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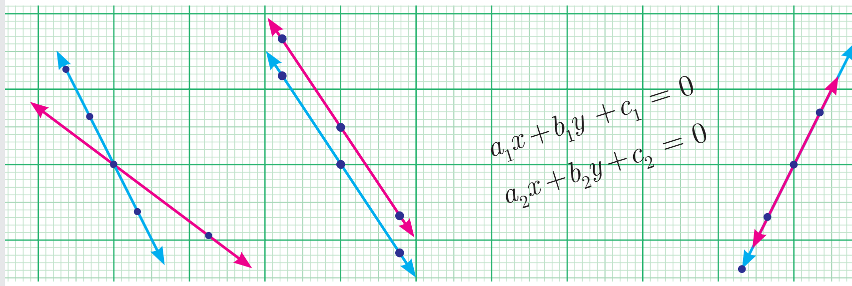

ALGEBRA

In real life, I assure you, there is no such things as Algebra.

- Fran Lebowitz

Diophantus

of Alexandria an Alexandrian Hellenistic mathematician who lived for about 84 years, was born between A.D(C.E) 201 and A.D(C.E) 215. Diophantus was the author of a series of books called Arithmetica. His texts deal with solving algebraic equations. He is also called as "the father of algebra".

Learning Outcomes

- To understand the classification of polynomials and to perform basic operations.
- To evaluate the value of a polynomial and understand the zeros of polynomial.
- To understand the remainder and factor theorems.
- To use Algebraic Identities in factorisation.
- To factorise a quadratic and a cubic polynomial.
- To use synthetic division to factorise a polynomial.
- To find GCD of polynomials.
- Able to draw graph for a given linear equation.
- To solve simultaneous linear equations in two variables by Graphical method and Algebraic method
- To understand consistency and inconsistency of linear equations in two variables.

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3.1 Introduction

Why study polynomials?

This chapter is going to be all about polynomial expressions in algebra. These are your friends, you have already met, without being properly introduced! We will properly introduce them to you, and they are going to be your friends in whatever mathematical journey you undertake from here on.

$(a+1)^2 = a^2 + 2a + 1$

Now that's a polynomial. That does not look very special, does it? We have seen a lots of algebraic expressions already, so why to bother about these? There are many reasons why polynomials are interesting and important in mathematics.

For now, we will just take one example showing their use. Remember, we studied lots of arithmetic and then came to algebra, thinking of variables as unknown numbers. Actually we can now get back to numbers and try to write them in the language of algebra.

Consider a number like 5418. It is actually 5 thousand 4 hundred and eighteen. Write it as:

$5 \times 1000 + 4 \times 100 + 1 \times 10 + 8$

which again can be written as:

$5 \times 10^3 + 4 \times 10^2 + 1 \times 10^1 + 8$

Now it should be clear what this is about. This is of the form $5x^3 + 4x^2 + x + 8$, which is a polynomial. How does writing in this form help? We always write numbers in decimal system, and hence always $x = 10$. Then what is the fun? Remember divisibility rules? Recall that a number is divisible by 3 only if the sum of its digits is divisible by 3. Now notice that if x divided by 3 gives 1 as remainder, then it is the same for x^2, x^3, etc. They all give remainder 1 when divided by 3. So you get each digit multiplied by 1, added together, which is the sum of digits. If that is divisible by 3, so is the whole number. You can check that the rule for divisibility by 9, or even divisibility by 2 or 5, can be proved similarly with great ease. Our purpose is not to prove divisibility rules but to show that representing numbers as polynomials shows us many new number patterns. In fact, many objects of study, not just numbers, can be represented as polynomials and then we can learn many things about them.

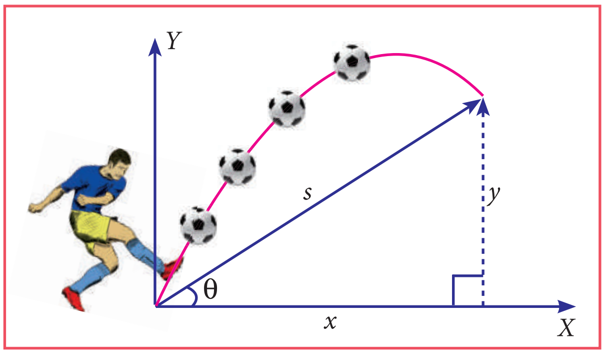


Fig. 3.1

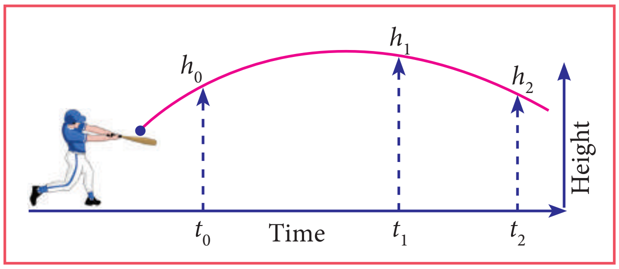


Fig. 3.2

In algebra we think of x^2, $5x^2-3$, $2x+7$ etc as *functions* of x. We draw pictures to see how the function varies as x varies, and this is very helpful to *understand* the function. And now, it turns out that a good number of functions that we encounter in science, engineering, business studies, economics, and of course in mathematics, all can be approximated by polynomials, if not actually be represented as polynomials. In fact, approximating functions using polynomials is a fundamental theme in all of higher mathematics and a large number of people make a living, simply by working on this idea.

Polynomials are extensively used in biology, computer science, communication systems ... the list goes on. The given pictures (Fig. 3.1, 3.2 & 3.3) may be repsresented as a quadratic polynomial. We will not only learn what polynomials are but also how we can use them like in numbers, we add them, multiply them, divide one by another, etc.,

Fig. 3.3

Observe the given figures.

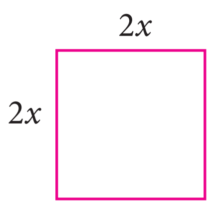


Fig. 3.4

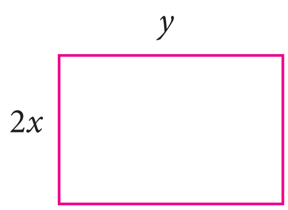


Fig. 3.5

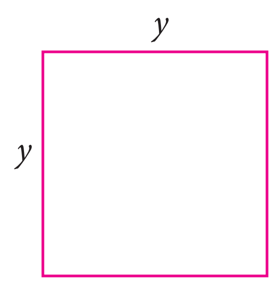


Fig. 3.6

The total area of the above 3 figures is $4x^2 + 2xy + y^2$, we call this expression as an algebraic expression. Here for different values of x and y we get different values of areas. Since the sides x and y can have different values, they are called variables. Thus, a variable is a symbol which can have various numerical values.

Variables are usually denoted by letters such as x, y, z, etc. In the above algebraic expression the numbers 4, 2 are called constants. Hence the constant is a symbol, which has a fixed numeric value.

Constants

Any real number is a constant. We can form numerical expressions using constants and the four arithmetical operations.

Examples of constant are 1, 5, −32, $\frac{3}{7}$, $-\sqrt{2}$, 8.432, 1000000 and so on.

Variables

The use of variables and constants together in expressions give us ways of representing a range of numbers, one for each value of the variable. For instance, we know the expression $2\pi r$, it stands for the circumference of a circle of radius r. As we vary r, say, 1cm, 4cm, 9cm etc, we get larger and larger circles of circumference 2π, 8π, 18π etc.,

The single expression $2\pi r$ is a short and compact description for the circumference of all these circles. We can use arithmetical operations to combine algebraic expressions and get a rich language of functions and numbers. Letters used for representing unknown real numbers called variables are x, y, a, b and so on.

Algebraic Expression

An **algebraic expression** is a combination of constants and variables combined together with the help of the four fundamental signs.

Examples of algebraic expression are

$$x^3-4x^2+8x-1,\ 4xy^2+3x^2y-\frac{5}{4}xy+9,\ 5x^2-7x+6$$

Coefficients

Any part of a term that is multiplied by the variable of the term is called the coefficient of the remaining term.

For example,

$x^2+5x-24$ is an algebraic expression containing three terms. The variable of this expression is x, coefficient of x^2 is 1, the coefficient of x is 5 and the constant is -24 (not 24).

Activity-1

Write the Variable, Coefficient and Constant in the given algebraic expression

Expression	$x+7$	$3y-2$	$5x^2$	$2xy+11$	$-\frac{1}{2}p+7$	$-8+3a$
Variable	x			x, y		
Coefficient	1				$-\frac{1}{2}$	
Constant	7					-8

3.2 Polynomials

A polynomial is an arithmetic expression consisting of variables and constants that involves four fundamental arithmetic operations and non-negative integer exponents of variables.

Polynomial in One Variable

An algebraic expression of the form $p(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0$ is called Polynomial in one variable x of degree 'n' where $a_0, a_1, a_2, \ldots a_n$ are constants $(a_n\neq 0)$ and n is a whole number.

In general polynomials are denoted by $f(x), g(x), p(t), q(z)$ and $r(x)$ and so on.

Note

The coefficient of variables in the algebraic expression may have any real numbers, where as the powers of variables in polynomial must have only non-negative integral powers that is, only whole numbers. Recall that $a^0=1$ for all a.

For example,

S.No	Given expression	Polynomial / not a polynomial	Reason
1	$4y^3 + 2y^2 + 3y + 6$	Polynomial	Non- negative integral power
2	$4x^{-4} + 5x^4$	Not a polynomial	One of the powers is negative (–4)
3	$m^2 + \frac{4}{5}m + 8$	Polynomial	Non- negative integral power
4	$\sqrt{5}\,y^2$	Polynomial	Non- negative integral power
5	$2r^2 + 3r - 1 + \frac{1}{r}$	Not a polynomial	One of the power is negative $\left(\frac{1}{r} = r^{-1}\right)$
6	$8 + \sqrt{q}$	Not a polynomial	power of q is fraction $(\sqrt{q} = q^{\frac{1}{2}})$
7	$\sqrt{8}\,p^2 + 5p - 7$	Polynomial	Non- negative integral power
8	$5n^{\frac{4}{5}} + 6n - 1$	Not a polynomial	One of the power of n is a fraction $\frac{4}{5}$

Standard Form of a Polynomial

We can write a polynomial $p(x)$ in the decreasing or increasing order of the powers of x. This way of writing the polynomial is called the standard form of a polynomial.

Activity-2

Write the following polynomials in standard form.

Sl.No.	Polynomial	Standard Form
1	$5m^4 - 3m + 7m^2 + 8$	
2	$\frac{2}{3}y + 8y^3 - 12 + \sqrt{5}\,y^2$	
3	$12p^2 - 8p^5 - 10p^4 - 7$	

For example: (i) $8x^4 + 4x^3 - 7x^2 - 9x + 6$ (ii) $5 - 3y + 6y^2 + 4y^3 - y^4$

Degree of the Polynomial

In a polynomial of one variable, the highest power of the variable is called the **degree of** the polynomial.

In case of a polynomial of more than one variable, the sum of the powers of the variables in each term is considered and the **highest sum** so obtained is called the degree of the polynomial.

This is intended as the most **significant** power of the polynomial. Obviously when we write x^2+5x the value of x^2 becomes much larger than $5x$ for large values of x. So we could think of $x^2 + 5x$ being almost the same as x^2 for large values of x. So the higher the power, the more it dominates. That is why we use the highest power as important information about the polynomial and give it a name.

Example 3.1 Find the degree of each term for the following polynomial and also find the degree of the polynomial $6ab^8 + 5a^2b^3c^2 - 7ab + 4b^2c + 2$

Solution

Given polynomial is $6ab^8 + 5a^2b^3c^2 - 7ab + 4b^2c + 2$

Degree of each of the terms is given below.

$6ab^8$ has degree $(1+8) = 9$

$5a^2b^3c^2$ has degree $(2+3+2) = 7$

$7ab$ has degree $(1+1) = 2$

$4b^2c$ has degree $(2+1) = 3$

The constant term 2 is always regarded as having degree Zero.

The **degree** of the polynomial $6ab^8 + 5a^2b^3c^2 - 7ab + 4b^2c + 2$.

= the largest exponent in the polynomial

= 9

A very Special Polynomial

We have said that coefficients can be any real numbers. What if the coefficient is zero? Well that term becomes zero, so we won't write it. What if all the coefficients are zero? We acknowledge that it exists and give it a name.

E5KX4F

It is the polynomial having all its coefficients to be zero.

$g(t) = 0t^4 + 0t^2 - 0t$, $h(p) = 0p^2 - 0p + 0$

From the above example we see that we cannot talk of the degree of the zero polynomial, since the above two have different degrees but both are zero polynomial. So we say that the degree of the zero polynomial is not defined.

The degree of the zero polynomial is not defined

Types of Polynomials

(i) Polynomial on the basis of number of terms	
MONOMIAL	A polynomial having one term is called a monomial Examples : 5, 6m, 12ab
BINOMIAL	A polynomial having two terms is called a Binomial Examples : $5x + 3$, $4a - 2$, $10p + 1$
TRINOMIAL	A polynomial having three terms is called a Trinomial Example : $4x^2 + 8x - 12$, $3a^2 + 4a + 10$

(ii) Polynomial based on degree	
CONSTANT	A polynomial of degree zero is called constant polynomial Examples : $5, -7, \frac{2}{3}, \sqrt{5}$
LINEAR	A polynomial of degree one is called linear polynomial Examples : $410x - 7$
QUADRATIC	A polynomial of degree two is called quadratic polynomial Example : $2\sqrt{5}x^2 + 8x - 4$
CUBIC	A polynomial of degree three is called cubic polynomial Example : $12y^3$, $6m^3 - 7m + 4$

Example 3.2 Classify the following polynomials based on number of terms.

S.No.	Polynomial	No of Terms	Type of polynomial based of terms
(i)	$5t^3 + 6t + 8t^2$	3 Terms	Trinomial
(ii)	$y-7$	2 Terms	Binomial
(iii)	$\frac{2}{3}r^4$	1 Term	Monomial
(iv)	$6y^5 + 3y - 7$	3 Terms	Trinomial
(v)	$8m^2 + 7m^2$	Like Terms. So, it is $15m^2$ which is 1 term only	Monomial

Example 3.3 Classify the following polynomials based on their degree.

S.No.	Polynomial	Degree	Type
(i)	$\sqrt[3]{4}\,z + 7$	Degree one	Linear polynomial
(ii)	$z^3 - z^2 + 3$	Degree three	Cubic polynomial
(iii)	$\sqrt{7}$	Degree zero	Constant polynomial
(iv)	$y^2 - \sqrt{8}$	Degree two	Quadratic polynomial

3.2.1 Arithmetic of Polynomials

We now have a rich language of polynomials, and we have seen that they can be classified in many ways as well. Now, what can we do with polynomials? Consider a polynomial on x.

We can evaluate the polynomial at a particular value of x. We can ask how the function given by the polynomial changes as x varies. Write the polynomial equation $p(x) = 0$ and solve for x. All this is interesting, and we will be doing plenty of all this as we go along. But there is something else we can do with polynomials, and that is *to treat them like numbers*! We already have a clue to this at the beginning of the chapter when we saw that every positive integer could be represented as a polynomial.

Following arithmetic, we can try to add polynomials, subtract one from another, multiply polynomials, divide one by another. As it turns out, the analogy between numbers and polynomials runs deep, with many interesting properties relating them. For now, it is fun to simply try and define these operations on polynomials and work with them.

Addition of Polynomials

The addition of two polynomials is also a polynomial.

Note

Only like terms can be added. $3x^2 + 5x^2$ gives $8x^2$ but unlike terms such as $3x^2$ and $5x^3$ when added gives $3x^2 + 5x^3$, a new polynomial.

Example 3.4

If $p(x) = 4x^2 - 3x + 2x^3 + 5$ and $q(x) = x^2 + 2x + 4$, then find $p(x) + q(x)$

Solution

Given Polynomial	Standard form
$p(x) = 4x^2 - 3x + 2x^3 + 5$	$2x^3 + 4x^2 - 3x + 5$
$q(x) = x^2 + 2x + 4$	$x^2 + 2x + 4$
$p(x) + q(x) = 2x^3 + 5x^2 - x + 9$	

We see that $p(x) + q(x)$ is also a polynomial. Hence the sum of any two polynomials is also a polynomial.

Subtraction of Polynomials

The subtraction of two polynomials is also a polynomial.

Note Only like terms can be subtracted. $8x^2 - 5x^2$ gives $3x^2$ but when $5x^3$ is subtracted from $3x^2$ we get, $3x^2 - 5x^3$, a new polynomial.

Example 3.5

If $p(x) = 4x^2 - 3x + 2x^3 + 5$ and $q(x) = x^2 + 2x + 4$, then find $p(x) - q(x)$

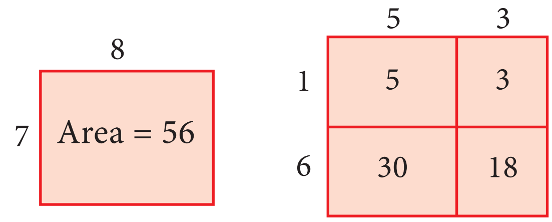
Solution

Given Polynomial	Standard form
$p(x) = 4x^2 - 3x + 2x^3 + 5$	$2x^3 + 4x^2 - 3x + 5$
$q(x) = x^2 + 2x + 4$	$x^2 + 2x + 4$
$p(x) - q(x) = 2x^3 + 3x^2 - 5x + 1$	

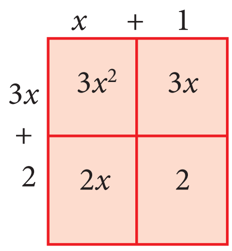
We see that $p(x) - q(x)$ is also a polynomial. Hence the subtraction of any two polynomials is also a polynomial.

Multiplication of Two Polynomials

Divide a rectangle with 8 units of length and 7 units of breadth into 4 rectangles as shown below, and observe that the area is same, this motivates us to study the multiplication of polynomials.



For example, considering length as $(x+1)$ and breadth as $(3x+2)$ the area of the rectangle can be found by the following way.



Hence the area of the rectangle is

$= 3x^2 + 3x + 2x + 2$

$= 3x^2 + 5x + 2$

If x is a variable and m, n are positive integers then $x^m \times x^n = x^{m+n}$.

When two polynomials are multiplied the product will also be a polynomial.

Example 3.6

Find the product $(4x - 5)$ and $(2x^2 + 3x - 6)$.

