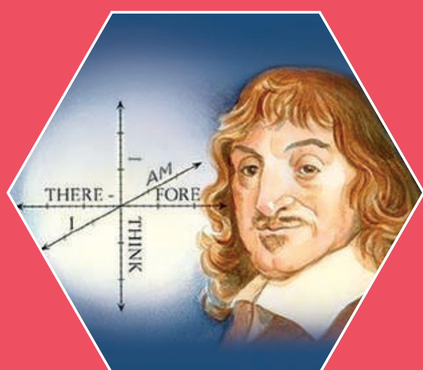
5

COORDINATE GEOMETRY

Divide each difficulty into as many parts as is feasible and necessary to resolve it.

- Rene Descartes



Rene Descartes
(AD(CE)1596-1650)

The French Mathematician Rene Descartes (pronounced "DAY- CART") developed a new branch of Mathematics known as Analytical Geometry or Coordinate Geometry which combined all arithmetic, algebra and geometry of the past ages in a single technique of visualising as points on a graph and equations as geometrical shapes. The fixing of a point position in the plane by assigning two numbers, coordinates, giving its distance from two lines perpendicular to each other, was entirely Descartes' invention.

Learning Outcomes

- To understand the Cartesian coordinate system.
- To identify the abscissa, ordinate and coordinates of any given point.
- To find the distance between any two points in the Cartesian plane using formula.
- To understand the mid-point formula and use it in problem solving.
- To derive the section formula and apply this in problem solving.
- To understand the centroid formula and to know its applications.

5.1 Mapping the Plane

How do you write your address? Here is one.

Sarakkalvilai Primary School
135, Sarakkalvilai,
Sarakkalvilai Housing Board Road,
Keezha Sarakkalvilai,
Nagercoil 629002, Kanyakumari Dist.
Tamil Nadu, India.

E6HCCN

Fig. 5.1

Somehow, this information is enough for anyone in the world from anywhere to locate the school one studied. Just consider there are crores and crores of buildings on the Earth. But yet, we can use an address system to locate a particular person's place of study, however interior it is.

How is this possible? Let us work out the procedure of locating a particular address. We know the World is divided into countries. One among them is India. Subsequently India is divided into States. Among these States we can locate our State Tamilnadu.

Further going deeper, we find our State is divided into Districts. Districts into taluks, taluks into villages proceeding further in this way, one could easily locate "Sarakkalvilai" among the villages in that Taluk. Further among the roads in that village, 'Housing Board Road' is the specific road which we are interested to explore. Finally we end up the search by locating Primary School building bearing the door number 135 to enable us precisely among the buildings in that road.

Fig. 5.2

In New York city of USA, there is an area called Manhattan. The map shows Avenues run in the North – South direction and the Streets run in the East – West direction. So, if you know that the place you are looking for is on 57th street between 9th and 10th Avenues, you can find it immediately on the map. Similarly it is easy to find a place on 2nd Avenue between 34th and 35th streets. In fact, New Yorkers make it even simpler. From the door number on a street, you can actually calculate which avenues it lies between, and from the door number on an avenue, you can calculate which streets it stands.

All maps do just this for us. They help us in finding our way and locate a place easily by using information of any landmark which is nearer to our search to make us understand whether we are near or far, how far are we, or what is in between etc. We use latitudes (east – west, like streets in Manhattan) and longitudes (north – south, like avenues in Manhattan) to pin point places on Earth. It is interesting to see how using numbers in maps helps us so much.

This idea, of using numbers to map places, comes from geometry. Mathematicians wanted to build maps of planes, solids and shapes of all kinds. Why would they want such maps? When we work with a geometric figure, we want to observe wheather a point lies inside the region or outside or on the boundary. Given two points on the boundary and a point outside, we would like to examine which of the two points on the boundary is closer to the one outside, and how much closer and so on. With solids like cubes, you can imagine how interesting and complicated such questions can be.

Mathematicians asked such questions and answered them only to develop their own understanding of circles, polygons and spheres. But the mathematical tools and techniques were used to find immense applications in day to day life. Mapping the world using latitudes and longitudes would not have been developed at all in 18th century, if the co-ordinate system had not been developed mathematically in the 17th century.

You already know the map of the real number system is the number line. It extends infinitely on both directions. In between any two points, on a number line, there lies infinite number of points. We are now going to build a map of the plane so that we can discuss about the points on the plane, of the distance between the points etc. We can then draw on the plane all the geometrical shapes we have discussed so far, precisely.

Arithmetic introduced us to the world of numbers and operations on them. Algebra taught us how to work with unknown values and find them using equations. Geometry taught us to describe shapes by their properties. Co-ordinate geometry will teach us how to use numbers and algebraic equations for studying geometry and beautiful integration of many techniques in one place. In a way, that is also great fun as an activity. Can't wait? Let us plunge in.

5.2 Devising a Coordinate System

You ask your friend to draw a rectangle on a blank sheet of paper, 5 cm by 3 cm. He says, "Sure, but where on this sheet?" How would you answer him?

Now look at the picture. How will you describe it to another person?

Fig. 5.3

Let us analyse the given picture (Fig. 5.3). Just like that a particular house is to be pointed out, it is going to be a difficult task. Instead, if any place or an object is fixed for identification then it is easy to identify any other place or object relative to it. For example, you fix the flag and talk about the house to the left of it, the hotel below it, the antenna on the house to right of it etc.

As shown in Fig. 5.4 draw two perpendicular lines in such a way that the flag is pointed out at near the intersection point. Now if you tell your friend the total length and width of the picture frame, keeping the flagpole as landmark, you can also say 2 cm to the right, 3 cm above etc. Since you know directions, you can also say 2 cm east, 3 cm north.

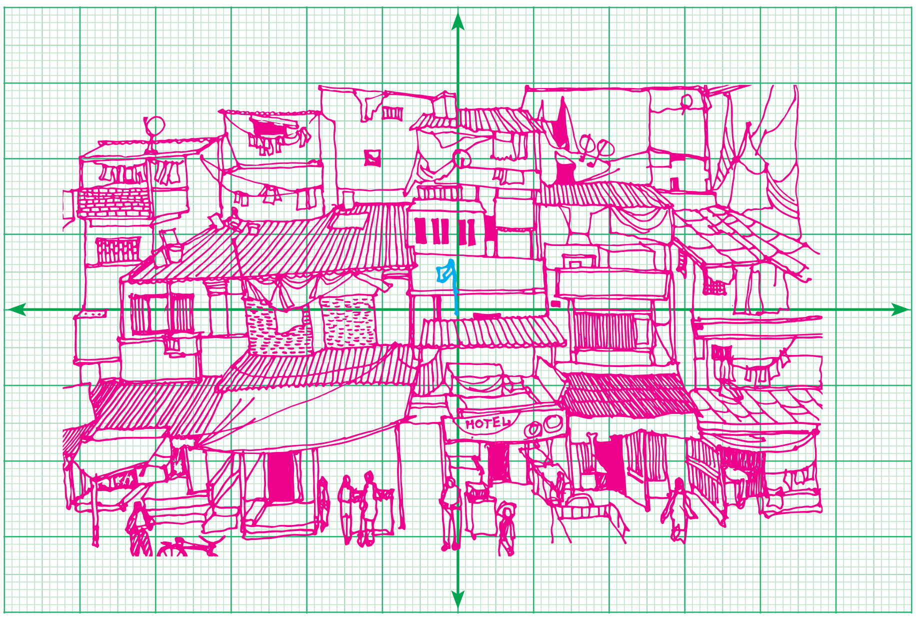


Fig. 5.4

This is what we are going to do. A number line is usually represented as horizontal line on which the positive numbers always lie on the right side of zero, negative on the left side of zero. Now consider another copy of the number line, but drawn vertically: the positive integers are represented above zero and the negative integers are below zero (fig 5.5).

Where do the two number lines meet? Obviously at zero for both lines. That will be our location where the "flag" is fixed. We can talk of other numbers relative to it, on both the lines. But now you see that we talk not only of numbers on the two number lines but lots more !

Suppose we go 2 to the right and then 3 to the top. We would call this place (→ 2, ↑ 3).

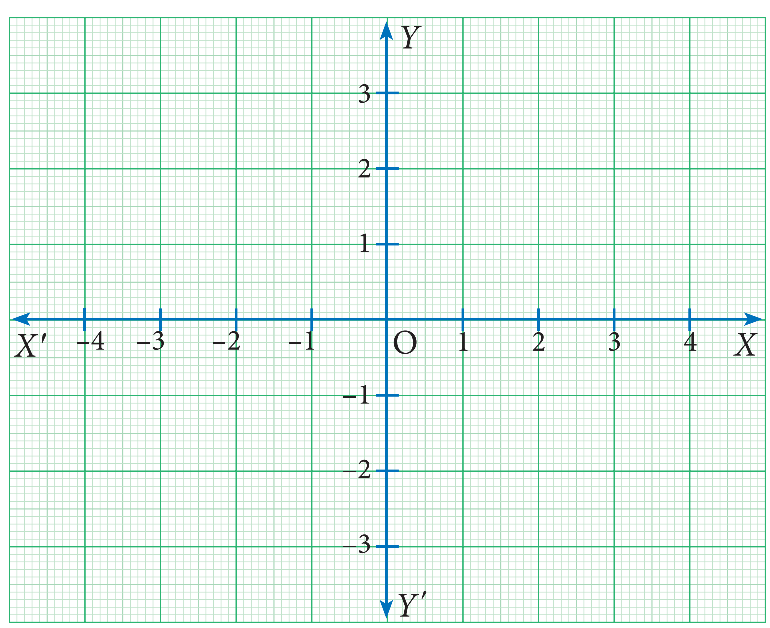


Fig. 5.5

All this vertical, horizontal, up, down etc is all very cumbersome. We simply say (2,3) and understand this as 2 to the right and then 3 up. Notice that we would reach the same place if we first went 3 up and then 2 right, so for us, the instruction (2,3) is not the same as the instruction (3,2).

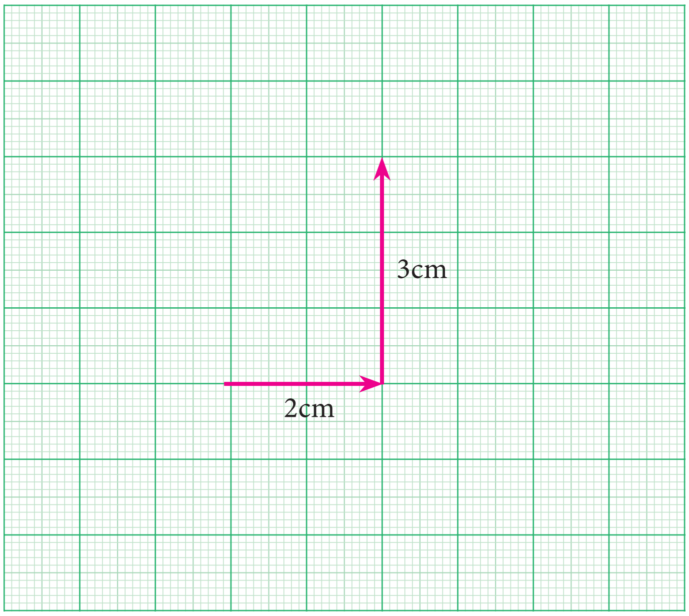


Fig. 5.6

What about (–2,3)? It would mean 2 left and 3 up. From where? Always from (0,0). What about the instruction (2,–3)? It would mean 2 right and 3 down. We need names for horizontal and vertical number lines too. We call the horizontal number line the x-axis and the vertical number line the y-axis. To the right we mark it as X, to the left as X', to the top as Y, to the bottom as Y'.

The x-co-ordinate is called the *abscissa* and the y-co-ordinate is called the *ordinate*. We call the meeting point of the axes (0,0) the origin.

Note

Whether we place (0,0) at the centre of the sheet, or somewhere else does not matter, (0,0) is always the origin for us, and all "instructions" are relative to that point. We usually denote the origin by the letter 'O'.

Now we can describe any point on a sheet of paper by a pair (x,y). However, what do (1, 2) etc mean on our paper? We need to choose some convenient unit and represent these numbers. For instance we can choose 1 unit to be 1 cm. Thus (2,3) is the instruction to move 2 cm to the right of (0,0) and then to move 3 cm up. Please remember that the choice of units is arbitrary: if we fix 1 unit to be 2 cm, our figures will be larger, but the relative distances will remain the same.

In fact, we now have a language to describe all the infinitely many points on the plane, not just our sheet of paper !

Since the x-axis and the y-axis divide the plane into four regions, we call them quadrants. (Remember, quadrilateral has 4 sides, quadrants are 4 regions.) They are usually numbered as I, II, III and IV, with I for upper east side, II for upper west side, III for lower west side and IV for lower east side, thus making an anti-clockwise tour of them all.

Note

Why this way (anti-clockwise) and not clockwise, or not starting from any of the other quadrants? It does not matter at all, but it is good to follow some convention, and this is what we have been doing for a few centuries now.

