Conditional Identities Involving the Angles Of a Triangle

EXERCISE 16 PAGE: 595

Q. 1. If A + B + C = π, prove that

sin 2A + sin 2B – sin 2C = 4cos A cos B sin C

Solution: = sin 2A + sin 2B – sin 2C

= 2 sin (B + C) cos A + 2 sin (A + C) cos B - 2 sin (A + B) cos C

Using formula, sin (A + B) = sin A cos B + cos A sin B

= sin 2A + sin 2B - sin 2C

Using formula

sin2A = 2sinAcosA

= 2sinAcosA + 2sinBcosB - 2sinCcosC

Since A + B + C = π

$\rightarrow B + C = 180 - A$

And sin(π – A) = sinA

= 2sin(B + C)cos A + 2sin(A + C)cosB - 2sin(A + B)cosC

= 2 (sin B cos C + cos B sin C) cos A + 2(sinAcosC + cosAsinC)cosB - 2(sinAcosB + cosAsinB)cosC

= 2cosAsinBcosC + 2cosAcosBsinC + 2sinAcosBcosC + 2cosAcosBsinC– 2sinAcosBcosC – 2cosAsinBcosC

= 2cosAcosBsinC + 2cosAcosBsinC

= 4cosAcosBsinC

= R.H.S

Q. 2. If A + B + C = π, prove that

cos 2A – cos 2B – cos 2C = -1 + 4 cos A sin B sin C

Solution: = cos2A – (cos2B + cos2C)

Using formula

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$= \cos 2A - \left\{2\cos\left(\frac{2B+2C}{2}\right)\cos\left(\frac{2B-2C}{`2}\right)\right\}$$

= cos2A - {2cos (B+C) cos (B-C)}

Since A + B + C = π

$$\rightarrow B + C = 180 - A$$

= cos2A – {2cos (π – A) cos (B-C)}

And cos(π – A) = -cosA

= cos2A – {-2cosAcos (B-C)}

= cos2A + 2cosAcos (B-C)

Using cos2A = $2\cos^2$A -1

= $2\cos^2$A – 1 + 2cosAcos (B-C)

= 2cosA {cosA + cos (B-C)} – 1

$$\text{Using}, \cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$= 2\cos A\left\{2\cos\left(\frac{A+B-C}{2}\right)\cos\left(\frac{A+C-B}{`2}\right)\right\} - 1$$

$$= 2\cos A\left\{2\cos\left(\frac{\pi-C-C}{2}\right)\cos\left(\frac{\pi-B-B}{`2}\right)\right\} - 1$$

$$\text{As}, \cos\left(\frac{\pi}{2} - A\right) = \sin A$$

$$= 2\cos A\left\{2\cos\left(\frac{\pi}{2} - \frac{2C}{2}\right)\cos\left(\frac{\pi}{`2} - \frac{2B}{2}\right)\right\} - 1$$

= 2cosA {2sinCsinB} – 1

= 4cosAsinBsinC – 1

= R.H.S

Q. 3. If A + B + C = π, prove that

cos 2A – cos 2B + cos 2C = 1 – 4sin A cos B sin C

Solution: = cos2A – cos2B + cos2C

Using,

$$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$$

$$= \cos 2A - \left\{2\sin\left(\frac{2B+2C}{2}\right)\sin\left(\frac{2B-2C}{`2}\right)\right\}$$

= cos2A - {2sin(B+C)sin(B-C)}

since A + B + C = π

$$\rightarrow B + C = 180 - A$$

And sin(π – A) = sinA

= cos2A – {2sin(π – A)sin(B-C)}

= cos2A – {2sinAsin(B-C)}

= cos2A - 2sinAsin(B-C)

Using, cos2A = 1 – 2sin^{2}A

= -2sin^{2}A + 1 – 2sinAsin(B-C)

= -2sinA{sinA + sin(B-C)} + 1

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$= -2\sin A\left\{2\sin\left(\frac{A+B-C}{2}\right)\cos\left(\frac{A+C-B}{`2}\right)\right\} + 1$$

$$= -2\sin A\left\{2\sin\left(\frac{\pi-C-C}{2}\right)\cos\left(\frac{\pi-B-B}{`2}\right)\right\} + 1$$

$= -2\sin A\left\{2\sin\left(\frac{\pi}{2} - \frac{2C}{2}\right)\cos\left(\frac{\pi}{`2} - \frac{2B}{2}\right)\right\} + 1$

As, $\sin\left(\frac{\pi}{2} - A\right) = \cos A$

= -2sinA{2cosCsinB} + 1

= -4sinAcosBsinC + 1

= R.H.S

Q. 4. If A + B + C = π, prove that

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

Solution: = sinA + sinB + sinC

Using,

$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$= \sin A + \left\{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{`2}\right)\right\}$

