Exercise 13B

Question 1: If a cos θ + b sin θ = m and a sin θ – b cos θ = n, prove that $(m^2 + n^2) = (a^2 + b^2)$.

Solution:

a cos θ + b sin θ = m

Squaring equation, we get

$a^2 \cos^2 θ + b^2 \sin^2 θ + 2ab \cos θ \sin θ = m^2$ ……(1)

Again Square equation, a sin θ – b cos θ = n

$a^2 \sin^2 θ + b^2 \cos^2 θ – 2ab \cos θ \sin θ = n^2$ ……(2)

Add (1) and (2)

$a^2 \cos^2 θ + b^2 \sin^2 θ + 2ab \cos θ \sin θ + a^2 \sin^2 θ + b^2 \cos^2 θ – 2ab \cos θ \sin θ = m^2 + n^2$

$a^2 (\cos^2 θ + \sin^2 θ) + b^2 (\cos^2 θ + \sin^2 θ) = m^2 + n^2$

(Using $\cos^2 θ + \sin^2 θ = 1$)

$a^2 + b^2 = m^2 + n^2$

Hence Proved.

Question 2: If x = a sec θ + b tan θ and y = a tan θ + b sec θ, prove that $(x^2 – y^2) = (a^2 – b^2)$.

Solution:

a sec θ + b tan θ = x

a tan θ + b sec θ = y

Squaring above equations:

$a^2 \sec^2 θ + b^2 \tan^2 θ + 2ab \sec θ \tan θ = x^2$ .....(1)

$a^2 \tan^2 θ + b^2 \sec^2 θ + 2ab \sec θ \tan θ = y^2$ ....(2)

Subtract equation (2) from (1):

a^2 (sec^2 θ – tan^2 θ) + b^2 (tan^2 θ – sec^2 θ) = x^2 – y^2

(using sec^2 θ = 1 + tan^2 θ)

or a^2 – b^2 = x^2 – y^2

Hence proved.

Question 3:

If $\left(\frac{x}{a}\sin\theta - \frac{y}{b}\cos\theta\right) = 1$ and $\left(\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta\right) = 1$,

prove that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$

Solution:
x/a sin θ – y/b cos θ = 1
x/a cos θ + y/b sin θ = 1

Squaring both the equations, we have
x^2/a^2 sin^2 θ + y^2/b^2 cos^2 θ – 2 cos θ sin θ = 1 ....(1)

x^2/a^2 cos2 θ + y^2/b^2 sin^2 θ + 2 cos θ sin θ = 1 ......(2)

Add (1) and (2), we get

x^2/a^2(sin^2 θ + cos^2 θ) + y^2/b^2 (sin^2 θ + cos^2 θ) = 1+1

(Using cos^2 θ + sin^2θ = 1)

x^2/a^2 + y^2/b^2 = 2

Question 4: If (sec θ + tan θ) = m and (sec θ − tan θ) = n, show that mn = 1.

Solution:
(sec θ + tan θ) = m ...(1) and

(sec θ − tan θ) = n ....(2)

Multiply (1) and (2), we have

$(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = mn$

$(\sec^2\theta - \tan^2\theta) = mn$

(Because $\sec^2\theta - \tan^2\theta = 1$)
$1 = mn$
Or $mn = 1$
Hence Proved

Question 5: If $(\operatorname{cosec}\theta + \cot\theta) = m$ and $(\operatorname{cosec}\theta - \cot\theta) = n$, show that $mn = 1$.

Solution:
$(\operatorname{cosec}\theta + \cot\theta) = m$ ...(1) and
$(\operatorname{cosec}\theta - \cot\theta) = n$ ...(2)

Multiply (1) and (2)

$(\operatorname{cosec}^2\theta - \cot^2\theta) = mn$

(Because $\operatorname{cosec}^2\theta - \cot^2\theta = 1$)
$1 = mn$
Or $mn = 1$
Hence Proved

Question 6: If $x = a\cos^3\theta$ and $y = b\sin^3\theta$, prove that

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$$

Solution:

$x = a\cos^3\theta$

$y = b\sin^3\theta$

L.H.S.

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3}$$

$$= \left(\frac{a\cos^3\theta}{a}\right)^{2/3} + \left(\frac{b\sin^3\theta}{b}\right)^{2/3}$$

$$= (\cos^3\theta)^{2/3} + (\sin^3\theta)^{2/3}$$

= cos^2 θ + sin^2 θ
= 1
= R.H.S.

Question 7: If (tan θ + sin θ) = m and (tan θ − sin θ) = n, prove that (m^2 − n^2)^2 = 16mn.

Solution:

(tan θ + sin θ) = m and (tan θ − sin θ) = n

To Prove: (m^2 − n^2)^2 = 16mn

L.H.S. = (m^2 − n^2)^2

= [(tan θ + sin θ)^2 - (tan θ − sin θ)^2]^2

= (4tan θ sin θ)^2

= 16 tan^2 θ sin^2 θ ...(1)

R.H.S. = 16mn

= 16(tan θ + sin θ)(tan θ − sin θ)

= 16(tan^2 θ − sin^2 θ)

= 16 [{sin^2 θ (1-cos^2 θ)/cos^2θ]

= 16 x sin^2 θ/cos^2θ x (1-cos^2 θ)

= 16 tan^2 θ sin^2 θ ...(2)
From (1) and (2)
L.H.S. = R.H.S.