Region	Quadrant	Nature of x,y	Signs of the coordinates
XOY	I	$x>0,\ y>0$	(+,+)
$X'OY$	II	$x<0,\ y>0$	(–,+)
$X'OY'$	III	$x<0,\ y<0$	(–,–)
XOY'	IV	$x>0,\ y<0$	(+,–)

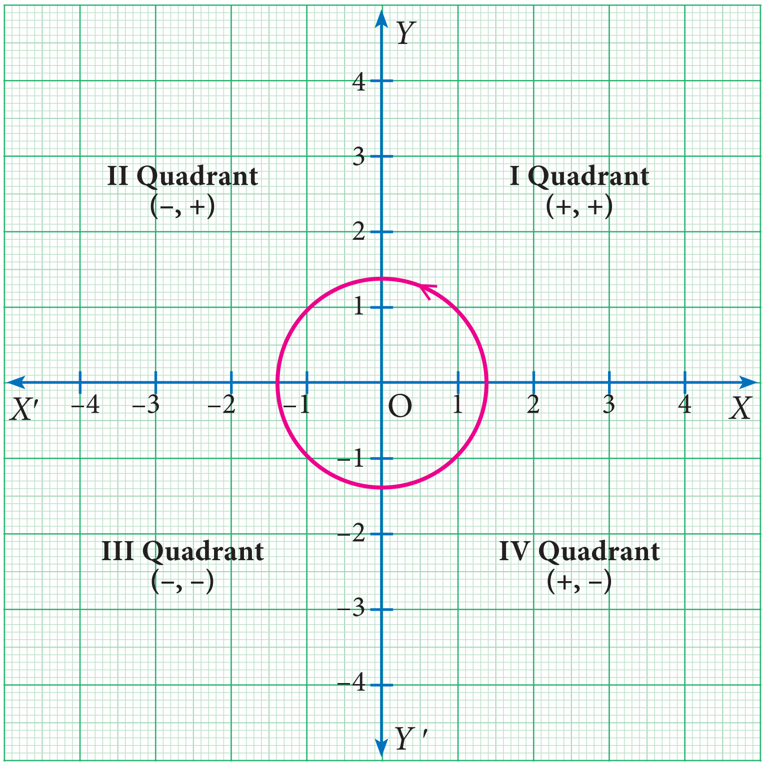


Fig. 5.7

Note

- For any point P on the x axis,the value of y coordinate (ordinate) is zero that is, P (x,0).
- For any point Q on the y axis, the value of x coordinate (abscissa) is zero. that is, Q (0, y)
- (x, y) ≠ (y, x) unless x= y
- A plane with the rectangular coordinate system is called the Cartesian plane.

5.2.1 Plotting Points in Cartesian Coordinate Plane

To plot the points (4, 5) in the Cartesian coordinate plane. We follow the x – axis until we reach 4 and draw a vertical line at $x = 4$.

Similarly, we follow the y – axis until we reach 5 and draw a horizontal line at $y = 5$.

The intersection of these two lines is the position of (4, 5) in the Cartesian plane.

This point is at a distance of 4 units from the y-axis and 5 units from the x-axis. Thus the position of (4, 5) is located in the Cartesian plane.

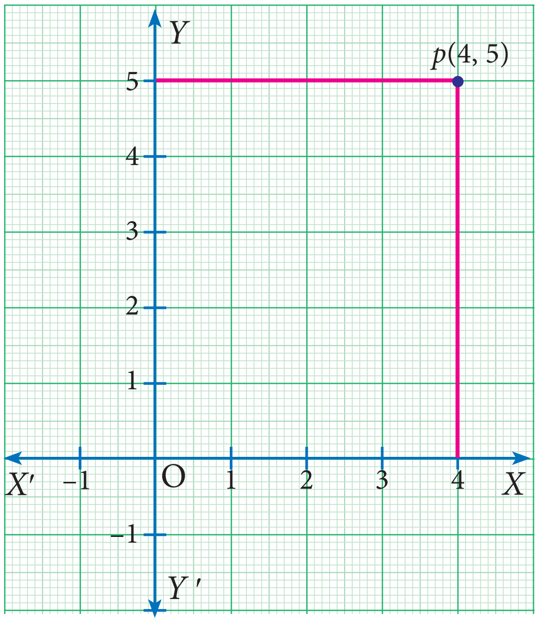


Fig. 5.8

Example 5.1 In which quadrant does the following points lie?

(a) (3,–8) (b) (–1,–3) (c) (2, 5) (d) (–7, 3)

Solution

(a) The x- coordinate is positive and y – coordinate is negative. So, point(3,–8) lies in the IV quadrant.

(b) The x-coordinate is negative and y – coordinate is negative. So, point(–1,–3) lies in the III quadrant.

(c) The x-coordinate is positive and y – coordinate is positive. So point(2,5) lies in the I quadrant.

(d) The x-coordinate is negative and y – coordinate is positive. So, point(–7,3) lies in the II quadrant

Example 5.2 Plot the points $A(2,4)$, $B(-3,5)$, $C(-4,-5)$, $D(4,-2)$ in the Cartesian plane.

Solution

(i) To plot (2, 4), draw a vertical line at $x = 2$ and draw a horizontal line at $y = 4$. The intersection of these two lines is the position of (2, 4) in the Cartesian plane. Thus, the Point A (2, 4) is located in the I quadrant of Cartesian plane.

(ii) To plot (–3, 5), draw a vertical line at $x = -3$ and draw a horizontal line at $y = 5$. The intersection of these two lines is the position of (–3, 5) in the Cartesian plane. Thus, the Point B (–3, 5) is located in the II quadrant of Cartesian plane.

(iii) To plot (–4, –5), draw a vertical line at $x = -4$ and draw a horizontal line at $y = -5$. The intersection of these two lines is the position of (-4, 5) in the Cartesian plane. Thus, the Point C (–4, –5) is located in the III quadrant of Cartesian plane.

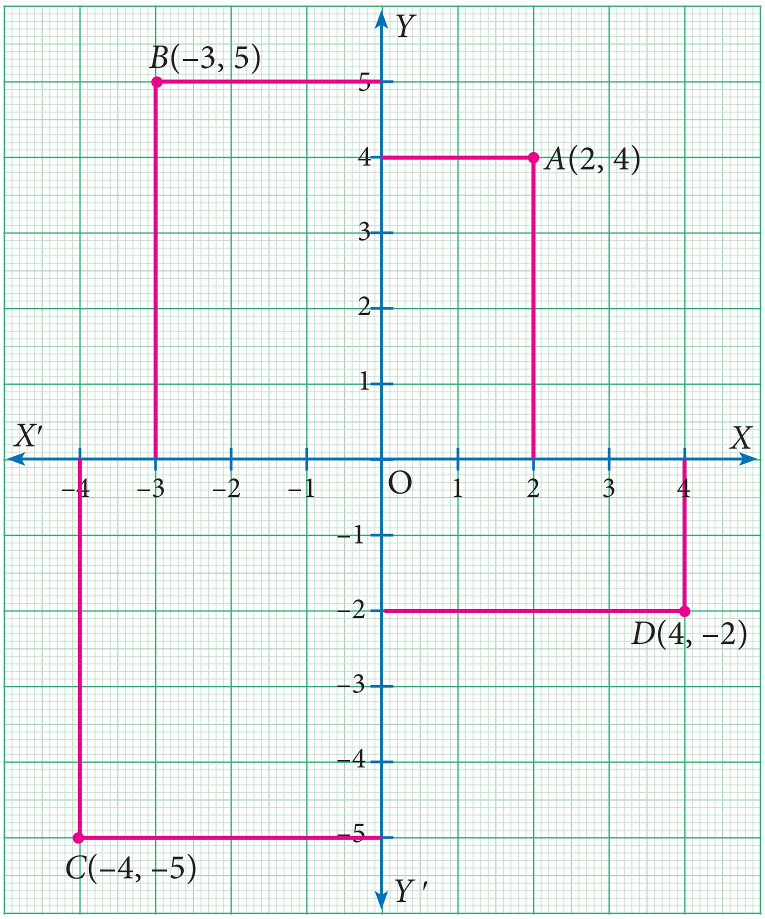


Fig. 5.9

(iv) To plot (4, –2), draw a vertical line at $x = 4$ and draw a horizontal line at $y = -2$. The Intersection of these two lines is the position of (4,–2) in the Cartesian plane. Thus, the Point D (4,–2) is located in the IV quadrant of Cartesian plane.

Example 5.3 Plot the following points $A(2,2)$, $B(-2,2)$, $C(-2,-1)$, $D(2,-1)$ in the Cartesian plane . Discuss the type of the diagram by joining all the points taken in order.

Solution

Point	*A*	*B*	*C*	*D*
Quadrant	I	II	III	IV

ABCD is a rectangle.

Can you find the length, breath and area of the rectangle?

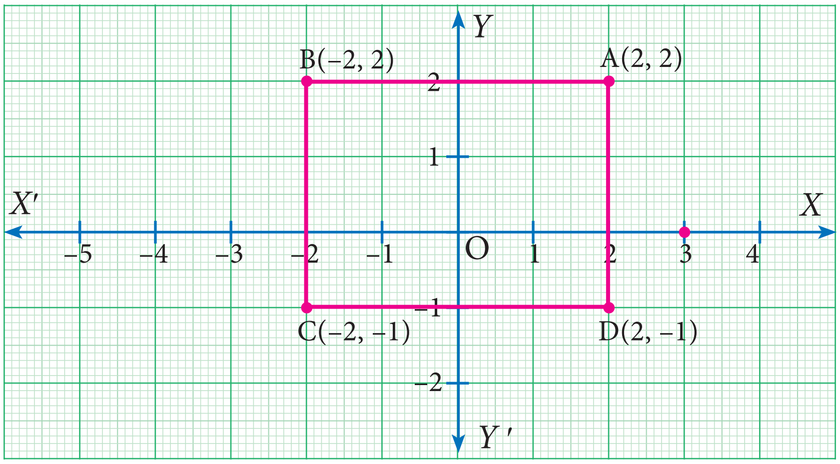


Fig. 5.10

Exercise 5.1

1. Plot the following points in the coordinate system and identify the quadrants $P(-7,6)$, $Q(7,-2)$, $R(-6,-7)$, $S(3,5)$ and $T(3,9)$

2. Write down the abscissa and ordinate of the following from fig 5.11.

   (i) P (ii) Q

   (iii) R (iv) S

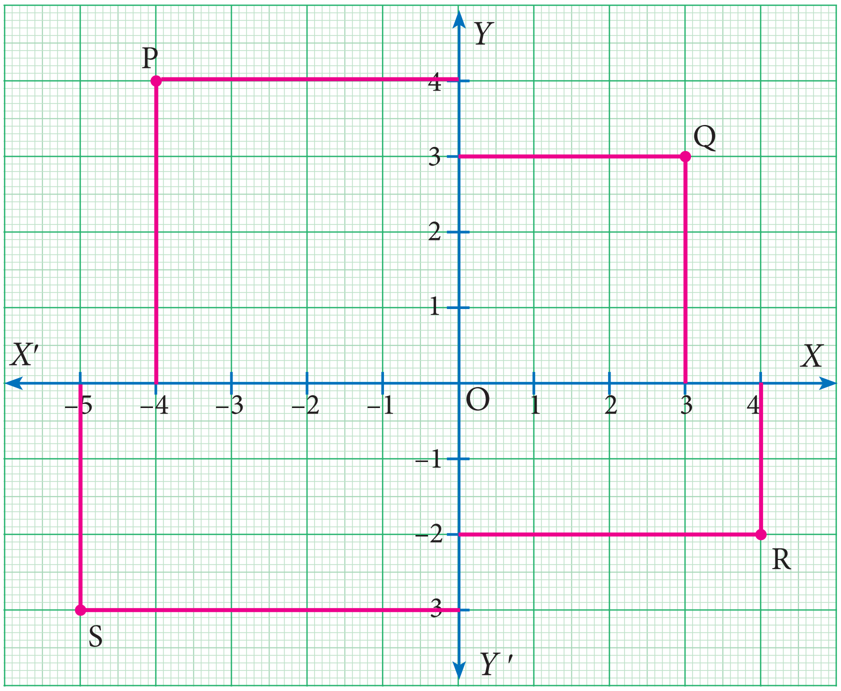


Fig. 5.11

3. Plot the following points in the coordinate plane and join them. What is your conclusion about the resulting figure?
   (i) (–5,3) (–1,3) (0,3) (5,3) (ii) (0,–4) (0,–2) (0,4) (0,5)

4. Plot the following points in the coordinate plane. Join them in order. What type of geometrical shape is formed?
   (i) (0,0) (–4,0) (–4,–4) (0,–4) (ii) (–3,3) (2,3) (–6,–1) (5,–1)

Activity - 1

Plot the following points on a graph sheet by taking the scale as 1cm = 1 unit.

Find how far the points are from each other?

A (1, 0) and *D* (4, 0). Find *AD* and also DA.

Is *AD* = *DA*?

You plot another set of points and verify your Result.