Solution

To multiply $(4x - 5)$ and $(2x^2 + 3x - 6)$ distribute each term of the first polynomial to every term of the second polynomial. In this case, we need to distribute the terms $4x$ and -5. Then gather the like terms and combine them:

$$(4x - 5)(2x^2 + 3x - 6) = 4x(2x^2 + 3x - 6) - 5(2x^2 + 3x - 6)$$

$$= 8x^3 + 12x^2 - 24x - 10x^2 - 15x + 30$$

$$= 8x^3 + 2x^2 - 39x + 30$$

Aliter: You may also use the method of detached coefficients:

	2	+3	-6
		+4	-5
	−10	−15	+30
8	+12	−24	
8	+2	−39	+30

$\therefore$ $(4x-5)(2x^2+3x-6) = 8x^3 + 2x^2 - 39x + 30$

Exercise 3.1

1. Which of the following expressions are polynomials. If not give reason:

   (i) $\frac{1}{x^2} + 3x - 4$ (ii) $x^2(x-1)$

   (iii) $\frac{1}{x}(x+5)$ (iv) $\frac{1}{x^{-2}} + \frac{1}{x^{-1}} + 7$

   (v) $\sqrt{5}x^2 + \sqrt{3}x + \sqrt{2}$ (vi) $m^2 - \sqrt[3]{m} + 7m - 10$

2. Write the coefficient of x^2 and x in each of the following polynomials.

   (i) $4 + \frac{2}{5}x^2 - 3x$ (ii) $6 - 2x^2 + 3x^3 - \sqrt{7}x$

   (iii) $\pi x^2 - x + 2$ (iv) $\sqrt{3}x^2 + \sqrt{2}x + 0.5$

   (v) $x^2 - \frac{7}{2}x + 8$

3. Find the degree of the following polynomials.

   (i) $1 - \sqrt{2}y^2 + y^7$ (ii) $\frac{x^3 - x^4 + 6x^6}{x^2}$

   (iii) $x^3(x^2 + x)$ (iv) $3x^4 + 9x^2 + 27x^6$

   (v) $2\sqrt{5}p^4 - \frac{8p^3}{\sqrt{3}} + \frac{2p^2}{7}$

4. Rewrite the following polynomial in standard form.

   (i) $x - 9 + \sqrt{7}x^3 + 6x^2$ (ii) $\sqrt{2}x^2 - \frac{7}{2}x^4 + x - 5x^3$

   (iii) $7x^3 - \frac{6}{5}x^2 + 4x - 1$ (iv) $y^2 + \sqrt{5}y^3 - 11 - \frac{7}{3}y + 9y^4$

5. Add the following polynomials and find the degree of the resultant polynomial.

   (i) $p(x) = 6x^2 - 7x + 2$ $q(x) = 6x^3 - 7x + 15$

   (ii) $h(x) = 7x^3 - 6x + 1$ $f(x) = 7x^2 + 17x - 9$

   (iii) $f(x) = 16x^4 - 5x^2 + 9$ $g(x) = -6x^3 + 7x - 15$

6. Subtract the second polynomial from the first polynomial and find the degree of the resultant polynomial.

   (i) $p(x) = 7x^2 + 6x - 1$ $q(x) = 6x - 9$

(ii) $f(y) = 6y^2 - 7y + 2$ $\quad$ $g(y) = 7y + y^3$

(iii) $h(z) = z^5 - 6z^4 + z$ $\quad$ $f(z) = 6z^2 + 10z - 7$

7. What should be added to $2x^3 + 6x^2 - 5x + 8$ to get $3x^3 - 2x^2 + 6x + 15$?

8. What must be subtracted from $2x^4 + 4x^2 - 3x + 7$ to get $3x^3 - x^2 + 2x + 1$?

9. Multiply the following polynomials and find the degree of the resultant polynomial:

   (i) $p(x) = x^2 - 9$ $\quad$ $q(x) = 6x^2 + 7x - 2$

   (ii) $f(x) = 7x + 2$ $\quad$ $g(x) = 15x - 9$

   (iii) $h(x) = 6x^2 - 7x + 1$ $\quad$ $f(x) = 5x - 7$

10. The cost of a chocolate is Rs. $(x + y)$ and Amir bought $(x + y)$ chocolates. Find the total amount paid by him in terms of x and y. If x =10, y =5 find the amount paid by him.

11. The length of a rectangle is $(3x+2)$ units and it's breadth is $(3x–2)$ units. Find its area in terms of x. What will be the area if $x = 20$ units.

12. $p(x)$is a polynomial of degree 1 and $q(x)$ is a polynomial of degree 2. What kind of the polynomial $p(x) \times q(x)$ is ?

3.2.2 Value and Zeros of a Polynomial

Consider the two graphs given below. The first is linear, the second is quadratic. The first intersects the X axis at one point ($x = -3$) and the second at two points (x = -1 and x = 2). They both intersect the Y axis only at one point. In general, every polynomial has a graph and the graph is shown as a picture (since we all like pictures more than formulas, don't we?). But also, the graph contains a lot of useful information like whether it is a straight line, what is the shape of the curve, how many places it cuts the x-axis, etc.

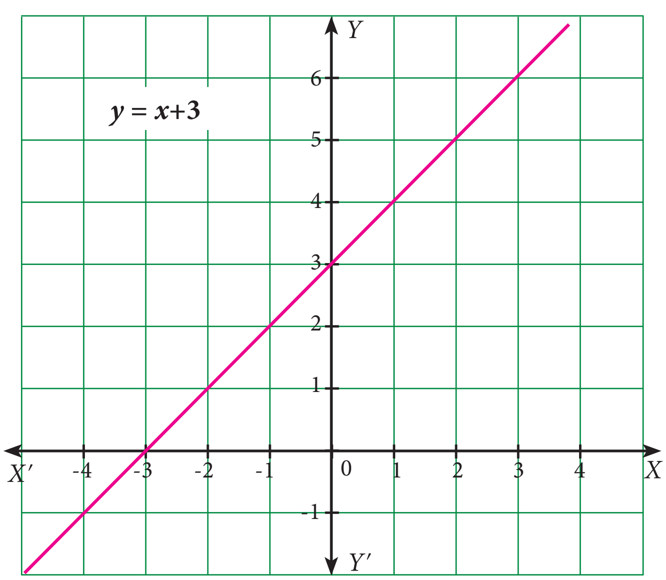


Fig. 3.7

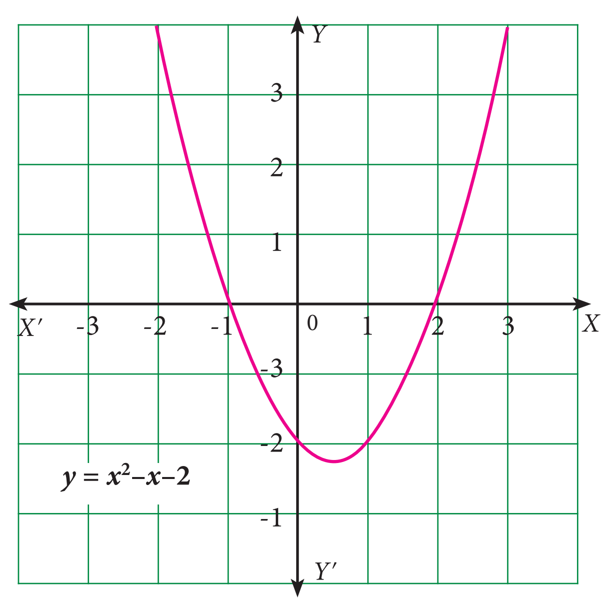


Fig. 3.8

In general, the value of a polynomial $p(x)$ at $x=a$, denoted $p(a)$, is obtained by replacing x by a, where a is any real number.

Notice that the value of $p(x)$ can be zero for many possible values of x as in the second graph 3.8. So it is interesting to ask, for how many values of x, does p(x) become zero, and for which values ? We call these values of x, the zeros of the polynomial $p(x)$.

Once we see that the values of the polynomial are what we plot in the graph of the polynomial, it is also easy to notice that the polynomial becomes zero exactly when the graph intersects the X-axis.

The number of zeros depends on the line or curves intersecting x-axis.

For Fig. 3.7, Number of zeros is equal to 1

For Fig. 3.8, Number of zeros is equal to 2

Note

Number of zeros of a polynomial $\leq$ the degree of the polynomial

Value of a Polynomial

Value of a polynomial $p(x)$ at $x = a$ is $p(a)$ obtained on replacing x by a $(a \in R)$

For example,

Consider $f(x) = x^2 + 3x - 1$.

The value of $f(x)$ at $x = 2$ is

$$f(2) = 2^2+3(2)-1 = 4+6-1 = 9.$$

Zeros of Polynomial

(i) Consider the polynomial $p(x) = 4x^3 - 6x^2 + 3x - 14$

The value of $p(x)$ at $x = 1$ is
$$p(1) = 4(1)^3 - 6(1)^2 + 3(1) - 14 = 4 - 6 + 3 - 14 = -13$$

Then, we say that the value of $p(x)$ at $x = 1$ is -13.

If we replace x by 0, we get
$$p(0) = 4(0)^3 - 6(0)^2 + 3(0) - 14 = 0 - 0 + 0 - 14 = -14$$

we say that the value of $p(x)$ at $x = 0$ is -14.

The value of $p(x)$ at $x = 2$ is
$$p(2) = 4(2)^3 - 6(2)^2 + 3(2) - 14 = 32 - 24 + 6 - 14 = 0$$

Since the value of $p(x)$ at $x = 2$ is zero, we can say that 2 is one of the zeros of $p(x)$ where $p(x) = 4x^3 - 6x^2 + 3x - 14$.

Roots of a Polynomial Equation

In general, if $p(a) = 0$ we say that a is zero of polynomial $p(x)$ or a is the root of polynomial equation $p(x) = 0$

Example 3.7 If $f(x) = x^2 - 4x + 3$, then find the values of $f(1)$, $f(-1)$, $f(2)$, $f(3)$. Also find the zeros of the polynomial $f(x)$.

Solution

$$f(x) = x^2 - 4x + 3$$

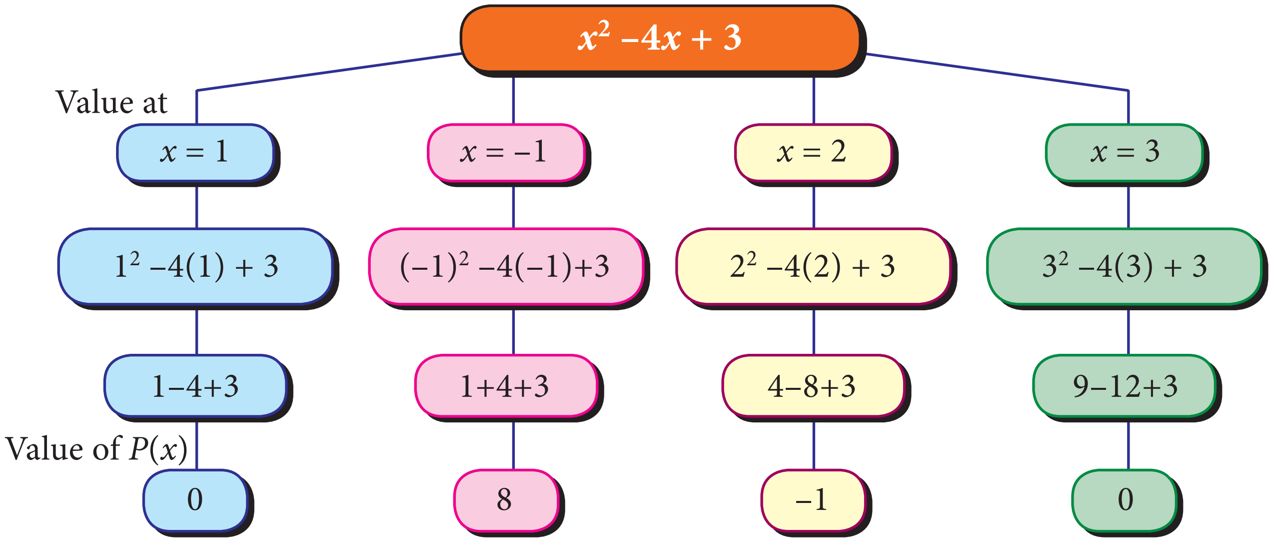


Fig. 3.9

Since the value of the polynomial $f(x)$ at $x = 1$ and $x = 3$ is zero, as the zeros of polynomial $f(x)$ are 1 and 3.

Example 3.8 Find the Zeros of the following polynomials.

(i) $f(x) = 2x + 1$ (ii) $f(x) = 3x - 5$

Solution

(i) Given that
$$f(x) = 2x + 1 = 2\left(x + \frac{1}{2}\right)$$
$$= 2\left(x - \left(-\frac{1}{2}\right)\right)$$
$$f\left(-\frac{1}{2}\right) = 2\left[-\frac{1}{2} - \left(-\frac{1}{2}\right)\right] = 2(0) = 0$$

Since $f\left(-\frac{1}{2}\right) = 0$, $x = -\frac{1}{2}$ is the zero of $f(x)$

(ii) Given that
$$f(x) = 3x - 5 = 3\left(x - \frac{5}{3}\right)$$
$$f\left(\frac{5}{3}\right) = 3\left(\frac{5}{3} - \frac{5}{3}\right) = 3(0) = 0$$

Since $f\left(\frac{5}{3}\right) = 0$, $x = \frac{5}{3}$ is the zero of $f(x)$

Example 3.9 Find the roots of the following polynomial equations.

(i) $5x - 3 = 0$ (ii) $-7 - 4x = 0$

Note

(i) A zero of a polynomial can be any real number not necessarily zero.

(ii) A non zero constant polynomial has no zero.

(iii) By convention, every real number is zero of the zero polynomial

Solution

(i) $5x - 3 = 0$

(or) $5x = 3$

Then, $x = \frac{3}{5}$

(ii) $-7 - 4x = 0$

(or) $4x = -7$

Then, $x = \frac{7}{-4} = -\frac{7}{4}$

Example 3.10 Check whether -3 and 3 are zeros of the polynomial $x^2 - 9$

Solution

Let $f(x) = x^2 - 9$

Then, $f(-3) = (-3)^2 - 9 = 9 - 9 = 0$

$f(+3) = 3^2 - 9 = 9 - 9 = 0$

$\therefore$ -3 and 3 are zeros of the polynomial $x^2 - 9$

Exercise 3.2

1. Find the value of the polynomial $f(y) = 6y - 3y^2 + 3$ at

   (i) $y = 1$ (ii) $y = -1$ (iii) $y = 0$

2. If $p(x) = x^2 - 2\sqrt{2}x + 1$, find $p(2\sqrt{2})$.

3. Find the zeros of the polynomial in each of the following :

   (i) $p(x) = x - 3$ (ii) $p(x) = 2x + 5$ (iii) $q(y) = 2y - 3$

   (iv) $f(z) = 8z$ (v) $p(x) = ax$ when $a \neq 0$ (vi) $h(x) = ax + b,\ a \neq 0,\ a, b \in R$

4. Find the roots of the polynomial equations .

   (i) $5x - 6 = 0$ (ii) $x + 3 = 0$

   (iii) $10x + 9 = 0$ (iv) $9x - 4 = 0$

5. Verify whether the following are zeros of the polynomial indicated against them, or not.

   (i) $p(x) = 2x - 1,\ x = \frac{1}{2}$ (ii) $p(x) = x^3 - 1,\ x = 1$

   (iii) $p(x) = ax + b,\ x = \frac{-b}{a}$ (iv) $p(x) = (x + 3)(x - 4),\ x = 4, x = -3$

6. Find the number of zeros of the following polynomials represented by their graphs.

(i)

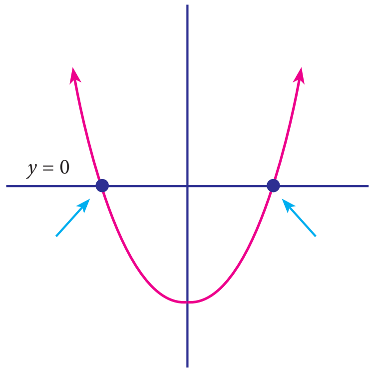


Fig. 3.10

(ii)

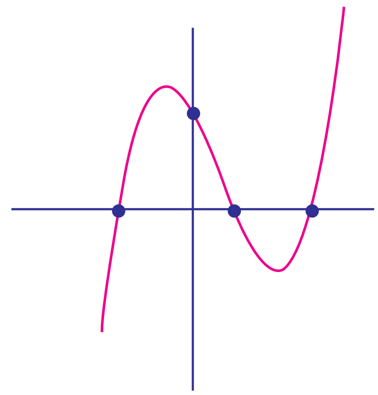


Fig. 3.11

(iii)

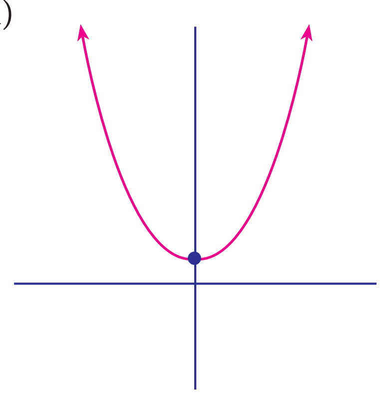


Fig. 3.12

(iv)

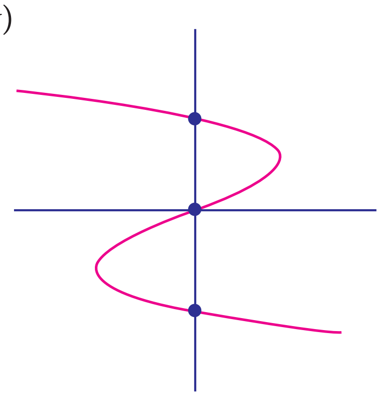


Fig. 3.13

(v)

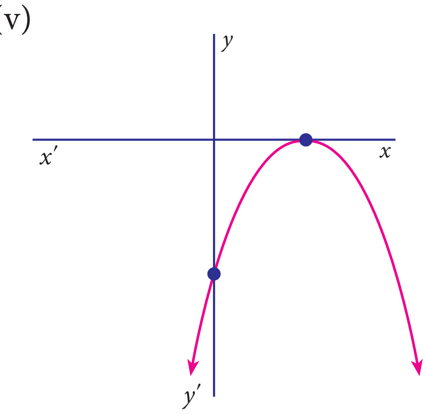


Fig. 3.14

3.3 Remainder Theorem

In this section , we shall study a simple and an elegant method of finding the remainder.

In the case of divisibility of a polynomial by a linear polynomial we use a well known theorem called **Remainder Theorem.**

If a polynomial $p(x)$ of degree greater than or equal to one is divided by a linear polynomial $(x-a)$ then the remainder is $p(a)$, where a is any real number.