Since A + B + C = π

$\rightarrow B + C = 180 - A$

And,

$\sin\left(\frac{\pi}{2} - A\right) = \cos A$

$= \sin A + \{2\sin\left(\frac{\pi-A}{2}\right)\cos\left(\frac{B-C}{2}\right)\}$

$= \sin A + \{2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)\}$

Using, sin2A = 2sinAcosA

$= 2\sin\frac{A}{2}\cos\frac{A}{2} + \{2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)\}$

$= 2\cos\frac{A}{2}\{\sin\frac{A}{2} + \cos\left(\frac{B-C}{2}\right)\}$

$\rightarrow B + C = 180 - A$

And,

$\sin\left(\frac{\pi}{2} - A\right) = \cos A$

$= 2\cos\frac{A}{2}\{\cos(\frac{B+C}{2}) + \cos\left(\frac{B-C}{2}\right)\}$

$= 2\cos\frac{A}{2}\{2\cos\left(\frac{B}{2}\right)\cos(\frac{C}{2})\}$

$= 4\cos\frac{A}{2}\cos\left(\frac{B}{2}\right)\cos(\frac{C}{2})$

= R.H.S

Q. 5. If A + B + C = π, prove that

$$\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

Solution: = cosA + cosB + cosC

Using,

$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos(\frac{A-B}{2})$

$= \cos A + \left\{2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{`2}\right)\right\}$

since A + B + C = π

$\rightarrow B + C = 180 - A$

And,

$\cos\left(\frac{\pi}{2} - A\right) = \sin A$

$= \cos A + \{2\cos\left(\frac{\pi-A}{2}\right)\cos\left(\frac{B-C}{2}\right)\}$

$= \cos A + \{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)\}$

Using , cos2A = 1 – 2sin^{2}A

$= 1 - 2\sin^2\frac{A}{2} + \{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)\}$

$= 2\sin\frac{A}{2}\left\{-\sin\frac{A}{2} + \cos\left(\frac{B-C}{2}\right)\right\} + 1$

$= 2\sin\frac{A}{2}\left\{\cos\left(\frac{-B-C}{2}\right) + \cos\left(\frac{B-C}{2}\right)\right\} + 1$

$= 2\sin\frac{A}{2}\left\{2\cos\left(\frac{-C}{2}\right)\cos\left(\frac{-B}{2}\right)\right\} + 1$

$= 4\sin\frac{A}{2}\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right) + 1$

= R.H.S

Q. 6. If A + B + C = π, prove that

$$\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

Solution: = sin2A + sin2B +

sin2C Using,

$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

Sin2A = 2sinAcosA

= 2sinAcosA + 2sin(B+C)cos(B - C)

since A + B + C = π

$\rightarrow B + C = 180 - A$

= 2sinAcosA + 2sin(π - A)cos(B - C)

= 2sinAcosA + 2sinAcos(B - C)

= 2sinA{cosA + cos (B-C)}

(but cos A = cos { 180 - (B + C) } = - cos (B + C)

And now using $\cos A - \cos B = 2\sin(\frac{A+B}{2})\sin(\frac{-A+B}{2})$

= 2sinA{2sinBsinC}

= 4sinAsinBsinC

$= 32\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}$

Now,

= sinA + sinB + sinC

Using,

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$= \sin A + \left\{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{`2}\right)\right\}$$

$$= \sin A + \{2\sin\left(\frac{\pi-A}{2}\right)\cos\left(\frac{B-C}{2}\right)\}$$

$$= \sin A + \{2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)\}$$

$$= 2\sin\frac{A}{2}\cos\frac{A}{2} + \{2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)\}$$

$$= 2\cos\frac{A}{2}\{\sin\frac{A}{2} + \cos\left(\frac{B-C}{2}\right)\}$$

$$= 2\cos\frac{A}{2}\{\cos(\frac{B+C}{2}) + \cos\left(\frac{B-C}{2}\right)\}$$

$$= 2\cos\frac{A}{2}\{2\cos\left(\frac{B}{2}\right)\cos(\frac{C}{2})\}$$

$= 4\cos\frac{A}{2}\cos\left(\frac{B}{2}\right)\cos(\frac{C}{2})$

Therefore,

$= \frac{32\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}}{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$

$= 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

= R.H.S

Q. 7. If A + B + C = π, prove that

sin (B + C – A) + sin (C + A – B) – sin (A + B – C) = 4cos A cos B sin C

Solution: = sin (B + C – A) + sin (C + A – B) – sin (A + B – C)

Using,

$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

= 2sinC cos(B-A) – sin(A+B-C)