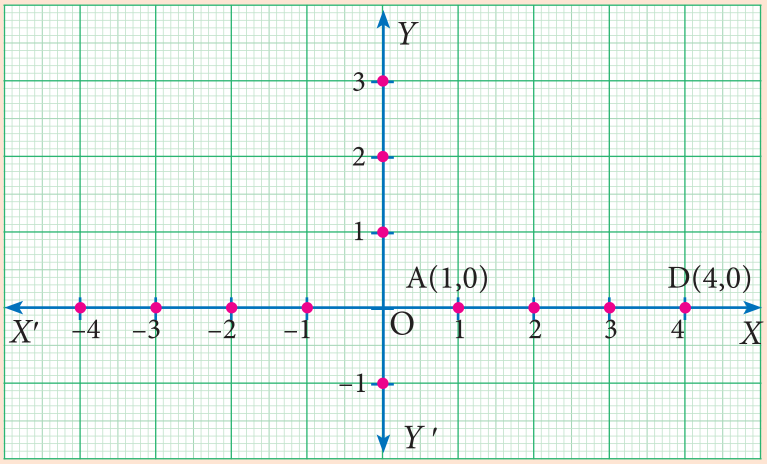


Fig. 5.12

5.3 Distance between any Two Points

Akila and Shanmugam are friends living on the same street in Sathyamangalam. Shanmugam's house is at the intersection of one street with another street on which there is a library. They both study in the same school, and that is not far from Shanmugam's house. Try to draw a picture of their houses, library and school by yourself before looking at the map below. Consider the school as the origin. (We can do this ! That is the whole point about the co-ordinate language we are using.)

Now fix the scale as 1 unit = 50 metres. Here are some questions for you to answer by studying the given figure (Fig 5.13).

1. How far is Akila's house from Shanmugam's house?
2. How far is the library from Shanmugam's house?
3. How far is the school from Shanmugam's and Akila's house?
4. How far is the library from Akila's house?
5. How far is Shanmugam's house from Akila's house?

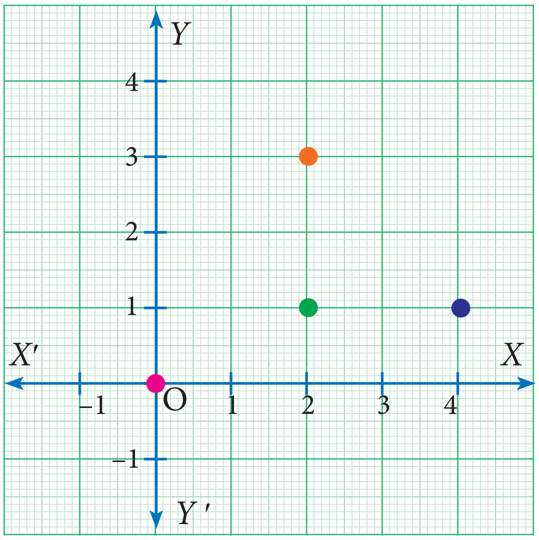


Fig. 5.13

Question 5 is not needed after answering question 1. Obviously, the distance from point A to B is the same as the distance from point B to A, and we usually call it the distance between points A and B. But as mathematicians we are supposed to note down properties as and when we see them, so it is better to note this too: distance (A,B) = distance (B,A). This is true for all points A and B on the plane, so of course question 5 is same as question 1.

Note

The equation distance(A,B)=distance(B,A) is not always obvious. Suppose that the road from A to B is a one-way street on which you cannot go the other way? Then the distance from B to A might be longer ! But we will avoid all these complications and assume that we can go both ways.

What about the other questions? They are not the same. Since we know that the two houses are on the same street which is running north – south, the y-distance tells us the answer to question 1. Similarly, we know that the library and Shanmugam's house are on the same street running east – west, we can take the x-distance to answer question 2.

Questions 3 and 4 depend on what kind of routes are available. If we assume that the only streets available are parallel to the x and y axes at the points marked 1, 2, 3 etc then we answer these questions by adding the x and y distances. But consider the large field east of Akila's house.

If she can walk across the field, of course she would prefer it. Now there are many ways of going from one place to another, so when we talk of the distance between them, it is not precise. We need some way to fix what we mean. When there are many routes between A and B, we will use distance(A,B) to denote the distance on the shortest route between A and B.

Once we think of distance(A,B) as the "straight line distance" between A and B, there is an elegant way of understanding it for any points A and B on the plane. This is the important reason for using the co-ordinate system at all ! Before that, 2 more questions from our example.

1. With the school as origin, define the coordinates of the two houses, the school and the library.
2. Use the coordinates to give the distance between any one of these and another.

The "straight line distance" is usually called "as the crow flies". This is to mean that we don't worry about any obstacles and routes on the ground, but how we would get from A to B if we could fly. No bird ever flies on straight lines, though.

We can give a systematic answer to this: given any two points $A = (x,y)$ and $B = (x',y')$ on the plane, find distance(A,B). It is easy to derive a formula in terms of the four numbers x, y, x' and y'. This is what we set out to do now

5.3.1 Distance Between Two Points on the Coordinate Axes

Points on x – axis: If two points lie on the x- axis, then the distance between them is equal to the difference between the x- coordinates.

Consider two points $A\ (x_1,0)$ and $B\ (x_2,0)$ on the x-axis .

The distance of B from A is

$$AB = OB - OA = x_2 - x_1 \text{ if } x_2 > x_1 \text{ or}$$
$$= x_1 - x_2 \text{ if } x_1 > x_2$$
$$AB = |x_2 - x_1|$$

(Read as modulus or absolute value of $x_2 - x_1$)

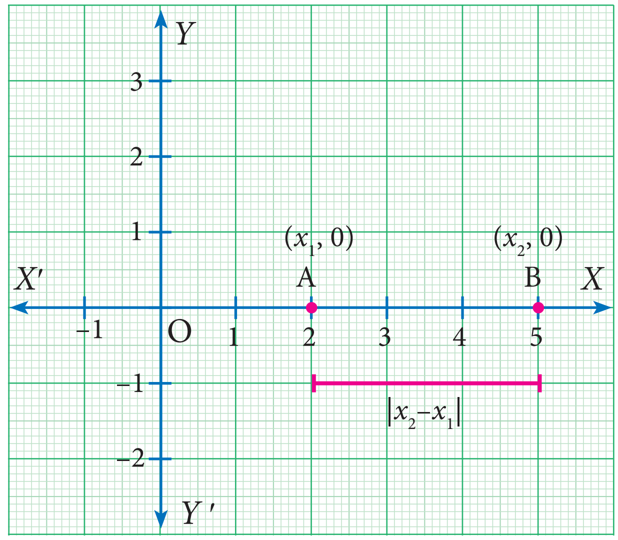


Fig. 5.14

Points on y – axis: If two points lie on y-axis then the distance between them is equal to the difference between the y-coordinates.

Consider two points $P(0,y_1)$ and $Q(0,y_2)$

The distance Q from P is

$$PQ = OQ - OP.$$
$$= y_2 - y_1 \text{ if } y_2 > y_1 \text{ or}$$
$$= y_1 - y_2 \text{ if } y_1 > y_2$$
$$PQ = |y_2 - y_1|$$

(Read as modulus or absolute value of $y_2 - y_1$)

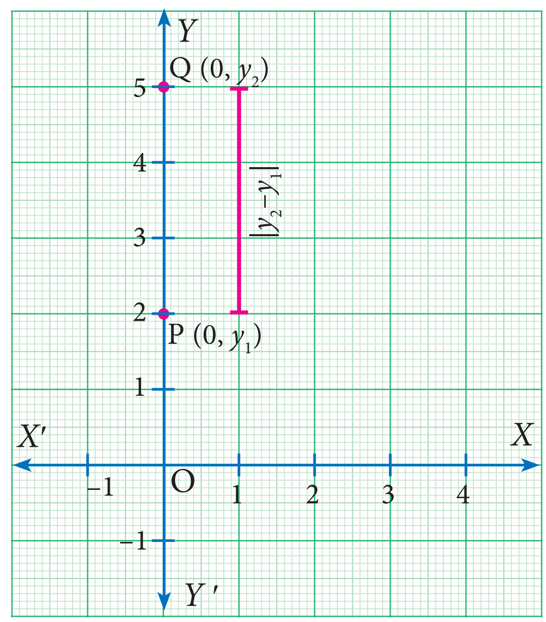


Fig. 5.15

5.3.2 Distance Between Two Points Lying on a Line Parallel to Coordinate Axes

Consider the points $A(x_1, y_1)$ and $B(x_2, y_1)$. Since the y - coordinates are equal the points lie on a line parallel to x- axis. From A and B draw AP and BQ perpendicular to x- axis respectively. Observe the given figure (Fig. 5.16), it is obvious that the distance AB is same as the distance PQ

Distance AB = Distance between $PQ = |x_2 - x_1|$
[The difference between x coordinates]

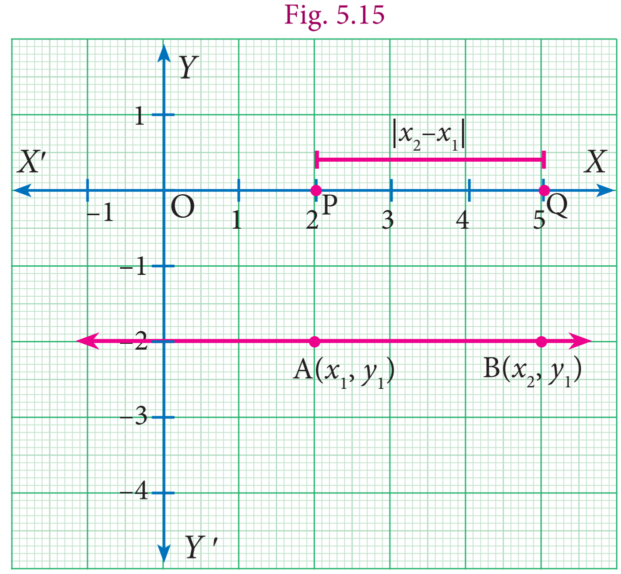


Fig. 5.16

Similarly consider the line joining the two points

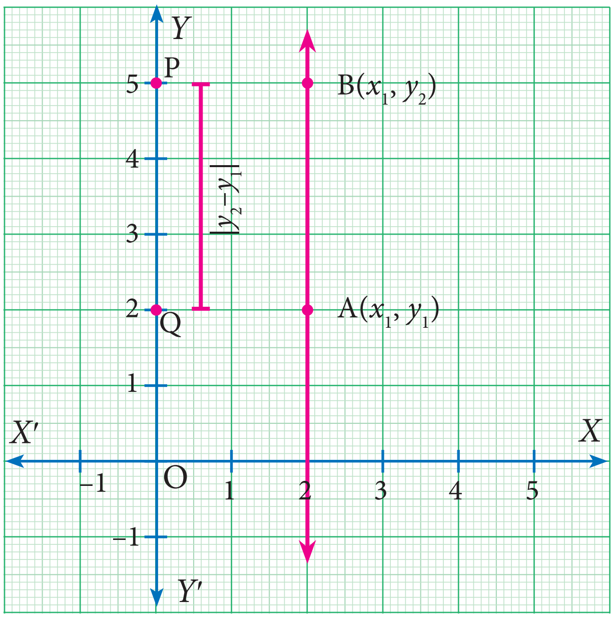


Fig. 5.17

$A(x_1, y_1)$ and $B(x_1, y_2)$, parallel to y – axis.

Then the distance between these two points is

$$|y_2 - y_1|$$

[The difference between y coordinates]

5.3.3 Distance Between Two Points on a Plane.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the Cartesian plane (or xy – plane), at a distance 'd' apart that is $d = PQ$.

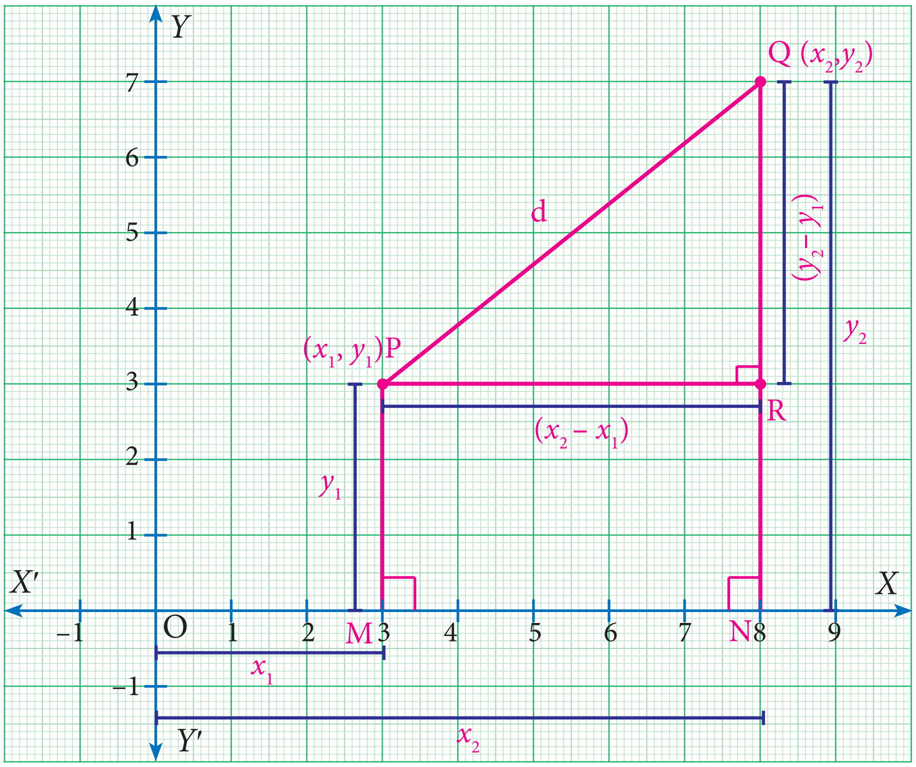


Fig. 5.18

Step 1

By the definition of coordinates,

$OM = x_1 \qquad MP = y_1$

$ON = x_2 \qquad NQ = y_2$

Now, $(PR \perp NQ)$

$PR = MN$ (Opposite sides of the rectangle $MNRP$)

Thinking Corner

A man goes 3 km towards north and then 4 km towards east. How far is he away from the initial position?