Significance of Remainder theorem : It enables us to find the remainder without actually following the cumbersome process of long division.

E5UT63

Note

(i) If $p(x)$ is divided by $(x+a)$, then the remainder is $p(-a)$

(ii) If $p(x)$ is divided by $(ax-b)$, then the remainder is $p\left(\frac{b}{a}\right)$

(iii) If $p(x)$ is divided by $(ax+b)$, then the remainder is $p\left(-\frac{b}{a}\right)$

Example 3.11

S.No.	Question	Solution	Hint
1	Find the remainder when $f(x) = x^3 + 3x^2 + 3x + 1$ is divided by $x+1$.	$f(x) = x^3 + 3x^2 + 3x + 1$ $f(-1) = (-1)^3+3(-1)^2+3(-1)+1$ $= -1+3-3+1 = 0$ Hence, the remainder is 0	$g(x) = x+1$ $g(x) = 0$ $x+1 = 0$ $x = -1$
2	Check whether $f(x) = x^3 - x + 1$ is a multiple of $g(x) = 2 - 3x$	$\therefore f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^3 - \frac{2}{3} + 1$ $= \frac{8}{27} - \frac{2}{3} + 1$ $= \frac{8 - 18 + 27}{27} = \frac{17}{27} \neq 0$ $\Rightarrow f(x)$ is not multiple of $g(x)$	$g(x) = 2-3x$ $= 0$ gives $x = \frac{2}{3}$
3	Find the remainder when $f(x) = x^3 - ax^2 + 6x - a$ is divided by $(x - a)$	We have $f(x) = x^3 - ax^2 + 6x - a$ $f(a) = a^3 - a(a)^2 + 6a - a$ $= a^3 - a^3 + 5a$ $= 5a$ Hence the required remainder is $5a$	Let $g(x) = x-a$ $g(x) = 0$ $x - a = 0$ $x = a$
4	For what value of k is the polynominal $2x^4 + 3x^3 + 2kx^2 + 3x + 6$ exactly divisible by $(x + 2)$?	Let $f(x) = 2x^4 + 3x^3 + 2kx^2 + 3x + 6$ If $f(x)$ is exactly divisible by $(x+2)$, then the remainder must be zero i.e ., $f(-2) = 0$ i.e ., $2(-2)^4 + 3(-2)^3 + 2k(-2)^2 + 3(-2)+6=0$ $2(16) + 3(-8) + 2k(4) -6 + 6=0$ $32-24 + 8k = 0$ $8k = -8$, $k = -1$ Hence $f(x)$ is exactly divisible by $(x-2)$ when $k = -1$	Let $g(x)=x+2$ $g(x)=0$ $x+2 = 0$ $x = -2$

Example 3.12 Without actual division , prove that $f(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$ is exactly divisible by $x^2 - 3x + 2$

Solution :

Let
$$f(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$$
$$\begin{aligned} g(x) &= x^2 - 3x + 2 \\ &= x^2 - 2x - x + 2 \\ &= x(x-2) - 1(x-2) \\ &= (x-2)(x-1) \end{aligned}$$

we show that $f(x)$ is exactly divisible by $(x-1)$ and $(x-2)$ using remainder theorem

$$\begin{aligned} f(1) &= 2(1)^4 - 6(1)^3 + 3(1)^2 + 3(1) - 2 \\ &= 2 - 6 + 3 + 3 - 2 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(2) &= 2(2)^4 - 6(2)^3 + 3(2)^2 + 3(2) - 2 \\ &= 32 - 48 + 12 + 6 - 2 \\ &= 0 \end{aligned}$$

$\therefore$ $f(x)$ is exactly divisible by $(x-1)(x-2)$

i.e., $f(x)$ is exactly divisible by $x^2 - 3x + 2$

If $p(x)$ is divided by $(x-a)$ with the remainder $p(a) = 0$, then $(x-a)$ is a factor of $p(x)$. Remainder Theorem leads to Factor Theorem.

3.3.1 Factor Theorem

If $p(x)$ is a polynomial of degree $n \geq 1$ and 'a' is any real number then

(i) $p(a) = 0$ implies $(x-a)$ is a factor of $p(x)$.

(ii) $(x-a)$ is a factor of $p(x)$ implies $p(a) = 0$.

Proof

If $p(x)$ is the dividend and $(x-a)$ is a divisor, then by division algorithm we write, $p(x) = (x-a)q(x) + p(a)$ where $q(x)$ is the quotient and $p(a)$ is the remainder.

(i) If $p(a) = 0$, we get $p(x) = (x-a)\,q(x)$ which shows that $(x-a)$ is a factor of $p(x)$.

(ii) Since $(x-a)$ is a factor of $p(x)$, $p(x) = (x-a)g(x)$ for some polynomial $g(x)$. In this case

$$\begin{aligned} p(a) &= (a-a)\,g(a) \\ &= 0 \times g(a) \\ &= 0 \end{aligned}$$

Hence, $p(a) = 0$, when $(x-a)$ is a factor of $p(x)$.

Thinking Corner

For any two integers a $(a \neq 0)$ and b, a divides b if $b = ax$, for some integer x.

Note

- $(x-a)$ is a factor of $p(x)$, if $p(a)=0$ $\quad (\because x-a=0,\ x=a)$
- $(x+a)$ is a factor of $p(x)$, if $p(-a)=0$ $\quad (\because x+a=0,\ x=-a)$
- $(ax+b)$ is a factor of $p(x)$, if $p\left(-\frac{b}{a}\right)=0$ $\quad \left(\because ax+b=0,\ ax=-b,\ x=-\frac{b}{a}\right)$
- $(ax-b)$ is a factor of $p(x)$, if $p\left(\frac{b}{a}\right)=0$ $\quad \left(\because ax-b=0,\ ax=b,\ x=\frac{b}{a}\right)$
- $(x-a)(x-b)$ is a factor of $p(x)$, if $p(a)=0$ and $p(b)=0$ $\left(\because \begin{array}{lcr} x-a=0 & \text{or} & x-b=0 \\ x=a & \text{or} & x=b \end{array}\right)$

Example 3.13 Show that $(x+2)$ is a factor of $x^3-4x^2-2x+20$

Solution

Let $p(x)=x^3-4x^2-2x+20$

By factor theorem, $(x+2)$ is factor of $p(x)$, *if* $p(-2)=0$

$$p(-2)=(-2)^3-4(-2)^2-2(-2)+20$$
$$=-8-4(4)+4+20$$
$$p(-2)=0$$

Therefore, $(x+2)$ is a factor of $x^3-4x^2-2x+20$

To find the zero of x+2;

put $x+2=0$

we get $x=-2$

Example 3.14 Is $(3x-2)$ a factor of $3x^3+x^2-20x+12$?

Solution

Let $p(x)=3x^3+x^2-20x+12$

By factor theorem, $(3x-2)$ is a factor, if $p\left(\frac{2}{3}\right)=0$

$$p\left(\frac{2}{3}\right)=3\left(\frac{2}{3}\right)^3+\left(\frac{2}{3}\right)^2-20\left(\frac{2}{3}\right)+12$$
$$=3\left(\frac{8}{27}\right)+\left(\frac{4}{9}\right)-20\left(\frac{2}{3}\right)+12$$
$$=\frac{8}{9}+\frac{4}{9}-\frac{120}{9}+\frac{108}{9}$$

To find the zero of $3x$–2;

put $3x-2=0$

$3x=2$

we get $x=\frac{2}{3}$

Progress Check

1. $(x+3)$ is a factor of $p(x)$, if $p(\_\_)=0$
2. $(3-x)$ is a factor of $p(x)$, if $p(\_\_)=0$
3. $(y-3)$ is a factor of $p(y)$, if $p(\_\_)=0$
4. $(-x-b)$ is a factor of $p(x)$, if $p(\_\_)=0$
5. $(-x+b)$ is a factor of $p(x)$, if $p(\_\_)=0$

$$p\left(\frac{2}{3}\right) = \frac{(120-120)}{9}$$

$$= 0$$

Therefore, $(3x-2)$ is a factor of $3x^3 + x^2 - 20x + 12$

Example 3.15 Find the value of m, if $(x-2)$ is a factor of the polynomial $2x^3 - 6x^2 + mx + 4$.

Solution

Let $p(x) = 2x^3 - 6x^2 + mx + 4$

By factor theorem, $(x-2)$ is a factor of $p(x)$, if $p(2) = 0$

$$\begin{aligned} p(2) &= 0 \\ 2(2)^3 - 6(2)^2 + m(2) + 4 &= 0 \\ 2(8) - 6(4) + 2m + 4 &= 0 \\ -4 + 2m &= 0 \\ m &= 2 \end{aligned}$$

> To find the zero of x–2;
>
> put $x - 2 = 0$
>
> we get $x = 2$

Exercise 3.3

1. Check whether $p(x)$ is a multiple of $g(x)$ or not .
   $p(x) = x^3 - 5x^2 + 4x - 3$; $g(x) = x - 2$

2. By remainder theorem, find the remainder when, $p(x)$ is divided by $g(x)$ where,
   (i) $p(x) = x^3 - 2x^2 - 4x - 1$; $g(x) = x + 1$
   (ii) $p(x) = 4x^3 - 12x^2 + 14x - 3$; $g(x) = 2x - 1$
   (iii) $p(x) = x^3 - 3x^2 + 4x + 50$; $g(x) = x - 3$

3. Find the remainder when $3x^3 - 4x^2 + 7x - 5$ is divided by $(x+3)$

4. What is the remainder when x^{2018} +2018 is divided by x–1

5. For what value of k is the polynomial
   $p(x) = 2x^3 - kx^2 + 3x + 10$ exactly divisible by $(x$–$2)$

6. If two polynomials $2x^3 + ax^2 + 4x - 12$ and $x^3 + x^2 - 2x + a$ leave the same remainder when divided by $(x - 3)$, find the value of a and also find the remainder.

7. Determine whether $(x-1)$ is a factor of the following polynomials:
   $i)$ $x^3 + 5x^2 - 10x + 4$ $ii)$ $x^4 + 5x^2 - 5x + 1$

8. Using factor theorem, show that $(x-5)$ is a factor of the polynomial $2x^3 - 5x^2 - 28x + 15$

9. Determine the value of m, if $(x+3)$ is a factor of $x^3 - 3x^2 - mx + 24$.

10. If both $(x-2)$ and $\left(x - \frac{1}{2}\right)$ are the factors of $ax^2 + 5x + b$, then show that $a = b$.

11. If $(x-1)$ divides the polynomial $kx^3 - 2x^2 + 25x - 26$ without remainder, then find the value of k.

12. Check if $(x+2)$ and $(x-4)$ are the sides of a rectangle whose area is $x^2 - 2x - 8$ by using factor theorem.

3.4 Algebraic Identities

An identity is an equality that remains true regardless of the values chosen for its variables.

We have already learnt about the following identities:

(1) $(a+b)^2 \equiv a^2 + 2ab + b^2$ (2) $(a-b)^2 \equiv a^2 - 2ab + b^2$

(3) $(a+b)(a-b) \equiv a^2 - b^2$ (4) $(x+a)(x+b) \equiv x^2 + (a+b)x + ab$

Note

(i) $a^2 + b^2 = (a+b)^2 - 2ab$ (ii) $a^2 + b^2 = (a-b)^2 + 2ab$

(iii) $a^2 + \frac{1}{a^2} = \left(a + \frac{1}{a}\right)^2 - 2$ (iv) $a^2 + \frac{1}{a^2} = \left(a - \frac{1}{a}\right)^2 + 2$

Example 3.16

Expand the following using identities: (i) $(3x+4y)^2$ (ii) $(2a-3b)^2$ (iii) $(5x+4y)(5x-4y)$ (iv) $(m+5)(m-8)$

Solution

(i) $(3x+4y)^2$ $\left[\text{we have } (a+b)^2 = a^2 + 2ab + b^2\right]$

$(3x+4y)^2 = (3x)^2 + 2(3x)(4y) + (4y)^2$ put $[a = 3x,\ b = 4y]$

$= 9x^2 + 24xy + 16y^2$

(ii) $(2a-3b)^2$ $\left[\text{we have } (a-b)^2 = a^2 - 2ab + b^2\right]$

$(2a-3b)^2 = (2a)^2 - 2(2a)(3b) + (3b)^2$ put $[a = 2a,\ b = 3b]$

$= 4a^2 - 12ab + 9b^2$

(iii) $(5x+4y)(5x-4y)$ $\left[\text{we have } (a+b)(a-b) = a^2-b^2\right]$

$(5x+4y)(5x-4y) = (5x)^2-(4y)^2$ put $[a=5x \quad b=4y]$

$= 25x^2-16y^2$

(iv) $(m+5)(m-8)$ $\left[\text{we have } (x+a)(x-b) = x^2+(a-b)x-ab\right]$

$(m+5)(m-8) = m^2+(5-8)m-(5)(8)$ put $[x=m,\ a=5,\ b=8]$

$= m^2-3m-40$

BCCTMP

3.4.1 Expansion of Trinomial $(a+b+c)^2$

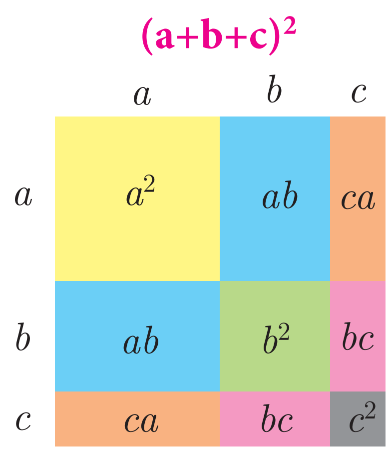
We know that $(x+y)^2 = x^2+2xy+y^2$

Put $x=a+b,\ y=c$

Then, $(a+b+c)^2 = (a+b)^2+2(a+b)(c)+c^2$

$= a^2+2ab+b^2+2ac+2bc+c^2$

$= a^2+b^2+c^2+2ab+2bc+2ca$



Thus, $(a+b+c)^2 \equiv a^2+b^2+c^2+2ab+2bc+2ca$

Example 3.17

Expand $(a-b+c)^2$

Solution

Replacing 'b' by '$-b$' in the expansion of

$(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$

$(a+(-b)+c)^2 = a^2+(-b)^2+c^2+2a(-b)+2(-b)c+2ca$

$= a^2+b^2+c^2-2ab-2bc+2ca$

Progress Check

Expand the following and verify :

$(a+b+c)^2 = (-a-b-c)^2$

$(-a+b+c)^2 = (a-b-c)^2$

$(a-b+c)^2 = (-a+b-c)^2$

$(a+b-c)^2 = (-a-b+c)^2$

Example 3.18

Expand $(2x+3y+4z)^2$

Solution

We know that,

$(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$

Substituting, $a=2x, b=3y$ and $c=4z$

$(2x+3y+4z)^2 = (2x)^2+(3y)^2+(4z)^2+2(2x)(3y)+2(3y)(4z)+2(4z)(2x)$

$= 4x^2+9y^2+16z^2+12xy+24yz+16xz$

Example 3.19 Find the area of square whose side length is $3m + 2n - 4l$

Solution

Area of square = side $\times$ side

$$= (3m + 2n - 4l) \times (3m + 2n - 4l)$$

$$= (3m + 2n - 4l)^2$$

We know that, $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

substituting
$a = 3m,$
$b = 2n$
$c = -4l$

$$[3m + 2n + (-4l)]^2 = (3m)^2 + (2n)^2 + (-4l)^2 + 2(3m)(2n) + 2(2n)(-4l) + 2(-4l)(3m)$$

$$= 9m^2 + 4n^2 + 16l^2 + 12mn - 16ln - 24lm$$

Therefore, Area of square $= [9m^2 + 4n^2 + 16l^2 + 12mn - 16ln - 24lm]$ sq.units.

3.4.2 Identities involving Product of Three Binomials

$$(x + a)(x + b)(x + c) = [(x + a)\ (x + b)]\ (x + c)$$

$$= [x^2 + (a + b)x + ab](x + c)$$

$$= x^2(x) + (a + b)(x)(x) + abx + x^2c + (a + b)(x)c + abc$$

$$= x^3 + ax^2 + bx^2 + abx + cx^2 + acx + bcx + abc$$

$$= x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$$

Thus, $(x + a)(x + b)(x + c) \equiv x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$

Example 3.20 Expand the following:

(i) $(x + 5)(x + 6)(x + 4)$ (ii) $(3x - 1)(3x + 2)(3x - 4)$

Solution

We know that $(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$ --(1)

(i) $(x + 5)(x + 6)(x + 4)$

$$= x^3 + (5 + 6 + 4)x^2 + (30 + 24 + 20)x + (5)(6)(4)$$

$$= x^3 + 15x^2 + 74x + 120$$

Replace
a by 5
b by 6
c by 4 in (1)

(ii) $(3x - 1)(3x + 2)(3x - 4)$

$$= (3x)^3 + (-1 + 2 - 4)(3x)^2 + (-2 - 8 + 4)(3x) + (-1)(2)(-4)$$

$$= 27x^3 + (-3)9x^2 + (-6)(3x) + 8$$

$$= 27x^3 - 27x^2 - 18x + 8$$

Replace
x by 3*x*, *a* by −1,
b by 2, *c* by −4
in (1)

3.4.3 Expansion of $(x+y)^3$ and $(x-y)^3$

$$(x+a)(x+b)(x+c) \equiv x^3+(a+b+c)x^2+(ab+bc+ca)x+abc$$

substituting $a=b=c=y$ in the identity

we get, $(x+y)(x+y)(x+y) = x^3+(y+y+y)x^2+(yy+yy+yy)x+yyy$

$$= x^3+(3y)x^2+(3y^2)x+y^3$$

Thus, $(x+y)^3 \equiv x^3+3x^2y+3xy^2+y^3$ (or) $(x+y)^3 \equiv x^3+y^3+3xy(x+y)$

by replacing y by $-y$, we get

$$(x-y)^3 \equiv x^3-3x^2y+3xy^2-y^3 \text{ (or) } (x-y)^3 \equiv x^3-y^3-3xy(x-y)$$

Example 3.21 Expand $(5a-3b)^3$

Solution

We know that, $(x-y)^3 = x^3-3x^2y+3xy^2-y^3$

$$\begin{aligned}(5a-3b)^3 &= (5a)^3-3(5a)^2(3b)+3(5a)(3b)^2-(3b)^3\\ &= 125a^3-3(25a^2)(3b)+3(5a)(9b^2)-(3b)^3\\ &= 125a^3-225a^2b+135ab^2-27b^3\end{aligned}$$

The following identity is also used:

$$x^3+y^3+z^3-3xyz \equiv (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$$

We can check this by performing the multiplication on the right hand side.

