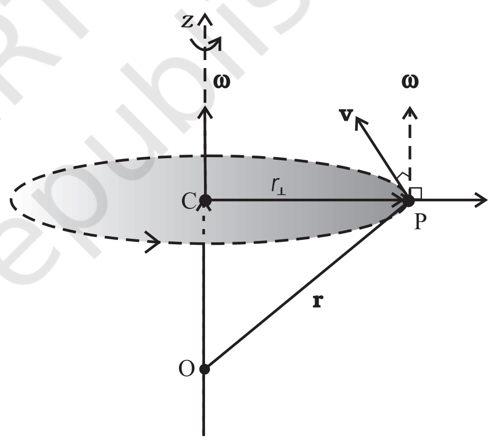


Fig. 7.17 (b) The angular velocity vector $\boldsymbol{\omega}$ *is directed along the fixed axis as shown. The linear velocity of the particle at P is* $\mathbf{v} = \boldsymbol{\omega}\ \mathbf{r}$*. It is perpendicular to both* $\boldsymbol{\omega}$ *and* $\mathbf{r}$ *and is directed along the tangent to the circle described by the particle.*

We shall now look at what the vector product $\boldsymbol{\omega}$ $\mathbf{r}$ corresponds to. Refer to Fig. 7.17(b) which is a part of Fig. 7.16 reproduced to show the path of the particle P. The figure shows the vector $\boldsymbol{\omega}$ directed along the fixed (z–) axis and also the position vector $\mathbf{r} = \mathbf{OP}$ of the particle at P of the rigid body with respect to the origin O. Note that the origin is chosen to be on the axis of rotation.