$= ON - OM$ (Measuring the distance from O)

$= x_2 - x_1$ ....(1)

And $RQ = NQ - NR$

$= NQ - MP$ (Opposite sides of the rectangle $MNRP$)

$= y_2 - y_1$ ....(2)

E768EB

Step 2

Triangle PQR is right angled at R. ($PR \perp NQ$)

$PQ^2 = PR^2 + RQ^2$ (By Pythagoras theorem)

$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$ (Taking positive square root)

Note

Distance between two points

- Given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the distance between these points is given by the formula $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$.
- The distance between PQ = The distance between QP
  i.e. $\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$
- The distance of a point $P(x_1, y_1)$ from the origin $O(0,0)$ is $OP = \sqrt{x_1^2 + y_1^2}$

5.3.4 Properties of Distances

We have already seen that distance (A,B) = distance (B,A) for any points A, B on the plane. What other properties have you noticed? In case you have missed them, here are some:

distance $(A,B) = 0$ exactly when A and B denote the identical point: $A = B$.

distance $(A,B) > 0$ for any two distinct points A and B.

Now consider three points A, B and C. If we are given their co-ordinates and we find that their x-co-ordinates are the same then we know that they are collinear, and lie on a line parallel to the y-axis. Similarly, if their y-co-ordinates are the same then we know that they are collinear, and lie on a line parallel to the x-axis. But these are not the only conditions. Points (0,0), (1,1) and (2,2) are collinear as well. Can you think of what relationship should exist between these coordinates for the points to be collinear?

The distance formula comes to our help here. We know that when A, B and C are the vertices of a triangle, we get,

distance(A,B) + distance(B,C) >distance(A,C) (after renaming the vertices suitably).

When do three points on the plane not form a triangle? When they are collinear, of course. In fact, we can show that when, distance(A,B) + distance(B,C) = distance(A,C), the points A, B and C must be collinear.

Similarly, when A, B and C are the vertices of a right angled triangle, $\angle ABC = 90^\circ$ we know that:

distance$(AB)^2$+ distance$(BC)^2$ = distance$(AC)^2$

with appropriate naming of vertices. We can also show that the converse holds: whenever the equality here holds for A, B and C, they must be the vertices of a right angled triangle.

The following examples illustrate how these properties of distances are useful for answering questions about specific geometric shapes.

Example 5.4

Find the distance between the points (–4, 3), (2,–3).

Solution

The distance between the points (-4, 3), (2,-3) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(2+4)^2 + (-3-3)^2}$$

$$= \sqrt{(6^2 + (-6)^2)} = \sqrt{(36+36)}$$

$$= \sqrt{(36 \times 2)}$$

$$= 6\sqrt{2}$$

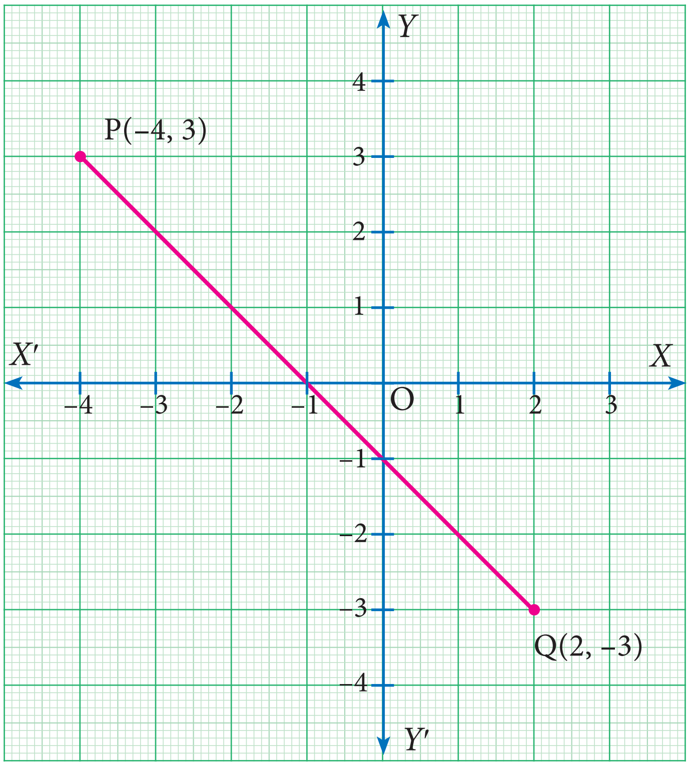


Fig. 5.19

Example 5.5 Show that the following points $A(3,1)$, $B(6,4)$ and $C(8,6)$ lies on a straight line.

Solution

Using the distance formula, we have

$$AB = \sqrt{(6-3)^2 + (4-1)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

$$BC = \sqrt{(8-6)^2 + (6-4)^2} = \sqrt{(4+4)} = \sqrt{8} = 2\sqrt{2}$$

$$AC = \sqrt{(8-3)^2 + (6-1)^2} = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2}$$

$$AB + BC = 3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2} = AC$$

Therefore the points lie on a straight line.

Collinear points

To show the collinearity of three points, we prove that the sum of the distance between two pairs of points is equal to the third pair of points.

In otherwords, points A, B, C are collinear if $AB + BC = AC$

Example 5.6 Show that the points $A(7,10)$, $B(-2,5)$, $C(3,-4)$ are the vertices of a right angled triangle.

Solution

Here $A = (7, 10)$, $B = (-2, 5)$, $C = (3,-4)$

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(-2-7)^2 + (5-10)^2}$$

$$= \sqrt{(-9)^2 + (-5)^2}$$

$$= \sqrt{(81+25)}$$

$$= \sqrt{106}$$

$$AB^2 = 106 \qquad \ldots (1)$$

$$BC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(3-(-2))^2 + (-4-5)^2} = \sqrt{(5)^2 + (-9)^2}$$

$$= \sqrt{25+81} = \sqrt{106}$$

$$BC^2 = 106 \qquad \ldots (2)$$

$$AC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(3-7)^2 + (-4-10)^2} = \sqrt{(-4)^2 + (-14)^2}$$

$$= \sqrt{16+196} = \sqrt{212}$$

$$AC^2 = 212 \qquad \ldots (3)$$

From (1), (2) & (3) we get,

$$AB^2 + BC^2 = 106 + 106 = 212 = AC^2$$

Since $AB^2 + BC^2 = AC^2$

$\therefore$ ΔABC is a right angled triangle, right angled at B.

Right angled triangle

We know that the sum of the squares of two sides is equal to the square of the third side,which is the hypotenuse of a right angled triangle.

Example 5.7 Show that the points $A(-4,-3)$, $B(3,1)$, $C(3,6)$, $D(-4,2)$ taken in that order form the vertices of a parallelogram.

Solution

Let $A(-4,\ -3)$, $B(3,\ 1)$, $C(3,\ 6)$, $D(-4,\ 2)$ be the four vertices of any quadrilateral $ABCD$. Using the distance formula,

Parallelogram

We know that opposite sides are equal

$$\text{Let} \quad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$AB = \sqrt{(3+4)^2 + (1+3)^2} = \sqrt{49+16} = \sqrt{65}$$

$$BC = \sqrt{(3-3)^2 + (6-1)^2}$$

$$= \sqrt{0+25} = \sqrt{25} = 5$$

$$CD = \sqrt{(-4-3)^2 + (2-6)^2}$$

$$= \sqrt{(-7)^2 + (-4)^2} = \sqrt{49+16} = \sqrt{65}$$

$$AD = \sqrt{(-4+4)^2 + (2+3)^2} = \sqrt{(0)^2 + (5)^2} = \sqrt{25} = 5$$

$AB = CD = \sqrt{65}$ and $BC = AD = 5$

Here, the opposite sides are equal. Hence $ABCD$ is a parallelogram.

Example 5.8

Calculate the distance between the points A (7, 3) and B which lies on the x-axis whose abscissa is 11.

Solution

Since B is on the x-axis, the y-coordinate of B is 0.

So, the coordinates of the point B is (11, 0)

By the distance formula the distance between the points A (7, 3), B (11, 0) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$AB = \sqrt{(11-7)^2 + (0-3)^2} = \sqrt{(4)^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25}$$

$$= 5$$

Example 5.9 Find the value of 'a' such that $PQ = QR$ where P, Q, and R are the points whose coordinates are $(6, -1)$, $(1, 3)$ and $(a, 8)$ respectively.

Solution

Given $P(6, -1)$, $Q(1, 3)$ and $R(a, 8)$

$$PQ = \sqrt{(1-6)^2+(3+1)^2} = \sqrt{(-5)^2+(4)^2} = \sqrt{41}$$

$$QR = \sqrt{(a-1)^2+(8-3)^2} = \sqrt{(a-1)^2+(5)^2}$$

$$\text{Given } PQ = QR$$

$$\text{Therefore } \sqrt{41} = \sqrt{(a-1)^2+(5)^2}$$

$$41 = (a-1)^2 + 25 \quad \text{[Squaring both sides]}$$

$$(a-1)^2 + 25 = 41$$

$$(a-1)^2 = 41 - 25$$

$$(a-1)^2 = 16$$

$$(a-1) = \pm 4 \quad \text{[taking square root on both sides]}$$

$$a = 1 \pm 4$$

$$a = 1 + 4 \text{ or } a = 1 - 4$$

$$a = 5 \text{ or } a = -3$$

Example 5.10 Let $A(2, 2)$, $B(8, -4)$ be two given points in a plane. If a point P lies on the X- axis (in positive side), and divides AB in the ratio 1: 2, then find the coordinates of P.

Solution

Given points are $A(2, 2)$ and $B(8, -4)$ and let $P = (x, 0)$ [P lies on x axis]

By the distance formula

$$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$AP = \sqrt{(x-2)^2+(0-2)^2} = \sqrt{x^2-4x+4+4} = \sqrt{x^2-4x+8}$$

$$BP = \sqrt{(x-8)^2+(0+4)^2} = \sqrt{x^2-16x+64+16} = \sqrt{x^2-16x+80}$$

Given $\quad AP : PB = 1 : 2$

i.e. $\quad \dfrac{AP}{BP} = \dfrac{1}{2} \quad (\because BP = PB)$

$$2AP = BP$$

squaring on both sides,

$$4AP^2 = BP^2$$

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$$
\begin{aligned}
4(x^2 - 4x + 8) &= (x^2 - 16x + 80) \\
4x^2 - 16x + 32 &= x^2 - 16x + 80 \\
3x^2 - 48 &= 0 \\
3x^2 &= 48 \\
x^2 &= 16 \\
x &= \pm 4
\end{aligned}
$$

E7F4FY

As the point P lies on x-axis (positive side), its x- coordinate cannot be -4.

Hence the coordinates of P is(4, 0)

Example 5.11 Show that (4, 3) is the centre of the circle passing through the points (9, 3), (7,–1), (–1,3). Also find its radius.

Solution

Let $P(4, 3)$, $A(9, 3)$, $B(7, -1)$ and $C(-1, 3)$

If P is the centre of the circle which passes through the points A, B, and C, then P is equidistant from A, B and C (i.e.) $PA = PB = PC$

By distance formula,

$$
\begin{aligned}
d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
AP = PA &= \sqrt{(4-9)^2 + (3-3)^2} = \sqrt{(-5)^2 + 0} = \sqrt{25} = 5 \\
BP = PB &= \sqrt{(4-7)^2 + (3+1)^2} = \sqrt{(3)^2 + (4)^2} = \sqrt{9+16} = \sqrt{25} = 5 \\
CP = PC &= \sqrt{(4+1)^2 + (3-3)^2} = \sqrt{(5)^2 + 0} = \sqrt{25} = 5
\end{aligned}
$$

$PA = PB = PC$

Therefore P is the centre of the circle, passing through A, B and C

Radius $= PA = 5$.

