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STANDARD NINE

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CONTENTS

Captions used in this Textbook

எண்ணென்ப ஏனை எழுத்தென்ப இவ்விரண்டும்
கண்ணென்ப வாழும் உயிர்க்கு – குறள் 392

Numbers and letters, they are known as
eyes to humans, they are. Kural 392

Learning Outcomes

To transform the classroom processes into learning centric with a set of bench marks

Note

To provide additional inputs for students in the content

Activity / Project

To encourage students to involve in activities to learn mathematics

ICT Corner

To encourage learner's understanding of content through application of technology

Thinking Corner

To kindle the inquisitiveness of students in learning mathematics. To make the students to have a diverse thinking

Points to Remember

To recall the points learnt in the topic

Multiple Choice Questions

To provide additional assessment items on the content

Progress Check

Self evaluation of the learner's progress

Exercise

To evaluate the learners' in understanding the content

"The essence of mathematics is not to make simple things complicated but to make complicated things simple" -S. Gudder

QR GUIDE

Let's use the QR code in the text books! How?

- Download the QR code scanner from the Google PlayStore/ Apple App Store into your smartphone
- Open the QR code scanner application
- Once the scanner button in the application is clicked, camera opens and then bring it closer to the QR code in the text book.
- Once the camera detects the QR code, a url appears in the screen.Click the url and go to the content page.

SYMBOLS

$=$	equal to
$\neq$	not equal to
$<$	less than
$\leq$	less than or equal to
$>$	greater than
$\geq$	greater than or equal to
$\approx$	equivalent to
$\cup$	union
$\cap$	intersection
$\mathbb{U}$	universal Set
$\in$	belongs to
$\notin$	does not belong to
$\subset$	proper subset of
$\subseteq$	subset of or is contained in
$\not\subset$	not a proper subset of
$\not\subseteq$	not a subset of or is not contained in
A' (or) A^c	complement of A
$\varnothing$ (or) $\{\ \}$	empty set or null set or void set
$n(A)$	number of elements in the set A
$P(A)$	power set of A
$\sum$	summation

$\|\|\|^{ly}$	similarly
Δ	symmetric difference
$\mathbb{N}$	natural numbers
$\mathbb{W}$	whole numbers
$\mathbb{Z}$	integers
$\mathbb{R}$	real numbers
$\triangle$	triangle
$\angle$	angle
$\perp$	perpendicular to
$\|\|$	parallel to
$\Longrightarrow$	implies
$\therefore$	therefore
$\because$	since (or) because
$\|\quad\|$	absolute value
$\simeq$	approximately equal to
$\|$ (or) :	such that
$\equiv$ (or) $\cong$	congruent
$\equiv$	identically equal to
π	pi
$\pm$	plus or minus
$P(E)$	probability of an event E

FYIDKS

E-book

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Evaluation

B566_9_MAT_EM_T3

DIGI Links

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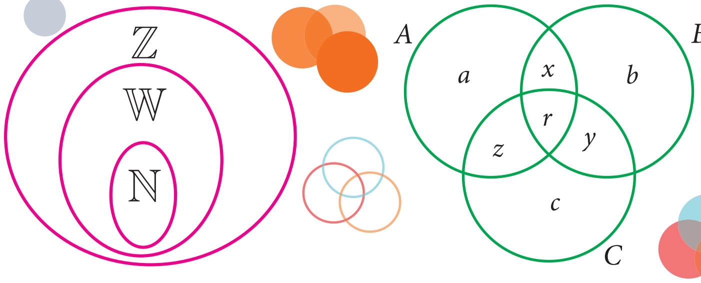
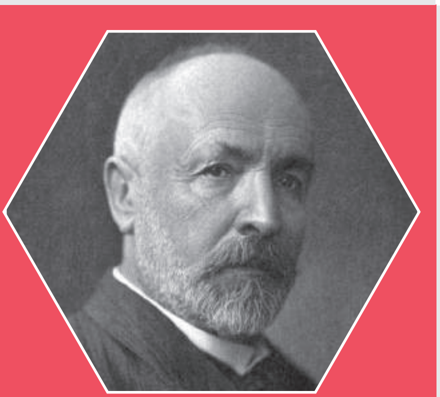


SET LANGUAGE

A set is a many that allows itself to thought of as a one

-Georg Cantor



Georg Cantor
(AD (CE) 1845 - 1918)

The theory of sets was developed by German mathematician Georg Cantor. Today it is used in almost every branch of Mathematics. In Mathematics, sets are convenient because all mathematical structures can be regarded as sets.