Note

(i) If $(x+y+z)=0$ then $x^3+y^3+z^3=3xyz$

Some identities involving sum, difference and product are stated without proof

(i) $x^3+y^3 \equiv (x+y)^3-3xy(x+y)$ (ii) $x^3-y^3 \equiv (x-y)^3+3xy(x-y)$

Example 3.22 Find the product of

$$(2x+3y+4z)(4x^2+9y^2+16z^2-6xy-12yz-8zx)$$

Solution

We know that, $(a+b+c)(a^2+b^2+c^2-ab-bc-ca) = a^3+b^3+c^3-3abc$

$$(2x+3y+4z)(4x^2+9y^2+16z^2-6xy-12yz-8zx)$$
$$= (2x)^3+(3y)^3+(4z)^3-3(2x)(3y)(4z)$$
$$= 8x^3+27y^3+64z^3-72xyz$$

Example 3.23 Evaluate $10^3-15^3+5^3$

Solution

Replace a by 10, b by −15, c by 5

We know that, if $a+b+c=0$, then $a^3+b^3+c^3=3abc$

Here, $a+b+c = 10-15+5=0$

$$\text{Therefore, } 10^3+(-15)^3+5^3 = 3(10)(-15)(5)$$
$$10^3-15^3+5^3 = -2250$$

Exercise 3.4

1. Expand the following:

   (i) $(2x+3y+4z)^2$ (ii) $(-p+2q+3r)^2$

   (iii) $(2p+3)(2p-4)(2p-5)$ (iv) $(3a+1)(3a-2)(3a+4)$

2. Using algebraic identity, find the coefficients of x^2, x and constant term without actual expansion.

   (i) $(x+5)(x+6)(x+7)$ (ii) $(2x+3)(2x-5)(2x-6)$

3. If $(x+a)(x+b)(x+c)=x^3+14x^2+59x+70$, find the value of

   (i) $a+b+c$ (ii) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ (iii) $a^2+b^2+c^2$ (iv) $\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}$

4. Expand:

   (i) $(3a-4b)^3$ (ii) $\left(x+\frac{1}{y}\right)^3$

5. Evaluate the following by using identities:

   (i) 98^3 (ii) 1001^3

6. If $(x+y+z)=9$ and $(xy+yz+zx)=26$, then find the value of $x^2+y^2+z^2$.

7. Find $27a^3+64b^3$, if $3a+4b=10$ and $ab=2$.

8. Find x^3-y^3, if $x-y=5$ and $xy=14$.

9. If $a+\frac{1}{a}=6$, then find the value of $a^3+\frac{1}{a^3}$.

10. If $x^2+\frac{1}{x^2}=23$, then find the value of $x+\frac{1}{x}$ *and* $x^3+\frac{1}{x^3}$.

11. If $\left(y-\frac{1}{y}\right)^3=27$, then find the value of $y^3-\frac{1}{y^3}$.

12. Simplify: (i) $(2a+3b+4c)(4a^2+9b^2+16c^2-6ab-12bc-8ca)$

(ii) $(x-2y+3z)(x^2+4y^2+9z^2+2xy+6yz-3xz)$

13. By using identity evaluate the following:

(i) $7^3-10^3+3^3$ (ii) $1+\frac{1}{8}-\frac{27}{8}$

14. If $2x-3y-4z=0$, then find $8x^3-27y^3-64z^3$.

3.5 Factorisation

Factorisation is the reverse of multiplication.

For Example : Multiply 3 and 5; we get product 15.

Factorise 15; we get factors 3 and 5.

For Example : Multiply $(x + 2)$ and $(x + 3)$; we get product x^2+5x+6.

Factorise x^2+5x+6 ; we get factors $(x + 2)$ and $(x + 3)$.

Multiply ↓		Factorise ↑	
	$(x+3)(x+2)$		$(x+3)(x+2)$
	$x(x+2)+3(x+2)$		$x(x+2)+3(x+2)$
	$x^2+2x+3x+6$		$x^2+2x+3x+6$
	x^2+5x+6		x^2+5x+6

Thus, the process of converting the given higher degree polynomial as the product of factors of its lower degree, which cannot be further factorised is called factorisation.

Two important ways of factorisation are :

(i) By taking common factor

$ab+ac$

$a\times b+a\times c$

$a(b+c)$ factored form

(ii) By grouping them

$a+b-pa-pb$

$(a+b)-p(a+b)$ group in pairs

$(a+b)(1-p)$ factored form

When a polynomial is factored, we "factored out" the common factor.

Example 3.24 Factorise the following:

(i) $am+bm+cm$ (ii) a^3-a^2b (iii) $5a-10b-4bc+2ac$ (iv) $x+y-1-xy$

Solutions

(i) $am+bm+cm$

$am+bm+cm$

$m(a+b+c)$ factored form

(ii) a^3-a^2b

$a^2\cdot a-a^2\cdot b$ group in pairs

$a^2\times(a-b)$ factored form

(iii) $5a-10b-4bc+2ac$

$5a-10b+2ac-4bc$

$5(a-2b)+2c(a-2b)$

$(a-2b)(5+2c)$

(iv) $x+y-1-xy$

$x-1+y-xy$

$(x-1)+y(1-x)$

$(x-1)-y(x-1)$

$(x-1)(1-y)$

$(a-b)=-(b-a)$

3.5.1 Factorisation using Identity

(i) $a^2+2ab+b^2 \equiv (a+b)^2$ (ii) $a^2-2ab+b^2 \equiv (a-b)^2$

(iii) $a^2-b^2 \equiv (a+b)(a-b)$ (iv) $a^2+b^2+c^2+2ab+2bc+2ca \equiv (a+b+c)^2$

(v) $a^3+b^3 \equiv (a+b)(a^2-ab+b^2)$ (vi) $a^3-b^3 \equiv (a-b)(a^2+ab+b^2)$

(vii) $a^3+b^3+c^3-3abc \equiv (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

Note

$(a+b)^2+(a-b)^2=2(a^2+b^2)$; $a^4-b^4=(a^2+b^2)(a+b)(a-b)$

$(a+b)^2-(a-b)^2=4ab$; $a^6-b^6=(a+b)(a-b)(a^2-ab+b^2)(a^2+ab+b^2)$

Progress Check

Prove: (i) $\left(a+\frac{1}{a}\right)^2+\left(a-\frac{1}{a}\right)^2=2\left(a^2+\frac{1}{a^2}\right)$ (ii) $\left(a+\frac{1}{a}\right)^2-\left(a-\frac{1}{a}\right)^2=4$

Example 3.25 Factorise the following:

(i) $9x^2+12xy+4y^2$ (ii) $25a^2-10a+1$ (iii) $36m^2-49n^2$

(iv) x^3-x (v) x^4-16 (vi) $x^2+4y^2+9z^2-4xy+12yz-6xz$

Solution

(i) $9x^2+12xy+4y^2 = (3x)^2+2(3x)(2y)+(2y)^2 \quad [\because a^2+2ab+b^2=(a+b)^2]$

$= (3x+2y)^2$

(ii) $25a^2-10a+1 = (5a)^2-2(5a)(1)+1^2$

$= (5a-1)^2 \quad \left[\because a^2-2ab+b^2=(a-b)^2\right]$

(iii) $36m^2-49n^2. = (6m)^2-(7n)^2$

$= (6m+7n)(6m-7n) \quad \left[\because a^2-b^2=(a+b)(a-b)\right]$

(iv) $x^3 - x = x(x^2 - 1)$

$= x(x^2 - 1^2)$

$= x(x+1)(x-1)$

(v) $x^4 - 16 = x^4 - 2^4$ $\left[\because a^4 - b^4 = (a^2+b^2)(a+b)(a-b)\right]$

$= (x^2 + 2^2)(x^2 - 2^2)$

$= (x^2 + 4)(x+2)(x-2)$

(vi) $x^2 + 4y^2 + 9z^2 - 4xy + 12yz - 6xz$

$= (-x)^2 + (2y)^2 + (3z)^2 + 2(-x)(2y) + 2(2y)(3z) + 2(3z)(-x)$

$= (-x + 2y + 3z)^2$ (or) $(x - 2y - 3z)^2$

Example 3.26 Factorise the following:

(i) $27x^3 + 125y^3$ (ii) $216m^3 - 343n^3$

(iii) $2x^4 - 16xy^3$ (iv) $8x^3 + 27y^3 + 64z^3 - 72xyz$

Solution

(i) $27x^3 + 125y^3 = (3x)^3 + (5y)^3$ $\left[\because (a^3+b^3) = (a+b)(a^2 - ab + b^2)\right]$

$= (3x+5y)\left((3x)^2 - (3x)(5y) + (5y)^2\right)$

$= (3x+5y)(9x^2 - 15xy + 25y^2)$

(ii) $216m^3 - 343n^3 = (6m)^3 - (7n)^3$ $\left[\because (a^3 - b^3) = (a-b)(a^2 + ab + b^2)\right]$

$= (6m - 7n)\left((6m)^2 + (6m)(7n) + (7n)^2\right)$

$= (6m - 7n)(36m^2 + 42mn + 49n^2)$

(iii) $2x^4 - 16xy^3 = 2x(x^3 - 8y^3)$

$= 2x\left(x^3 - (2y)^3\right)$ $\left[\because (a^3 - b^3) = (a-b)(a^2 + ab + b^2)\right]$

$= 2x\left((x-2y)(x^2 + (x)(2y) + (2y)^2)\right)$

$= 2x(x-2y)(x^2 + 2xy + 4y^2)$

(iv) $8x^3 + 27y^3 + 64z^3 - 72xyz$

$= (2x)^3 + (3y)^3 + (4z)^3 - 3(2x)(3y)(4z)$

$= (2x + 3y + 4z)(4x^2 + 9y^2 + 16z^2 - 6xy - 12yz - 8xz)$

Thinking Corner

Check 15 divides the following

(i) $2017^3 + 2018^3$

(ii) $2018^3 - 1973^3$

Exercise 3.5

1. Factorise the following expressions:
   (i) $2a^2 + 4a^2b + 8a^2c$ (ii) $ab - ac - mb + mc$

2. Factorise the following:
   (i) $x^2 + 4x + 4$ (ii) $3a^2 - 24ab + 48b^2$ (iii) $x^5 - 16x$
   (iv) $m^2 + \frac{1}{m^2} - 23$ (v) $6 - 216x^2$ (vi) $a^2 + \frac{1}{a^2} - 18$

3. Factorise the following:
   (i) $4x^2 + 9y^2 + 25z^2 + 12xy + 30yz + 20xz$
   (ii) $25x^2 + 4y^2 + 9z^2 - 20xy + 12yz - 30xz$

4. Factorise the following:
   (i) $8x^3 + 125y^3$ (ii) $27x^3 - 8y^3$ (iii) $a^6 - 64$

5. Factorise the following:
   (i) $x^3 + 8y^3 + 6xy - 1$ (ii) $l^3 - 8m^3 - 27n^3 - 18lmn$

3.5.2 Factorising the Quadratic Polynomial (Trinomial) of the type $ax^2 + bx + c,\ a \neq 0$

The linear factors of $ax^2 + bx + c$ will be in the form $(kx + m)$ and $(lx + n)$

$$\text{Thus, } ax^2 + bx + c = (kx + m)(lx + n) = klx^2 + (lm + kn)x + mn$$

Comparing coefficients of x^2, x and constant term c on both sides.
We have, $a = kl$, $b = (lm + kn)$ and $c = mn$, where ac is the product of kl and mn that is, equal to the product of lm and kn which are the coefficient of x. Therefore $(kl \times mn) = (lm \times kn)$.

Steps to be followed to factorise $ax^2 + bx + c$:

Step 1 : Multiply the coefficient of x^2 and constant term, that is ac.
Step 2 : Split ac into two factors whose sum and product is equal to b and ac respectively.
Step 3 : The terms are grouped into two pairs and factorise.

Example 3.27

Factorise $2x^2 + 15x + 27$

Solution

Compare with $ax^2 + bx + c$
we get, $a = 2,\ b = 15,\ c = 27$

product $ac = 2 \times 27 = 54$ and sum $b = 15$

We find the pair 6, 9 only satisfies "$b = 15$"

Product of factors $ac = 54$	Sum of factors $b = 15$	Product of factors $ac = 54$	Sum of factors $b = 15$
1×54	55	-1×-54	-55
2×27	29	-2×-27	-29
3×18	21	-3×-18	-21
6×9	**15**	-6×-9	-15
The required factors are **6** and **9**			

and also "$ac = 54$".

$\therefore$ we split the middle term as $6x$ and $9x$

$$2x^2+15x+27 = 2x^2+6x+9x+27$$
$$= 2x(x+3)+9(x+3)$$
$$= (x+3)(2x+9)$$

Therefore, $(x+3)$ and $(2x+9)$ are the factors of $2x^2+15x+27$.

Example 3.28 Factorise $2x^2-15x+27$

Solution

Compare with ax^2+bx+c

$a=2,\ b=-15,\ c=27$

product $ac = 2\times 27 = 54$, sum $b=-15$

$\therefore$ we split the middle term as $-6x$ and $-9x$

$$2x^2-15x+27 = 2x^2-6x-9x+27$$
$$= 2x(x-3)-9(x-3)$$
$$= (x-3)(2x-9)$$

Product of factors $ac = 54$	Sum of factors $b = -15$	Product of factors $ac = 54$	Sum of factors $b = -15$
1×54	55	-1×-54	-55
2×27	29	-2×-27	-29
3×18	21	-3×-18	-21
6×9	15	-6×-9	-15
The required factors are -6 and -9			

Therefore, $(x-3)$ and $(2x-9)$ are the factors of $2x^2-15x+27$.

Example 3.29 Factorise $2x^2+15x-27$

Solution

Compare with ax^2+bx+c

Here, $a=2,\ b=15,\ c=-27$

product $ac = 2\times -27 = -54$, sum $b=15$

$\therefore$ we split the middle term as $18x$ and $-3x$

$$2x^2+15x-27 = 2x^2+18x-3x-27$$
$$= 2x(x+9)-3(x+9)$$
$$= (x+9)(2x-3)$$

Product of factors $ac = -54$	Sum of factors $b = 15$	Product of factors $ac = -54$	Sum of factors $b = 15$
-1×54	53	1×-54	-53
-2×27	25	2×-27	-25
-3×18	15	3×-18	-15
-6×9	3	6×-9	-3
The required factors are -3 and 18			

Therefore, $(x+9)$ and $(2x-3)$ are the factors of $2x^2+15x-27$.

Example 3.30 Factorise $2x^2-15x-27$

Solution

Compare with ax^2+bx+c

Here, $a=2,\ b=-15,\ c=-27$

product $ac = 2\times -27 = -54$, sum $b=-15$

$\therefore$ we split the middle term as $-18x$ and $3x$

Product of factors $ac = -54$	Sum of factors $b = -15$	Product of factors $ac = -54$	Sum of factors $b = -15$
-1×54	53	1×-54	-53
-2×27	25	2×-27	-25
-3×18	15	3×-18	-15
-6×9	3	6×-9	-3
The required factors are 3 and -18			

$$
\begin{aligned}
2x^2 - 15x - 27 &= 2x^2 - 18x + 3x - 27 \\
&= 2x(x-9) + 3(x-9) \\
&= (x-9)(2x+3)
\end{aligned}
$$

Therefore, $(x-9)$ and $(2x+3)$ are the factors of $2x^2 - 15x - 27$

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Example 3.31 Factorise $(x+y)^2 + 9(x+y) + 20$

Solution

Let $x + y = p$, we get $p^2 + 9p + 20$

Compare with $ax^2 + bx + c$,

We get $a = 1,\ b = 9,\ c = 20$

product $ac = 1 \times 20 = 20$, sum $b = 9$

$\therefore$ we split the middle term as $4p$ and $5p$

Product of factors $ac = 20$	Sum of factors $b = 9$	Product of factors $ac = 20$	Sum of factors $b = 9$
1×20	21	-1×-20	-21
2×10	12	-2×-10	-12
4×5	9	-4×-5	-9
The required factors are 4 and 5			

$$
\begin{aligned}
p^2 + 9p + 20 &= p^2 + 4p + 5p + 20 \\
&= p(p+4) + 5(p+4) \\
&= (p+4)(p+5)
\end{aligned}
$$

Put, $p = x + y$ we get, $(x+y)^2 + 9(x+y) + 20 = (x+y+4)(x+y+5)$

Exercise 3.6

1. Factorise the following:

   (i) $x^2 + 10x + 24$ (ii) $z^2 + 4z - 12$ (iii) $p^2 - 6p - 16$

   (iv) $t^2 + 72 - 17t$ (v) $y^2 - 16y - 80$ (vi) $a^2 + 10a - 600$

2. Factorise the following:

   (i) $2a^2 + 9a + 10$ (ii) $5x^2 - 29xy - 42y^2$ (iii) $9 - 18x + 8x^2$

   (iv) $6x^2 + 16xy + 8y^2$ (v) $12x^2 + 36x^2y + 27y^2x^2$

   (vi) $\left(a+b\right)^2 + 9\left(a+b\right) + 18$

3. Factorise the following:

   (i) $(p-q)^2 - 6(p-q) - 16$ (ii) $m^2 + 2mn - 24n^2$ (iii) $\sqrt{5}a^2 + 2a - 3\sqrt{5}$

   (iv) $a^4 - 3a^2 + 2$ (v) $8m^3 - 2m^2n - 15mn^2$ (vi) $\dfrac{1}{x^2} + \dfrac{1}{y^2} + \dfrac{2}{xy}$

3.6 Division of Polynomials

Let us consider the numbers 13 and 5. When 13 is divided by 5 what is the quotient and remainder.?

Yes, of course, the quotient is 2 and the remainder is 3. We write 13 = (5×2)+3

Let us try.

Divide	Expressed as	Remainder	Divisor
11 by 4	(4×2)+3	3	4
22 by 11	(11×2)+0	0	11

Dividend = (Divisor × Quotient) + Remainder.

From the above examples, we observe that the remainder is less than the divisor.

3.6.1 Division Algorithm for Polynomials

Let $p(x)$ and $g(x)$ be two polynomials such that degree of $p(x) \geq$ degree of $g(x)$ and $g(x) \neq 0$. Then there exists unique polynomials $q(x)$ and $r(x)$ such that

$$p(x) = g(x) \times q(x) + r(x) \qquad \dots (1)$$

where $r(x) = 0$ or degree of $r(x) <$ degree of $g(x)$.