Exercise 5.2

1. Find the distance between the following pairs of points.
   (i) (1, 2) and (4, 3) (ii) (3,4) and (– 7, 2)
   (ii) (a, b) and (c, b) (iv) (3,– 9) and (–2, 3)

2. Determine whether the given set of points in each case are collinear or not.
   (i) (7,–2),(5,1),(3,4) (ii) $(a,-2)$, $(a,3)$, $(a,0)$

3. Show that the following points taken in order form an isosceles triangle.
(i) $A\ (5,4), B(2,0), C\ (-2,3)$ (ii) $A\ (6,-4), B\ (-2, -4), C\ (2,10)$

4. Show that the following points taken in order form an equilateral triangle in each case.
(i) $A(2, 2), B(-2, -2), C\left(-2\sqrt{3}, 2\sqrt{3}\right)$ (ii) $A\left(\sqrt{3}, 2\right), B\ (0,1), C(0,3)$

5. Show that the following points taken in order form the vertices of a parallelogram.
(i) $A(-3, 1), B(-6, -7), C\ (3, -9), D(6, -1)$ (ii) $A\ (-7, -3), B(5,10), C(15,8), D(3, -5)$

6. Verify that the following points taken in order form the vertices of a rhombus.
(i) $A(3,-2), B\ (7,6), C\ (-1,2), D\ (-5, -6)$ (ii) $A\ (1,1), B(2,1), C\ (2,2), D(1,2)$

7. $A\ (-1, 1), B\ (1, 3)$ and $C\ (3, a)$ are points and if $AB = BC$, then find 'a'.

8. The abscissa of a point A is equal to its ordinate, and its distance from the point $B(1, 3)$ is 10 units, What are the coordinates of A?

9. The point (x, y) is equidistant from the points (3,4)and(−5,6). Find a relation between x and y.

10. Let $A(2, 3)$ and $B(2,-4)$ be two points. If P lies on the x-axis, such that $AP = \frac{3}{7}AB$, find the coordinates of P.

11. Show that the point (11,2) is the centre of the circle passing through the points (1,2), (3,−4) and (5,−6)

12. The radius of a circle with centre at origin is 30 units. Write the coordinates of the points where the circle intersects the axes. Find the distance between any two such points.

5.4 The Mid-point of a Line Segment

Fig. 5.20

Imagine a person riding his two-wheeler on a straight road towards East from his college to village A and then to village B. At some point in between A and B, he suddenly realises that there is not enough petrol for the journey. On the way there is no petrol bunk in between these two places. Should he travel back to A or just try his luck moving towards B? Which would be the shorter distance? There is a dilemma. He has to know whether he crossed the half way mid-point or not.

Fig. 5.21

The above Fig. 5.21 illustrates the situation. Imagine college as origin O from which the distances of village A and village B are respectively x_1 and x_2 $(x_1 < x_2)$. Let M be the mid-point of AB then x can be obtained as follows.

$AM = MB$ and so, $\quad x - x_1 = x_2 - x$

and this is simplified to $\quad x = \dfrac{x_1 + x_2}{2}$

Now it is easy to discuss the general case. If $A(x_1, y_1)$, $B(x_2, y_2)$ are any two points and $M(x, y)$ is the mid-point of the line segment AB, then M' is the mid-point of AC (in the Fig. 5.22). In a right triangle the perpendicular bisectors of the sides intersect at the mid-point of the hypotenuse. (Also, this property is due to similarity among the two coloured triangles shown; In such triangles, the corresponding sides will be proportional).

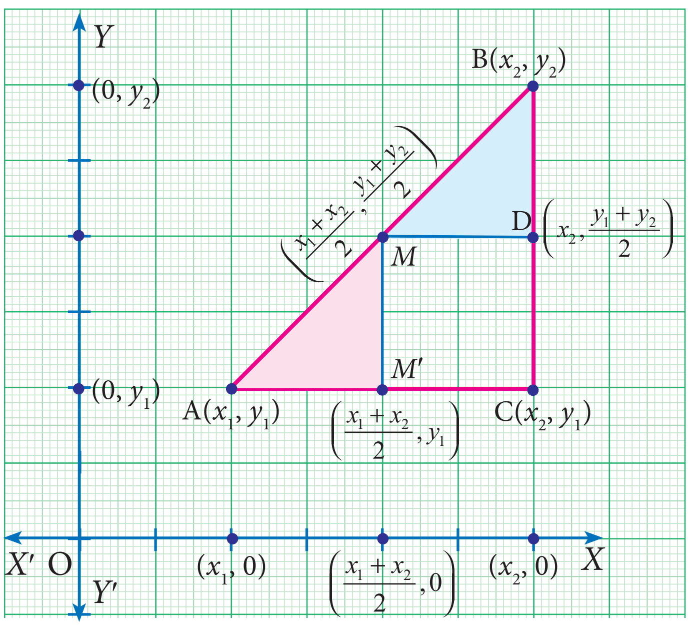


Fig. 5.22

Another way of solving

(Using similarity property)

Let us take the point M as $M(x,y)$

Now, $\Delta AMM'$ and ΔMBD are similar. Therefore,

$$\frac{AM'}{MD} = \frac{MM'}{BD} = \frac{AM}{MB}$$

$$\frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y} = \frac{1}{1} \quad (AM = MB)$$

Consider, $\dfrac{x - x_1}{x_2 - x} = 1$

$2x = x_2 + x_1 \Rightarrow x = \dfrac{x_2 + x_1}{2}$

Similarly, $y = \dfrac{y_2 + y_1}{2}$

The x-coordinate of M = the average of the x-coordinates of A and C = $\dfrac{x_1 + x_2}{2}$ and

similarly, the y-coordinate of M = the average of the y-coordinates of B and C = $\dfrac{y_1 + y_2}{2}$

The mid-point M of the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Thinking Corner

If D is the mid-point of AC and C is the mid-point of AB, then find the length of AB if $AD = 4cm$.

For example, The mid-point of the line segment joining the points $(-8,-10)$ and $(4,-2)$ is given by $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$ where $x_1=-8$, $x_2=4$, $y_1=-10$ and $y_2=-2$.

The required mid-point is $\left(\frac{-8+4}{2},\frac{-10-2}{2}\right)$ or $(-2,-6)$.

Let us now see the application of mid-point formula in our real life situation, consider the longitude and latitude of the following cities.

Name of the city	Longitude	Latitude
Chennai (Besant Nagar)	80.27° E	13.00° N
Mangaluru (Kuthethoor)	74.85° E	13.00° N
Bengaluru (Rajaji Nagar)	77.56° E	13.00° N

Fig. 5.23

Let us take the longitude and latitude of Chennai (80.27° E, 13.00° N) and Mangaluru (74.85° E, 13.00° N) as pairs. Since Bengaluru is located in the middle of Chennai and Mangaluru, we have to find the average of the coordinates, that is $\left(\frac{80.27+74.85}{2},\frac{13.00+13.00}{2}\right)$. This gives (77.56°E, 13.00°N) which is the longitude and latitude of Bengaluru. In all the above examples, the point exactly in the middle is the mid-point and that point divides the other two points in the same ratio.

Example 5.12

The point $(3,-4)$ is the centre of a circle. If AB is a diameter of the circle and B is $(5,-6)$, find the coordinates of A.

Solution Let the coordinates of A be (x_1, y_1) and the given point is $B(5,-6)$. Since the centre is the mid-point of the diameter AB, we have

$$\frac{x_1+x_2}{2}=3 \qquad \frac{y_1+y_2}{2}=-4$$
$$x_1+5=6 \qquad y_1-6=-8$$
$$x_1=6-5 \qquad y_1=-8+6$$
$$x_1=1 \qquad y_1=-2$$

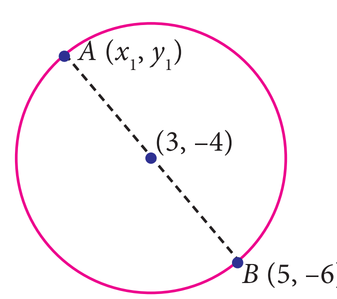


Fig. 5.24

Therefore, the coordinates of A is $(1,-2)$.

Progress Check

(i) Let X be the mid-point of the line segment joining $A(3,0)$ and $B(-5,4)$ and Y be the mid-point of the line segment joining $P(-11,-8)$ and $Q(8,-2)$. Find the mid-point of the line segment XY.

(ii) If $(3,x)$ is the mid-point of the line segment joining the points $A(8,-5)$ and $B(-2,11)$, then find the value of 'x'.

Example 5.13

If $(x,3)$, $(6,y)$, $(8,2)$ and $(9,4)$ are the vertices of a parallelogram taken in order, then find the value of x and y.

Solution Let $A(x,3)$, $B(6,y)$, $C(8,2)$ and $D(9,4)$ be the vertices of the parallelogram $ABCD$. By definition, diagonals AC and BD bisect each other.

Mid-point of AC = Mid-point of BD

$$\left(\frac{x+8}{2},\frac{3+2}{2}\right)=\left(\frac{6+9}{2},\frac{y+4}{2}\right)$$

equating the coordinates on both sides, we get

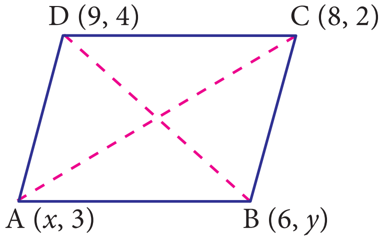


Fig. 5.25

$$\frac{x+8}{2}=\frac{15}{2} \quad \text{and} \quad \frac{5}{2}=\frac{y+4}{2}$$

$$x+8=15 \qquad 5=y+4$$

$$x=7 \qquad y=1$$

FZ7E2T

Hence, $x = 7$ and $y = 1$.

Thinking Corner

$A(6,1)$, $B(8,2)$ and $C(9,4)$ are three vertices of a parallelogram $ABCD$ taken in order. Find the fourth vertex D. If (x_1,y_1), (x_2,y_2), (x_3,y_3) and (x_4,y_4) are the four vertices of the parallelogram, then using the given points, find the value of $(x_1+x_3-x_2, y_1+y_3-y_2)$ and state the reason for your result.

Example 5.14

Find the points which divide the line segment joining $A(-11,4)$ and $B(9,8)$ into four equal parts.

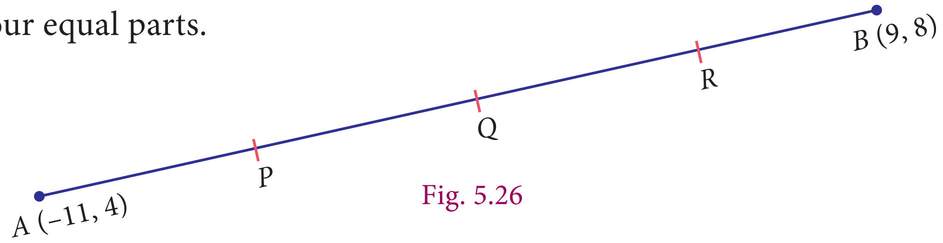


Fig. 5.26

Solution

Let P, Q, R be the points on the line segment joining $A(-11,4)$ and $B(9,8)$ such that $AP = PQ = QR = RB$.

Here Q is the mid-point of AB, P is the mid-point of AQ and R is the mid-point of QB.

$$Q \text{ is the mid-point of } AB = \left(\frac{-11+9}{2}, \frac{4+8}{2}\right) = \left(\frac{-2}{2}, \frac{12}{2}\right) = (-1,6)$$

$$P \text{ is the mid-point of } AQ = \left(\frac{-11-1}{2}, \frac{4+6}{2}\right) = \left(\frac{-12}{2}, \frac{10}{2}\right) = (-6,5)$$

$$R \text{ is the mid-point of } QB = \left(\frac{-1+9}{2}, \frac{6+8}{2}\right) = \left(\frac{8}{2}, \frac{14}{2}\right) = (4,7)$$

Hence the points which divides AB into four equal parts are $P(-6, 5)$, $Q(-1, 6)$ and $R(4, 7)$.

Example 5.15

The mid-points of the sides of a triangle are $(5,1)$, $(3,-5)$ and $(-5,-1)$. Find the coordinates of the vertices of the triangle.