Since A + B + C = π

$\rightarrow B + A = 180 - C$

= 2sinCcos(B-A) – sin(π – C – C)

= 2sinCcos(B-A) – sin2C

Since , sin2A = 2sinAcosA,

= 2sinCcos(B-A) – 2sinCcosC

= 2sinC{cos(B-A) – cosC}

Using ,

$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$

$= 2\sin C\{2\sin(\frac{B-A+C}{2})\sin(\frac{C-B+A}{2})\}$

$= 2\sin C\{2\sin(\frac{\pi-A-A}{2})\sin(\frac{\pi-B-B}{2})\}$

= 4cosAcosBsinC

= R.H.S

Q. 8. If A + B + C = π, prove that

$$\frac{\cos A}{\sin B \sin C}+\frac{\cos B}{\sin C \sin A}+\frac{\cos C}{\sin A \sin B}=2$$

Solution:

$= \frac{\cos A}{\sin B \sin C}+\frac{\cos B}{\sin C \sin A}+\frac{\cos C}{\sin A \sin B}$

Taking L.C.M

$= \frac{\cos A\sin A + \cos B\sin B+\cos C\sin C}{\sin B\sin C\sin A}$

Multiplying and divide the above equation by 2, we get

$= \frac{2\cos A\sin A + 2\cos B\sin B+2\cos C\sin C}{2\sin B\sin C\sin A}$

Since , sin2A = 2sinAcosA

$= \frac{\sin 2A + \sin 2B+\sin 2C}{2\sin B\sin C\sin A}$

Now,

= sin2A + sin2B + sin2C

= 2sinAcosA + 2sin(B+C)cos(B - C)

since A + B + C = π

$\rightarrow B + A = 180 - C$

= 2sinAcosA + 2sin(π - A)cos(B - C)

= 2sinAcosA + 2sinAcos(B - C)

= 2sinA{cosA + cos (B-C)}

(but cos A = cos { 180 - (B + C) } = - cos (B + C)

And now using $\cos A - \cos B = 2\sin(\frac{A+B}{2})\sin(\frac{-A+B}{2})$

= 2sinA{2sinBsinC}

= 4sinAsinBsinC

Putting the above value in the equation, we get

$= \frac{4\sin A\sin B\sin C}{2\sin B\sin C\sin A}$

= 2

= R.H.S

Q. 9. If A + B + C = π, prove that

$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$

Solution: $= \cos^2 A + \cos^2 B + \cos^2 C$

Using formula,

$\frac{1+\cos 2A}{2} = \cos^2 A$

$= \frac{1+\cos 2A}{2} + \frac{1+\cos 2B}{2} + \frac{1+\cos 2C}{2}$

$= \frac{1 + \cos 2A + 1 + \cos 2B + 1 + \cos 2C}{2}$

$= \frac{3 + \cos 2A + \cos 2B + \cos 2C}{2}$

Using,

$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$= \frac{3 + \cos 2A + 2\cos\left(\frac{2B+2C}{2}\right)\cos\left(\frac{2B-2C}{2}\right)}{2}$

$= \frac{3 + \cos 2A + 2\cos(B+C)\cos(B-C)}{2}$

Using, since A + B + C = π

$\rightarrow B + C = 180 - A$

And, cos(π – A) = -cosA

$= \frac{3 + \cos 2A + 2\cos(\pi-A)\cos(B-C)}{2}$

$= \frac{3 + \cos 2A - 2\cos(A)\cos(B-C)}{2}$

Using cos2A = $2\cos^2$A -1

$= \frac{3 + 2\cos^2 A - 1 - 2\cos(A)\cos(B-C)}{2}$

$= \frac{2 + 2\cos^2 A - 2\cos(A)\cos(B-C)}{2}$

= 1 + $\cos^2$A – cosAcos(B-C)

= 1 + cosA{cosA - cos(B-C)}

Using,

$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$

$= 1 + \cos A\left(2\sin\left(\frac{A+B-C}{2}\right)\sin\left(\frac{B-C-A}{2}\right)\right)$

Since , A + B + C = π

$= 1 + \cos A\left(2\sin\left(\frac{\pi-C-C}{2}\right)\sin\left(\frac{B-(\pi-B)}{2}\right)\right)$

