6 TRIGONOMETRY

"The deep study of nature is the most fruitful source of mathematical discoveries" - Joseph Fourier.

Francois Viete
(1540–1603 AD(CE))

French mathematician Francois Viete used trigonometry in the study of Algebra for solving certain equations by making suitable trigonometric substitutions. His famous formula for π can be derived with repeated use of trigonometric ratios. One of his famous works titled Canon Mathematicus covers trigonometry; it contains trigonometric tables, it also gives the mathematics behind the construction of the tables, and it details how to solve both plane and spherical triangles. He also provided the means for extracting roots and solutions of equations of degree atmost six. Viete introduced the term "coefficient" in mathematics. He provided a simple formula relating the roots of a equation with its coefficients. He also provided geometric methods to solve doubling the cube and trisecting the angle problems. He was also involved in deciphering codes.

Learning Outcomes

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- To recall trigonometric ratios.
- To recall fundamental relations between the trigonometric ratios of an angle.
- To recall trigonometric ratios of complementary angles.
- To understand trigonometric identities.
- To know methods of solving problems concerning heights and distances of various objects.

6.1 Introduction

From very ancient times surveyors, navigators and astronomers have made use of triangles to determine distances that could not be measured directly. This gave birth to the branch of mathematics what we call today as "Trigonometry".

Hipparchus of Rhodes around 200 BC(BCE), constructed a table of chord lengths for a circle of circumference $360 \times 60 = 21600$ units which corresponds to one unit of circumference for each minute of arc. For this achievement, Hipparchus is considered as "The Father of Trigonometry" since it became the basis for further development.

Indian scholars of the 5th century AD(CE), realized that working with half-chords for half-angles greatly simplified the theory of chords and its application to astronomy. Mathematicians like Aryabhata, the two Bhaskaras and several others developed astonishingly sophisticated techniques for calculating half-chord (Jya) values.

Mathematician Abu Al-Wafa of Baghdad believed to have invented the tangent function, which he called the "Shadow". Arabic scholars did not know how to translate the word Jya, into their texts and simply wrote jiba as a close approximate word.

Misinterpreting the Arabic word 'jiba' for 'cove' or 'bay', translators wrote the Arabic word 'jiba' as 'sinus' in Latin to represent the half-chord. From this, we have the name 'sine' used to this day. The word "Trigonometry" itself was invented by German mathematician Bartholomaeus Pitiscus in the beginning of 17th century AD(CE).

Recall

Trigonometric Ratios

Let $0° < \theta < 90°$

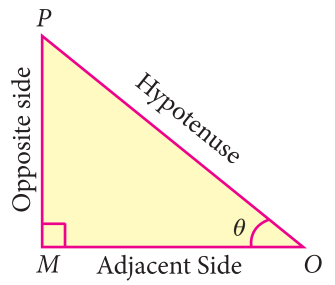


Fig. 6.1

Let us take right triangle OMP

$$\sin\theta = \frac{\text{opposite side}}{\text{Hypotenuse}} = \frac{MP}{OP}$$

$$\cos\theta = \frac{Adjacent\ side}{Hypotenuse} = \frac{OM}{OP}$$

From the above two ratios we can obtain other four trigonometric ratios as follows.

$$\tan\theta = \frac{\sin\theta}{\cos\theta};\quad \cot\theta = \frac{\cos\theta}{\sin\theta};$$

$$\text{cosec}\theta = \frac{1}{\sin\theta};\quad \sec\theta = \frac{1}{\cos\theta}$$

Note

All right triangles with θ as one of the angle are similar. Hence the trigonometric ratios defined through such right angle triangles do not depend on the triangle chosen.

Table of Trigonometric Ratios for $0°, 30°, 45°, 60°, 90°$

Trigonometric Ratio \ θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined
$\text{cosec}\theta$	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined
$\cot\theta$	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Complementary angle

$\sin(90° - \theta) = \cos\theta$	$\cos(90° - \theta) = \sin\theta$	$\tan(90° - \theta) = \cot\theta$
$\text{cosec}(90° - \theta) = \sec\theta$	$\sec(90° - \theta) = \text{cosec}\,\theta$	$\cot(90° - \theta) = \tan\theta$

Visual proof of Trigonometric complementary angle

Consider a semicircle of radius 1 as shown in the figure.

Let $\angle QOP = \theta$.

Then $\angle QOR = 90° - \theta$, so that $OPQR$ forms a rectangle.

From triangle OPQ, $\frac{OP}{OQ} = \cos\theta$

But $OQ = \text{radius} = 1$

Therefore $OP = OQ \cos\theta = \cos\theta$

Similarly, $\frac{PQ}{OQ} = \sin\theta$

Gives, $PQ = OQ\sin\theta = \sin\theta$ (since $OQ = 1$)

$OP = \cos\theta$, $PQ = \sin\theta$ ... (1)

Now, from triangle QOR,

we have $\frac{OR}{OQ} = \cos(90° - \theta)$

Therefore, $OR = OQ \cos(90° - \theta)$

So, $OR = \cos(90° - \theta)$

Similarly, $\frac{RQ}{OQ} = \sin(90° - \theta)$

Then, $RQ = \sin(90° - \theta)$

$OR = \cos(90° - \theta)$, $RQ = \sin(90° - \theta)$ ... (2)

Since $OPQR$ is a rectangle,

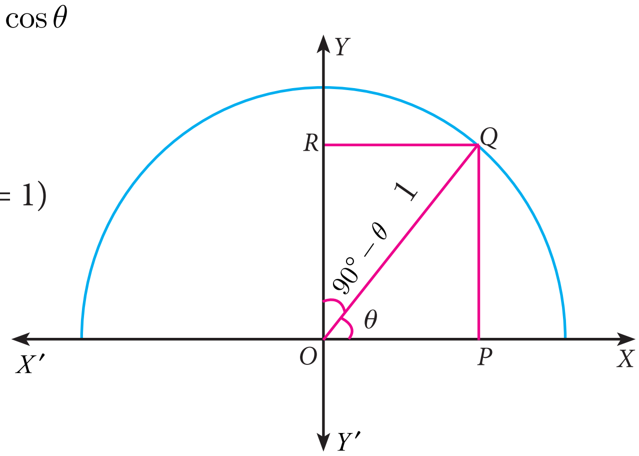


Fig. 6.2

$OP = RQ$ and $OR = PQ$

Therefore from (1) and (2)

we get,

$\sin(90° - \theta) = \cos\theta$ and $\cos(90° - \theta) = \sin\theta$

Note

$(\sin\theta)^2 = \sin^2\theta$	$(\text{cosec}\theta)^2 = \text{cosec}^2\theta$
$(\cos\theta)^2 = \cos^2\theta$	$(\sec\theta)^2 = \sec^2\theta$
$(\tan\theta)^2 = \tan^2\theta$	$(\cot\theta)^2 = \cot^2\theta$

Thinking Corner

1. When will the values of $\sin\theta$ and $\cos\theta$ be equal?
2. For what values of θ, $\sin\theta = 2$?
3. Among the six trigonometric quantities, as the value of angle θincrease from $0°$ to $90°$, which of the six trigonometric quantities has undefined values?
4. Is it possible to have eight trigonometric ratios?
5. Let $0° \le \theta \le 90°$. For what values of θ does
   (i) $\sin\theta > \cos\theta$ (ii) $\cos\theta > \sin\theta$ (iii) $\sec\theta = 2\tan\theta$ (iv) $\text{cosec}\,\theta = 2\cot\theta$

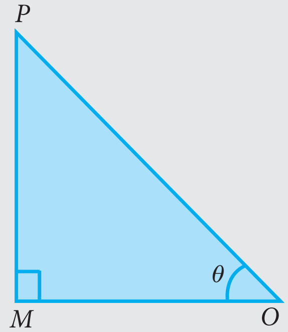
6.2 Trigonometric identities

For all real values of θ, we have the following three identities.

(i) $\sin^2\theta + \cos^2\theta = 1$ (ii) $1 + \tan^2\theta = \sec^2\theta$ (iii) $1 + \cot^2\theta = \text{cosec}^2\theta$

These identities are termed as three fundamental identities of trigonometry.

We will now prove them as follows.

Picture	identity	Proof
 Fig. 6.3	$\sin^2\theta + \cos^2\theta = 1$	In right triangle OMP, we have $\frac{OM}{OP} = \cos\theta,\quad \frac{PM}{OP} = \sin\theta$ ...(1) By Pythagoras theorem $MP^2 + OM^2 = OP^2$ ...(2) Dividing each term on both sides of (2) by OP^2, (since $OP \neq 0$) we get, $\frac{MP^2}{OP^2} + \frac{OM^2}{OP^2} = \frac{OP^2}{OP^2}$ Gives, $\left(\frac{MP}{OP}\right)^2 + \left(\frac{OM}{OP}\right)^2 = \left(\frac{OP}{OP}\right)^2$ From (1), $(\sin\theta)^2 + (\cos\theta)^2 = 1^2$ Hence $\sin^2\theta + \cos^2\theta = 1$

	$1+\tan^2\theta=\sec^2\theta$	In right triangle OMP, we have $\frac{MP}{OM}=\tan\theta$, $\frac{OP}{OM}=\sec\theta$ ...(3) From (2), $MP^2+OM^2=OP^2$ Dividing each term on both sides of (2) by OM^2, (since $OM\neq 0$) we get, $\frac{MP^2}{OM^2}+\frac{OM^2}{OM^2}=\frac{OP^2}{OM^2}$ Gives, $\left(\frac{MP}{OM}\right)^2+\left(\frac{OM}{OM}\right)^2=\left(\frac{OP}{OM}\right)^2$ From (3), $(\tan\theta)^2+1^2=(\sec\theta)^2$ Hence $1+\tan^2\theta=\sec^2\theta$.
	$1+\cot^2\theta=\text{cosec}^2\theta$	In right triangle OMP, we have $\frac{OM}{MP}=\cot\theta$, $\frac{OP}{MP}=\text{cosec}\,\theta$ ...(4) From (2), $MP^2+OM^2=OP^2$ Dividing each term on both sides of (2) by MP^2, (since $MP\neq 0$) we get, $\frac{MP^2}{MP^2}+\frac{OM^2}{MP^2}=\frac{OP^2}{MP^2}$ Gives, $\left(\frac{MP}{MP}\right)^2+\left(\frac{OM}{MP}\right)^2=\left(\frac{OP}{MP}\right)^2$ From (4), $1^2+(\cot\theta)^2=(\text{cosec}\theta)^2$ Hence, $1+\cot^2\theta=\text{cosec}^2\theta$

These identities can also be rewritten as follows.