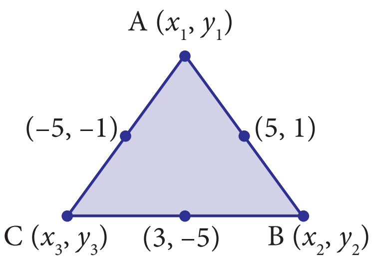


Fig. 5.27

Solution Let the vertices of the ΔABC be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ and the given mid-points of the sides AB, BC and CA are $(5,1)$, $(3,-5)$ and $(-5,-1)$ respectively. Therefore,

$$\frac{x_1+x_2}{2} = 5 \Rightarrow x_1 + x_2 = 10 \quad \ldots(1)$$

$$\frac{x_2+x_3}{2} = 3 \Rightarrow x_2 + x_3 = 6 \quad \ldots(2)$$

$$\frac{x_3+x_1}{2} = -5 \Rightarrow x_3 + x_1 = -10 \quad \ldots(3)$$

Adding (1), (2) and (3)

$$2x_1 + 2x_2 + 2x_3 = 6$$

$$x_1 + x_2 + x_3 = 3 \quad \ldots(4)$$

$(4) - (2) \Rightarrow x_1 = 3 - 6 = -3$

$(4) - (3) \Rightarrow x_2 = 3 + 10 = 13$

$(4) - (1) \Rightarrow x_3 = 3 - 10 = -7$

$$\frac{y_1+y_2}{2} = 1 \Rightarrow y_1 + y_2 = 2 \quad \ldots(5)$$

$$\frac{y_2+y_3}{2} = -5 \Rightarrow y_2 + y_3 = -10 \quad \ldots(6)$$

$$\frac{y_3+y_1}{2} = -1 \Rightarrow y_3 + y_1 = -2 \quad \ldots(7)$$

Adding (5), (6) and (7),

$$2y_1 + 2y_2 + 2y_3 = -10$$

$$y_1 + y_2 + y_3 = -5 \quad \ldots(8)$$

$(8) - (6) \Rightarrow y_1 = -5 + 10 = 5$

$(8) - (7) \Rightarrow y_2 = -5 + 2 = -3$

$(8) - (5) \Rightarrow y_3 = -5 - 2 = -7$

Therefore the vertices of the triangles are $A(-3,5)$, $B(13,-3)$ and $C(-7,-7)$.

Thinking Corner

If (a_1,b_1), (a_2,b_2) and (a_3,b_3) are the mid-points of the sides of a triangle, using the mid-points given in example 5.15 find the value of $(a_1+a_3-a_2,\ b_1+b_3-b_2)$, $(a_1+a_2-a_3,\ b_1+b_2-b_3)$ and $(a_2+a_3-a_1,\ b_2+b_3-b_1)$. Compare the results. What do you observe? Give reason for your result?

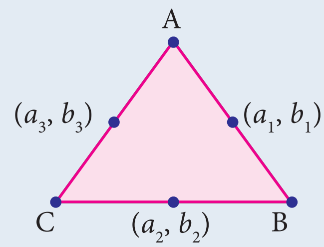


Fig. 5.28

Exercise 5.3

1. Find the mid-points of the line segment joining the points

   (i) $(-2,3)$ and $(-6,-5)$ (ii) $(8,-2)$ and $(-8,0)$

   (iii) (a,b) and $(a+2b,2a-b)$ (iv) $\left(\frac{1}{2},-\frac{3}{7}\right)$ and $\left(\frac{3}{2},\frac{-11}{7}\right)$

2. The centre of a circle is $(-4,2)$. If one end of the diameter of the circle is $(-3,7)$, then find the other end.

3. If the mid-point (x,y) of the line joining $(3,4)$ and $(p,7)$ lies on $2x+2y+1=0$, then what will be the value of p?

4. The mid-point of the sides of a triangle are $(2,4)$, $(-2,3)$ and $(5,2)$. Find the coordinates of the vertices of the triangle.

5. $O(0,0)$ is the centre of a circle whose one chord is AB, where the points A and B are $(8,6)$ and $(10,0)$ respectively. OD is the perpendicular from the centre to the chord AB. Find the coordinates of the mid-point of OD.

6. The points $A(-5,4)$, $B(-1,-2)$ and $C(5,2)$ are the vertices of an isosceles right-angled triangle where the right angle is at B. Find the coordinates of D so that $ABCD$ is a square.

7. The points $A(-3,6)$, $B(0,7)$ and $C(1,9)$ are the mid-points of the sides DE, EF and FD of a triangle DEF. Show that the quadrilateral $ABCD$ is a parallelogram.

8. $A(-3,2)$, $B(3,2)$ and $C(-3,-2)$ are the vertices of the right triangle, right angled at A. Show that the mid-point of the hypotenuse is equidistant from the vertices.

5.5 Points of Trisection of a Line Segment

The mid-point of a line segment is the point of bisection, which means dividing into two parts of equal length. Suppose we want to divide a line segment into three parts of equal length, we have to locate points suitably to effect a trisection of the segment.

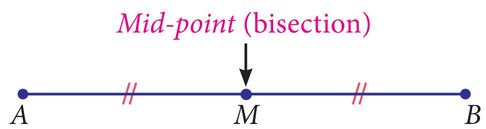


Fig. 5.29

$AM = MB$

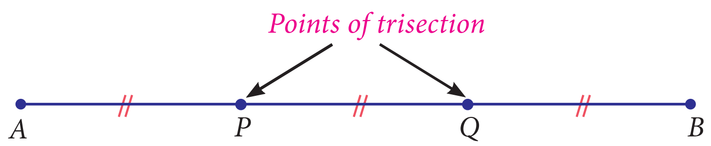


Fig. 5.30

$AP = PQ = QB$

For a given line segment, there are two points of trisection. The method of obtaining this is similar to that of what we did in the case of locating the point of bisection (i.e., the mid-point). Observe the given Fig. 5.31. Here P and Q are the points of trisection of the line segment AB where A is (x_1, y_1) and B is (x_2, y_2). Clearly we know that, P is the mid-point of AQ and Q is the mid-point of PB. Now consider the ΔACQ and ΔPDB (Also, can be verified using similarity property of triangles which will be dealt in detail in higher classes).

$$A'P' = P'Q' = Q'B'$$

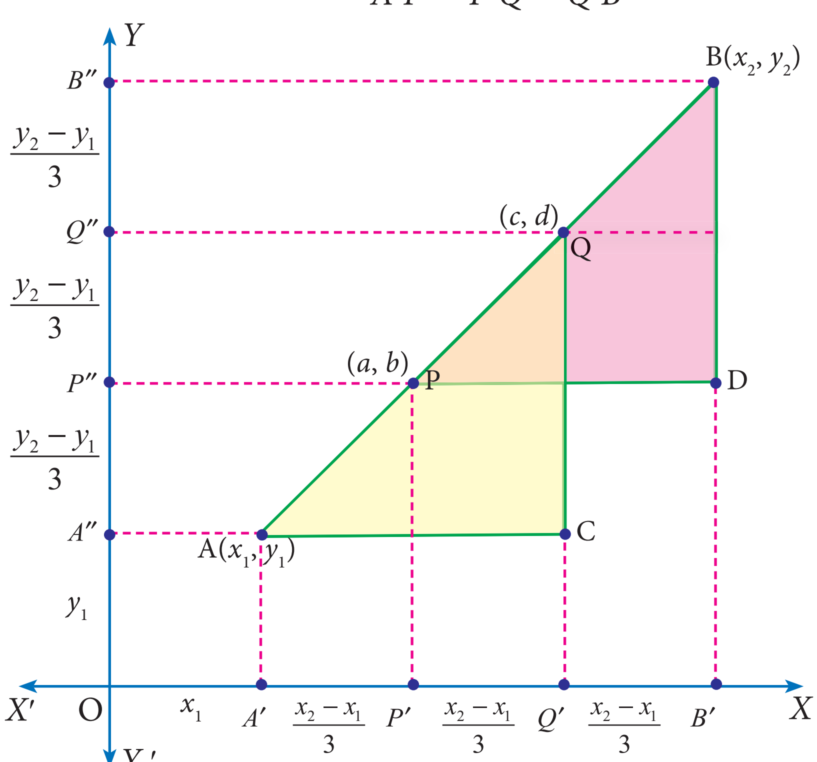


Fig. 5.31

Note that when we divide the segment into 3 equal parts, we are also dividing the horizontal and vertical legs into three equal parts.

If P is (a, b), then

$$a = OP' = OA' + A'P'$$

$$= x_1 + \frac{x_2 - x_1}{3} = \frac{x_2 + 2x_1}{3};$$

$$b = PP' = OA'' + A''P''$$

$$= y_1 + \frac{y_2 - y_1}{3} = \frac{y_2 + 2y_1}{3}$$

Thus we get the point P as $\left(\dfrac{x_2 + 2x_1}{3}, \dfrac{y_2 + 2y_1}{3}\right)$

If Q is (c, d), then

$$c = OQ' = OB' - Q'B' = x_2 - \left(\frac{x_2 - x_1}{3}\right) = \frac{2x_2 + x_1}{3};$$

$$d = OQ'' = OB'' - Q''B'' = y_2 - \left(\frac{y_2 - y_1}{3}\right) = \frac{2y_2 + y_1}{3}$$

Thus the required point Q is $\left(\frac{2x_2 + x_1}{3}, \frac{2y_2 + y_1}{3}\right)$

Example 5.16 Find the points of trisection of the line segment joining $(-2,-1)$ and $(4,8)$.

Solution Let $A(-2,-1)$ and $B(4,8)$ are the given points.

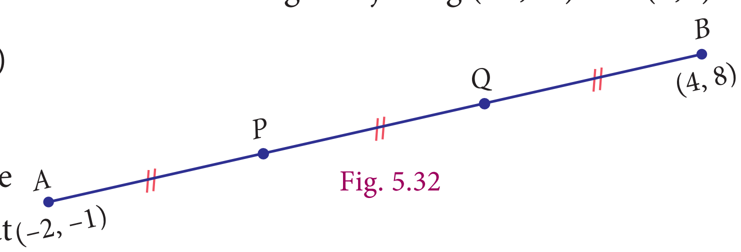


Fig. 5.32

Let $P(a,b)$ and $Q(c,d)$ be the points of trisection of AB, so that $AP = PQ = QB$.

By the formula proved above,

P is the point

$$\left(\frac{x_2 + 2x_1}{3}, \frac{y_2 + 2y_1}{3}\right) = \left(\frac{4 + 2(-2)}{3}, \frac{8 + 2(-1)}{3}\right)$$

$$= \left(\frac{4-4}{3}, \frac{8-2}{3}\right) = (0, 2)$$

Q is the point

$$\left(\frac{2x_2 + x_1}{3}, \frac{2y_2 + y_1}{3}\right) = \left(\frac{2(4) - 2}{3}, \frac{2(8) - 1}{3}\right)$$

$$= \left(\frac{8-2}{3}, \frac{16-1}{3}\right) = (2, 5)$$

Progress Check

(i) Find the coordinates of the points of trisection of the line segment joining $(4,-1)$ and $(-2,-3)$.

(ii) Find the coordinates of points of trisection of the line segment joining the point $(6,-9)$ and the origin.

5.6 Section Formula

We studied bisection and trisection of a given line segment. These are only particular cases of the general problem of dividing a line segment joining two points (x_1, y_1) and (x_2, y_2) in the ratio $m : n$.

Given a segment AB and a positive real number r.

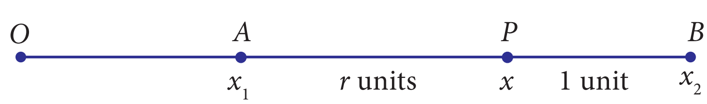


Fig.5.33

We wish to find the coordinate of point P which divides AB in the ratio r :1. This means $\frac{AP}{PB} = \frac{r}{1}$ or $AP = r(PB)$.

This means that $x - x_1 = r(x_2 - x)$

Solving this, $x = \dfrac{rx_2 + x_1}{r+1}$ ….. (1)

We can use this result for points on a line to the general case as follows.

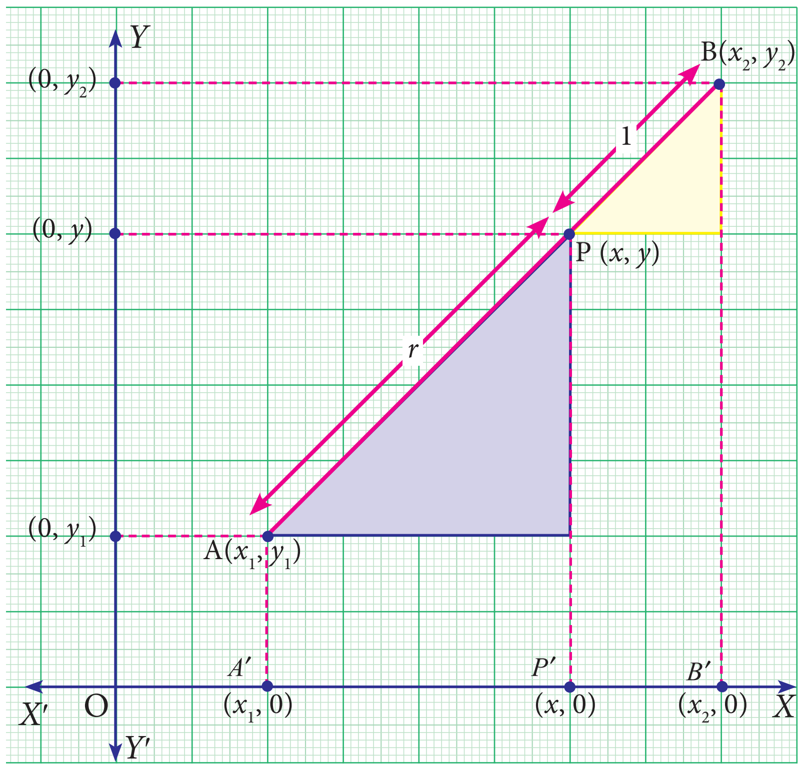


Fig. 5.34

Taking $AP:PB = r:1$, we get $A'P':P'B' = r:1$.