The polynomial $p(x)$ is the Dividend, $g(x)$ is the Divisor, $q(x)$ is the Quotient and $r(x)$ is the Remainder. Now (1) can be written as

Dividend = (Divisor × Quotient) + Remainder.

If $r(x)$ is zero, then we say $p(x)$ is a multiple of $g(x)$. In other words, $g(x)$ divides $p(x)$.

If it looks complicated, don't worry! it is important to know how to divide polynomials, and that comes easily with practice. The examples below will help you.

Example 3.32

Divide $x^3 - 4x^2 + 6x$ by x, where , $x \neq 0$

Solution

We have

$$\frac{x^3 - 4x^2 + 6x}{x} = \frac{x^3}{x} - \frac{4x^2}{x} + \frac{6x}{x}, x \neq 0$$

$$= x^2 - 4x + 6$$

Example 3.33

Find the quotient and the remainder when $(5x^2 - 7x + 2) \div (x - 1)$

Solution

$(5x^2 - 7x + 2) \div (x - 1)$

$$
\begin{array}{r|l}
 & \quad 5x-2 \\ \hline
x-1 & 5x^2-7x+2 \\
 & 5x^2-5x \\
 & (-)\ \ (+) \\ \hline
 & \qquad -2x+2 \\
 & \qquad -2x+2 \\
 & \qquad (+)\ \ (-) \\ \hline
 & \qquad\quad 0
\end{array}
$$

(i) $\dfrac{5x^2}{x} = 5x$

(ii) $5x(x-1) = 5x^2-5x$

(iii) $-\dfrac{2x}{x} = -2$

(iv) $-2(x-1) = (-2x+2)$

$\therefore$ Quotient $= 5x-2$ Remainder $= 0$

Example 3.34

Find quotient and the remainder when $f(x)$ is divided by $g(x)$

(i) $f(x) = (8x^3-6x^2+15x-7)$, $g(x) = 2x+1$. (ii) $f(x) = x^4-3x^3+5x^2-7$, $g(x) = x^2+x+1$

Solution

(i) $f(x) = (8x^3-6x^2+15x-7)$, $g(x) = 2x+1$

$$
\begin{array}{r|l}
 & 4x^2-5x+10 \\ \hline
2x+1 & 8x^3-6x^2+15x-7 \\
 & 8x^3+4x^2 \\
 & (-)\ \ (-) \\ \hline
 & \quad -10x^2+15x \\
 & \quad -10x^2-5x \\
 & \quad\ \ (+)\qquad (+) \\ \hline
 & \qquad\qquad 20x-7 \\
 & \qquad\qquad 20x+10 \\
 & \qquad\qquad (-)\quad (-) \\ \hline
 & \qquad\qquad\qquad -17
\end{array}
$$

$\therefore$ Quotient $= 4x^2-5x+10$ and

Remainder $= -17$

(ii) $f(x) = x^4-3x^3+5x^2-7$, $g(x) = x^2+x+1$

$$
\begin{array}{r|l}
 & x^2-4x+8 \\ \hline
x^2+x+1 & x^4-3x^3+5x^2+0x-7 \\
 & x^4+x^3+x^2 \\
 & (-)\ \ (-)\ \ (-) \\ \hline
 & \qquad -4x^3+4x^2+0x \\
 & \qquad -4x^3-4x^2-4x \\
 & \qquad\ \ (+)\quad (+)\quad (+) \\ \hline
 & \qquad\qquad\quad 8x^2+4x-7 \\
 & \qquad\qquad\quad 8x^2+8x+8 \\
 & \qquad\qquad\quad (-)\ \ (-)\ \ (-) \\ \hline
 & \qquad\qquad\qquad -4x-15
\end{array}
$$

$\therefore$ Quotient $= x^2-4x+8$ and

Remainder $= -4x-15$

3.6.2 Synthetic Division

Synthetic Division is a shortcut method of polynomial division. The advantage of synthetic division is that it allows one to calculate without writing variables, than long division.

Example 3.35

Find the quotient and remainder when $p(x) = (3x^3-2x^2-5+7x)$ is divided by $d(x) = x+3$ using synthetic division.

Solution

Step 1 Arrange dividend and the divisor in standard form.

$3x^3 - 2x^2 + 7x - 5$ (standard form of dividend)

$x + 3$ (standard form of divisor)

Write the coefficients of dividend in the first row. Put '0' for missing term(s).

3 −2 7 −5 (first row)

Step 2 Find out the zero of the divisor.

$x + 3 = 0$ implies $x = -3$

Step 3 Write the zero of divisor in front of dividend in the first row. Put '0' in the first column of second row.

$$\begin{array}{r|cccc l} -3 & 3 & -2 & 7 & -5 & \text{(first row)} \\ & 0 & & & & \text{(second row)} \\ \hline \end{array}$$

Step 4 Complete the second row and third row as shown below.

$$\begin{array}{r|ccccccc l} -3 & 3 & & -2 & & 7 & & -5 & \text{(first row)} \\ & 0 & -3\times 3 & -9 & -3\times -11 & 33 & -3\times 40 & -120 & \text{(second row)} \\ \hline & 3 & & -11 & & 40 & & -125 & \text{(third row)} \end{array}$$

All the entries except the last one in the third row are the coefficients of the quotient.

Then quotient is $3x^2 - 11x + 40$ and remainder is -125.

Example 3.36 Find the quotient and remainder when $(3x^3 - 4x^2 - 5)$ is divided by $(3x+1)$ using synthetic division.

Solution

Let $p(x) = 3x^3 - 4x^2 - 5,\ d(x) = (3x + 1)$

Standard form: $p(x) = 3x^3 - 4x^2 + 0x - 5$ and $d(x) = 3x + 1$

To find the zero of $3x+1$;

put $3x + 1 = 0$

we get $3x = -1$

$x = -\frac{1}{3}$

$$\begin{array}{r|cccc l} -\frac{1}{3} & 3 & -4 & 0 & -5 & \\ & 0 & -1 & \frac{5}{3} & \frac{-5}{9} & \\ \hline & 3 & -5 & \frac{5}{3} & \frac{-50}{9} & \text{(remainder)} \end{array}$$

$$3x^3 - 4x^2 - 5 = \left(x + \frac{1}{3}\right)\left(3x^2 - 5x + \frac{5}{3}\right) - \frac{50}{9}$$

$$3x^3 - 4x^2 - 5 = \frac{(3x+1)}{3} \times 3\left(x^2 - \frac{5}{3}x + \frac{5}{9}\right) - \frac{50}{9}$$

$$= (3x+1)\left(x^2 - \frac{5}{3}x + \frac{5}{9}\right) - \left(\frac{50}{9}\right) \qquad \text{(since, } p(x) = d(x)q(x) + r\text{)}$$

Hence the quotient is $\left(x^2 - \frac{5}{3}x + \frac{5}{9}\right)$ and remainder is $\frac{-50}{9}$

Example 3.37 If the quotient on dividing $x^4 + 10x^3 + 35x^2 + 50x + 29$ by $(x+4)$ is $x^3 - ax^2 + bx + 6$, then find the value of a, b and also remainder.

Solution

Let $p(x) = x^4 + 10x^3 + 35x^2 + 50x + 29$

Standard form $= x^4 + 10x^3 + 35x^2 + 50x + 29$

Coefficient are 1 10 35 50 29

> To find the zero of $x+4$;
> put $x + 4 = 0$
> we get $x = -4$

–4	1	10	35	50	29	
	0	–4	–24	–44	–24	
	1	6	11	6	5	(remainder)

quotient $x^3 + 6x^2 + 11x + 6$ is compared with given quotient $x^3 - ax^2 + bx + 6$

coefficient of x^2 is $6 = -a$ and coefficient of x is $11 = b$

Therefore, $a = -6$, $b = 11$ and remainder$= 5$.

Exercise 3.7

1. Find the quotient and remainder of the following.

   (i) $(4x^3 + 6x^2 - 23x + 18) \div (x+3)$ (ii) $(8y^3 - 16y^2 + 16y - 15) \div (2y-1)$

   (iii) $(8x^3 - 1) \div (2x-1)$ (iv) $(-18z + 14z^2 + 24z^3 + 18) \div (3z+4)$

2. The area of a rectangle is $x^2 + 7x + 12$. If its breadth is $(x+3)$, then find its length.

3. The base of a parallelogram is $(5x+4)$. Find its height, if the area is $25x^2-16$.

4. The sum of $(x+5)$ observations is (x^3+125). Find the mean of the observations.

5. Find the quotient and remainder for the following using synthetic division:

   (i) $(x^3 + x^2 - 7x - 3) \div (x - 3)$ (ii) $(x^3 + 2x^2 - x - 4) \div (x + 2)$

   (iii) $(3x^3 - 2x^2 + 7x - 5) \div (x + 3)$ (iv) $(8x^4 - 2x^2 + 6x + 5) \div (4x + 1)$

6. If the quotient obtained on dividing $(8x^4 - 2x^2 + 6x - 7)$ by $(2x + 1)$ is $(4x^3 + px^2 - qx + 3)$, then find p, q and also the remainder.

7. If the quotient obtained on dividing $3x^3 + 11x^2 + 34x + 106$ by $x - 3$ is $3x^2 + ax + b$, then find a, b and also the remainder.

3.6.3 Factorisation using Synthetic Division

In this section, we use the synthetic division method that helps to factorise a cubic polynomial into linear factors. If we identify one linear factor of cubic polynomial $p(x)$ then using synthetic division we can get the quadratic factor of $p(x)$. Further if possible one can factorise the quadratic factor into linear factors.

Note

- For any non constant polynomial $p(x)$, $x = a$ is zero if and only if $p(a) = 0$
- $x-a$ is a factor for $p(x)$ if and only if $p(a) = 0$ (Factor theorem)

To identify $(x-1)$ and $(x+1)$ are the factors of a polynomial

- $(x-1)$ is a factor of $p(x)$ if and only if the sum of coefficients of $p(x)$ is 0.
- $(x+1)$ is a factor of $p(x)$ if and only if the sum of the coefficients of even power of x, including constant is equal to the sum of the coefficients of odd powers of x

Example 3.38

(i) Prove that $(x-1)$ is a factor of $x^3 - 7x^2 + 13x - 7$

(ii) Prove that $(x+1)$ is a factor of $x^3 + 7x^2 + 13x + 7$

Solution

(i) Let $p(x) = x^3 - 7x^2 + 13x - 7$

Sum of coefficients $= 1 - 7 + 13 - 7 = 0$

Thus $(x-1)$ is a factor of $p(x)$

(ii) Let $q(x) = x^3 + 7x^2 + 13x + 7$

Sum of coefficients of even powers of x and constant term = 7 + 7 = 14

Sum of coefficients of odd powers of x = 1 + 13 = 14

Hence, $(x+1)$ is a factor of $q(x)$

Example 3.39

Factorise $x^3 + 13x^2 + 32x + 20$ into linear factors.

Solution

Let, $p(x) = x^3 + 13x^2 + 32x + 20$

Sum of all the coefficients $= 1 + 13 + 32 + 20 = 66 \neq 0$

Hence, $(x-1)$ is not a factor.

Sum of coefficients of even powers and constant term = 13 + 20 = 33

Sum of coefficients of odd powers = 1 + 32 = 33

Hence, $(x+1)$ is a factor of $p(x)$

Now we use synthetic division to find the other factors

Method I	Method II
$\begin{array}{r\|rrrl} -1 & 1 & 13 & 32 & 20 \\ & 0 & -1 & -12 & -20 \\ \hline -2 & 1 & 12 & 20 & \boxed{0 \text{ (remainder)}} \\ & 0 & -2 & -20 & \\ \hline & 1 & 10 & \boxed{0 \text{ (remainder)}} & \end{array}$	$\begin{array}{r\|rrrl} -1 & 1 & 13 & 32 & 20 \\ & 0 & -1 & -12 & -20 \\ \hline & 1 & 12 & 20 & \boxed{0 \text{ (remainder)}} \end{array}$
$p(x) = (x+1)(x+2)(x+10)$	Then $p(x) = (x+1)(x^2+12x+20)$
Hence,	Now $x^2+12x+20 = x^2+10x+2x+20$
$x^3+13x^2+32x+20$	$= x(x+10)+2(x+10)$
$= (x+1)(x+2)(x+10)$	$= (x+2)(x+10)$
	Hence, $x^3+13x^2+32x+20$ $= (x+1)(x+2)(x+10)$

Example 3.40 Factorise $x^3-5x^2-2x+24$

Solution

Let $p(x) = x^3-5x^2-2x+24$

When $x = 1, p(1) = 1-5-2+24 = 18 \neq 0$ $\quad (x-1)$ is not a factor.

When $x = -1,\ p(-1) = -1-5+2+24 = 20 \neq 0$ $\quad (x+1)$ is not a factor.

Therefore, we have to search for different values of x by trial and error method.

When $x = 2$

$p(2) = 2^3-5(2)^2-2(2)+24$

$= 8-20-4+24$

$= 8 \neq 0$ Hence, $(x–2)$ is not a factor

When $x = -2$

$p(-2) = (-2)^3-5(-2)^2-2(-2)+24$

$= -8-20+4+24$

$p(-2) = 0$

Hence, $(x+2)$ is a factor

Note

Check whether 3 is a zero of $x^2-7x+12$. If it is not, then check for –3 or 4 or –4 and so on.

$$\begin{array}{r|rrrl} -2 & 1 & -5 & -2 & 24 \\ & 0 & -2 & +14 & -24 \\ \hline 3 & 1 & -7 & 12 & \boxed{0 \text{ (remainder)}} \\ & 0 & 3 & -12 & \\ \hline & 1 & -4 & \boxed{0 \text{ (remainder)}} & \end{array}$$

Thus, $(x+2)(x-3)(x-4)$ are the factors.

Therefore, $x^3-5x^2-2x^2+24 = (x+2)(x-3)(x-4)$

Exercise 3.8

1. Factorise each of the following polynomials using synthetic division:

(i) $x^3 - 3x^2 - 10x + 24$ (ii) $2x^3 - 3x^2 - 3x + 2$

(iii) $-7x + 3 + 4x^3$ (iv) $x^3 + x^2 - 14x - 24$

(v) $x^3 - 7x + 6$ (vi) $x^3 - 10x^2 - x + 10$

3.7 Greatest Common Divisor (GCD)

The Greatest Common Divisor, abbreviated as GCD, of two or more polynomials is a polynomial, of the highest common possible degree, that is a factor of the given two or more polynomials. It is also known as the Highest Common Factor (HCF).

This concept is similar to the greatest common divisor of two integers.

For example, Consider the expressions $14xy^2$ and $42xy$. The common divisors of 14 and 42 are 2, 7 and 14. Their GCD is thus 14. The only common divisors of xy^2 and xy *are* x, y and xy; their GCD is thus xy.

$$14xy^2 = 1 \times 2 \times 7 \times x \times y \times y$$

$$42xy = 1 \times 2 \times 3 \times 7 \times x \times y$$

Therefore the requried GCD of $14xy^2$ and $42xy$ is $14xy$.

To find the GCD by Factorisation

(i) Each expression is to be resolved into factors first.

(ii) The product of factors having the highest common powers in those factors will be the GCD.

(iii) If the expression have numerical coefficient, find their GCD separately and then prefix it as a coefficient to the GCD for the given expressions.

Example 3.41 Find GCD of the following:

(i) $16x^3y^2, \ 24xy^3z$

(ii) $(y^3 + 1)$ and $(y^2 - 1)$

(iii) $2x^2 - 18$ and $x^2 - 2x - 3$

(iv) $(a-b)^2, (b-c)^3, (c-a)^4$

Solutions

(i) $16x^3y^2 = 2 \times 2 \times 2 \times 2 \times x^3y^2 = 2^4 \times x^3 \times y^2 = 2^3 \times 2 \times x^2 \times x \times y^2$

$$24xy^3z = 2\times2\times2\times3\times x\times y^3\times z = 2^3\times3\times x\times y^3\times z = 2^3\times3\times x\times y\times y^2\times z$$

Therefore, $GCD = 2^3xy^2$

(ii) $y^3+1 = y^3+1^3 = (y+1)(y^2-y+1)$

$y^2-1 = y^2-1^2 = (y+1)(y-1)$

Therefore, $GCD = (y+1)$

(iii) $2x^2-18 = 2(x^2-9) = 2(x^2-3^2) = 2(x+3)(x-3)$

$$x^2-2x-3 = x^2-3x+x-3$$

$$= x(x-3)+1(x-3)$$

$$= (x-3)(x+1)$$

Therefore, $GCD = (x-3)$

(iv) $(a-b)^2,\ (b-c)^3,\ (c-a)^4$

There is no common factor other than one.

Therefore, $GCD = 1$

Exercise 3.9

1. Find the GCD for the following:

(i) $p^5,\ p^{11},\ p^9$ (ii) $4x^3,\ y^3,\ z^3$

(iii) $9a^2b^2c^3,\ 15a^3b^2c^4$ (iv) $64x^8,\ 240x^6$

(v) $ab^2c^3,\ a^2b^3c,\ a^3bc^2$ (vi) $35x^5y^3z^4,\ 49x^2yz^3,\ 14xy^2z^2$

(vii) $25ab^3c,\ 100a^2bc,\ 125ab$ (viii) $3abc, 5xyz, 7pqr$

2. Find the GCD of the following:

(i) $(2x+5),\ (5x+2)$ (ii) $a^{m+1},\ a^{m+2},\ a^{m+3}$

(iii) $2a^2+a,\ 4a^2-1$ (iv) $3a^2,\ 5b^3,\ 7c^4$

(v) $x^4-1,\ x^2-1$ (vi) $a^3-9ax^2,\ (a-3x)^2$

3.8 Linear Equation in Two Variables

A linear equation in two variables is of the form $ax + by + c = 0$ where a, b and c are real numbers, both a and b are not zero (The two variables are denoted here by x and y and c is a constant).

Examples

Linear equation in two variables	Not a linear equation in two variables.
$2x + y = 4$	$xy + 2x = 5$ (Why?)
$-5x + \frac{1}{2} = y$	$\sqrt{x} + \sqrt{y} = 25$ (Why?)
$5x = 35y$	$x(x+1) = y$ (Why?)

If an equation has two variables each of which is in first degree such that the variables are not multiplied with each other, then it is a linear equation in two variables (If the degree of an equation in two variables is 1, then it is called a linear equation in two variables).

An understanding of linear equation in two variables will be easy if it is done along with a geometrical visualization (through graphs). We will make use of this resource.

Why do we classify, for example, the equation $2x + y = 4$ is a linear equation? You are right; because its graph will be a line. Shall we check it up?