Identity	Equal forms
$\sin^2\theta+\cos^2\theta=1$	$\sin^2\theta=1-\cos^2\theta$ (or) $\cos^2\theta=1-\sin^2\theta$
$1+\tan^2\theta=\sec^2\theta$	$\tan^2\theta=\sec^2\theta-1$ (or) $\sec^2\theta-\tan^2\theta=1$
$1+\cot^2\theta=\text{cosec}^2\theta$	$\cot^2\theta=\text{cosec}^2\theta-1$ (or) $\text{cosec}^2\theta-\cot^2\theta=1$

Note

Though the above identities are true for any angle θ, we will consider the six trigonometric ratios only for $0°<\theta<90°$

Activity 1

Take a white sheet of paper. Construct two perpendicular lines OX, OY which meet at O, as shown in the Fig. 6.4(a).

Considering OX as X-axis OY as Y-axis.

We will verify the values of $\sin\theta$ and $\cos\theta$ for certain angles θ.

Let $\theta = 30°$

Construct a line segment OA of any length such that $\angle AOX = 30°$, as shown in the Fig. 6.4(b).

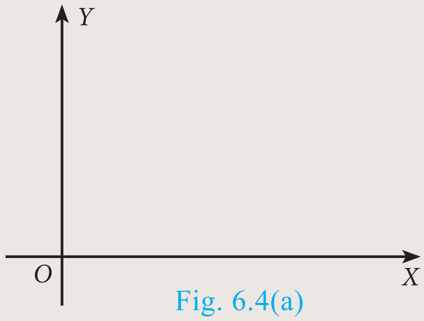


Fig. 6.4(a)

Draw a perpendicular from A to OX, meeting at B.

Now using scale, measure the lengths of AB, OB and OA.

Find the ratios $\frac{AB}{OA}$, $\frac{OB}{OA}$ and $\frac{AB}{OB}$.

What do you get? Can you compare these values with the trigonometric table values? What is your conclusion? Carry out the same procedure for $\theta = 45°$ and $\theta = 60°$. What are your conclusions?

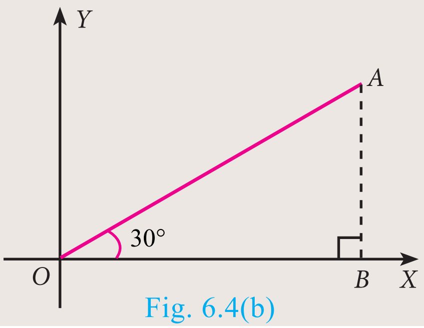


Fig. 6.4(b)

Example 6.1 Prove that $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$

Solution $\tan^2\theta - \sin^2\theta = \tan^2\theta - \frac{\sin^2\theta}{\cos^2\theta}.\cos^2\theta$

$$= \tan^2\theta(1-\cos^2\theta) = \tan^2\theta\sin^2\theta$$

Example 6.2 Prove that $\frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A}$

Solution $\frac{\sin A}{1+\cos A} = \frac{\sin A}{1+\cos A}\times\frac{1-\cos A}{1-\cos A}$ [multiply numerator and denominator by the conjugate of $1+\cos A$]

$$= \frac{\sin A(1-\cos A)}{(1+\cos A)(1-\cos A)} = \frac{\sin A(1-\cos A)}{1-\cos^2 A}$$

$$= \frac{\sin A(1-\cos A)}{\sin^2 A} = \frac{1-\cos A}{\sin A}$$

Example 6.3 Prove that $1+\frac{\cot^2\theta}{1+\operatorname{cosec}\theta} = \operatorname{cosec}\theta$

Solution $1+\frac{\cot^2\theta}{1+\operatorname{cosec}\theta} = 1+\frac{\operatorname{cosec}^2\theta-1}{\operatorname{cosec}\theta+1}$ $\left[\text{since } \operatorname{cosec}^2\theta - 1 = \cot^2\theta\right]$

$$= 1+\frac{(\operatorname{cosec}\theta+1)(\operatorname{cosec}\theta-1)}{\operatorname{cosec}\theta+1}$$

$$= 1+(\operatorname{cosec}\theta-1) = \operatorname{cosec}\theta$$

Example 6.4 Prove that $\sec\theta - \cos\theta = \tan\theta\sin\theta$

Solution
$$\sec\theta - \cos\theta = \frac{1}{\cos\theta} - \cos\theta = \frac{1-\cos^2\theta}{\cos\theta}$$
$$= \frac{\sin^2\theta}{\cos\theta} \qquad \left[\text{since } 1-\cos^2\theta = \sin^2\theta\right]$$
$$= \frac{\sin\theta}{\cos\theta}\times\sin\theta = \tan\theta\sin\theta$$

Example 6.5 Prove that $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}} = \operatorname{cosec}\theta + \cot\theta$

Solution
$$\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta}\times\frac{1+\cos\theta}{1+\cos\theta}}$$
[multiply numerator and denominator by the conjugate of $1-\cos\theta$]
$$= \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}} = \frac{1+\cos\theta}{\sqrt{\sin^2\theta}} \qquad \left[\text{since } \sin^2\theta+\cos^2\theta = 1\right]$$
$$= \frac{1+\cos\theta}{\sin\theta} = \operatorname{cosec}\theta+\cot\theta$$

Example 6.6 Prove that $\dfrac{\sec\theta}{\sin\theta} - \dfrac{\sin\theta}{\cos\theta} = \cot\theta$

Solution
$$\frac{\sec\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{\cos\theta}}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = \frac{1}{\sin\theta\cos\theta} - \frac{\sin\theta}{\cos\theta}$$
$$= \frac{1-\sin^2\theta}{\sin\theta\cos\theta} = \frac{\cos^2\theta}{\sin\theta\cos\theta} = \cot\theta$$

Example 6.7 Prove that $\sin^2 A\cos^2 B + \cos^2 A\sin^2 B + \cos^2 A\cos^2 B + \sin^2 A\sin^2 B = 1$

Solution $\sin^2 A\cos^2 B + \cos^2 A\sin^2 B + \cos^2 A\cos^2 B + \sin^2 A\sin^2 B$
$$= \sin^2 A\cos^2 B + \sin^2 A\sin^2 B + \cos^2 A\sin^2 B + \cos^2 A\cos^2 B$$
$$= \sin^2 A(\cos^2 B + \sin^2 B) + \cos^2 A(\sin^2 B + \cos^2 B)$$
$$= \sin^2 A(1) + \cos^2 A(1) \qquad (\text{since } \sin^2 B + \cos^2 B = 1)$$
$$= \sin^2 A + \cos^2 A = 1$$

Example 6.8 If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, then prove that $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$

Solution Now, $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$.

Squaring both sides,
$$(\cos\theta + \sin\theta)^2 = (\sqrt{2}\cos\theta)^2$$
$$\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta = 2\cos^2\theta$$

$$2\cos^2\theta - \cos^2\theta - \sin^2\theta = 2\sin\theta\cos\theta$$
$$\cos^2\theta - \sin^2\theta = 2\sin\theta\cos\theta$$
$$(\cos\theta + \sin\theta)(\cos\theta - \sin\theta) = 2\sin\theta\cos\theta$$
$$\cos\theta - \sin\theta = \frac{2\sin\theta\cos\theta}{\cos\theta + \sin\theta} = \frac{2\sin\theta\cos\theta}{\sqrt{2}\cos\theta} \quad [\text{since } \cos\theta + \sin\theta = \sqrt{2}\cos\theta]$$
$$= \sqrt{2}\sin\theta$$

Therefore $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$.

Example 6.9 Prove that $(\text{cosec}\,\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta) = 1$

Solution $(\text{cosec}\,\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta)$

$$= \left(\frac{1}{\sin\theta} - \sin\theta\right)\left(\frac{1}{\cos\theta} - \cos\theta\right)\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$$
$$= \frac{1-\sin^2\theta}{\sin\theta} \times \frac{1-\cos^2\theta}{\cos\theta} \times \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$$
$$= \frac{\cos^2\theta\sin^2\theta \times 1}{\sin^2\theta\cos^2\theta} = 1$$

Example 6.10 Prove that $\frac{\sin A}{1+\cos A} + \frac{\sin A}{1-\cos A} = 2\text{cosec}A$.

Solution $\frac{\sin A}{1+\cos A} + \frac{\sin A}{1-\cos A}$

$$= \frac{\sin A(1-\cos A) + \sin A(1+\cos A)}{(1+\cos A)(1-\cos A)}$$
$$= \frac{\sin A - \sin A\cos A + \sin A + \sin A\cos A}{1-\cos^2 A}$$
$$= \frac{2\sin A}{1-\cos^2 A} = \frac{2\sin A}{\sin^2 A} = 2\,\text{cosec A}$$

Example 6.11 If $\text{cosec}\,\theta + \cot\theta = P$, then prove that $\cos\theta = \frac{P^2-1}{P^2+1}$

Solution Given $\text{cosec}\,\theta + \cot\theta = P$ ...(1)

$$\text{cosec}^2\theta - \cot^2\theta = 1 \text{ (identity)}$$
$$\text{cosec}\,\theta - \cot\theta = \frac{1}{\text{cosec}\,\theta + \cot\theta}$$
$$\text{cosec}\,\theta - \cot\theta = \frac{1}{P} \qquad ...(2)$$

Adding (1) and (2) we get, $2\text{cosec}\,\theta = P + \frac{1}{P}$

$$2\text{cosec}\,\theta = \frac{P^2+1}{P} \qquad ...(3)$$

Subtracting (2) from (1), we get, $2\cot\theta = P - \frac{1}{P}$

$$2\cot\theta = \frac{P^2-1}{P} \qquad ...(4)$$

Dividing (4) by (3) we get, $\frac{2\cot\theta}{2\,\text{cosec}\,\theta} = \frac{P^2-1}{P} \times \frac{P}{P^2+1}$ gives, $\cos\theta = \frac{P^2-1}{P^2+1}$

Example 6.12 Prove that $\tan^2 A - \tan^2 B = \frac{\sin^2 A - \sin^2 B}{\cos^2 A\cos^2 B}$

Solution $\tan^2 A - \tan^2 B = \frac{\sin^2 A}{\cos^2 A} - \frac{\sin^2 B}{\cos^2 B}$

$$= \frac{\sin^2 A\cos^2 B - \sin^2 B\cos^2 A}{\cos^2 A\cos^2 B}$$

$$= \frac{\sin^2 A(1-\sin^2 B) - \sin^2 B(1-\sin^2 A)}{\cos^2 A\ \cos^2 B}$$

$$= \frac{\sin^2 A - \sin^2 A\sin^2 B - \sin^2 B + \sin^2 A\sin^2 B}{\cos^2 A\cos^2 B} = \frac{\sin^2 A - \sin^2 B}{\cos^2 A\cos^2 B}$$

Example 6.13 Prove that $\left(\frac{\cos^3 A - \sin^3 A}{\cos A - \sin A}\right) - \left(\frac{\cos^3 A + \sin^3 A}{\cos A + \sin A}\right) = 2\sin A\cos A$