Therefore $A'P' = r(P'B')$

Thus, $(x - x_1) = r(x_2 - x)$

which gives $x = \dfrac{rx_2 + x_1}{r+1}$ … [see (1)]

Precisely in the same way we can have $y = \dfrac{ry_2 + y_1}{r+1}$

If P is between A and B, and $\dfrac{AP}{PB} = r$, then we have the formula,

$$P \text{ is } \left(\frac{rx_2 + x_1}{r+1}, \frac{ry_2 + y_1}{r+1}\right).$$

Thinking Corner

(i) What happens when $m = n = 1$? Can you identify it with a result already proved?

(ii) $AP : PB = 1 : 2$ and $AQ : QB = 2:1$. What is $AP : AB$? What is $AQ : AB$?

If r is taken as $\dfrac{m}{n}$, then the section formula is $\left(\dfrac{mx_2 + nx_1}{m+n}, \dfrac{my_2 + ny_1}{m+n}\right)$, which is the standard form.

Note

- The line joining the points (x_1, y_1) and (x_2, y_2) is divided by x-axis in the ratio $\frac{-y_1}{y_2}$ and by y-axis in the ratio $\frac{-x_1}{x_2}$.
- If three points are collinear, then one of the points divide the line segment joining the other two points in the ratio $r : 1$.
- Remember that the section formula can be used only when the given three points are collinear.
- This formula is helpful to find the centroid, incenter and excenters of a triangle. It has applications in physics too; it helps to find the center of mass of systems, equilibrium points and many more.

Example 5.17 Find the coordinates of the point which divides the line segment joining the points (3,5) and (8,–10) internally in the ratio 3:2.

Solution

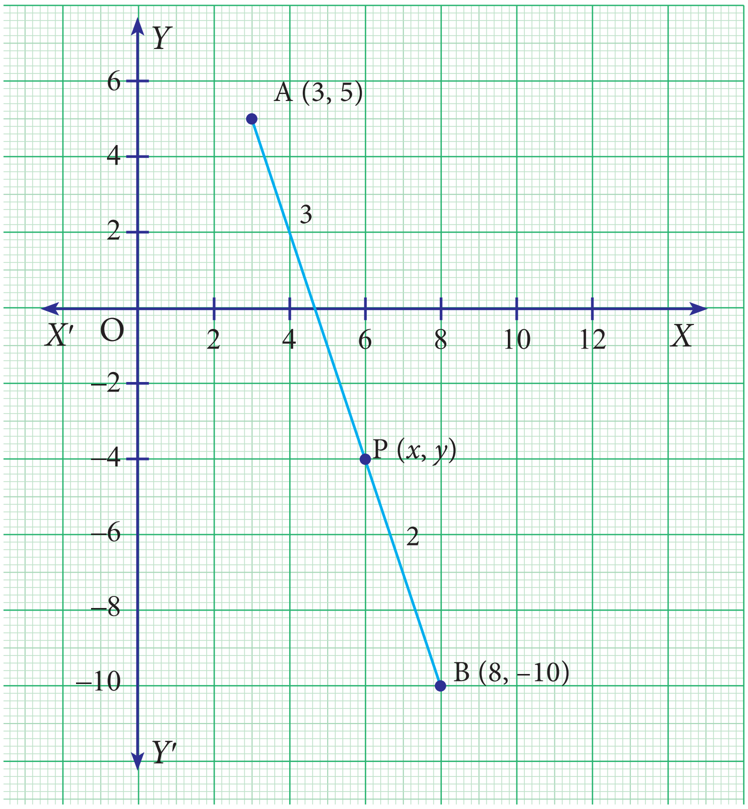


Fig. 5.35

Let $A(3,5)$, $B(8,–10)$ be the given points and let the point $P(x,y)$ divides the line segment AB internally in the ratio 3:2.

By section formula, $P(x, y) = P\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$

Here $x_1 = 3,\ y_1 = 5,\ x_2 = 8,\ y_2 = -10$ and $m = 3,\ n = 2$

Therefore $P(x, y) = P\left(\frac{3(8)+2(3)}{3+2}, \frac{3(-10)+2(5)}{3+2}\right) = P\left(\frac{24+6}{5}, \frac{-30+10}{5}\right) = P(6, -4)$

Example 5.18

In what ratio does the point $P(-2, 4)$ divide the line segment joining the points $A(-3, 6)$ and $B(1, -2)$ internally?

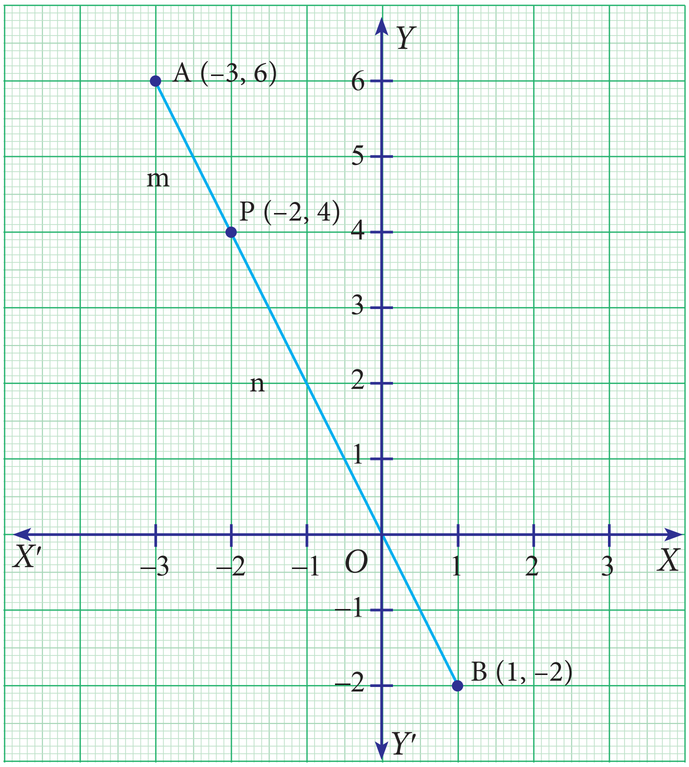


Fig. 5.36

Solution

Given points are $A(-3, 6)$ and $B(1, -2)$. $P(-2, 4)$ divide AB internally in the ratio $m : n$.

By section formula,

$$P(x, y) = P\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$

$$= P(-2, 4) \quad .....(1)$$

Here $x_1 = -3,\ y_1 = 6,\ x_2 = 1,\ y_2 = -2$

$$(1) \Rightarrow \left(\frac{m(1)+n(-3)}{m+n}, \frac{m(-2)+n(6)}{m+n}\right) = P(-2, 4)$$

Equating x-coordinates, we get

$$\frac{m-3n}{m+n} = -2 \text{ or } m-3n = -2m-2n$$

$$3m = n$$

$$\frac{m}{n} = \frac{1}{3}$$

$$m : n = 1 : 3$$

Hence P divides AB internally in the ratio 1:3.

Note

We may arrive at the same result by also equating the y-coordinates.

Try it.

Example 5.19

What are the coordinates of B if point $P(-2,3)$ divides the line segment joining $A(-3,5)$ and B internally in the ratio 1:6?

Solution

Let $A(-3,5)$ and $B(x_2, y_2)$ be the given two points.

Given $P(-2,3)$ divides AB internally in the ratio 1:6.

By section formula, $P\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right) = P(-2,3)$

$$P\left(\frac{1(x_2)+6(-3)}{1+6}, \frac{1(y_2)+6(5)}{1+6}\right) = P(-2,3)$$

Equating the coordinates

$\frac{x_2-18}{7}=-2$	$\frac{y_2+30}{7}=3$
$x_2-18=-14$	$y_2+30=21$
$x_2=4$	$y_2=-9$

Therefore, the coordinate of B is $(4,-9)$

Exercise 5.4

1. Find the coordinates of the point which divides the line segment joining the points $A(4,-3)$ and $B(9,7)$ in the ratio 3:2.
2. In what ratio does the point $P(2,-5)$ divide the line segment joining $A(-3,5)$ and $B(4,-9)$.
3. Find the coordinates of a point P on the line segment joining $A(1,2)$ and $B(6,7)$ in such a way that $AP=\frac{2}{5}AB$.
4. Find the coordinates of the points of trisection of the line segment joining the points $A(-5,6)$ and $B(4,-3)$.
5. The line segment joining $A(6,3)$ and $B(-1,-4)$ is doubled in length by adding half of AB to each end. Find the coordinates of the new end points.
6. Using section formula, show that the points $A(7,-5)$, $B(9,-3)$ and $C(13,1)$ are collinear.
7. A line segment AB is increased along its length by 25% by producing it to C on the side of B. If A and B have the coordinates $(-2,-3)$ and $(2,1)$ respectively, then find the coordinates of C.

5.7 The Coordinates of the Centroid

Consider a ΔABC whose vertices are $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$.

Let AD, BE and CF be the medians of the ΔABC.

The mid-point of BC is $D\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$

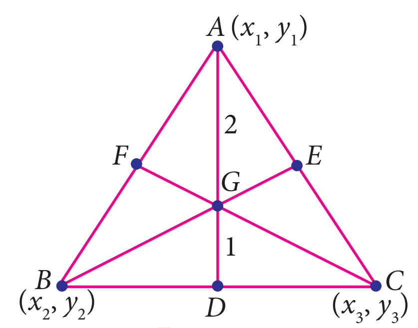


Fig. 5.37

The centroid G divides the median AD internally in the ratio 2:1 and therefore by section formula, the centroid

$$G(x,y) \text{ is } \left(\frac{\frac{2(x_2+x_3)}{2}+1(x_1)}{2+1}, \frac{\frac{2(y_2+y_3)}{2}+1(y_1)}{2+1} \right) = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3} \right)$$

The centroid G of the triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is

$$G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3} \right)$$

Activity - 2

1. Draw ΔABC with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ on the graph sheet.
2. Draw medians and locate the centroid of ΔABC

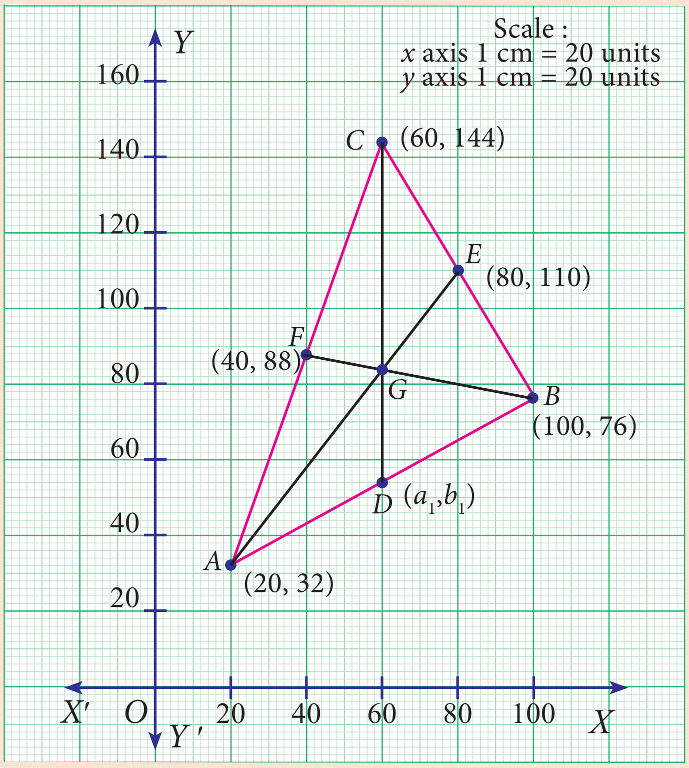


Fig. 5.38

Observation

(i) The coordinates of the vertices of ΔABC where

$A(x_1, y_1) =$ ______________,

$B(x_2, y_2) =$ ______________

and $C(x_3, y_3) =$ __________

(ii) The coordinates of the centroid G= ___________

(iii) Use the formula to locate the centroid, whose coordinates are = __________.

(iv) Mid-point of AB is ____________.

(v) Find the point which divides the line segment joining (x_3, y_3) and the mid-point of AB internally in the ratio 2:1 is ____________.

Note

- The medians of a triangle are concurrent and the point of concurrence, the centroid G, is one-third of the distance from the opposite side to the vertex along the median.
- The centroid of the triangle obtained by joining the mid-points of the sides of a triangle is the same as the centroid of the original triangle.
- If (a_1, b_1), (a_2, b_2) and (a_3, b_3) are the mid-points of the sides of a triangle ABC then its centroid G is given by

$$G\left(\frac{a_1+a_2+a_3}{3}, \frac{b_1+b_2+b_3}{3}\right)$$

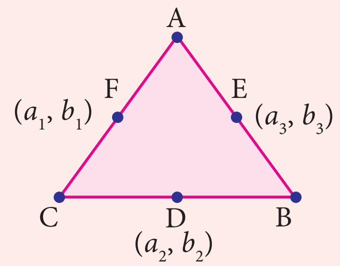


Fig. 5.39

Example 5.20 Find the centroid of the triangle whose veritices are $A(6,-1)$, $B(8,3)$ and $C(10,-5)$.