E3RQRH

Learning Outcomes

- To describe and represent a set in different forms.
- To identify different types of sets.
- To understand and perform set operations and apply this in Venn diagram.
- To know the commutative, associative and distributive properties among sets.
- To understand and verify De Morgan's laws.
- To use set language in solving life oriented word problems.

1.1 Introduction

In our daily life, we often deal with collection of objects like books, stamps, coins, etc. Set language is a mathematical way of representing a collection of objects.

Let us look at the following pictures. What do they represent?

Here, Fig.1.1 represents a collection of fruits and Fig. 1.2 represents a collection of house- hold items.

We observe in these cases, our attention turns from one individual object to a collection of objects based on their characteristics. Any such collection is called a set.

Fig. 1.1

Fig. 1.2

1.2 Set

A set is a well-defined collection of objects.

Here “well-defined collection of objects” means that given a specific object it must be possible for us to decide whether the object is an element of the given collection or not.

The objects of a set are called its members or elements.

For example,

1. The collection of all books in a District Central Library.
2. The collection of all colours in a rainbow.
3. The collection of prime numbers.

Which of the following are sets ?

1. Collection of Natural numbers.
2. Collection of English alphabets.
3. Collection of good students in a class.
4. Collection of States in our country.
5. Collection of beautiful flowers in a garden.

We see that in the adjacent box, statements (1), (2), and (4) are well defined and therefore they are sets. Whereas (3) and (5) are not well defined because the words good and beautiful are difficult to agree on. I might consider a student to be good and you may not. I might consider the Jasmine is the beautiful flower but you may not. So we will consider only those collections to be sets where there is no such ambiguity.

Therefore (3) and (5) are not sets.

Activity - 1

Discuss and give as many examples of collections from your daily life situations, which are sets and which are not sets.

Note

- Elements of a set are listed only once.
- The order of listing the elements of the set does not change the set.

For example, the collection 1,2,3,4,5,6,7,8, … as well as the collection 1, 3, 2, 4, 5, 7, 6, 8, … are the same though listed in different order. Since it is necessary to know whether an object is an element in the set or not, we do not want to list that element many times.

Notation

A set is usually denoted by capital letters of the English Alphabets A, B, P, Q, X, Y, etc.

The elements of a set is written within curly brackets "{ }"

If x is an element of a set A or x belongs to A, we write $x \in A$.

If x is not an element of a set A or x does not belongs to A, we write $x \notin A$.

For example,

Consider the set $A = \{2,3,5,7\}$ then

2 is an element of A; we write $2 \in A$

5 is an element of A; we write $5 \in A$

6 is not an element of A; we write $6 \notin A$

Example 1.1

Consider the set A = {Ashwin, Murali Vijay, Vijay Shankar, Badrinath }.

Fill in the blanks with the appropriate symbol $\in$ or $\notin$.

(i) Murali Vijay ______ A. (ii) Ashwin ______ A. (iii) Badrinath ______ A.

(iv) Ganguly ______ A. (v) Tendulkar ______ A

Solution

(i) Murali Vijay $\in A$. (ii) Ashwin $\in A$ (iii) Badrinath $\in A$

(iv) Ganguly $\notin A$. (v) Tendulkar $\notin A$.

1.3 Representation of a Set

The collection of odd numbers can be described in many ways:

(1) "The set of odd numbers" is a fine description, we understand it well.

(2) It can be written as $\{1, 3, 5, \ldots\}$

(3) Also, it can be said as the collection of all numbers x where x is an odd number.

All of them are equivalent and useful. For instance,the two descriptions "The collection of all solutions to the equation $x-5 = 3$" and $\{8\}$ refer to the same set.