Solution $\left(\frac{\cos^3 A - \sin^3 A}{\cos A - \sin A}\right) - \left(\frac{\cos^3 A + \sin^3 A}{\cos A + \sin A}\right)$

$$= \left(\frac{(\cos A - \sin A)(\cos^2 A + \sin^2 A + \cos A\sin A)}{\cos A - \sin A}\right) \quad \left[\begin{array}{l}\text{since } a^3 - b^3 = (a-b)(a^2+b^2+ab) \\ \qquad\quad a^3 + b^3 = (a+b)(a^2+b^2-ab)\end{array}\right]$$

$$- \left(\frac{(\cos A + \sin A)(\cos^2 A + \sin^2 A - \cos A\sin A)}{\cos A + \sin A}\right)$$

$$= (1 + \cos A\sin A) - (1 - \cos A\sin A)$$

$$= 2\ \cos A\sin A$$

Example 6.14 Prove that $\frac{\sin A}{\sec A + \tan A - 1} + \frac{\cos A}{\text{cosecA+cot}\,A - 1} = 1$

Solution $\frac{\sin A}{\sec A + \tan A - 1} + \frac{\cos A}{\text{cosecA+cot}\,A - 1}$

$$= \frac{\sin A(\cos\text{ec}A + \cot A - 1) + \cos A(\sec A + \tan A - 1)}{(\sec A + \tan A - 1)(\cos\text{ec}A + \cot A - 1)}$$

$$= \frac{\sin A\cos\text{ec}A + \sin A\cot A - \sin A + \cos A\sec A + \cos A\tan A - \cos A}{(\sec A + \tan A - 1)(\cos\text{ec}A + \cot A - 1)}$$

$$= \frac{1 + \cos A - \sin A + 1 + \sin A - \cos A}{\left(\frac{1}{\cos A} + \frac{\sin A}{\cos A} - 1\right)\left(\frac{1}{\sin A} + \frac{\cos A}{\sin A} - 1\right)}$$

$$= \frac{2}{\left(\frac{1+\sin A-\cos A}{\cos A}\right)\left(\frac{1+\cos A-\sin A}{\sin A}\right)}$$

$$= \frac{2\sin A\cos A}{(1+\sin A-\cos A)(1+\cos A-\sin A)}$$

$$= \frac{2\sin A\cos A}{[1+(\sin A-\cos A)][1-(\sin A-\cos A)]} = \frac{2\sin A\cos A}{1-(\sin A-\cos A)^2}$$

$$= \frac{2\sin A\cos A}{1-(\sin^2 A+\cos^2 A-2\sin A\cos A)} \qquad = \frac{2\sin A\cos A}{1-(1-2\sin A\cos A)}$$

$$= \frac{2\sin A\cos A}{1-1+2\sin A\cos A} = \frac{2\sin A\cos A}{2\sin A\cos A} = 1.$$

Example 6.15 Show that $\left(\frac{1+\tan^2 A}{1+\cot^2 A}\right) = \left(\frac{1-\tan A}{1-\cot A}\right)^2$

Solution

LHS

$$\left(\frac{1+\tan^2 A}{1+\cot^2 A}\right) = \frac{1+\tan^2 A}{1+\frac{1}{\tan^2 A}}$$

$$= \frac{1+\tan^2 A}{\frac{\tan^2 A+1}{\tan^2 A}} = \tan^2 A \quad \dots (1)$$

RHS

$$\left(\frac{1-\tan A}{1-\cot A}\right)^2 = \left(\frac{1-\tan A}{1-\frac{1}{\tan A}}\right)^2$$

$$= \left(\frac{1-\tan A}{\frac{\tan A-1}{\tan A}}\right)^2 = (-\tan A)^2 = \tan^2 A \dots (2)$$

From (1) and (2), $\left(\frac{1+\tan^2 A}{1+\cot^2 A}\right) = \left(\frac{1-\tan A}{1-\cot A}\right)^2$

Example 6.16 Prove that $\frac{(1+\cot A+\tan A)(\sin A-\cos A)}{\sec^3 A-\cos ec^3 A} = \sin^2 A\cos^2 A$

Solution $\frac{(1+\cot A+\tan A)(\sin A-\cos A)}{\sec^3 A-\cos ec^3 A}$

$$= \frac{\left(1+\frac{\cos A}{\sin A}+\frac{\sin A}{\cos A}\right)(\sin A-\cos A)}{(\sec A-\cos\mathrm{ec} A)(\sec^2 A+\sec A\cos\mathrm{ec} A+\cos ec^2 A)}$$

$$= \frac{\frac{(\sin A\cos A+\cos^2 A+\sin^2 A)(\sin A-\cos A)}{\sin A\cos A}}{(\sec A-\mathrm{cosec} A)\left(\frac{1}{\cos^2 A}+\frac{1}{\cos A\sin A}+\frac{1}{\sin^2 A}\right)}$$

$$= \frac{(\sin A\cos A + 1)\left(\frac{\sin A}{\sin A\cos A} - \frac{\cos A}{\sin A\cos A}\right)}{(\sec A - \text{cosec}A)\left(\frac{\sin^2 A + \sin A\cos A + \cos^2 A}{\sin^2 A\cos^2 A}\right)}$$

$$= \frac{(\sin A\cos A + 1)(\sec A - \cos ecA)}{(\sec A - \cos ecA)(1 + \sin A\cos A)} \times \sin^2 A\cos^2 A = \sin^2 A\cos^2 A$$

Example 6.17 If $\frac{\cos^2\theta}{\sin\theta} = p$ and $\frac{\sin^2\theta}{\cos\theta} = q$, then prove that $p^2q^2(p^2 + q^2 + 3) = 1$

Solution We have $\frac{\cos^2\theta}{\sin\theta} = p$ ...(1) and $\frac{\sin^2\theta}{\cos\theta} = q$ ...(2)

$$p^2q^2(p^2 + q^2 + 3) = \left(\frac{\cos^2\theta}{\sin\theta}\right)^2\left(\frac{\sin^2\theta}{\cos\theta}\right)^2 \times \left[\left(\frac{\cos^2\theta}{\sin\theta}\right)^2 + \left(\frac{\sin^2\theta}{\cos\theta}\right)^2 + 3\right] \quad \text{[from (1) and (2)]}$$

$$= \left(\frac{\cos^4\theta}{\sin^2\theta}\right)\left(\frac{\sin^4\theta}{\cos^2\theta}\right) \times \left[\frac{\cos^4\theta}{\sin^2\theta} + \frac{\sin^4\theta}{\cos^2\theta} + 3\right]$$

$$= (\cos^2\theta \times \sin^2\theta) \times \left[\left(\frac{\cos^6\theta + \sin^6\theta + 3\sin^2\theta\cos^2\theta}{\sin^2\theta\cos^2\theta}\right)\right]$$

$$= \cos^6\theta + \sin^6\theta + 3\sin^2\theta\cos^2\theta$$

$$= (\cos^2\theta)^3 + (\sin^2\theta)^3 + 3\sin^2\theta\cos^2\theta$$

$$= [(\cos^2\theta + \sin^2\theta)^3 - 3\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)] + 3\sin^2\theta\cos^2\theta$$

$$= 1 - 3\cos^2\theta\sin^2\theta(1) + 3\cos^2\theta\sin^2\theta = 1$$

Progress Check

1. The number of trigonometric ratios is ______.
2. $1 - \cos^2\theta$ is ______.
3. $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta)$ is ______.
4. $(\cot\theta + \text{cosec}\theta)(\cot\theta - \text{cosec}\theta)$ is ______.
5. $\cos 60^\circ \sin 30^\circ + \cos 30^\circ \sin 60^\circ$ is ______.
6. $\tan 60^\circ \cos 60^\circ + \cot 60^\circ \sin 60^\circ$ is ______.
7. $(\tan 45^\circ + \cot 45^\circ) + (\sec 45^\circ\ \text{cosec}45^\circ)$ is ______.
8. (i) $\sec\theta = \text{cosec}\theta$ if θ is ______. (ii) $\cot\theta = \tan\theta$ if θ is _____.

Exercise 6.1

1. Prove the following identities.
   (i) $\cot\theta + \tan\theta = \sec\theta\,\text{cosec}\theta$ (ii) $\tan^4\theta + \tan^2\theta = \sec^4\theta - \sec^2\theta$
2. Prove the following identities.
   (i) $\frac{1 - \tan^2\theta}{\cot^2\theta - 1} = \tan^2\theta$ (ii) $\frac{\cos\theta}{1 + \sin\theta} = \sec\theta - \tan\theta$

3. Prove the following identities.

(i) $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} = \sec\theta + \tan\theta$ (ii) $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} + \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = 2\sec\theta$

4. Prove the following identities.

(i) $\sec^6\theta = \tan^6\theta + 3\tan^2\theta\sec^2\theta + 1$

(ii) $(\sin\theta + \sec\theta)^2 + (\cos\theta + \operatorname{cosec}\theta)^2 = 1 + (\sec\theta + \operatorname{cosec}\theta)^2$

5. Prove the following identities.

(i) $\sec^4\theta(1-\sin^4\theta) - 2\tan^2\theta = 1$ (ii) $\dfrac{\cot\theta - \cos\theta}{\cot\theta + \cos\theta} = \dfrac{\operatorname{cosec}\theta - 1}{\operatorname{cosec} + 1}$

6. Prove the following identities.

(i) $\dfrac{\sin A - \sin B}{\cos A + \cos B} + \dfrac{\cos A - \cos B}{\sin A + \sin B} = 0$ (ii) $\dfrac{\sin^3 A + \cos^3 A}{\sin A + \cos A} + \dfrac{\sin^3 A - \cos^3 A}{\sin A - \cos A} = 2$

7. (i) If $\sin\theta + \cos\theta = \sqrt{3}$, then prove that $\tan\theta + \cot\theta = 1$.

(ii) If $\sqrt{3}\sin\theta - \cos\theta = 0$, then show that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

8. (i) If $\dfrac{\cos\alpha}{\cos\beta} = m$ and $\dfrac{\cos\alpha}{\sin\beta} = n$, then prove that $(m^2 + n^2)\cos^2\beta = n^2$

(ii) If $\cot\theta + \tan\theta = x$ and $\sec\theta - \cos\theta = y$, then prove that $(x^2y)^{\frac{2}{3}} - (xy^2)^{\frac{2}{3}} = 1$

9. (i) If $\sin\theta + \cos\theta = p$ and $\sec\theta + \operatorname{cosec}\theta = q$, then prove that $q(p^2 - 1) = 2p$

(ii) If $\sin\theta(1 + \sin^2\theta) = \cos^2\theta$, then prove that $\cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4$

10. If $\dfrac{\cos\theta}{1+\sin\theta} = \dfrac{1}{a}$, then prove that $\dfrac{a^2 - 1}{a^2 + 1} = \sin\theta$

6.3 Heights and Distances

In this section, we will see how trigonometry is used for finding the heights and distances of various objects without actually measuring them. For example, the height of a tower, mountain, building or tree, distance of a ship from a light house, width of a river, etc. can be determined by using knowledge of trigonometry. The process of finding Heights and Distances is the best example of applying trigonometry in real-life situations. We would explain these applications through some examples. Before studying methods to find heights and distances, we should understand some basic definitions.