FZB1QQ

Solution

The centroid $G(x, y)$ of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$G(x, y) = G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

We have $(x_1, y_1) = (6,-1)$; $(x_2, y_2) = (8,3)$;
$(x_3, y_3) = (10,-5)$

The centroid of the triangle

$$G(x, y) = G\left(\frac{6+8+10}{3}, \frac{-1+3-5}{3}\right)$$

$$= G\left(\frac{24}{3}, \frac{-3}{3}\right) = G(8,-1)$$

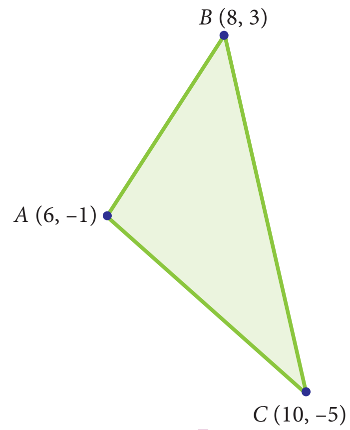


Fig. 5.40

Note

- The Euler line of a triangle is the line that passes through the orthocenter (H), centroid (G) and the circumcenter (S). G divides the line segment $\overline{HS}$ in the ratio 2:1 from the orthocenter. That is centroid divides orthocenter and circumcenter internally in the ratio 2:1 from the Orthocentre.
- In an equilateral triangle, orthocentre, incentre, centroid and circumcentre are all the same.

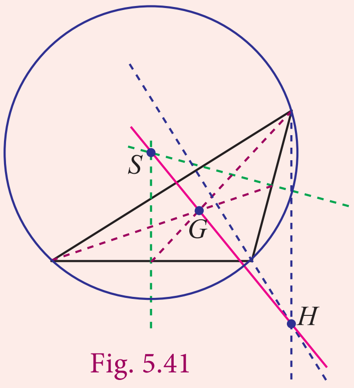


Fig. 5.41

Example 5.21 If the centroid of a triangle is at $(-2,1)$ and two of its vertices are $(1,-6)$ and $(-5,2)$, then find the third vertex of the triangle.

Solution Let the vertices of a triangle be

$A(1,-6),\ B(-5,2)$ and $C(x_3, y_3)$

Given the centroid of a triangle as $(-2,1)$ we get,

$$\frac{x_1+x_2+x_3}{3} = -2 \qquad\qquad \frac{y_1+y_2+y_3}{3} = 1$$

$$\frac{1-5+x_3}{3} = -2 \qquad\qquad \frac{-6+2+y_3}{3} = 1$$

$$-4+x_3 = -6 \qquad\qquad -4+y_3 = 3$$

$$x_3 = -2 \qquad\qquad y_3 = 7$$

Therefore, third vertex is $(-2,7)$.

Thinking Corner

(i) Master gave a trianglular plate with vertices A(5, 8), B(2, 4), C(8, 3) and a stick to a student. He wants to balance the plate on the stick. Can you help the boy to locate that point which can balance the plate.

(ii) Which is the centre of gravity for this triangle? why?

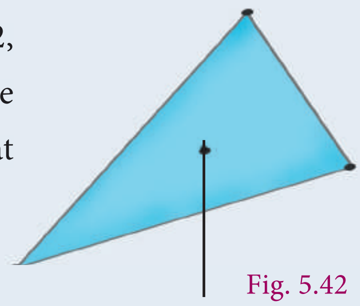


Fig. 5.42

Exercise 5.5

1. Find the centroid of the triangle whose vertices are

   (i) $(2,-4)$, $(-3,-7)$ and $(7,2)$ (ii) $(-5,-5)$, $(1,-4)$ and $(-4,-2)$

2. If the centroid of a triangle is at $(4,-2)$ and two of its vertices are $(3,-2)$ and $(5,2)$ then find the third vertex of the triangle.

3. Find the length of median through A of a triangle whose vertices are $A(-1,3)$, $B(1,-1)$ and $C(5,1)$.

4. The vertices of a triangle are $(1,2)$, $(h,-3)$ and $(-4,k)$. If the centroid of the triangle is at the point $(5,-1)$ then find the value of $\sqrt{(h+k)^2+(h+3k)^2}$.

5. Orthocentre and centroid of a triangle are $A(-3,5)$ and $B(3,3)$ respectively. If C is the circumcentre and AC is the diameter of this circle, then find the radius of the circle.

6. ABC is a triangle whose vertices are $A(3,4)$, $B(-2,-1)$ and $C(5,3)$. If G is the centroid and $BDCG$ is a parallelogram then find the coordinates of the vertex D.

7. If $\left(\frac{3}{2},5\right)$, $\left(7,\frac{-9}{2}\right)$ and $\left(\frac{13}{2},\frac{-13}{2}\right)$ are mid-points of the sides of a triangle, then find the centroid of the triangle.

Exercise 5.6

FZGA4G

Multiple Choice Questions

1. If the y-coordinate of a point is zero, then the point always lies ______

   (1)in the I quadrant (2) in the II quadrant (3)on x-axis (4) on y-axis

2. The points (–5, 2) and (2, –5) lie in the ________

   (1) same quadrant (2) II and III quadrant respectively
   (3) II and IV quadrant respectively (4) IV and II quadrant respectively

3. On plotting the points $O(0,0)$, $A(3, -4)$, $B(3, 4)$ and $C(0, 4)$ and joining OA, AB, BC and CO, which of the following figure is obtained?

   (1) Square (2) Rectangle (3) Trapezium (4) Rhombus

4. If $P(-1,1)$, $Q(3,-4)$, $R(1, -1)$, $S(-2, -3)$ and $T(-4, 4)$ are plotted on a graph paper, then the points in the fourth quadrant are __________

   (1) P and T (2) Q and R (3) only S (4) P and Q

5. The point whose ordinate is 4 and which lies on the y-axis is _______________

   (1)(4, 0) (2) (0, 4) (3) (1, 4) (4) (4, 2)

6. The distance between the two points (2, 3) and (1, 4) is ______

   (1) 2 (2) $\sqrt{56}$ (3) $\sqrt{10}$ (4) $\sqrt{2}$

7. If the points A (2,0), B (-6,0), C (3, a–3) lie on the x-axis then the value of a is _____

   (1) 0 (2) 2 (3) 3 (4) –6

8. If (x+2, 4) = (5, y–2), then the coordinates (x,y) are _____

   (1) (7, 12) (2) (6, 3) (3) (3, 6) (4) (2, 1)

9. If Q_1, Q_2, Q_3, Q_4 are the quadrants in a Cartesian plane then $Q_2 \cap Q_3$ is ___________

   (1) $Q_1 \cup Q_2$ (2) $Q_2 \cup Q_3$ (3) Null set (4) Negative x-axis.

10. The distance between the point (5, –1) and the origin is _________

    (1) $\sqrt{24}$ (2) $\sqrt{37}$ (3) $\sqrt{26}$ (4) $\sqrt{17}$

11. The coordinates of the point C dividing the line segment joining the points $P(2,4)$ and $Q(5,7)$ internally in the ratio 2:1 is

(1) $\left(\frac{7}{2},\frac{11}{2}\right)$ (2) (3,5) (3) (4,4) (4) (4,6)

12. If $P\left(\frac{a}{3},\frac{b}{2}\right)$ is the mid-point of the line segment joining $A(-4,3)$ and $B(-2,4)$ then (a,b) is

(1) $(-9,7)$ (2) $\left(-3,\frac{7}{2}\right)$ (3) $(9,-7)$ (4) $\left(3,-\frac{7}{2}\right)$

13. In what ratio does the point $Q(1,6)$ divide the line segment joining the points $P(2,7)$ and $R(-2,3)$

(1) 1:2 (2) 2:1 (3) 1:3 (4) 3:1

14. If the coordinates of one end of a diameter of a circle is (3,4) and the coordinates of its centre is (−3,2), then the coordinate of the other end of the diameter is

(1) (0,−3) (2) (0,9) (3) (3,0) (4) (−9,0)

15. The ratio in which the x-axis divides the line segment joining the points $A(a_1,b_1)$ and $B(a_2,b_2)$ is

(1) $b_1:b_2$ (2) $-b_1:b_2$ (3) $a_1:a_2$ (4) $-a_1:a_2$

16. The ratio in which the x-axis divides the line segment joining the points (6,4) and (1, −7) is

(1) 2:3 (2) 3:4 (3) 4:7 (4) 4:3

17. If the coordinates of the mid-points of the sides AB, BC and CA of a triangle are (3,4), (1,1) and (2,−3) respectively, then the vertices A and B of the triangle are

(1) (3,2), (2,4) (2) (4,0), (2,8) (3) (3,4), (2,0) (4) (4,3), (2,4)

18. The mid-point of the line joining $(-a,2b)$ and $(-3a,-4b)$ is

(1) $(2a,3b)$ (2) $(-2a,-b)$ (3) $(2a,b)$ (4) $(-2a,-3b)$

19. In what ratio does the y-axis divides the line joining the points (−5,1) and (2,3) internally

(1) 1:3 (2) 2:5 (3) 3:1 (4) 5:2

20. If (1,−2), (3,6), $(x,10)$ and (3,2) are the vertices of the parallelogram taken in order, then the value of x is

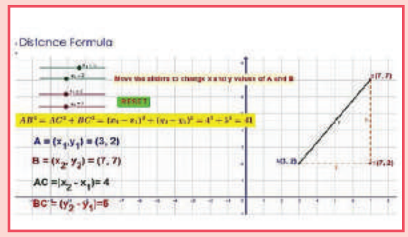
(1) 6 (2) 5 (3) 4 (4) 3

Points to remember

- If x_1, x_2 are the x-coordinates of two points on the x-axis then the distance between them is $x_2 - x_1$, if $x_2 > x_1$.
- If y_1, y_2 are the y-coordinates of two points on the y-axis then the distance between them is $|y_1 - y_2|$.
- Distance between the two points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$
- Distance between (x_1, y_1) and the origin $(0, 0)$ is $\sqrt{x_1^2+y_1^2}$
- The mid-point M of the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$
- The point P which divides the line segment joining the two points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m:n$ is $P\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$
- The centroid G of the triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is $G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$
- The centroid of the triangle obtained by joining the mid-points of the sides of a triangle is the same as the centroid of the original triangle.

ICT Corner-1

Expected Result is shown in this picture



Step – 1

Open the Browser and copy and paste the Link given below (or) by typing the URL given (or) Scan the QR Code.

Step - 2

GeoGebra work book called "IX Analytical Geometry" will open. There are several worksheets given. Select the one you want. For example, open" Distance Formula"

Step-3

Move the sliders x_1, x_2, y_1, y_2 to change the co-ordinates of A and B. Now you calculate the distance AB using theDistance formula in a piece of paper and check your answer

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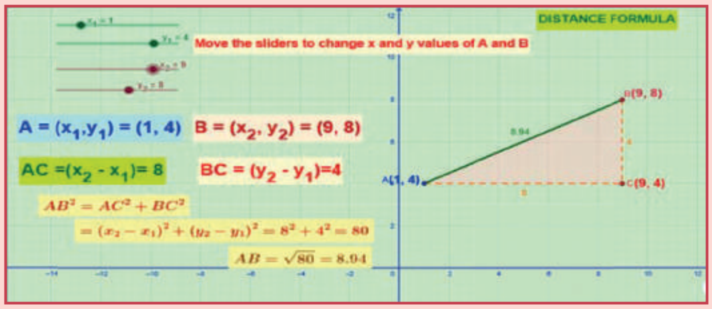
ICT Corner-2

Expected Result is shown in this picture



Step – 1

Open the Browser by typing the URL Link given below (or) Scan the QR Code. GeoGebra work sheet named "Co-ordinate Geometry" will open. There are two worksheets under the title Distance Formula and Section Formula.

Step - 2

Move the sliders of the respective values to change the points and ratio. Work out the solution and check and click on the respective check box and check the answer.

Scan the QR Code.

B566_9_MAT_EM_T3

Activity

Plot the points $A(1, 0)$, $B(-7, 2)$, $C(-3, 7)$ on a graph sheet and join them to form a triangle.

Plot the point $G(-3, 3)$.

Join AG and extend it to intersect BC at D.

Join BG and extend it to intersect AC at E.

What do you infer when you measure the distance between BD and DC and the distance between CE and EA?

Using distance formula find the lengths of CG and GF, where F in on AB.

Write your inference about $AG: GD$, $BG: GE$ and $CG: GF$.

Note: ***G*** is the **centroid** of the triangle and AD, BE and CF are the three medians of the triangle.