A set can be represented in any one of the following three ways or forms:

1.3.1 Descriptive Form

In descriptive form, a set is described in words.

For example,

(i) The set of all vowels in English alphabets.

(ii) The set of whole numbers.

1.3.2 Set Builder Form or Rule Form

In set builder form, all the elements are described by a rule.

For example,

(i) $A = \{x : x \text{ is a vowel in English alphabets}\}$

(ii) $B = \{x | x \text{ is a whole number}\}$

Note

The symbol ':' or '|' stands for "such that".

1.3.3 Roster Form or Tabular Form

A set can be described by listing all the elements of the set.

For example,

(i) $A = \{a, e, i, o, u\}$

(ii) $B = \{0,1,2,3,\ldots\}$

Can this form of representation be possible always?

Note

Three dots (…) in the example (ii) is called ellipsis. It indicates that the pattern of the listed elements continues in the same manner.

Activity-2

Write the following sets in respective forms.

S.No.	Descriptive Form	Set Builder Form	Roster Form
1	The set of all natural numbers less than 10		
2		$\{x : x \text{ is a multiple of } 3, x \in \mathbb{N}\}$	
3			$\{2,4,6,8,10\}$
4	The set of all days in a week.		
5			$\{\ldots-3,-2,-1,0,1,2,3\ldots\}$

Example 1.2

Write the set of letters of the following words in Roster form

(i) ASSESSMENT (ii) PRINCIPAL

Solution

(i) ASSESSMENT (ii) PRINCIPAL

$X = \{A, S, E, M, N, T\}$ $Y = \{P, R, I, N, C, A, L\}$

Exercise 1.1

1. Which of the following are sets?

   (i) The collection of prime numbers upto 100.
   (ii) The collection of rich people in India.
   (iii) The collection of all rivers in India.
   (iv) The collection of good Hockey players.

2. List the set of letters of the following words in Roster form.

   (i) INDIA
   (ii) PARALLELOGRAM
   (iii) MISSISSIPPI
   (iv) CZECHOSLOVAKIA

3. Consider the following sets $A = \{0, 3, 5, 8\}$, $B = \{2, 4, 6, 10\}$ and $C = \{12, 14,18, 20\}$.

   (a) State whether True or False:

   (i) $18 \in C$
   (ii) $6 \notin A$
   (iii) $14 \notin C$
   (iv) $10 \in B$
   (v) $5 \in B$
   (vi) $0 \in B$

   (b) Fill in the blanks:

   (i) $3 \in$ ____
   (ii) $14 \in$ _____
   (iii) 18 ____ B
   (iv) 4 _____ B

4. Represent the following sets in Roster form.

   (i) A = The set of all even natural numbers less than 20.
   (ii) $B = \{y : y = \frac{1}{2n}, n \in \mathbb{N}, n \le 5\}$
   (iii) $C = \{x : x$ is perfect cube, $27 < x < 216\}$
   (iv) $D = \{x : x \in \mathbb{Z}, -5 < x \le 2\}$

5. Represent the following sets in set builder form.

   (i) B = The set of all Cricket players in India who scored double centuries in One Day Internationals.
   (ii) $C = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$
   (iii) D = The set of all tamil months in a year.
   (iv) E = The set of odd Whole numbers less than 9.

6. Represent the following sets in descriptive form.

(i) P ={ January, June, July}

(ii) Q = {7,11,13,17,19,23,29}

(iii) R = $\{x : x \in \mathbb{N}, x < 5\}$

(iv) S = {x : x is a consonant in English alphabets}

1.4 Types of Sets

There is a very special set of great interest: the empty collection ! Why should one care about the empty collection? Consider the set of solutions to the equation $x^2+1 = 0$. It has no elements at all in the set of Real Numbers. Also consider all rectangles with one angle greater than 90 degrees. There is no such rectangle and hence this describes an empty set.

So, the empty set is important, interesting and deserves a special symbol too.

1.4.1 Empty Set or Null Set

A set consisting of no element is called the *empty set* or *null set* or *void set*.

It is denoted by $\varnothing$ or { }.