Line of Sight

The line of sight is the line drawn from the eye of an observer to the point in the object viewed by the observer.

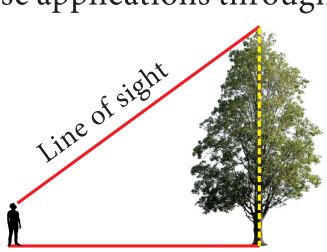


Fig. 6.5

Theodolite

Theodolite is an instrument which is used in measuring the angle between an object and the eye of the observer. A theodolite consists of two graduated wheels placed at right angles to each other and a telescope. The wheels are used for the measurement of horizontal and vertical angles. The angle to the desired point is measured by positioning the telescope towards that point. The angle can be read on the telescope scale.

Fig. 6.6

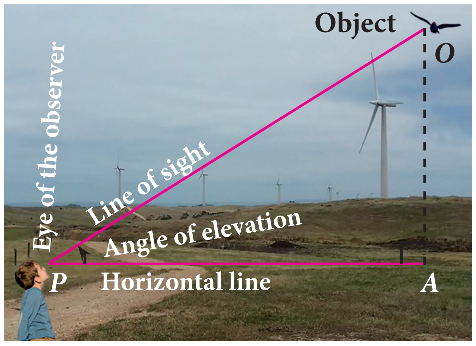


Fig. 6.7

Angle of Elevation

The angle of elevation is an angle formed by the line of sight with the horizontal when the point being viewed is above the horizontal level. That is, the case when we raise our head to look at the object. (see Fig. 6.7)

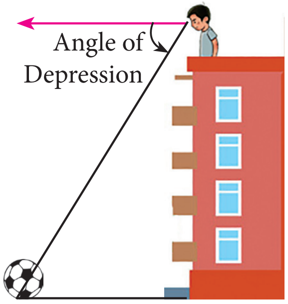


Fig. 6.8

Angle of Depression

The angle of depression is an angle formed by the line of sight with the horizontal when the point is below the horizontal level. That is, the case when we lower our head to look at the point being viewed. (see Fig. 6.8)

Clinometer

The angle of elevation and depression are usually measured by a device called clinometer.

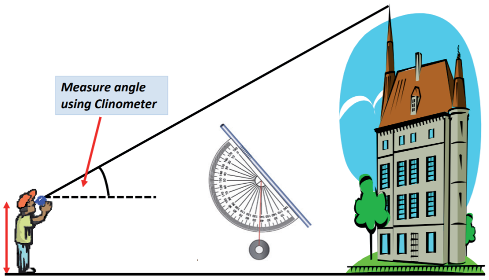


Fig. 6.9

Note

(i) From a given point, when height of a object increases the angle of elevation increases.

If $h_1 > h_2$ then $\alpha > \beta$

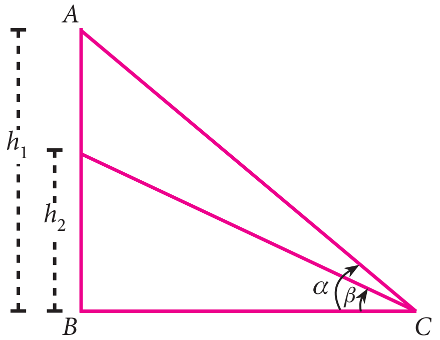


Fig. 6.10(a)

(ii) The angle of elevation increases as we move towards the foot of the vertical object like tower or building.

If $d_2 < d_1$ then $\beta > \alpha$

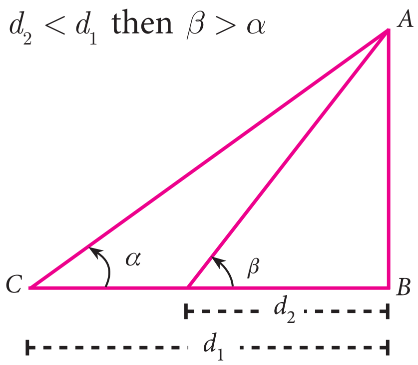
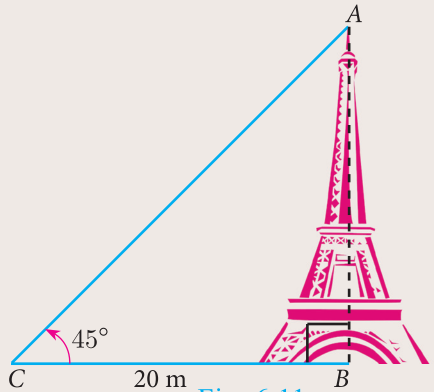


Fig. 6.10(b)

Activity 2

Representation of situations through right triangles. Draw a figure to illustrate the situation.

Situations	Draw a figure
A tower stands vertically on the ground. From a point on the ground, which is 20m away from the foot of the tower, the angle of elevation of the top of the tower is found to be 45°.	 Fig. 6.11
An observer 2.8 m tall is 25.2 m away from a chimney. The angle of elevation of the top of the chimney from her eyes is 45°.	………………………………..
From a point P on the ground the angle of elevation of the top of a 20 m tall building is 30°. A flag is hoisted at the top of the building and the angle of elevation of the top of the flagstaff from P is 55° .	………………………………..
The shadow of a tower standing on a level ground is found to be 40 m longer when the Sun's altitude is 30° than when it is 60°.	…………………………………

6.3.1 Problems involving Angle of Elevation

In this section, we try to solve problems when Angle of elevation are given.

Example 6.18

Calculate the size of $\angle BAC$ in the given triangles.

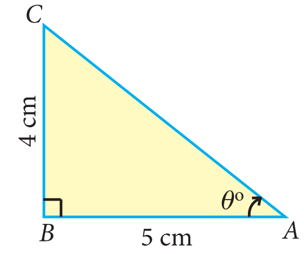


Fig. 6.12(a)

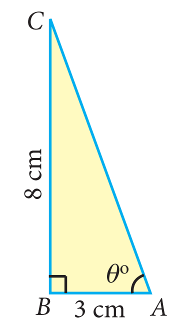


Fig. 6.12(b)

Solution

(i) In right triangle ABC [see Fig.4.12(a)]

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{4}{5}$$

$$\theta = \tan^{-1}\left(\frac{4}{5}\right) = \tan^{-1}(0.8)$$

$$\theta = 38.7° \text{ (since } \tan\ 38.7° = 0.8011)$$

$$\angle BAC = 38.7°$$

(ii) In right triangle ABC [see Fig.4.12(b)]

$$\tan\theta = \frac{8}{3}$$

$$\theta = \tan^{-1}\left(\frac{8}{3}\right) = \tan^{-1}(2.66)$$

$$\theta = 69.4° \text{ (since } \tan\ 69.4° = 2.6604)$$

$$\angle BAC = 69.4°$$

Example 6.19 A tower stands vertically on the ground. From a point on the ground, which is 48 m away from the foot of the tower, the angle of elevation of the top of the tower is 30°. Find the height of the tower.

Solution Let PQ be the height of the tower.

Take $PQ = h$ and QR is the distance between the tower and the point R. In right triangle PQR, $\angle PRQ = 30^\circ$

$$\tan\theta = \frac{PQ}{QR}$$

$$\tan 30^\circ = \frac{h}{48} \quad \text{gives,} \quad \frac{1}{\sqrt{3}} = \frac{h}{48} \quad \text{so,} \quad h = 16\sqrt{3}$$

Therefore the height of the tower is $16\sqrt{3}$ m

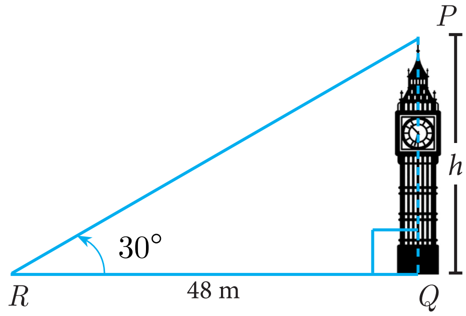


Fig. 6.13

Example 6.20 A kite is flying at a height of 75 m above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is 60°. Find the length of the string, assuming that there is no slack in the string.

Solution Let AB be the height of the kite above the ground. Then, $AB = 75$.

Let AC be the length of the string.

In right triangle ABC, $\angle ACB = 60^\circ$

$$\sin\theta = \frac{AB}{AC}$$

$$\sin 60^\circ = \frac{75}{AC}$$

$$\text{gives} \quad \frac{\sqrt{3}}{2} = \frac{75}{AC} \quad \text{so,} \quad AC = \frac{150}{\sqrt{3}} = 50\sqrt{3}$$

Hence, the length of the string is $50\sqrt{3}$ m.

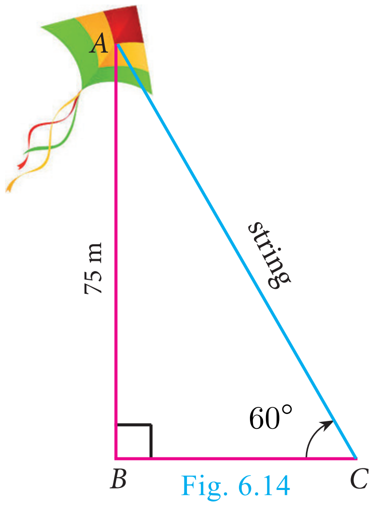


Fig. 6.14

Example 6.21 Two ships are sailing in the sea on either sides of a lighthouse. The angle of elevation of the top of the lighthouse as observed from the ships are 30° and 45° respectively. If the lighthouse is 200 m high, find the distance between the two ships. ($\sqrt{3} = 1.732$)

Solution Let AB be the lighthouse and C and D the positions of the two ships.

Then, $AB = 200$ m.

$\angle ACB = 30^\circ$, $\angle ADB = 45^\circ$

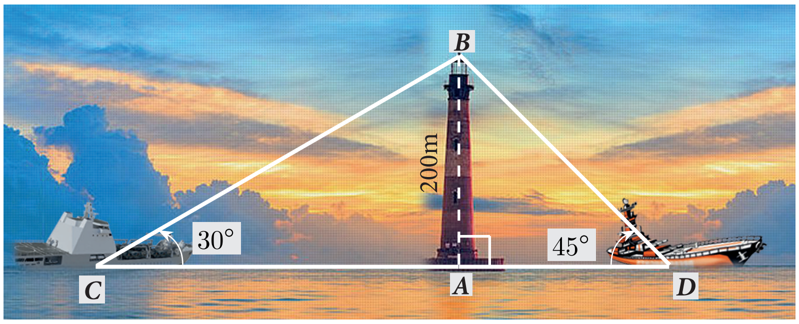


Fig. 6.15

In right triangle BAC, $\tan 30° = \frac{AB}{AC}$

$$\frac{1}{\sqrt{3}} = \frac{200}{AC} \quad \text{gives } AC = 200\sqrt{3} \qquad ...(1)$$

In right triangle BAD, $\tan 45° = \frac{AB}{AD}$

$$1 = \frac{200}{AD} \quad \text{gives } AD = 200 \qquad ...(2)$$

Now, $CD = AC + AD \; = 200\sqrt{3} + 200$ [by (1) and (2)]

$$CD = 200(\sqrt{3}+1) \; = 200 \times 2.732 \; = 546.4$$

Distance between two ships is 546.4 m.