$= 1 + \cos A\left(2\cos C\sin\left(\frac{B}{2} - \frac{\pi}{2}\right)\right)$

= 1 – 2cosAcosCcosC

= R.H.S

Q. 10. If A + B + C = π, prove that

$\sin^2 A - \sin^2 B + \sin^2 C = 2\sin A \cos B \sin C$

Solution: $= \sin^2 A - \sin^2 B + \sin^2 C$

Using formula,

$$\frac{1-\cos 2A}{2} = \sin^2 A$$

$$= \frac{1-\cos 2A}{2} - \frac{1-\cos 2B}{2} + \frac{1-\cos 2C}{2}$$

$$= \frac{1-\cos 2A - 1 + \cos 2B + 1 - \cos 2C}{2}$$

$$= \frac{1-\cos 2A + \cos 2B - \cos 2C}{2}$$

Using ,

$$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$$

$$= \frac{1-\cos 2A + \left\{2\sin\left(\frac{2B+2C}{2}\right)\sin\left(\frac{2C-2B}{2}\right)\right\}}{2}$$

$$= \frac{1-\cos 2A + 2\sin(B+C)\sin(C-B)}{2}$$

since A + B + C = π

$$\to B + C = 180 - A$$

And sin(π – A) = sinA

$$= \frac{1-\cos 2A + 2\sin(\pi - A)\sin(C-B)}{2}$$

$$= \frac{1-\cos 2A + 2\sin A\sin(C-B)}{2}$$

Using , $\cos 2A = 1 - 2\sin^2 A$

$$= \frac{1-1+2\sin^2 A + 2\sin A\sin(C-B)}{2}$$

$$= \frac{2\sin A\{\sin A + \sin(C-B)\}}{2}$$

$$= \frac{2\sin A\{\sin A + \sin(C-B)\}}{2}$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$= \frac{2\sin A\{2\sin\left(\frac{A+C-B}{2}\right)\cos(\frac{A-C+B}{`2})\}}{2}$$

$$= \frac{1-2\sin A\{2\sin\left(\frac{\pi-B-B}{2}\right)\cos(\frac{\pi-C-C}{`2})\}}{2}$$

$$= \frac{2\sin A\{2\sin\left(\frac{\pi}{2}-\frac{2B}{2}\right)\cos(\frac{\pi}{2}-\frac{2C}{2})\}}{2}$$

$$\text{As, } \sin\left(\frac{\pi}{2}-A\right) = \cos A$$

$$= \frac{2\sin A\{2\cos B\sin C\}}{2}$$

= 2sinAcosBsinC

= R.H.S

Q. 11. If A + B + C = π, prove that

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

Solution:

$$= \sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2}$$

Using formula ,

$$\frac{1-\cos 2A}{2} = \sin^2 A$$

$= \frac{1-\cos A}{2} + \frac{1-\cos B}{2} + \frac{1-\cos C}{2}$

$= \frac{1-\cos A + 1-\cos B + 1-\cos C}{2}$

$= \frac{3-\cos A - \cos B - \cos C}{2}$

Using,

$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$= \frac{3-\cos A - \{2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)\}}{2}$

$= \frac{3-\cos A - 2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2}$

Using , since A + B + C = π

$\rightarrow B + C = 180 - A$

And, cos(π – A) = -cosA

$= \frac{3-\cos A - 2\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2}$

$= \frac{3-\cos A - 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2}$

Using , $\cos 2A = 1 - 2\sin^2 A$

$= \frac{3-1 + 2\sin^2\frac{A}{2} - 2\sin\frac{A}{2}\cos\frac{B-C}{2})}{2}$

$= \frac{2 - 2\sin\frac{A}{2}\{\sin\frac{A}{2} - \cos\left(\frac{B-C}{2}\right)\}}{2}$

Since A + B + C = π

And Using ,

$$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$$

$$= \frac{2 - 2\sin\frac{A}{2}\{2\sin\left(\frac{\frac{B+C}{2}+\frac{B-C}{2}}{2}\right)\sin(\frac{\frac{B+C}{2}-(\frac{B-C}{2})}{2})\}}{2}$$

$$= \frac{2 - 2\sin\frac{A}{2}\{2\sin\left(\frac{\frac{2B}{2}}{2}\right)\sin\left(\frac{\frac{2C}{2}}{2}\right)\}}{2}$$

Using, since A + B + C = π

$$= \frac{2 - 2\sin\frac{A}{2}\{2\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\}}{2}$$

$$= 1 - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

= R.H.S

Q. 12. If A + B + C = π, prove that

tan 2A + tan 2B + tan 2C = tan 2A tan 2B tan 2C

Solution: = tan 2A + tan 2B + tan 2C

Since A + B + C = π

A + B = π – C

2A + 2B = 2π – 2C

Tan (2A+2B) = tan (2π – 2C)

Since tan (2π – C) = -tan C

Tan (2A + 2B) = -tan 2C

Now using formula,

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\frac{\tan 2A + \tan 2B}{1 - \tan 2A \tan 2B} = -\tan 2C$$

Tan 2A + tan 2B = -tan 2C + tan 2C tan 2B tan 2A

Tan 2A + tan 2B + tan 2C = tan 2A tan 2B tan 2C

= R.H.S