Thinking Corner

Are the sets {0} and {$\varnothing$} empty sets?

For example,

(i) A={x : x is an odd integer and divisible by 2}

$\therefore A=\{\ \}$ or $\varnothing$

(ii) The set of all integers between 1 and 2.

1.4.2.Singleton Set

A set which has only one element is called a *singleton set*.

For example,

(i) $A = \{\mathbf{x : 3 < x < 5, x} \in \mathbb{N}\}$ (ii) The set of all even prime numbers.

1.4.3 Finite Set

A set with finite number of elements is called a *finite set*.

Note

An empty set has no elements, so $\varnothing$ is a finite set.

For example,

1. The set of family members.
2. The set of indoor/outdoor games you play.
3. The set of curricular subjects you learn in school.
4. $A = \{x : x \text{ is a factor of } 36\}$

1.4.4 Infinite Set

A set which is not finite is called an *infinite set*.

Thinking Corner

Is the set of natural numbers a finite set?

For example,

(i) $\{5,10,15,...\}$ (ii) The set of all points on a line.

To discuss further about the types of sets, we need to know the cardinality of sets.

Cardinal number of a set : When a set is finite, it is very useful to know how many elements it has. The number of elements in a set is called the Cardinal number of the set.

The cardinal number of a set A is denoted by $n(A)$

Example 1.3 If $A = \{1,2,3,4,5,7,9,11\}$, find $n(A)$.

Solution

$A = \{1,2,3,4,5,7,9,11\}$

Since set A contains 8 elements, $n(A) = 8$.

Thinking Corner

If $A = \{1, b, b, \{4, 2\}, \{x, y, z\}, d, \{d\}\}$, then $n(A)$ is____

1.4.5 Equivalent Sets

Two finite sets A and B are said to be equivalent if they contain the same number of elements. It is written as $A \approx B$.

If A and B are equivalent sets, then $n(A) = n(B)$

For example,

Consider $A = \{\text{ball, bat}\}$ and

$B = \{\text{history, geography}\}$.

Here A is equivalent to B because $n(A) = n(B) = 2$.

Thinking Corner

Let $A=\{x : x \text{ is a colour in national flag of India}\}$ and $B=\{\text{Red, Blue, Green}\}$. Are these two sets equivalent?

Example 1.4 Are $P = \{x : -3 \leq x \leq 0, x \in \mathbb{Z}\}$ and $Q =$ The set of all prime factors of 210, equivalent sets?

Solution

$P = \{-3, -2, -1, 0\}$, The prime factors of 210 are 2,3,5,and 7 and so, $Q = \{2, 3, 5, 7\}$

$n(P) = 4$ and $n(Q) = 4$. Therefore P and Q are equivalent sets.

1.4.6 Equal Sets

Two sets are said to be equal if they contain exactly the same elements, otherwise they are said to be unequal.

In other words, two sets A and B are said to be equal, if

(i) every element of A is also an element of B

(ii) every element of B is also an element of A

Thinking Corner

Are the sets $\varnothing$, $\{0\}$, $\{\varnothing\}$ equal or equivalent?

For example,

Consider the sets $A = \{1, 2, 3, 4\}$ and $B = \{4, 2, 3, 1\}$

Since A and B contain exactly the same elements, A and B are equal sets.

Note

- If A and B are equal sets, we write $A = B$.
- If A and B are unequal sets, we write $A \neq B$.

A set does not change, if one or more elements of the set are repeated.

For example, if we are given

$A=\{a, b, c\}$ and $B=\{a, a, b, b, b, c\}$ then, we write $B = \{ a, b, c, \}$. Since, every element of A is also an element of B and every element of B is also an element of A, the sets A and B are equal.

Example 1.5

Are $A = \{x : x \in \mathbb{N}, 4 \leq x \leq 8\}$ and

$B = \{ 4, 5, 6, 7, 8\}$ equal sets?

Solution

$A = \{ 4, 5, 6, 7, 8\}$, $B = \{ 4, 5, 6, 7, 8\}$

A and B are equal sets.