Example 6.22 From a point on the ground, the angles of elevation of the bottom and top of a tower fixed at the top of a 30 m high building are $45°$ and $60°$ respectively. Find the height of the tower. ($\sqrt{3} = 1.732$)

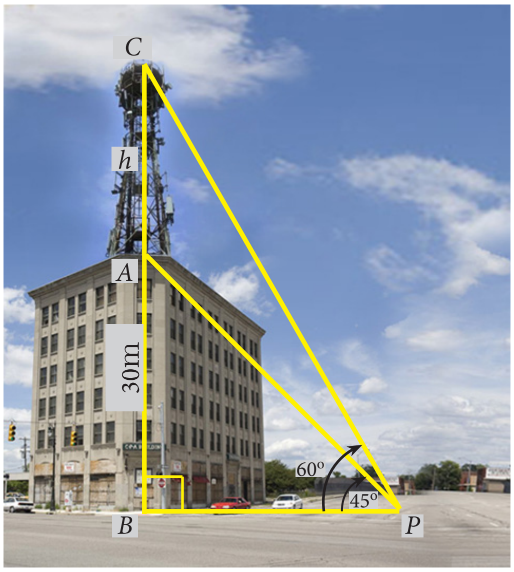


Fig. 6.16

Solution Let AC be the height of the tower.

Let AB be the height of the building.

Then, $AC = h$ metres, $AB = 30$ m

In right triangle CBP, $\angle CPB = 60°$

$$\tan\theta = \frac{BC}{BP}$$

$$\tan 60° = \frac{AB + AC}{BP} \quad \text{so, } \sqrt{3} = \frac{30+h}{BP} \qquad ...(1)$$

In right triangle ABP, $\angle APB = 45°$

$$\tan\theta = \frac{AB}{BP}$$

$$\tan 45° = \frac{30}{BP} \text{ gives } BP = 30 \qquad ...(2)$$

Substituting (2) in (1), we get $\sqrt{3} = \frac{30+h}{30}$

$$h = 30(\sqrt{3}-1) = 30(1.732-1) = 30(0.732) \; = 21.96$$

Hence, the height of the tower is 21.96 m.

Example 6.23 A TV tower stands vertically on a bank of a canal. The tower is watched from a point on the other bank directly opposite to it. The angle of elevation of the top of the tower is $58°$. From another point 20 m away from this point on the line joining this point to the foot of the tower, the angle of elevation of the top of the tower is $30°$. Find the height of the tower and the width of the canal. ($\tan 58° = 1.6003$)

Solution

Let AB be the height of the TV tower.

$CD = 20$ m.

Let BC be the width of the canal.

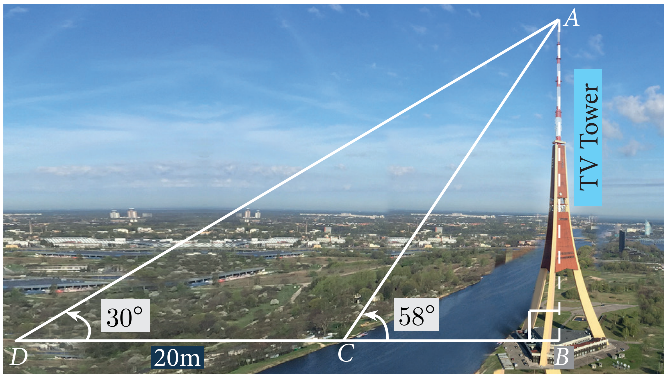


Fig. 6.17

In right triangle ABC, $\tan 58° = \frac{AB}{BC}$

$$1.6003 = \frac{AB}{BC} \quad ...(1)$$

In right triangle ABD, $\tan 30° = \frac{AB}{BD} = \frac{AB}{BC + CD}$

$$\frac{1}{\sqrt{3}} = \frac{AB}{BC + 20} \quad ...(2)$$

Dividing (1) by (2) we get, $\frac{1.6003}{\frac{1}{\sqrt{3}}} = \frac{BC + 20}{BC}$

$$BC = \frac{20}{1.7791} = 11.24 \text{ m} \quad ...(3)$$

$$1.6003 = \frac{AB}{11.24} \quad \text{[from (1) and (3)]}$$

$$AB = 17.99$$

Hence, the height of the tower is 17.99 m and the width of the canal is 11.24 m.

Example 6.24 An aeroplane sets off from G on a bearing of 24° towards H, a point 250 km away. At H it changes course and heads towards J on a bearing of 55° and a distance of 180 km away.

(i) How far is H to the North of G? (ii) How far is H to the East of G?

(iii) How far is J to the North of H? (iv) How far is J to the East of H?

$$\begin{pmatrix} \sin 24° = 0.4067 & \sin 11° = 0.1908 \\ \cos 24° = 0.9135 & \cos 11° = 0.9816 \end{pmatrix}$$

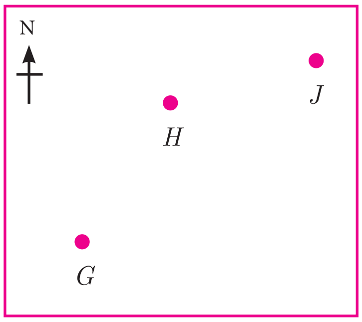


Fig. 6.18 (a)

Solution

(i) In right triangle GOH, $\cos 24° = \frac{OG}{GH}$

$$0.9135 = \frac{OG}{250}; \quad OG = 228.38 \text{ km}$$

Distance of H to the North of $G = 228.38$ km

(ii) In right triangle GOH,

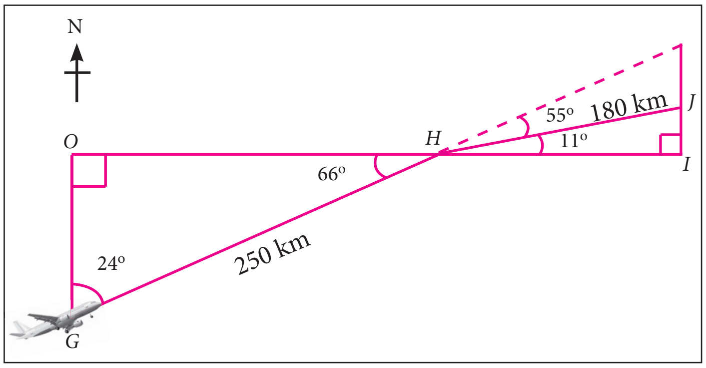


Fig. 6.18 (b)

$$\sin 24° = \frac{OH}{GH}$$

$$0.4067 = \frac{OH}{250}\ ;\ OH = 101.68$$

Distance of H to the East of $G = 101.68$ km

(iii) In right triangle HIJ,

$$\sin 11° = \frac{IJ}{HJ}$$

$$0.1908 = \frac{IJ}{180};\ IJ = 34.34 \text{ km}$$

Distance of J to the North of $H = 34.34$ km

(iv) In right triangle HIJ,

$$\cos 11° = \frac{HI}{HJ}$$

$$0.9816 = \frac{HI}{180};\ HI = 176.69 \text{ km}$$

Distance of J to the East of $H = 176.69$ km

Example 6.25 Two trees are standing on flat ground. The angle of elevation of the top of both the trees from a point X on the ground is $40°$. If the horizontal distance between X and the smaller tree is 8 m and the distance of the top of the two trees is 20 m, calculate

(i) the distance between the point X and the top of the smaller tree.

(ii) the horizontal distance between the two trees.

$(\cos 40° = 0.7660)$

Solution Let AB be the height of the bigger tree and CD be the height of the smaller tree and X is the point on the ground.

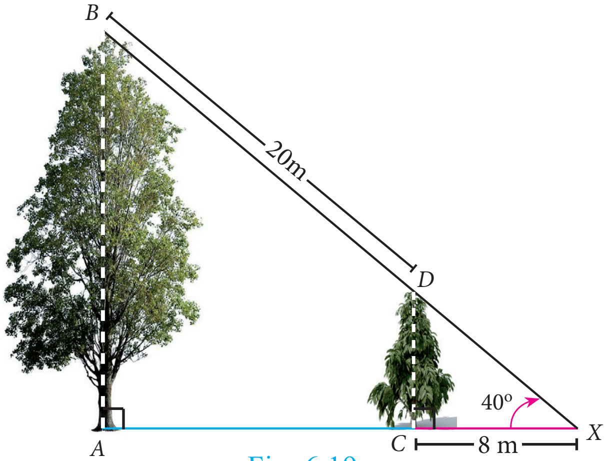


Fig. 6.19

(i) In right triangle XCD, $\cos 40° = \frac{CX}{XD}$

$$XD = \frac{8}{0.7660} = 10.44 \text{ m}$$

Therefore the distance between X and top of the smaller tree = XD = 10.44 m

(ii) In right triangle XAB,

$$\cos 40° = \frac{AX}{BX} = \frac{AC + CX}{BD + DX}$$

$$0.7660 = \frac{AC+8}{20+10.44} \text{ gives } AC = 23.32 - 8 = 15.32 \text{ m}$$

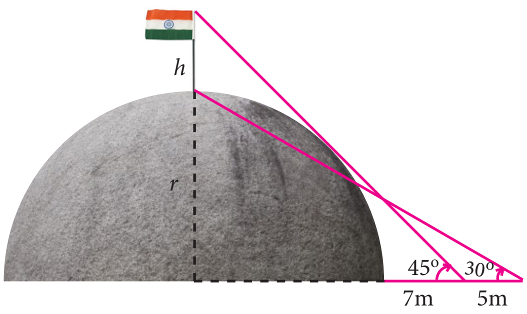
Therefore the horizontal distance between two trees $= AC = 15.32$ m

Thinking Corner

1. What type of triangle is used to calculate heights and distances?
2. When the height of the building and distances from the foot of the building is given, which trigonometric ratio is used to find the angle of elevation?
3. If the line of sight and angle of elevation is given, then which trigonometric ratio is used
   (i) to find the height of the building
   (ii) to find the distance from the foot of the building.

Exercise 6.2

1. Find the angle of elevation of the top of a tower from a point on the ground, which is 30 m away from the foot of a tower of height $10\sqrt{3}$ m.
2. A road is flanked on either side by continuous rows of houses of height $4\sqrt{3}$ m with no space in between them. A pedestrian is standing on the median of the road facing a row house. The angle of elevation from the pedestrian to the top of the house is $30°$. Find the width of the road.
3. To a man standing outside his house, the angles of elevation of the top and bottom of a window are 60° and 45° respectively. If the height of the man is 180 cm and if he is 5 m away from the wall, what is the height of the window? $(\sqrt{3} = 1.732)$
4. A statue 1.6 m tall stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of the statue is $60°$ and from the same point the angle of elevation of the top of the pedestal is $40°$. Find the height of the pedestal. $(\tan 40° = 0.8391,\ \sqrt{3} = 1.732)$
5. A flag pole 'h' metres is on the top of the hemispherical dome of radius 'r' metres. A man is standing 7 m away from the dome. Seeing the top of the pole at an angle $45°$ and moving 5 m away from the dome and seeing the bottom of the pole at an angle $30°$. Find (i) the height of the pole (ii) radius of the dome. $(\sqrt{3} = 1.732)$



6. The top of a 15 m high tower makes an angle of elevation of 60° with the bottom of an electronic pole and angle of elevation of 30° with the top of the pole. What is the height of the electric pole?