We try to draw its graph. To draw the graph of $2x + y = 4$, we need some points on the line so that we can join them. (These are the ordered pairs satisfying the equation).

To prepare table giving ordered pairs for $2x + y = 4$. It is better, to take it as

$y = 4 - 2x$. (Why? How?)

When $x = -4$, $y = 4 - 2(-4) = 4 + 8 = 12$

When $x = -2$, $y = 4 - 2(-2) = 4 + 4 = 8$

When $x = 0$, $y = 4 - 2(0) = 4 + 0 = 4$

When $x = +1$, $y = 4 - 2(+1) = 4 - 2 = 2$

When $x = +3$, $y = 4 - 2(+3) = 4 - 6 = -2$

Thus the values are tabulated as follows:

x-value	-4	-2	0	1	3
y-value	12	8	4	2	-2

(To fix a line, do we need so many points? It is enough if we have two and probably one more for verification.)

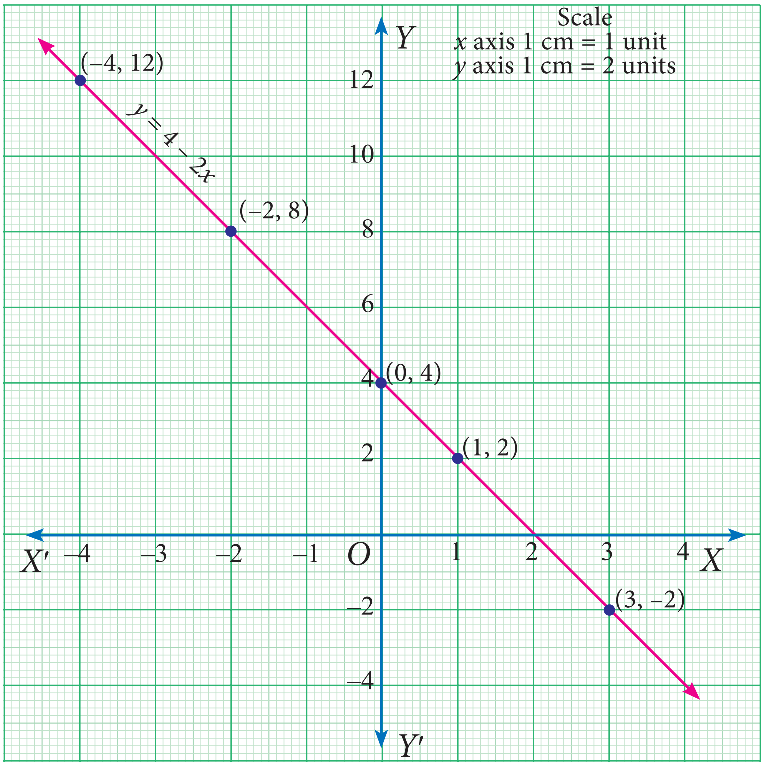


Fig. 3.15

When you plot the points (–4,12), (–2,8), (0,4), (1,2) and (3,–2), you find that they all lie on a line.

This clearly shows that the equation $2x + y = 4$ represents a line (and hence said to be linear).

All the points on the line satisfy this equation and hence the ordered pairs of all the points on the line are the solutions of the equation.

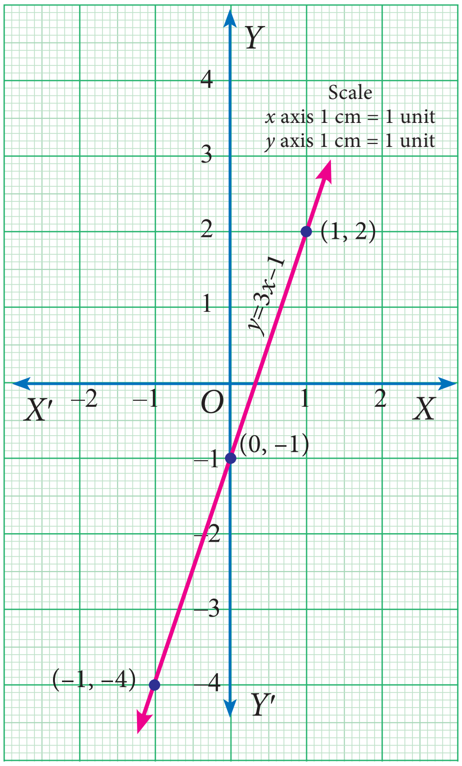


Fig. 3.16

Example 3.42

Draw the graph for the following:

(i) $y = 3x - 1$ (ii) $y = \left(\dfrac{2}{3}\right)x + 3$

Solution

(i) Let us prepare a table to find the ordered pairs of points for the line $y = 3x - 1$.

We shall assume any value for x, for our convenience let us take –1, 0 and 1.

When $x = -1$, $y = 3(-1) - 1 = -4$

When $x = 0$, $y = 3(0) - 1 = -1$

When $x = 1$, $y = 3(1) - 1 = 2$

x	-1	0	1
y	-4	-1	2

The points (x,y) to be plotted :

$(-1,-4)$, $(0,-1)$ and $(1,2)$.

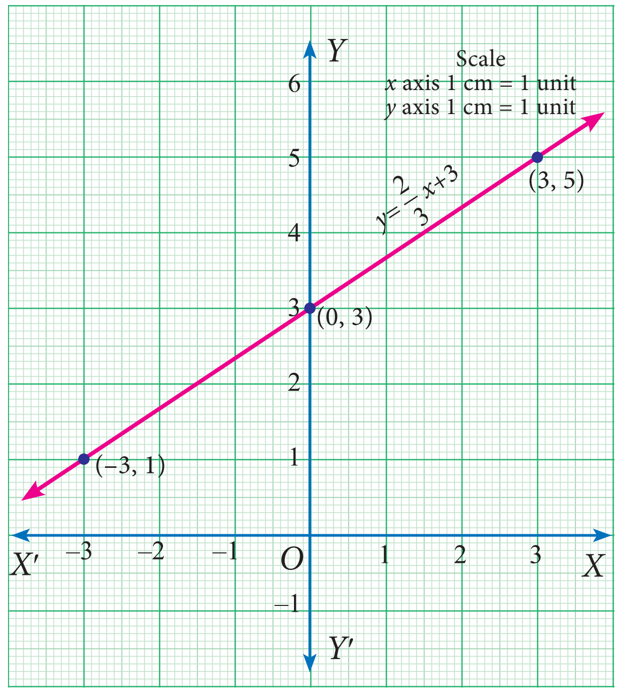


Fig. 3.17

(ii) Let us prepare a table to find the ordered pairs of points

for the line $y=\left(\frac{2}{3}\right)x+3$.

Let us assume $-3, 0, 3$ as x values. (why?)

When $x=-3$, $y=\frac{2}{3}(-3)+3=1$

When $x=0$, $y=\frac{2}{3}(0)+3 = 3$

When $x=3$, $y=\frac{2}{3}(3)+3 = 5$

x	-3	0	3
y	1	3	5

The points (x,y) to be plotted :

$(-3,1)$, $(0,3)$ and $(3,5)$.

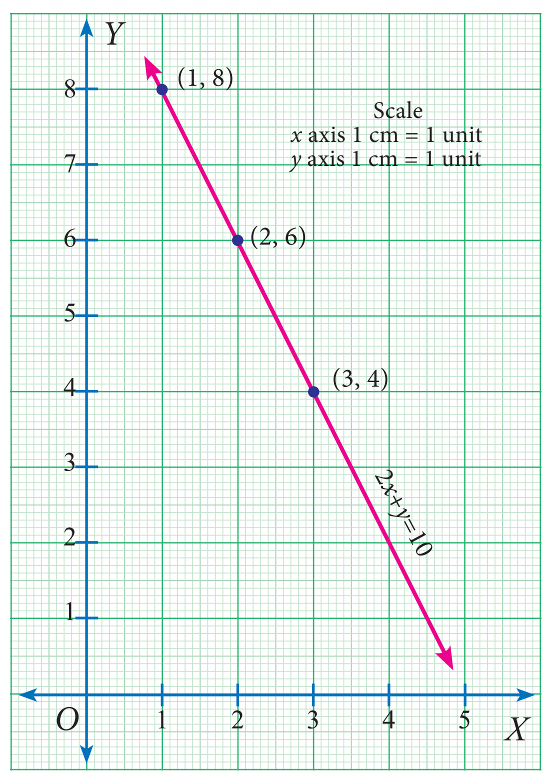


Fig. 3.18

3.8.1 Simultaneous Linear Equations

With sufficient background of graphing an equation, now we are set to study about system of equations, particularly pairs of simultaneous equations.

What are simultaneous linear equations? These consists of two or more linear equations with the same variables.

Why do we need them? A single equation like $2x+y=10$ has an unlimited number of solutions. The points (1,8), (2,6), (3,4) and many more lie on the graph of the equation, which means these are some of its endless list of solutions. To be able to solve an equation like this, another equation needs to be used alongside it; then it is possible to find a single ordered pair that solves both equations at the same time.

The equations we consider together in such settings make a meaningful situation and are known as simultaneous linear equations.

Real life Situation to understand the simultaneous linear equations

Consider the situation, Anitha bought two erasers and a pencil for ₹10. She does not know the individual cost of each. We shall form an equation by considering the cost of eraser as 'x' and that of pencil as 'y'.

That is $2x + y = 10$ ... (1)

Now, Anitha wants to know the individual cost of an eraser and a pencil. She tries to solve the first equation, assuming various values of x and y.

2 × cost of eraser + 1 × cost of pencil = 10

$2(1)+8 = 10$

$2(1.5)+7 = 10$

$2(2)+6 = 10$

$2(2.5)+5 = 10$

$2(3)+4 = 10$

⋮ ⋮

Points to be plotted :

x	1	1.5	2	2.5	3	...
y	8	7	6	5	4	...

She gets infinite number of answers. So she tries to find the cost with the second equation.

Again, Anitha needs some more pencils and erasers. This time, she bought 3 erasers and 4 pencils and the shopkeeper received ₹30 as the total cost from her. We shall form an equation like the previous one.

The equation is $3x + 4y = 30$ ...(2)

Even then she arrives at an infinite number of answers.

3 × cost of eraser + 4 × cost of pencil = 30

$3(2)+4(6) = 30$

$3(4)+4(4.5) = 30$

$3(6)+4(3) = 30$

$3(8)+4(1.5) = 30$

⋮ ⋮

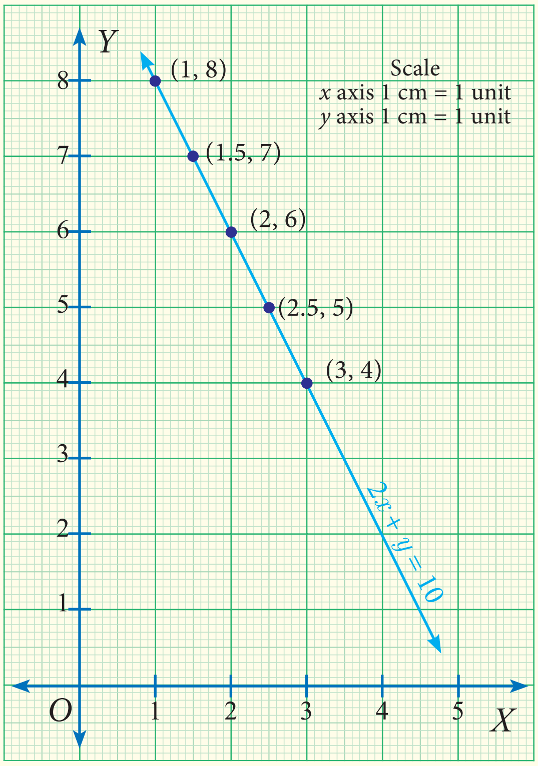


Fig. 3.19

Points to be plotted :

x	2	4	6	8	...
y	6	4.5	3	1.5	...

While discussing this with her teacher, the teacher suggested that she can get a unique answer if she solves both the equations together.

By solving equations (1) and (2) we have the cost of an eraser as ₹2 and cost of a pencil as ₹6. It can be visualised in the graph.

The equations we consider together in such settings make a meaningful situation and are known as simultaneous linear equations.

Thus a system of linear equations consists of two or more linear equations with the same variables. Then such equations are called Simultaneous linear equations or System of linear equations or a Pair of linear equations.

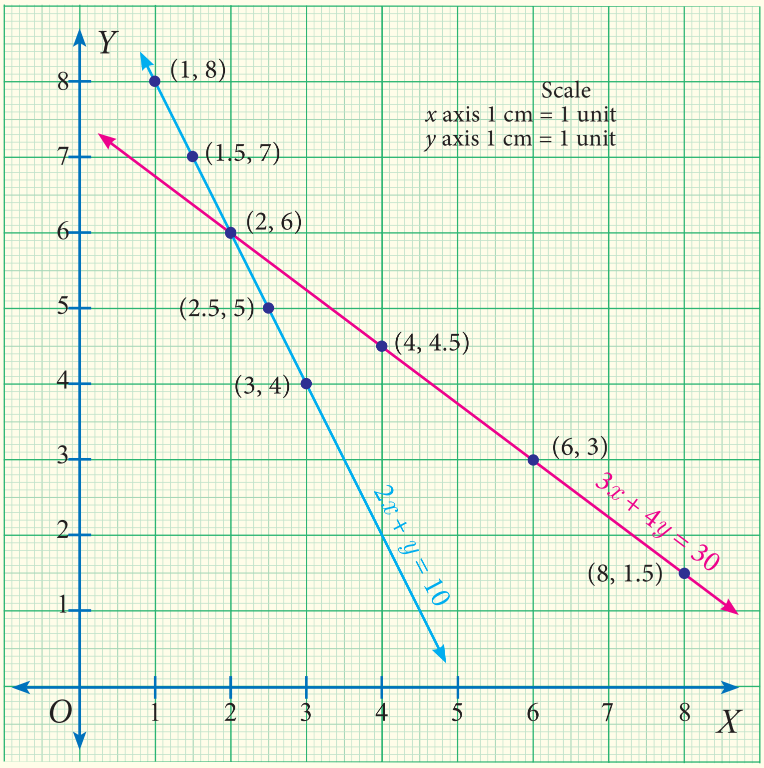


Fig. 3.20

Example 3.43

Check whether (5, −1) is a solution of the simultaneous equations $x - 2y = 7$ and $2x + 3y = 7$.

Solution

Given $\quad x - 2y = 7 \quad \ldots(1)$

$2x + 3y = 7 \quad \ldots(2)$

When $x = 5,\ y = -1$ we get

From (1) $\quad x - 2y = 5 - 2(-1) = 5 + 2 = 7$ which is RHS of (1)

From (2) $\quad 2x + 3y = 2(5) + 3(-1) = 10-3 = 7$ which is RHS of (2)

Thus the values x = 5, y = −1 satisfy both (1) and (2) simultaneously. Therefore (5,−1) is a solution of the given equations.

Progress Check

Examine if (3,3) will be a solution for the simultaneous linear equations $2x - 5y - 2 = 0$ and $x + y - 6 = 0$ by drawing a graph.

3.8.2 Methods of solving simultaneous linear equations

There are different methods to find the solution of a pair of simultaneous linear equations. It can be broadly classified as geometric way and algebraic ways.

Geometric way	Algebraic ways
1. Graphical method	1. Substitution method 2. Elimination method 3. Cross multiplication method

Solving by Graphical Method

Already we have seen graphical representation of linear equation in two variables. Here we shall learn, how we are graphically representing a pair of linear equations in two variables and find the solution of simultaneous linear equations.

Example 3.44 Use graphical method to solve the following system of equations: $x + y = 5$; $2x - y = 4$.

Solution

Given $x + y = 5$ ...(1)

$2x - y = 4$ ...(2)

To draw the graph (1) is very easy. We can find the x and y values and thus two of the points on the line (1).

When $x = 0$, (1) gives $y = 5$.

Thus $A(0,5)$ is a point on the line.

When $y = 0$, (1) gives $x = 5$.

Thus $B(5,0)$ is another point on the line.

Plot A and B; join them to produce the line (1).

To draw the graph of (2), we can adopt the same procedure.

When $x = 0$, (2) gives $y = -4$.

Thus P(0,–4) is a point on the line.

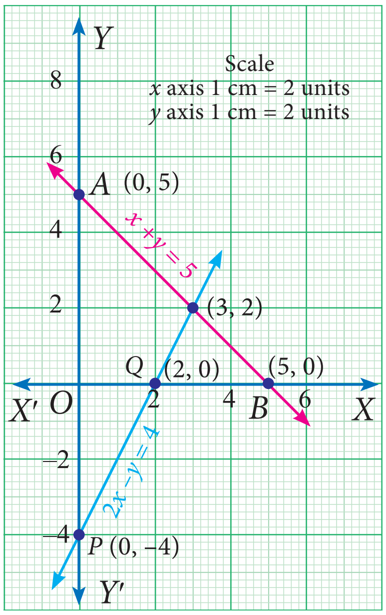


Fig. 3.21

When $y = 0$, (2) gives $x = 2$.

Thus $Q(2,0)$ is another point on the line.

Plot P and Q; join them to produce the line (2).

The point of intersection (3, 2) of lines (1) and (2) is a solution.

The solution is the point that is common to both the lines. Here we find it to be (3,2). We can give the solution as $x = 3$ and $y = 2$.

Note

It is always good to verify if the answer obtained is correct and satisfies both the given equations.

Example 3.45 Use graphical method to solve the following system of equations: $3x + 2y = 6$; $6x + 4y = 8$

Solution

Let us form table of values for each line and then fix the ordered pairs to be plotted.

Graph of $3x + 2y = 6$

x	-2	0	2
y	6	3	0

Points to be plotted :

$(-2,6)$, $(0,3)$, $(2,0)$

Graph of $6x + 4y = 8$

x	-2	0	2
y	5	2	-1

Points to be plotted :

$(-2,5)$, $(0,2)$, $(2,-1)$

When we draw the graphs of these two equations, we find that they are parallel and they fail to meet to give a point of intersection. As a result there is no ordered pair that can be common to both the equations. In this case there is no solution to the system.