Note

Equal sets are equivalent sets but equivalent sets need not be equal sets. For example, if $A = \{ p,q,r,s,t\}$ and $B= \{ 4,5,6,7,8\}$. Here $n(A)=n(B)$, so A and B are equivalent but not equal.

1.4.7 Universal Set

A Universal set is a set which contains all the elements of all the sets under consideration and is usually denoted by U.

For example,

(i) If we discuss about elements in Natural numbers, then the universal set U is the set of all Natural numbers. $U=\{x : x \in \mathbb{N}\}$.

(ii) If A={earth, mars, jupiter}, then the universal set U is the planets of solar system.

1.4.8 Subset

Let A and B be two sets. If every element of A is also an element of B, then A is called a subset of B. We write $A \subseteq B$.

$A \subseteq B$ is read as "A is a subset of B"

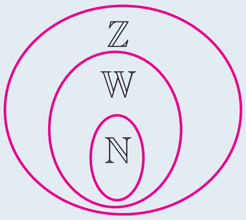
Thus $A \subseteq B$, if $a \in A$ implies $a \in B$.

If A is not a subset of B, we write $A \nsubseteq B$

Clearly, if A is a subset of B, then $n(A) \leq n(B)$.

Since every element of A is also an element of B, the set B must have at least as many elements as A, thus $n(A) \leq n(B)$.

Thinking Corner



Is $\mathbb{W}$ subset of $\mathbb{N}$ or $\mathbb{Z}$?

The other way is also true. Suppose that $n(A) > n(B)$, then A has more elements than B, and hence there is at least one element in A that cannot be in B, so A is not a subset of B.

For example,

(i) $\{1\} \subseteq \{1,2,3\}$ (ii) $\{2,4\} \nsubseteq \{1,2,3\}$

Example 1.6

Insert the appropriate symbol $\subseteq$ or $\nsubseteq$ in each blank to make a true statement. (i) $\{10, 20, 30\}$ ____ $\{10, 20, 30, 40\}$ (ii) $\{p, q, r\}$ _____ $\{w, x, y, z\}$

Solution

(i) $\{10, 20, 30\}$ ____ $\{10, 20, 30, 40\}$

Since every element of $\{10, 20, 30\}$ is also an element of $\{10, 20, 30, 40\}$, we get $\{10, 20, 30\} \subseteq \{10, 20, 30, 40\}$.

(ii) $\{p, q, r\}$ _____ $\{w, x, y, z\}$

Since the element p belongs to $\{p, q, r\}$ but does not belong to $\{w, x, y, z\}$, shows that $\{p, q, r\} \nsubseteq \{w, x, y, z\}$.

Activity-3

Discuss with your friends and give examples of subsets of sets from your daily life situation.

Example 1.7

Write all the subsets of $A = \{a, b\}$.

Solution

$A = \{a,b\}$

Subsets of A are $\varnothing, \{a\}, \{b\}$ and $\{a, b\}$.

Note

- If $A \subseteq B$ and $B \subseteq A$, then $A=B$.
  In fact this is how we defined equality of sets.
- Empty set is a subset of every set.

  This is not easy to see ! Let A be any set. The only way for the empty set to be not a subset of A would be to have an element x in it but with x not in A. But how can x be in the empty set ? That is impossible. So this only way being impossible, the empty set must be a subset of A. (Is your head spinning? Think calmly, explain it to a friend, and you will agree it is alright !)
- Every set is a subset of itself. (Try and argue why?)

1.4.9. Proper Subset

Let A and B be two sets. If A is a subset of B and $A \neq B$, then A is called a proper subset of B and we write $A \subset B$.

For example,

If $A=\{1,2,5\}$ and $B=\{1,2,3,4,5\}$ then A is a proper subset of B ie. $A \subset B$.

1.4.10 Disjoint Sets

Two sets A and B are said to be disjoint if they do not have common elements.

In other words, if $A \cap B=\varnothing$, then A and B are said to be disjoint sets.