7. A vertical pole fixed to the ground is divided in the ratio 1:9 by a mark on it with lower part shorter than the upper part. If the two parts subtend equal angles at a place on the ground, 25 m away from the base of the pole, what is the height of the pole?

8. A traveler approaches a mountain on highway. He measures the angle of elevation to the peak at each milestone. At two consecutive milestones the angles measured are 4° and 8°. What is the height of the peak if the distance between consecutive milestones is 1 mile. $(\tan 4^\circ = 0.0699, \quad \tan 8^\circ = 0.1405)$

6.3.2 Problems involving Angle of Depression

In this section, we try to solve problems when Angles of depression are given.

Note

Angle of Depression and Angle of Elevation are equal become they are alternative angles.

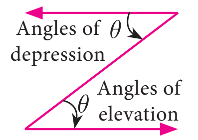


Fig. 6.20

Example 6.26 A player sitting on the top of a tower of height 20 m observes the angle of depression of a ball lying on the ground as 60°. Find the distance between the foot of the tower and the ball. $(\sqrt{3} = 1.732)$

Solution Let BC be the height of the tower and A be the position of the ball lying on the ground. Then,

$$BC = 20 \text{ m and } \angle XCA = 60^\circ = \angle CAB$$

Let $AB = x$ metres.

In right triangle ABC,

$$\tan 60^\circ = \frac{BC}{AB}$$

$$\sqrt{3} = \frac{20}{x}$$

$$x = \frac{20 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{20 \times 1.732}{3} = 11.54 \text{ m}$$

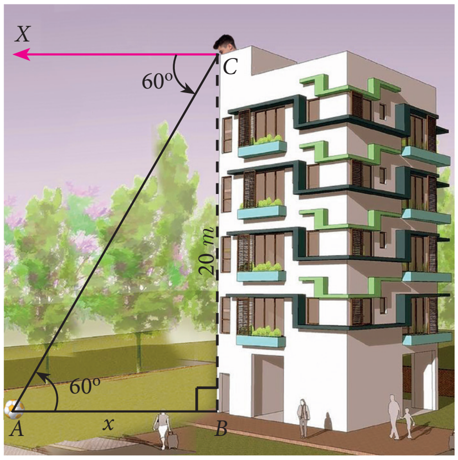


Fig. 6.21

Hence, the distance between the foot of the tower and the ball is 11.54 m.

Example 6.27 The horizontal distance between two buildings is 140 m. The angle of depression of the top of the first building when seen from the top of the second building is 30°. If the height of the first building is 60 m, find the height of the second building. $(\sqrt{3} = 1.732)$

Solution The height of the first building $AB = 60$ m. Now, $AB = MD = 60$ m

Let the height of the second building $CD = h$. Distance $BD = 140$ m

Now, $AM = BD = 140$ m

From the diagram,

$$\angle XCA = 30^\circ = \angle CAM$$

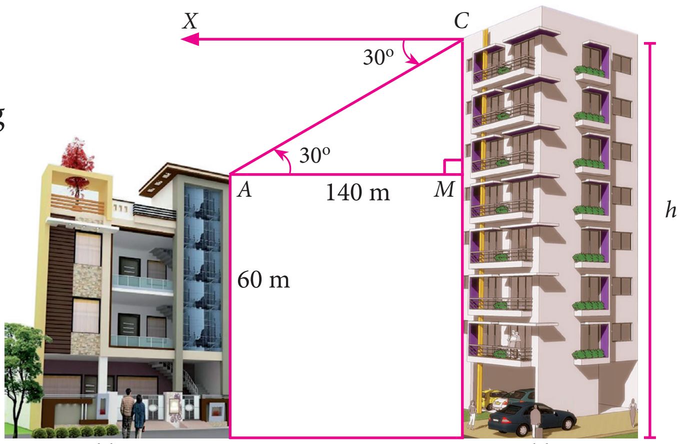


Fig. 6.22

In right triangle AMC, $\tan 30^\circ = \frac{CM}{AM}$

$$\frac{1}{\sqrt{3}} = \frac{CM}{140}$$

$$CM = \frac{140}{\sqrt{3}} = \frac{140\sqrt{3}}{3}$$

$$= \frac{140 \times 1.732}{3}$$

$$CM = 80.78$$

Now, $h = CD = CM + MD = 80.78 + 60 = 140.78$

Therefore the height of the second building is 140.78 m

Example 6.28 From the top of a tower 50 m high, the angles of depression of the top and bottom of a tree are observed to be 30° and 45° respectively. Find the height of the tree. $(\sqrt{3} = 1.732)$

Solution The height of the tower $AB = 50$ m

Let the height of the tree $CD = y$ and $BD = x$

From the diagram, $\angle XAC = 30^\circ = \angle ACM$ and $\angle XAD = 45^\circ = \angle ADB$

In right triangle ABD,

$$\tan 45^\circ = \frac{AB}{BD}$$

$$1 = \frac{50}{x} \text{ gives } x = 50 \text{ m}$$

In right triangle AMC,

$$\tan 30^\circ = \frac{AM}{CM}$$

$$\frac{1}{\sqrt{3}} = \frac{AM}{50} \quad [\text{since } DB = CM]$$

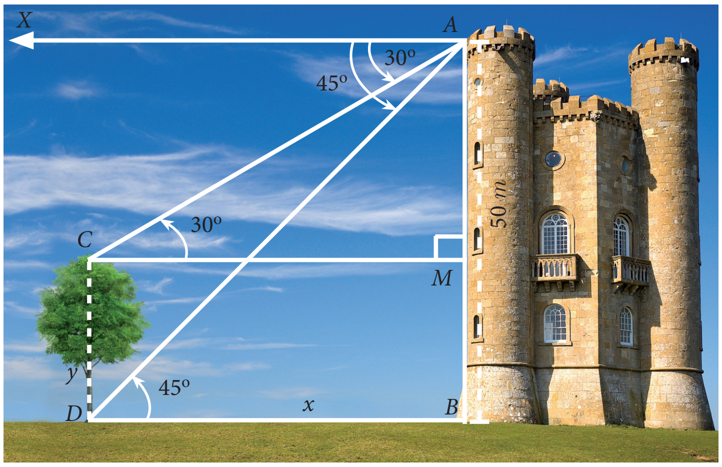


Fig. 6.23

$$AM = \frac{50}{\sqrt{3}} = \frac{50\sqrt{3}}{3} = \frac{50 \times 1.732}{3} = 28.85 \text{ m}.$$

Therefore, height of the tree $= CD = MB = AB - AM = 50 - 28.85 = 21.15$ m

Example 6.29 As observed from the top of a 60 m high light house from the sea level, the angles of depression of two ships are $28°$ and $45°$. If one ship is exactly behind the other on the same side of the lighthouse, find the distance between the two ships. ($\tan 28° = 0.5317$)

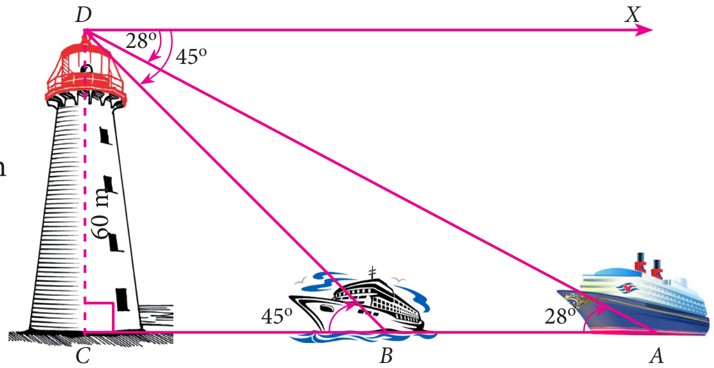


Fig. 6.24

Solution Let the observer on the lighthouse CD be at D.

Height of the lighthouse $CD = 60$ m

From the diagram,

$\angle XDA = 28° = \angle DAC$ and

$\angle XDB = 45° = \angle DBC$

In right triangle DCB, $\tan 45° = \frac{DC}{BC}$

$$1 = \frac{60}{BC} \text{ gives } BC = 60 \text{ m}$$

In right triangle DCA, $\tan 28° = \frac{DC}{AC}$

$$0.5317 = \frac{60}{AC} \text{ gives } AC = \frac{60}{0.5317} = 112.85$$

Distance between the two ships $AB = AC - BC = 52.85$ m

Example 6.30 A man is watching a boat speeding away from the top of a tower. The boat makes an angle of depression of $60°$ with the man's eye when at a distance of 200 m from the tower. After 10 seconds, the angle of depression becomes $45°$. What is the approximate speed of the boat (in km / hr), assuming that it is sailing in still water? ($\sqrt{3} = 1.732$)

Solution Let AB be the tower.

Let C and D be the positions of the boat.

From the diagram,

$\angle XAC = 60° = \angle ACB$ and

$\angle XAD = 45° = \angle ADB$, $BC = 200$ m

In right triangle ABC, $\tan 60° = \frac{AB}{BC}$

gives $\sqrt{3} = \frac{AB}{200}$

we get $AB = 200\sqrt{3}$ ...(1)

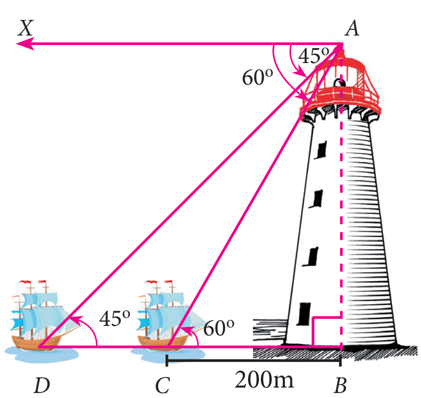


Fig. 6.25

In right triangle ABD, $\tan 45° = \frac{AB}{BD}$

gives $1 = \frac{200\sqrt{3}}{BD}$ [by (1)]

we get, $BD = 200\sqrt{3}$

Now, $CD = BD - BC$

$$CD = 200\sqrt{3} - 200 = 200(\sqrt{3} - 1) = 146.4$$

It is given that the distance CD is covered in 10 seconds.

That is, the distance of 146.4 m is covered in 10 seconds.