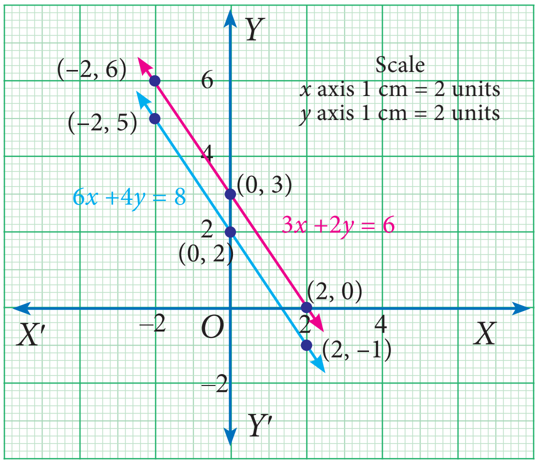


Fig. 3.22

Example 3.46 Use graphical method to solve the following system of equations: $y = 2x + 1$; $-4x + 2y = 2$

Solution

Let us form table of values for each line and then fix the ordered pairs to be plotted.

Graph of $y = 2x + 1$

Graph of $-4x + 2y = 2$

$$2y = 4x + 2$$

$$y = 2x + 1$$

x	-2	-1	0	1	2
$2x$	-4	-2	0	2	4
1	1	1	1	1	1
$y = 2x+1$	-3	-1	1	3	5

Points to be plotted :

$(-2, -3), (-1, -1), (0,1), (1,3), (2, 5)$

x	-2	-1	0	1	2
$2x$	-4	-2	0	2	4
1	1	1	1	1	1
$y = 2x+1$	-3	-1	1	3	5

Points to be plotted :

$(-2, -3), (-1, -1), (0, 1), (1, 3), (2, 5)$

Here both the equations are identical; they were only represented in different forms. Since they are identical, their solutions are same. All the points on one line are also on the other!

This means we have an infinite number of solutions which are the ordered pairs of all the points on the line.

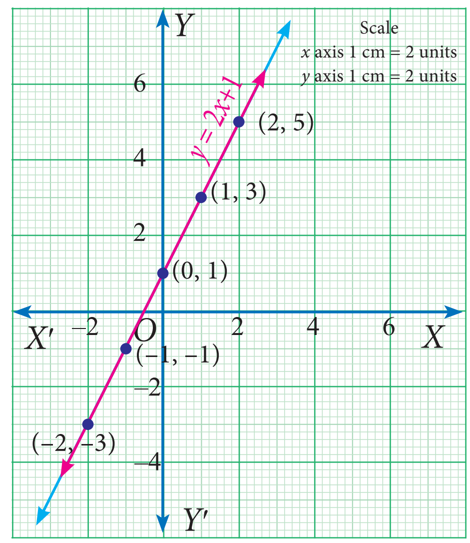


Fig. 3.23

Example 3.47

The perimeter of a rectangle is 36 metres and the length is 2 metres more than three times the width. Find the dimension of rectangle by using the method of graph.

Solution

Let us form equations for the given statement.

Let us consider l and b as the length and breadth of the rectangle respectively.

Now let us frame the equation for the first statement

Perimeter of rectangle = 36

$$2(l+b) = 36$$

$$l+b = \frac{36}{2}$$

$$l = 18 - b \qquad \ldots (1)$$

b	2	4	5	8
18	18	18	18	18
$-$b	-2	-4	-5	-8
l = 18$-b$	16	14	13	10

Points: (2,16), (4,14), (5,13), (8,10)

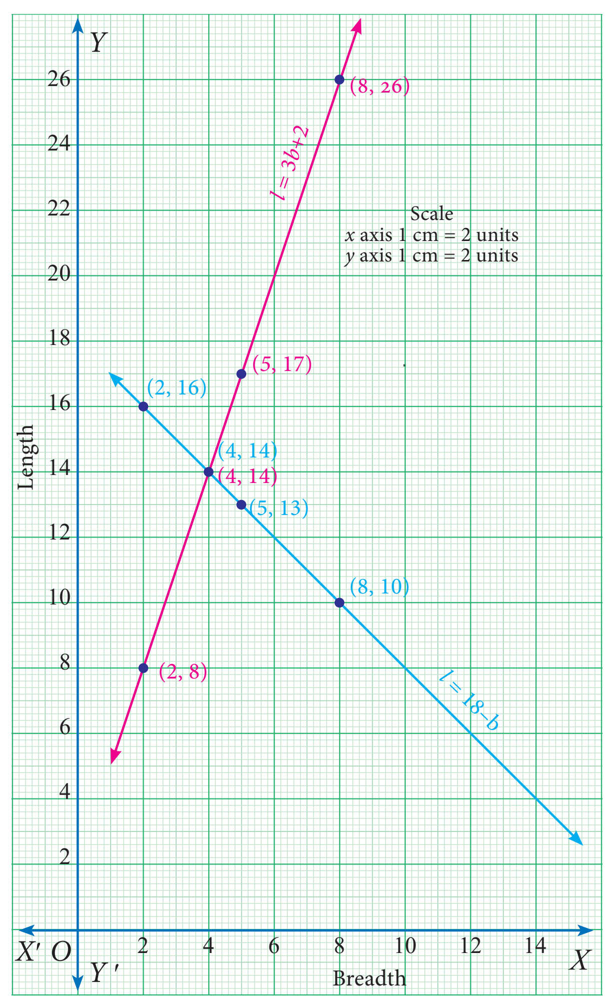


Fig. 3.24

The second statement states that the length is 2 metres more than three times the width which is a straight line written as $l = 3b + 2$ ... (2)

Now we shall form table for the above equation (2).

b	2	4	5	8
$3b$	6	12	15	24
2	2	2	2	2
$l = 3b + 2$	8	14	17	26

Points: (2,8), (4,14), (5,17), (8,26)

The solution is the point that is common to both the lines. Here we find it to be (4,14). We can give the solution to be $b = 4$, $l = 14$.

Verification :

$2(l+b) = 36$ ...(1)	$l = 3b + 2$ ...(2)
$2(14+4) = 36$	$14 = 3(4) + 2$
$2 \times 18 = 36$	$14 = 12 + 2$
$36 = 36$ true	$14 = 14$ true

Exercise 3.10

1. Draw the graph for the following

   (i) $y = 2x$ (ii) $y = 4x - 1$ (iii) $y = \left(\frac{3}{2}\right)x + 3$ (iv) $3x + 2y = 14$

2. Solve graphically

   (i) $x + y = 7$; $x - y = 3$ (ii) $3x + 2y = 4$; $9x + 6y - 12 = 0$

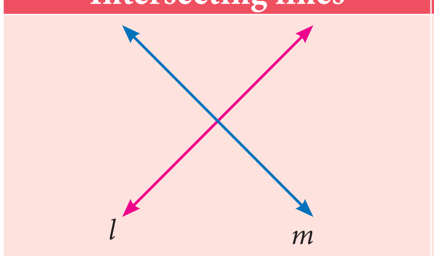
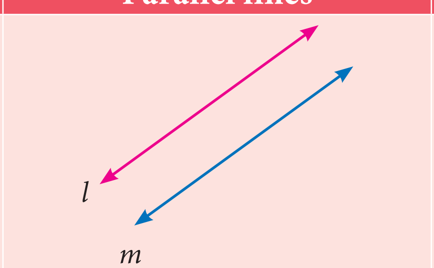
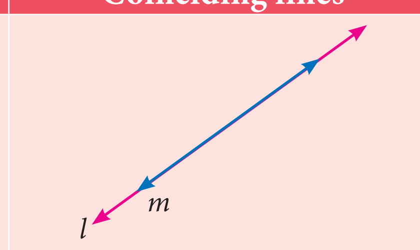
   (iii) $\frac{x}{2} + \frac{y}{4} = 1$; $\frac{x}{2} + \frac{y}{4} = 2$ (iv) $x - y = 0$; $y + 3 = 0$

   (v) $y = 2x + 1$; $y + 3x - 6 = 0$ (vi) $x = -3$; $y = 3$

3. Two cars are 100 miles apart. If they drive towards each other they will meet in 1 hour. If they drive in the same direction they will meet in 2 hours. Find their speed by using graphical method.

Some special terminology

We found that the graphs of equations within a system tell us how many solutions are there for that system. Here is a visual summary.

Intersecting lines	Parallel lines	Coinciding lines
		
One single solution	*No solution*	*Infinite number of solutions*

- When a system of linear equation has one solution (the graphs of the equations intersect once), the system is said to be a consistent system.
- When a system of linear equation has no solution (the graphs of the equations don't intersect at all), the system is said to be an inconsistent system.
- When a system of linear equation has infinitely many solutions, the lines are the same (the graph of lines are identical at all points), the system is consistent.

Solving by Substitution Method

In this method we substitute the value of one variable, by expressing it in terms of the other variable to reduce the given equation of two variables into equation of one variable (in order to solve the pair of linear equations). Since we are substituting the value of one variable in terms of the other variable, this method is called substitution method.

The procedure may be put shortly as follows:

Step 1: From any of the given two equations, find the value of one variable in terms of the other.

Step 2: Substitute the value of the variable, obtained in step 1 in the other equation and solve it.

Step 3: Substitute the value of the variable obtained in step 2 in the result of step 1 and get the value of the remaining unknown variable.

Example 3.48 Solve the system of linear equations $x+3y=16$ and $2x-y=4$ by substitution method.

Solution

Given $x+3y=16$ ... (1)

$2x-y=4$ ... (2)

Step 1	Step 2	Step 3	Solution
From equation (2) $2x-y=4$ $-y=4-2x$ $y=2x-4$ ...(3)	Substitute (3) in (1) $x+3y=16$ $x+3(2x-4)=16$ $x+6x-12=16$ $7x=28$ $x=4$	Substitute $x=4$ in (3) $y=2x-4$ $y=2(4)-4$ $y=4$	$x=4$ and $y=4$

Example 3.49 The sum of the digits of a given two digit number is 5. If the digits are reversed, the new number is reduced by 27. Find the given number.

Solution

Let x be the digit at ten's place and y be the digit at unit place.

Given that $x+y=5$ …… (1)

	Tens	Ones	Value
Given Number	x	y	$10x + y$
New Number (after reversal)	y	x	$10y+x$

Given, Original number − reversing number $= 27$

$$(10x+y)-(10y+x)=27$$
$$10x-x+y-10y=27$$
$$9x-9y=27$$
$$\Rightarrow \quad x-y=3 \quad \text{... (2)}$$

Also from (1), $y = 5 - x$ ... (3)

Substitute (3) in (2) to get $x-(5-x)=3$

$$x-5+x=3$$
$$2x=8$$
$$x=4$$

Substituting $x=4$ in (3), we get $y=5-x=5-4$

$$y=1$$

Thus, $10x+y=10\times 4+1=40+1=41$.

Therefore, the given two-digit number is 41.

Verification :

sum of the digits = 5

$x + y = 5$

$4 + 1 = 5$

$5 = 5$ true

Original number − reversed number = 27

$41 - 14 = 27$

$27 = 27$ true

Exercise 3.11

1. Solve, using the method of substitution

   (i) $2x-3y=7;\ 5x+y=9$

   (ii) $1.5x+0.1y=6.2;\ 3x-0.4y=11.2$

   (iii) $10\%\ of\ x+20\%\ of\ y=24;\ 3x-y=20$

   (iv) $\sqrt{2}\,x-\sqrt{3}\,y=1;\ \sqrt{3}\,x-\sqrt{8}\,y=0$

2. Raman's age is three times the sum of the ages of his two sons. After 5 years his age will be twice the sum of the ages of his two sons. Find the age of Raman.

3. The middle digit of a number between 100 and 1000 is zero and the sum of the other digit is 13. If the digits are reversed, the number so formed exceeds the original number by 495. Find the number.

Solving by Elimination Method

This is another algebraic method for solving a pair of linear equations. This method is more convenient than the substitution method. Here we eliminate (i.e. remove) one of the two variables in a pair of linear equations, so as to get a linear equation in one variable which can be solved easily.

The various steps involved in the technique are given below:

Step 1: Multiply one or both of the equations by a suitable number(s) so that either the coefficients of first variable or the coefficients of second variable in both the equations become numerically equal.

Step 2: Add both the equations or subtract one equation from the other, as obtained in step 1, so that the terms with equal numerical coefficients cancel mutually.

Step 3: Solve the resulting equation to find the value of one of the unknowns.

Step 4: Substitute this value in any of the two given equations and find the value of the other unknown.

Example 3.50 Given $4a+3b=65$ and $a+2b=35$ solve by elimination method.

Solution

Given, $4a+3b=65$ .....(1)

$a+2b=35$ .....(2)

$$\begin{array}{lrcrcr} (2)\times 4 \text{ gives} & 4a & + & 8b & = & 140 \\ & (-) & & (-) & & (-) \\ \text{Already (1) is} & \underline{4a} & \underline{+} & \underline{3b} & \underline{=} & \underline{65} \\ & & & 5b & = & 75 \end{array}$$

which gives $b=15$

Put $b=15$ in (2):

$a+2(15)=35$ which simplifies to $a=5$

Thus the solution is $a=5$, $b=15$.

Verification :

$4a+3b = 65$ ...(1)

$4(5)+3(15) = 65$

$20+45 = 65$

$65 = 65$ True

$a+2b = 35$ ...(2)

$5+2(15) = 35$

$5+30 = 35$

$35 = 35$ True

Example 3.51 Solve for x and y: $8x-3y=5xy$, $6x-5y=-2xy$ by the method of elimination.

Solution

The given system of equations are $8x-3y=5xy$ ...(1)

$6x-5y=-2xy$ ...(2)

Observe that the given system is not linear because of the occurrence of xy term. Also note that if $x=0$, then $y=0$ and vice versa. So, (0,0) is a solution for the system and any other solution would have both $x\neq 0$ and $y\neq 0$.

Let us take up the case where $x \neq 0,\ y \neq 0$.

Dividing both sides of each equation by xy,

$$\frac{8x}{xy} - \frac{3y}{xy} = \frac{5xy}{xy} \qquad \text{we get,} \qquad \frac{8}{y} - \frac{3}{x} = 5 \qquad \ldots(3)$$

$$\frac{6x}{xy} - \frac{5y}{xy} = \frac{-2xy}{xy} \qquad\qquad \frac{6}{y} - \frac{5}{x} = -2 \qquad \ldots(4)$$

Let $a = \frac{1}{x},\ b = \frac{1}{y}$.

(3)&(4) respectively become, $8b - 3a = 5$ ...(5)

$$6b - 5a = -2 \qquad \ldots(6)$$

which are linear equations in a and b.

To eliminate a, we have, $(5)\times 5 \Rightarrow \quad 40b - 15a = 25$ .....(7)

$$(6)\times 3 \Rightarrow \quad 18b - 15a = -6 \quad .....(8)$$

Now proceed as in the previous example to get the solution $\left(\frac{11}{23}, \frac{22}{31}\right)$.

Thus, the system have two solutions $\left(\frac{11}{23}, \frac{22}{31}\right)$ and $(0,0)$.

Exercise 3.12

1. Solve by the method of elimination

   (i) $2x - y = 3;\quad 3x + y = 7$ (ii) $x - y = 5;\quad 3x + 2y = 25$

   (iii) $\frac{x}{10} + \frac{y}{5} = 14;\quad \frac{x}{8} + \frac{y}{6} = 15$ (iv) $3(2x + y) = 7xy;\quad 3(x + 3y) = 11xy$

   (v) $\frac{4}{x} + 5y = 7;\ \frac{3}{x} + 4y = 5$ (vi) $13x + 11y = 70;\ 11x + 13y = 74$

2. The monthly income of A and B are in the ratio 3:4 and their monthly expenditures are in the ratio 5:7. If each saves ₹ 5,000 per month, find the monthly income of each.

3. Five years ago, a man was seven times as old as his son, while five year hence, the man will be four times as old as his son. Find their present age.

Solving by Cross Multiplication Method

The substitution and elimination methods involves many arithmetic operations, whereas the cross multiplication method utilize the coefficients effectively, which simplifies the procedure to get the solution. This method of cross multiplication is so called because we draw cross ways between the numbers in the denominators and cross multiply the coefficients along the arrows ahead. Now let us discuss this method as follows:

Suppose we are given a pair of linear simultaneous equations such as

$$a_1x+b_1y+c_1=0 \qquad ...(1)$$

$$a_2x+b_2y+c_2=0 \qquad ...(2)$$

such that $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$. We can solve them as follows :

$(1)\times b_2 - (2)\times b_1$ gives $b_2(a_1x+b_1y+c_1)-b_1(a_2x+b_2y+c_2)=0$

$$\Rightarrow \quad x(a_1b_2-a_2b_1)=(b_1c_2-b_2c_1)$$

$$\Rightarrow \quad x=\frac{(b_1c_2-b_2c_1)}{(a_1b_2-a_2b_1)}$$

$(1)\times a_2 - (2)\times a_1$ similarly can be considered and that will simplify to

$$y=\frac{(c_1a_2-c_2a_1)}{(a_1b_2-a_2b_1)}$$

Hence the solution for the system is

$$\boxed{x=\frac{(b_1c_2-b_2c_1)}{(a_1b_2-a_2b_1)}, \quad y=\frac{(c_1a_2-c_2a_1)}{(a_1b_2-a_2b_1)}}$$

This can also be written as

$$\boxed{\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}}$$

which can be remembered as

$$\begin{array}{ccccccc} & x & & y & & 1 & \\ b_1 & & c_1 & & a_1 & & b_1 \\ b_2 & & c_2 & & a_2 & & b_2 \end{array}$$

Example 3.52 Solve $3x-4y=10$ and $4x+3y=5$ by the method of cross multiplication.

Solution

The given system of equations are

$$3x-4y=10 \quad \Rightarrow \quad 3x-4y-10=0 \quad .....(1)$$

$$4x+3y=5 \quad \Rightarrow \quad 4x+3y-5=0 \quad .....(2)$$

For the cross multiplication method, we write the co-efficients as

$$\begin{array}{ccccccc} & x & & y & & 1 & \\ -4 & & -10 & & 3 & & -4 \\ 3 & & -5 & & 4 & & 3 \end{array}$$

$$\frac{x}{(-4)(-5)-(3)(-10)} = \frac{y}{(-10)(4)-(-5)(3)} = \frac{1}{(3)(3)-(4)(-4)}$$

$$\frac{x}{(20)-(-30)} = \frac{y}{(-40)-(-15)} = \frac{1}{(9)-(-16)}$$

$$\frac{x}{20+30} = \frac{y}{-40+15} = \frac{1}{9+16}$$

$$\frac{x}{50} = \frac{y}{-25} = \frac{1}{25}$$

Therefore, we get $x = \frac{50}{25}$; $y = \frac{-25}{25}$

$x = 2$; $y = -1$

Thus the solution is $x = 2$, $y = -1$.