Question 8: If (cot θ + tan θ) = m and (sec θ − cos θ) = n, prove that (m^2n)^(2/3) − (mn^2)^(2/3) = 1.

Solution:

(cot θ + tan θ) = m and (sec θ − cos θ) = n

m = 1/tanθ + tan θ = (1+ tan^2 θ)/ tan θ = sec^2 θ / tan θ
= 1/sinθcosθ

or m = 1/sinθcosθ

Again, n = sec θ − cos θ
= 1/cosθ − cos θ
= (1 − cos^2 θ)/cosθ

= sin^2 θ/cos θ

or n = sin^2 θ/cos θ

To prove: (m^2n)^(2/3) − (mn^2)^(2/3) = 1

L.H.S.

(m^2n)^(2/3) − (mn^2)^(2/3)

Substituting the values of m and n, we have

$$= \left[\left(\frac{1}{\sin\theta\cos\theta}\right)^2 \times \frac{\sin^2\theta}{\cos\theta}\right]^{2/3} - \left[\left(\frac{1}{\sin\theta\cos\theta}\right) \times \left(\frac{\sin^2\theta}{\cos\theta}\right)^2\right]^{2/3}$$

$$= \left[\frac{\sin^2\theta}{\sin^2\theta\cos^3\theta}\right]^{2/3} - \left[\frac{\sin^4\theta}{\sin\theta\cos^3\theta}\right]^{2/3}$$

$$= \left[\frac{1}{\cos^3\theta}\right]^{2/3} - \left[\frac{\sin^3\theta}{\cos^3\theta}\right]^{2/3}$$

$$= \frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}$$

= (1 − sin^2 θ)cos^2 θ

(We know, 1 − sin^2 θ = cos^2 θ)

= cos^2 θ/ co^2 θ

= 1

=R.H.S.

Hence proved.

Question 9: If (cosec θ − sin θ) = a^3 and (sec θ − cos θ) = b^3, prove that a^2b^2(a^2 + b^2) = 1.

Solution:
(cosec θ − sin θ) = a^3 and (sec θ − cos θ) = b^3

(cosec θ − sin θ) = a^3

(1/sinθ − sin θ) = a^3

cos^2θ/sinθ = a^3

And a^2 = (a^3)^(2/3) = (cos^2θ/sinθ)^(2/3) .....(1)

Again

(sec θ − cos θ) = b^3

(1/cosθ − cos θ) = b^3

= sin^2 θ/cosθ = b^3

And, b^2 = (b^3)^(2/3) = (sin^2 θ/cosθ)^(2/3)

To Prove:a^2b^2(a^2 + b^2) = 1

L.H.S.

a^2b^2(a^2 + b^2)

$= \left[\left(\frac{\cos^2\theta}{\sin\theta}\right)\right]^{2/3} \times \left[\left(\frac{\sin^2\theta}{\cos\theta}\right)\right]^{\frac{2}{3}} \times \left(\left[\left(\frac{\cos^2\theta}{\sin\theta}\right)\right]^{\frac{2}{3}} + \left[\left(\frac{\sin^2\theta}{\cos\theta}\right)\right]^{\frac{2}{3}}\right)$

$= \left(\frac{\cos^2\theta\sin^2\theta}{\cos\theta\sin\theta}\right)^{2/3} \times \left(\left[\left(\frac{\cos^2\theta}{\sin\theta}\right)\right]^{\frac{2}{3}} + \left[\left(\frac{\sin^2\theta}{\cos\theta}\right)\right]^{\frac{2}{3}}\right)$

$= [\cos^3\theta]^{2/3} + [\sin^3\theta]^{2/3}$

= sin^2 θ + cos^2 θ
= 1
=R.H.S.

Hence proved.

Question 10: If (2 sin θ + 3 cos θ) = 2, show that (3 sin θ − 2 cos θ) = ± 3.

Solution:

(2 sin θ + 3 cos θ) = 2 ...(1)

(2 sin θ + 3 cos θ)^2 + (3 sin θ – 2 cos θ)^2
= 4sin^2 θ + 9 cos^2 θ + 12sin θ cos θ + 9 sin^2 θ + 4 cos^2 θ – 12 sin θ cos θ

= 13sin^2 θ + 13 cos^2 θ

= 13(sin^2 θ + cos^2 θ)

= 13

(Because (sin^2 θ + cos^2 θ) = 1)

=> (2 sin θ + 3 cos θ)^2 + (3 sin θ – 2 cos θ)^2 = 13

Using equation (1)

=> (2)^2 + (3 sin θ – 2 cos θ)^2 = 13
=> (3 sin θ – 2 cos θ)^2 = 9
or (3 sin θ – 2 cos θ) = ± 3

Hence Proved.