Therefore, speed of the boat $= \frac{\text{distance}}{\text{time}}$

$$= \frac{146.4}{10} = 14.64 \text{ m/s gives } 14.64 \times \frac{3600}{1000} \text{ km/hr} = 52.704 \text{ km/hr}$$

Exercise 6.3

1. From the top of a rock $50\sqrt{3}$ m high, the angle of depression of a car on the ground is observed to be 30°. Find the distance of the car from the rock.
2. The horizontal distance between two buildings is 70 m. The angle of depression of the top of the first building when seen from the top of the second building is 45°. If the height of the second building is 120 m, find the height of the first building.
3. From the top of the tower 60 m high the angles of depression of the top and bottom of a vertical lamp post are observed to be 38° and 60° respectively. Find the height of the lamp post. ($\tan 38° = 0.7813$, $\sqrt{3} = 1.732$)
4. An aeroplane at an altitude of 1800 m finds that two boats are sailing towards it in the same direction. The angles of depression of the boats as observed from the aeroplane are 60° and 30° respectively. Find the distance between the two boats. ($\sqrt{3} = 1.732$)
5. From the top of a lighthouse, the angle of depression of two ships on the opposite sides of it are observed to be 30° and 60°. If the height of the lighthouse is h meters and the line joining the ships passes through the foot of the lighthouse, show that the distance between the ships is $\frac{4h}{\sqrt{3}}$ m.
6. A lift in a building of height 90 feet with transparent glass walls is descending from the top of the building. At the top of the building, the angle of depression to a fountain in the garden is 60°. Two minutes later, the angle of depression reduces to 30°. If the fountain is $30\sqrt{3}$ feet from the entrance of the lift, find the speed of the lift which is descending.

6.3.3 Problems involving Angle of Elevation and Depression

Let us consider the following situation.

A man standing at a top of lighthouse located in a beach watch on aeroplane flying above the sea. At the same instant he watch a ship sailing in the sea. The angle with which he watch the plane correspond to angle of elevation and the angle of watching the ship corresponding to angle of depression. This is one example were one oberseves both angle of elevation and angle of depression.

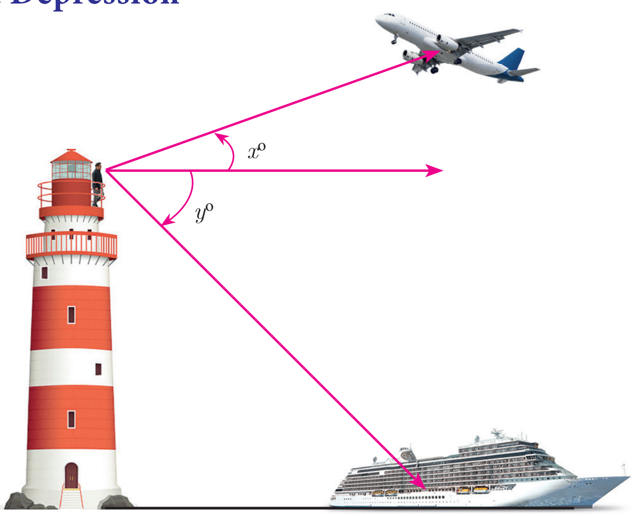


Fig. 6.26

In the Fig.4.26, $x°$ is the angle of elevation and $y°$ is the angle of depression.

In this section, we try to solve problems when Angles of elevation and depression are given.

Example 6.31 From the top of a 12 m high building, the angle of elevation of the top of a cable tower is 60° and the angle of depression of its foot is 30°. Determine the height of the tower.

Solution As shown in Fig.4.27, OA is the building, O is the point of observation on the top of the building OA. Then, $OA = 12$ m.

PP' is the cable tower with P as the top and P' as the bottom.

Then the angle of elevation of P, $\angle MOP = 60°$.

And the angle of depression of P', $\angle MOP' = 30°$.

Suppose, height of the cable tower $PP' = h$ metres.

Through O, draw $OM \perp PP'$

$MP = PP' - MP' = h - OA = h - 12$

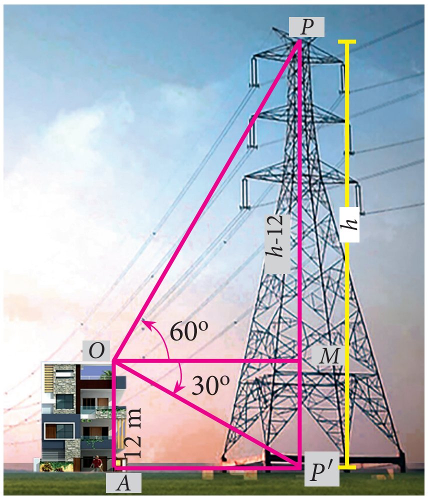


Fig. 6.27

In right triangle OMP, $\dfrac{MP}{OM} = \tan 60°$

gives $\dfrac{h-12}{OM} = \sqrt{3}$

so, $OM = \dfrac{h-12}{\sqrt{3}}$ ...(1)

In right triangle OMP', $\dfrac{MP'}{OM} = \tan 30°$

gives $\dfrac{12}{OM} = \dfrac{1}{\sqrt{3}}$

so, $OM = 12\sqrt{3}$ ...(2)

From (1) and (2) we have, $\dfrac{h-12}{\sqrt{3}} = 12\sqrt{3}$

gives $h - 12 = 12\sqrt{3} \times \sqrt{3}$ we get, $h = 48$

Hence, the required height of the cable tower is 48 m.

Example 6.32 A pole 5 m high is fixed on the top of a tower. The angle of elevation of the top of the pole observed from a point 'A' on the ground is 60° and the angle of depression to the point 'A' from the top of the tower is 45°. Find the height of the tower. $(\sqrt{3} = 1.732)$

Solution Let BC be the height of the tower and CD be the height of the pole.

Let 'A' be the point of observation.

Let $BC = x$ and $AB = y$.

From the diagram,

$\angle BAD = 60°$ and $\angle XCA = 45° = \angle BAC$

In right triangle ABC, $\tan 45° = \dfrac{BC}{AB}$

gives $1 = \dfrac{x}{y}$ so, $x = y$ ...(1)

In right triangle ABD, $\tan 60° = \dfrac{BD}{AB} = \dfrac{BC + CD}{AB}$

gives $\sqrt{3} = \dfrac{x+5}{y}$ so, $\sqrt{3}\, y = x + 5$

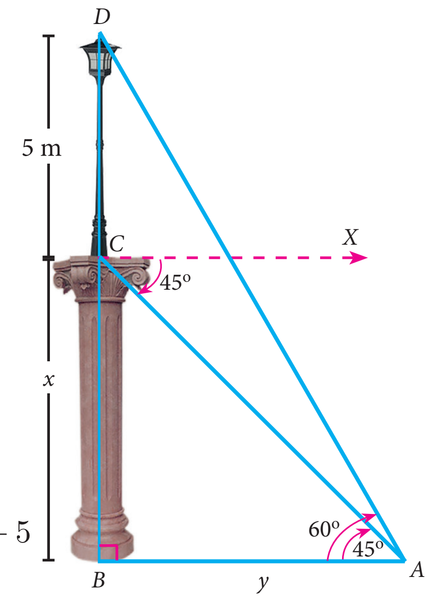


Fig. 6.28

we get, $\sqrt{3}\, x = x + 5$ [From (1)]

so, $x = \dfrac{5}{\sqrt{3}-1} = \dfrac{5}{\sqrt{3}-1} \times \dfrac{\sqrt{3}+1}{\sqrt{3}+1} = \dfrac{5(1.732+1)}{2} = 6.83$

Hence, height of the tower is 6.83 m.

Example 6.33 From a window (h metres high above the ground) of a house in a street, the angles of elevation and depression of the top and the foot of another house on the opposite side of the street are θ_1 and θ_2 respectively. Show that the height of the opposite house is $h\left(1 + \dfrac{\cot\theta_2}{\cot\theta_1}\right)$.

Solution Let W be the point on the window where the angles of elevation and depression are measured. Let PQ be the house on the opposite side.

Then WA is the width of the street.

Height of the window = h metres

$= AQ\ (WR = AQ)$

Let $PA = x$ metres.

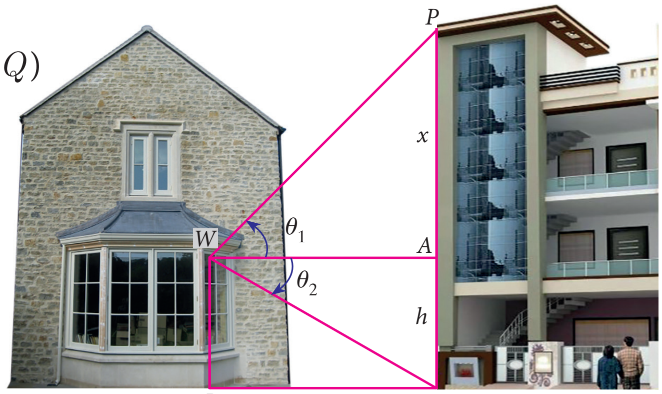


Fig. 6.29

In right triangle PAW, $\tan\theta_1 = \dfrac{AP}{AW}$

gives $\tan\theta_1 = \dfrac{x}{AW}$

so, $AW = \dfrac{x}{\tan\theta_1}$

we get, $AW = x\cot\theta$ ...(1)

In right triangle QAW, $\tan\theta_2 = \dfrac{AQ}{AW}$

gives $\tan\theta_2 = \dfrac{h}{AW}$

we get, $AW = h\cot\theta_2$ ...(2)

From (1) and (2) we get, $x\cot\theta_1 = h\cot\theta_2$

gives, $x = h\dfrac{\cot\theta_2}{\cot\theta_1}$

Therefore, height of the opposite house = $PA+AQ = x + h = h\dfrac{\cot\theta_2}{\cot\theta_1} + h = h\left(1+\dfrac{\cot\theta_2}{\cot\theta_1}\right)$

Hence Proved.

Thinking Corner

What is the minimum number of measurements required to determine the height or distance or angle of elevation?

Progress Check

1. The line drawn from the eye of an observer to the point of object is ___________.
2. Which instrument is used in measuring the angle between an object and the eye of the observer?
3. When the line of sight is above the horizontal level, the angle formed is ________.
4. The angle of elevation ____________ as we move towards the foot of the vertical object (tower).
5. When the line of sight is below the horizontal level, the angle formed is ________.

Exercise 6.4

1. From the top of a tree of height 13 m the angle of elevation and depression of the top and bottom of another tree are 45° and 30° respectively. Find the height of the second tree. $(\sqrt{3} = 1.732)$

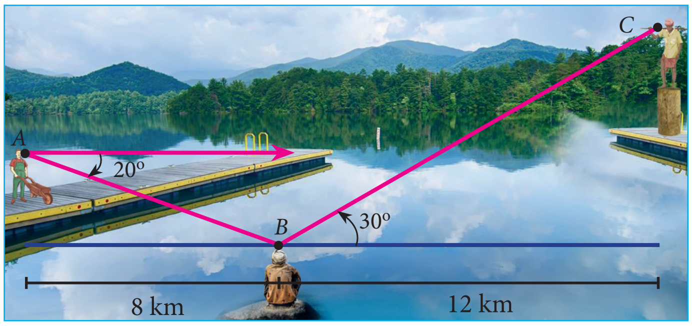
2. A man is standing on the deck of a ship, which is 40 m above water level. He observes the angle of elevation of the top of a hill as $60°$ and the angle of depression of the base of the hill as $30°$. Calculate the distance of the hill from the ship and the height of the hill. $(\sqrt{3} = 1.732)$

3. If the angle of elevation of a cloud from a point 'h' metres above a lake is θ_1 and the angle of depression of its reflection in the lake is θ_2. Prove that the height that the cloud is located from the ground is $\dfrac{h(\tan\theta_1 + \tan\theta_2)}{\tan\theta_2 - \tan\theta_1}$.