Verification :

$$\begin{aligned} 3x-4y &= 10 \quad ...(1) \\ 3(2)-4(-1) &= 10 \\ 6+4 &= 10 \\ 10 &= 10 \quad \text{True} \end{aligned}$$

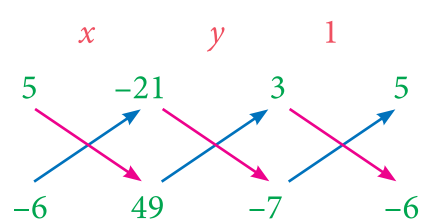
$$\begin{aligned} 4x+3y &= 5 \quad ...(2) \\ 4(2)+3(-1) &= 5 \\ 8-3 &= 5 \\ 5 &= 5 \quad \text{True} \end{aligned}$$

Example 3.53 Solve by cross multiplication method : $3x+5y=21$; $-7x-6y=-49$

Solution

The given system of equations are $3x+5y-21=0$; $-7x-6y+49=0$

Now using the coefficients for cross multiplication, we get,



$$\Rightarrow \frac{x}{(5)(49)-(-6)(-21)} = \frac{y}{(-21)(-7)-(49)(3)} = \frac{1}{(3)(-6)-(-7)(5)}$$

$$\frac{x}{119} = \frac{y}{0} = \frac{1}{17}$$

$$\Rightarrow \quad \frac{x}{119} = \frac{1}{17}, \quad \frac{y}{0} = \frac{1}{17}$$

$$\Rightarrow \quad x = \frac{119}{17}, \quad y = \frac{0}{17}$$

$$\Rightarrow \quad x = 7, \quad y = 0$$

Verification :

$$\begin{aligned} 3x+5y &= 21 \quad ...(1) \\ 3(7)+5(0) &= 21 \\ 21+0 &= 21 \\ 21 &= 21 \quad \text{True} \end{aligned}$$

$$\begin{aligned} -7x-6y &= -49 \quad ...(2) \\ -7(7)-6(0) &= -49 \\ -49 &= -49 \\ -49 &= -49 \quad \text{True} \end{aligned}$$

Note

Here $\frac{y}{0} = \frac{1}{17}$ is to mean $y = \frac{0}{17}$. Thus, $\frac{y}{0}$ is only a notation and it is not division by zero. It is always true that division by zero is not defined.

Exercise 3.13

1. Solve by cross-multiplication method

   (i) $8x-3y=12\,;\; 5x=2y+7$ (ii) $6x+7y-11=0\,;\; 5x+2y=13$

   (iii) $\dfrac{2}{x}+\dfrac{3}{y}=5\,;\dfrac{3}{x}-\dfrac{1}{y}+9=0$

2. Akshaya has 2 rupee coins and 5 rupee coins in her purse. If in all she has 80 coins totalling ₹ 220, how many coins of each kind does she have.

3. It takes 24 hours to fill a swimming pool using two pipes. If the pipe of larger diameter is used for 8 hours and the pipe of the smaller diameter is used for 18 hours. Only half of the pool is filled. How long would each pipe take to fill the swimming pool.

3.8.3 Consistency and Inconsistency of Linear Equations in Two Variables

Consider linear equations in two variables say

$$a_1x+b_1y+c_1=0 \quad ...(1)$$

$$a_2x+b_2y+c_2=0 \quad ...(2)$$ where a_1, a_2, b_1, b_2, c_1 and c_2 are real numbers.

Then the system has :

(i) a unique solution if $\dfrac{a_1}{a_2}\neq\dfrac{b_1}{b_2}$ (Consistent)

(ii) an Infinite number of solutions if $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{c_1}{c_2}$ (Consistent)

(iii) no solution if $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}\neq\dfrac{c_1}{c_2}$ (Inconsistent)

Example 3.54 Check whether the following system of equation is consistent or inconsistent and say how many solutions we can have if it is consistent.

(i) $2x-4y=7$
$x-3y=-2$

(ii) $4x+y=3$
$8x+2y=6$

(iii) $4x+7=2y$
$2x+9=y$

Solution

Sl. No	Pair of lines	$\frac{a_1}{a_2}$	$\frac{b_1}{b_2}$	$\frac{c_1}{c_2}$	Compare the ratios	Graphical representation	Algebraic interpretation
(i)	$2x-4y=7$ $x-3y=-2$	$\frac{2}{1}=2$	$\frac{-4}{-3}=\frac{4}{3}$	$\frac{7}{-2}=\frac{-7}{2}$	$\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$	Intersecting lines	Unique solution
(ii)	$4x+y=3$ $8x+2y=6$	$\frac{4}{8}=\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{6}=\frac{1}{2}$	$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$	Coinciding lines	Infinite many solutions
(iii)	$4x+7=2y$ $2x+9=y$	$\frac{4}{2}=2$	$\frac{2}{1}=2$	$\frac{7}{9}$	$\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$	Parallel lines	No solution

Example 3.55 Check the value of k for which the given system of equations $kx+2y=3;\ 2x-3y=1$ has a unique solution.

Solution Given linear equations are

$$\begin{aligned} kx+2y&=3 \ .....(1) \\ 2x-3y&=1 \ .....(2) \end{aligned} \quad \begin{bmatrix} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \end{bmatrix}$$

Here $a_1=k,\ b_1=2,\ a_2=2,\ b_2=-3;$

For unique solution we take $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$; therefore $\frac{k}{2}\neq\frac{2}{-3}$; $k\neq\frac{4}{-3}$, that is $k\neq-\frac{4}{3}$.

Example 3.56 Find the value of k, for the following system of equation has infinitely many solutions. $2x-3y=7;\ (k+2)x-(2k+1)y=3(2k-1)$

Solution Given two linear equations are

$$\begin{aligned} &2x-3y=7 \\ &(k+2)x-(2k+1)y=3(2k-1) \end{aligned} \quad \begin{bmatrix} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \end{bmatrix}$$

Here $a_1=2,\ b_1=-3,\ a_2=(k+2),\ b_2=-(2k+1),\ c_1=7,\ c_2=3(2k-1)$

For infinite number of solution we consider $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

$$\frac{2}{k+2}=\frac{-3}{-(2k+1)}=\frac{7}{3(2k-1)}$$

$$\frac{2}{k+2}=\frac{-3}{-(2k+1)}$$

$$2(2k+1)=3(k+2)$$

$$4k+2=3k+6$$

$$k=4$$

$$\frac{-3}{-(2k+1)}=\frac{7}{3(2k-1)}$$

$$9(2k-1)=7(2k+1)$$

$$18k-9=14k+7$$

$$4k=16$$

$$k=4$$

Example 3.57 Find the value of k for which the system of linear equations $8x+5y=9;$ $kx+10y=15$ has no solution.

Solution Given linear equations are

$$\begin{aligned} 8x+5y&=9 \\ kx+10y&=15 \end{aligned} \quad \begin{bmatrix} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \end{bmatrix}$$

Here $a_1=8,\ b_1=5,\ c_1=9,\ a_2=k, b_2=10, c_2=15$

For no solution, we know that $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$ and so, $\frac{8}{k}=\frac{5}{10}\neq\frac{9}{15}$

$$80=5k$$

$$k=16$$

Activity - 3

1. Find the value of k for the given system of linear equations satisfying the condition below:
   (i) $2x+ky=1;\ 3x-5y=7$ has a unique solution
   (ii) $kx+3y=3;\ 12x+ky=6$ has no solution
   (iii) $(k-3)x+3y=k;\ kx+ky=12$ has infinite number of solution
2. Find the value of a and b for which the given system of linear equation has infinite number of solutions $3x-(a+1)y=2b-1,\quad 5x+(1-2a)y=3b$

Activity - 4

For the given linear equations, find another linear equation satisfying each of the given condition

Given linear equation	Another linear equation		
	Unique Solution	Infinite many solutions	No solution
$2x+3y=7$	$3x+4y=8$	$4x+6y=14$	$6x+9y=15$
$3x-4y=5$			
$y-4x=2$			
$5y-2x=8$			

Exercise 3.14

Solve by any one of the methods

1. The sum of a two digit number and the number formed by interchanging the digits is 110. If 10 is subtracted from the first number, the new number is 4 more than 5 times the sums of the digits of the first number. Find the first number.
2. The sum of the numerator and denominator of a fraction is 12. If the denominator is increased by 3, the fraction becomes $\frac{1}{2}$. Find the fraction.
3. ABCD is a cyclic quadrilateral such that $\angle A = (4y+20)^\circ$, $\angle B = (3y-5)^\circ$, $\angle C = (4x)^\circ$ and $\angle D = (7x+5)^\circ$. Find the four angles.
4. On selling a T.V. at 5% gain and a fridge at 10% gain, a shopkeeper gains ₹2000. But if he sells the T.V. at 10% gain and the fridge at 5% loss, he gains Rs.1500 on the transaction. Find the actual price of the T.V. and the fridge.
5. Two numbers are in the ratio 5 : 6. If 8 is subtracted from each of the numbers, the ratio becomes 4 : 5. Find the numbers.
6. 4 Indians and 4 Chinese can do a piece of work in 3 days. While 2 Indians and 5 Chinese can finish it in 4 days. How long would it take for 1 Indian to do it? How long would it take for 1 Chinese to do it?

Exercise 3.15

Multiple choice questions

1. If $x^3 + 6x^2 + kx + 6$ is exactly divisible by $(x + 2)$, then $k=$?
   (1) -6 (2) -7 (3) -8 (4) 11

2. The root of the polynomial equation $2x + 3 = 0$ is
   (1) $\frac{1}{3}$ (2) $-\frac{1}{3}$ (3) $-\frac{3}{2}$ (4) $-\frac{2}{3}$

3. The type of the polynomial $4-3x^3$ is
   (1) constant polynomial (2) linear polynomial
   (3) quadratic polynomial (4) cubic polynomial.

4. If $x^{51} + 51$ is divided by $x + 1$, then the remainder is
   (1) 0 (2) 1 (3) 49 (4) 50

5. The zero of the polynomial $2x+5$ is
   (1) $\frac{5}{2}$ (2) $-\frac{5}{2}$ (3) $\frac{2}{5}$ (4) $-\frac{2}{5}$

6. The sum of the polynomials $p(x) = x^3 - x^2 - 2$, $q(x) = x^2-3x+ 1$
   (1) $x^3 - 3x - 1$ (2) $x^3 + 2x^2 - 1$ (3) $x^3 - 2x^2 - 3x$ (4) $x^3 - 2x^2 + 3x -1$

7. Degree of the polynomial $(y^3-2)(y^3 + 1)$ is
   (1) 9 (2) 2 (3) 3 (4) 6

8. Let the polynomials be
   (A) $-13q^5 + 4q^2 + 12q$ (B) $(x^2 +4)(x^2 + 9)$
   (C) $4q^8 - q^6 + q^2$ (D) $-\frac{5}{7}y^{12} + y^3 + y^5$
   Then ascending order of their degree is
   (1) A,B,D,C (2) A,B,C,D (3) B,C,D,A (4) B,A,C,D

9. If $p(a) = 0$ then $(x - a)$ is a ___________ of $p(x)$
   (1) divisor (2) quotient (3) remainder (4) factor

10. Zeros of $(2 - 3x)$ is ___________
    (1) 3 (2) 2 (3) $\frac{2}{3}$ (4) $\frac{3}{2}$

11. Which of the following has $x - 1$ as a factor?
    (1) $2x - 1$ (2) $3x - 3$ (3) $4x - 3$ (4) $3x - 4$

12. If $x - 3$ is a factor of $p(x)$, then the remainder is
    (1) 3 (2) -3 (3) $p(3)$ (4) $p(-3)$

13. $(x + y)(x^2 - xy + y^2)$ is equal to
    (1) $(x + y)^3$ (2) $(x - y)^3$ (3) $x^3 + y^3$ (4) $x^3 - y^3$

14. $(a+b-c)^2$ is equal to __________

(1) $(a-b+c)^2$ (2) $(-a-b+c)^2$ (3) $(a+b+c)^2$ (4) $(a-b-c)^2$

15. In an expression ax^2+bx+c the sum and product of the factors respectively,

(1) a, bc (2) b, ac (3) ac, b (4) bc, a

16. If $(x+5)$ and $(x-3)$ are the factors of ax^2+bx+c, then values of a, b and c are

(1) 1,2,3 (2) 1,2,15 (3) 1,2, −15 (4) 1, −2,15

17. Cubic polynomial may have maximum of ___________ linear factors

(1) 1 (2) 2 (3) 3 (4) 4

18. Degree of the constant polynomial is __________

(1) 3 (2) 2 (3) 1 (4) 0

19. Find the value of m from the equation $2x+3y=m$. If its one solution is $x=2$ and $y=-2$.

(1) 2 (2) −2 (3) 10 (4) 0

20. Which of the following is a linear equation

(1) $x+\frac{1}{x}=2$ (2) $x(x-1)=2$ (3) $3x+5=\frac{2}{3}$ (4) $x^3-x=5$

21. Which of the following is a solution of the equation $2x-y=6$

(1) (2,4) (2) (4,2) (3) (3, −1) (4) (0,6)

22. If (2,3) is a solution of linear equation $2x+3y=k$ then, the value of k is

(1) 12 (2) 6 (3) 0 (4) 13

23. Which condition does not satisfy the linear equation $ax+by+c=0$

(1) $a\neq 0$, $b=0$ (2) $a=0$, $b\neq 0$

(3) $a=0$, $b=0, c\neq 0$ (4) $a\neq 0$, $b\neq 0$

24. Which of the following is not a linear equation in two variable

(1) $ax+by+c=0$ (2) $0x+0y+c=0$

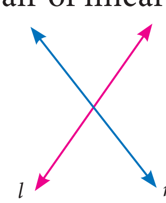
(3) $0x+by+c=0$ (4) $ax+0y+c=0$

25. The value of k for which the pair of linear equations $4x+6y-1=0$ and $2x+ky-7=0$ represents parallel lines is

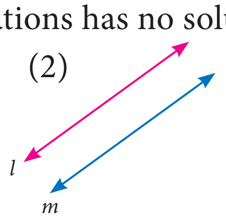
(1) $k=3$ (2) $k=2$ (3) $k=4$ (4) $k=-3$

26. A pair of linear equations has no solution then the graphical representation is

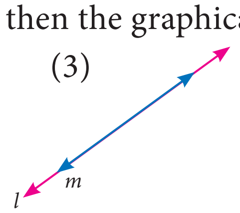
(1)



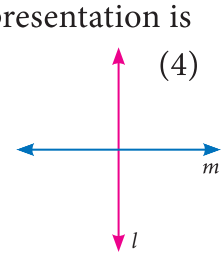
(2)



(3)



(4)



27. If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ where $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ then the given pair of linear equation has __________ solution(s)

(1) no solution (2) two solutions (3) unique (4) infinite

28. If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ where $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ then the given pair of linear equation has __________ solution(s)

(1) no solution (2) two solutions (3) infinite (4) unique

29. GCD of any two prime numbers is __________

(1) -1 (2) 0 (3) 1 (4) 2

30. The GCD of $x^4 - y^4$ and $x^2 - y^2$ is

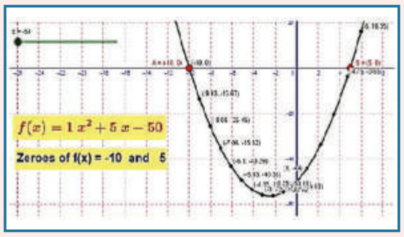
(1) $x^4 - y^4$ (2) $x^2 - y^2$ (3) $(x+y)^2$ (4) $(x+y)^4$

Points to Remember

- An algebraic expression of the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ is called Polynomial in one variable x of degree 'n' where $a_0, a_1, a_2, \ldots a_n$ are constants ($a_n \neq 0$) and n is a whole number.
- Let $p(x)$ be a polynominal. If $p(a) = 0$ then we say that 'a' is a zero of the polynomial $p(x)$
- If $x = a$ statisfies the polynominal $p(x) = 0$ then $x = a$ is called a root of the polynominal equation $p(x) = 0$.
- **Remainder Theorem:** If a polynomial $p(x)$ of degree greater than or equal to one is divided by a linear polynomial $(x–a)$, then the remainder is $p(a)$, where a is any real number.
- **Factor Theorem**
  If $p(x)$ is divided by $(x-a)$ and the remainder $p(a) = 0$, then $(x-a)$ is a factor of the polynomial $p(x)$
- Solution of an equation is the set of all values that when substituted for unknowns make an equation true.
- An equation with two variable each with exponent as 1 and not multiplied with each other is called a linear equation with two variables.
- Linear equation in two variables has infinite number of solutions.
- The graph of a linear equation in two variables is a straight line.
- Simultaneous linear equations consists of two or more linear equations with the same variables.

ICT Corner-1

Expected Result is shown in this picture



Step – 1 Open the Browser and copy and paste the Link given below (or) by typing the URL given (or) Scan the QR Code.

Step - 2 GeoGebra Work Book called "Polynomials and Quadratic Equations" will appear. There are several work sheets in this work Book. Open the worksheet named "Zeroes: Quadratic Polynomial".

Step-3 Drag the sliders a, b and c to change the quadratic co-efficient. Follow the changes in points A and B where the curve cuts the x-axis. These points are called Zeroes of a Polynomial.

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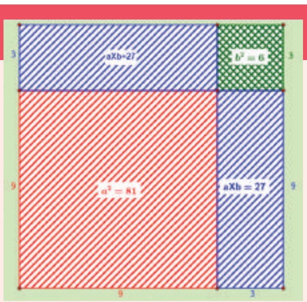
ICT Corner-2



Expected Result is shown in this picture

Step – 1 Open the Browser, type the URL Link given below (or) Scan the QR Code. GeoGebra work sheet named "Algebraic Identities" will open. In the work sheet, there are many activities on Algebraic Identities.

In the first activity diagrammatic approach for $(a+b)^2$ is given. Move the sliders a and b and compare the areas with the Identity given.

Step - 2 Similarly move the sliders a and b and compare the areas with the remaining Identities.

Scan the QR Code.

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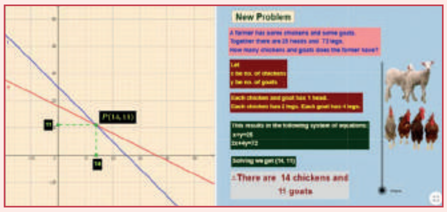
ICT Corner-3

Expected Result is shown in this picture



Step – 1 Open the Browser by typing the URL Link given below (or) Scan the QR Code. GeoGebra work sheet named "Algebra" will open. There are three worksheets under the title Solving by rule of cross multiplication, Graphical method and Chick-Goat puzzle.

Step - 2 Move the sliders or type the respective values in the respective boxes to change the equations. Work out the solution and check the solutions. In third title click on new problem and solve. Move the slider to see the steps.

Scan the QR Code.

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