4. The angle of elevation of the top of a cell phone tower from the foot of a high apartment is $60°$ and the angle of depression of the foot of the tower from the top of the apartment is $30°$. If the height of the apartment is 50 m, find the height of the cell phone tower. According to radiations control norms, the minimum height of a cell phone tower should be 120 m. State if the height of the above mentioned cell phone tower meets the radiation norms.

5. The angles of elevation and depression of the top and bottom of a lamp post from the top of a 66 m high apartment are $60°$ and $30°$ respectively. Find
   (i) The height of the lamp post.
   (ii) The difference between height of the lamp post and the apartment.
   (iii) The distance between the lamp post and the apartment. $(\sqrt{3} = 1.732)$

6. Three villagers A, B and C can see each other across a valley. The horizontal distance between A and B is 8 km and the horizontal distance between B and C is 12 km. The angle of depression of B from A is $20°$ and the angle of elevation of C from B is $30°$. Calculate : (i) the vertical height between A and B. (ii) the vertical height between B and C. $(\tan 20° = 0.3640,\ \sqrt{3} = 1.732)$



Exercise 6.5

LDJS3Y

Multiple choice questions

1. The value of $\sin^2\theta + \dfrac{1}{1+\tan^2\theta}$ is equal to

   (1) $\tan^2\theta$ (2) 1 (3) $\cot^2\theta$ (4) 0

2. $\tan\theta\,\text{cosec}^2\theta - \tan\theta$ is equal to

   (1) $\sec\theta$ (2) $\cot^2\theta$ (3) $\sin\theta$ (4) $\cot\theta$

3. If $(\sin\alpha + \text{cosec}\,\alpha)^2 + (\cos\alpha + \sec\alpha)^2 = k + \tan^2\alpha + \cot^2\alpha$, then the value of k is equal to

   (1) 9 (2) 7 (3) 5 (4) 3

4. If $\sin\theta + \cos\theta = a$ and $\sec\theta + \text{cosec}\theta = b$, then the value of $b(a^2 - 1)$ is equal to

(1) $2a$ (2) $3a$ (3) 0 (4) $2ab$

5. If $5x = \sec\theta$ and $\frac{5}{x} = \tan\theta$, then $x^2 - \frac{1}{x^2}$ is equal to

(1) 25 (2) $\frac{1}{25}$ (3) 5 (4) 1

6. If $\sin\theta = \cos\theta$, then $2\tan^2\theta + \sin^2\theta - 1$ is equal to

(1) $\frac{-3}{2}$ (2) $\frac{3}{2}$ (3) $\frac{2}{3}$ (4) $\frac{-2}{3}$

7. If $x = a\tan\theta$ and $y = b\sec\theta$ then

(1) $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ (2) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (3) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (4) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

8. $(1 + \tan\theta + \sec\theta)(1 + \cot\theta - \text{cosec}\theta)$ is equal to

(1) 0 (2) 1 (3) 2 (4) -1

9. $a\cot\theta + b\ \text{cosec}\theta = p$ and $b\cot\theta + a\ \text{cosec}\theta = q$ then $p^2 - q^2$ is equal to

(1) $a^2 - b^2$ (2) $b^2 - a^2$ (3) $a^2 + b^2$ (4) $b - a$

10. If the ratio of the height of a tower and the length of its shadow is $\sqrt{3}:1$, then the angle of elevation of the sun has measure

(1) $45°$ (2) $30°$ (3) $90°$ (4) $60°$

11. The electric pole subtends an angle of $30°$ at a point on the same level as its foot. At a second point 'b' metres above the first, the depression of the foot of the tower is $60°$. The height of the tower (in metres) is equal to

(1) $\sqrt{3}\,b$ (2) $\frac{b}{3}$ (3) $\frac{b}{2}$ (4) $\frac{b}{\sqrt{3}}$

12. A tower is 60 m height. Its shadow is x metres shorter when the sun's altitude is $45°$ than when it has been $30°$, then x is equal to

(1) 41.92 m (2) 43.92 m (3) 43 m (4) 45.6 m

13. The angle of depression of the top and bottom of 20 m tall building from the top of a multistoried building are $30°$ and $60°$ respectively. The height of the multistoried building and the distance between two buildings (in metres) is

(1) 20, $10\sqrt{3}$ (2) 30, $5\sqrt{3}$ (3) 20, 10 (4) 30, $10\sqrt{3}$

14. Two persons are standing 'x' metres apart from each other and the height of the first person is double that of the other. If from the middle point of the line joining their feet an observer finds the angular elevations of their tops to be complementary, then the height of the shorter person (in metres) is

(1) $\sqrt{2}\,x$ (2) $\frac{x}{2\sqrt{2}}$ (3) $\frac{x}{\sqrt{2}}$ (4) $2x$

15. The angle of elevation of a cloud from a point h metres above a lake is β. The angle of depression of its reflection in the lake is $45°$. The height of location of the cloud from the lake is

(1) $\dfrac{h(1+\tan\beta)}{1-\tan\beta}$ (2) $\dfrac{h(1-\tan\beta)}{1+\tan\beta}$ (3) $h\tan(45°-\beta)$ (4) none of these

Unit Exercise - 6

1. Prove that (i) $\cot^2 A\left(\dfrac{\sec A-1}{1+\sin A}\right)+\sec^2 A\left(\dfrac{\sin A-1}{1+\sec A}\right)=0$ (ii) $\dfrac{\tan^2\theta-1}{\tan^2\theta+1}=1-2\cos^2\theta$

2. Prove that $\left(\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}\right)^2=\dfrac{1-\cos\theta}{1+\cos\theta}$

3. If $x\sin^3\theta+y\cos^3\theta=\sin\theta\cos\theta$ and $x\sin\theta=y\cos\theta$, then prove that $x^2+y^2=1$.

4. If $a\cos\theta-b\sin\theta=c$, then prove that $(a\sin\theta+b\cos\theta)=\pm\sqrt{a^2+b^2-c^2}$.

5. A bird is sitting on the top of a 80 m high tree. From a point on the ground, the angle of elevation of the bird is $45°$. The bird flies away horizontally in such away that it remained at a constant height from the ground. After 2 seconds, the angle of elevation of the bird from the same point is $30°$. Determine the speed at which the bird flies. $(\sqrt{3}=1.732)$

6. An aeroplane is flying parallel to the Earth's surface at a speed of 175 m/sec and at a height of 600 m. The angle of elevation of the aeroplane from a point on the Earth's surface is $37°$ at a given point. After what period of time does the angle of elevation increase to $53°$? $(\tan 53°=1.3270,\ \tan 37°=0.7536)$

7. A bird is flying from A towards B at an angle of $35°$, a point 30 km away from A. At B it changes its course of flight and heads towards C on a bearing of $48°$ and distance 32 km away.

   (i) How far is B to the North of A? (ii) How far is B to the West of A?

   (iii) How far is C to the North of B? (iv) How far is C to the East of B?

   $(\sin 55°=0.8192,\ \cos 55°=0.5736,\ \sin 42°=0.6691,\ \cos 42°=0.7431)$

8. Two ships are sailing in the sea on either side of the lighthouse. The angles of depression of two ships as observed from the top of the lighthouse are $60°$ and $45°$ respectively. If the distance between the ships is $200\left(\dfrac{\sqrt{3}+1}{\sqrt{3}}\right)$ metres, find the height of the lighthouse.

9. A building and a statue are in opposite side of a street from each other 35 m apart. From a point on the roof of building the angle of elevation of the top of statue is $24°$ and the angle of depression of base of the statue is $34°$. Find the height of the statue. $(\tan 24°=0.4452,\ \tan 34°=0.6745)$

Points to Remember

- An equation involving trigonometric ratios of an angle is called a trigonometric identity if it is true for all values of the angle.
- Trigonometric identities

  (i) $\sin^2\theta + \cos^2\theta = 1$ (ii) $1+\tan^2\theta = \sec^2\theta$ (iii) $1+\cot^2\theta = \text{cosec}^2\theta$
- The line of sight is the line drawn from the eye of an observer to the point in the object viewed by the observer.
- The angle of elevation of an object viewed is the angle formed by the line of sight with the horizontal when it is above the horizontal level.
- The angle of depression of an object viewed is the angle formed by the line of sight with the horizontal when it is below the horizontal level.
- The height or length of an object or distance between two distant objects can be determined with the help of trigonometric ratios.

ICT CORNER

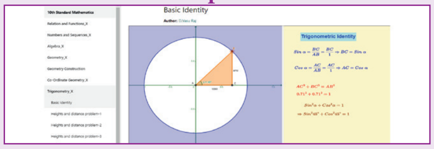
ICT 6.1

Step 1: Open the Browser type the URL Link given below (or) Scan the QR Code. Chapter named **"Trigonometry"** will open. Select the work sheet **"Basic Identity"**

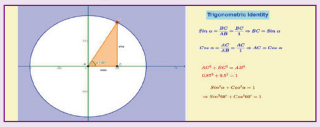
Step 2: In the given worksheet you can change the triangle by dragging the point "B". Check the identity for each angle of the right angled triangle in the unit circle.

Step 1

Step 2



Expected results

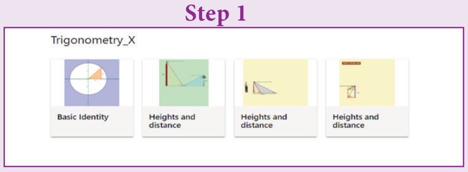


ICT 6.2

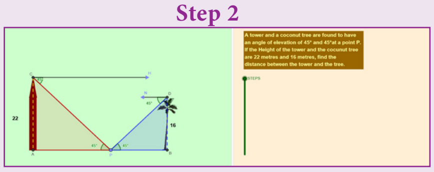
Step 1: Open the Browser type the URL Link given below (or) Scan the QR Code. Chapter named **"Trigonometry"** will open. Select the work sheet **"Heights and distance problem-1"**

Step 2: In the given worksheet you can change the Question by clicking on "New Problem". Move the slider, to view the steps. Workout the problem yourself and verify the answer.

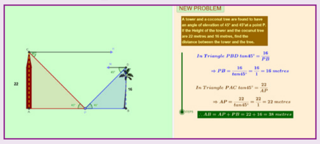
Step 1



Step 2



Expected results



Scan the QR Code.

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