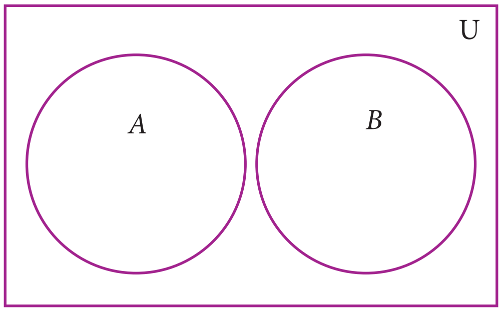


Fig. 1.3

Example 1.8

Verify whether $A=\{20, 22, 23, 24\}$ and $B=\{25, 30, 40, 45\}$ are disjoint sets.

Solution

$$A = \{20,22, 23, 24\} , B=\{25, 30, 40, 45\}$$

$$A \cap B = \{20,22, 23, 24\} \cap \{25, 30, 40, 45\}$$

$$= \{ \ \}$$

Since $A \cap B = \varnothing$, A and B are disjoint sets.

Note

If $A \cap B \neq \varnothing$, then A and B are said to be overlapping sets .Thus if two sets have atleast one common element, they are called overlapping sets.

1.4.11 Power Set

The set of all subsets of a set A is called the power set of 'A'. It is denoted by $P(A)$.

For example,

(i) If $A=\{2, 3\}$, then find the power set of A.

The subsets of A are $\varnothing$, $\{2\},\{3\},\{2,3\}$.

The power set of A,

$P(A) = \{\varnothing ,\{2\},\{3\},\{2,3\}\}$

(ii) If $A = \{\varnothing , \{\varnothing\}\}$, then the power set of A is $\{ \varnothing , \{\varnothing , \{\varnothing\}\}, \{\varnothing\} , \{\{\varnothing\}\} \}$.

An important property.

We already noted that $n(A) \leq n[P(A)]$. But how big is $P(A)$? Think about this a bit, and see whether you come to the following conclusion:

(i) If $n(A) = m$, then $n[P(A)] = 2^m$

(ii) The number of proper subsets of a set A is $n[P(A)]-1 = 2^m-1$.

Example 1.9 Find the number of subsets and the number of proper subsets of a set $X=\{a, b, c, x, y, z\}$.

Solution Given $X=\{a, b, c, x, y, z\}$.Then, $n(X) =6$

The number of subsets $= n[P(X)]$	$= 2^6$	$= 64$
The number of proper subsets	$= n[P(X)]$-1	$= 2^6-1$
	$= 64 - 1$	$= 63$

Thinking Corner

Every set has only one improper subset. Verify this fact using any set.

Exercise 1.2

1. Find the cardinal number of the following sets.

(i) $M = \{p, q, r, s, t, u\}$

(ii) $P = \{x : x = 3n+2, n\in \mathbb{W}$ and $x< 15\}$

(iii) $Q = \{y : y = \frac{4}{3n}, n \in \mathbb{N}$ and $2 < n \leq 5\}$

(iv) $R = \{x : x$ is an integers, $x\in \mathbb{Z}$ and $-5 \leq x <5\}$

(v) S = The set of all leap years between 1882 and 1906.

2. Identify the following sets as finite or infinite.

(i) X = The set of all districts in Tamilnadu.

(ii) Y = The set of all straight lines passing through a point.

(iii) $A = \{ x : x \in \mathbb{Z}$ and $x <5\}$

(iv) $B = \{ x : x^2-5x+6 = 0, x \in \mathbb{N} \}$

3. Which of the following sets are equivalent or unequal or equal sets?

(i) A = The set of vowels in the English alphabets.

B = The set of all letters in the word “VOWEL”

(ii) $C = \{2,3,4,5\}$ $D = \{ x : x \in \mathbb{W}, 1< x<5\}$

(iii) $X = \{ x : x$ is a letter in the word “LIFE”$\}$ $Y = \{ F, I, L, E\}$

(iv) $G = \{ x : x$ is a prime number and $3 < x < 23\}$ $H = \{ x : x$ is a divisor of $18\}$

4. Identify the following sets as null set or singleton set.

(i) $A = \{x : x \in \mathbb{N}, 1 < x < 2\}$

(ii) B = The set of all even natural numbers which are not divisible by 2

(iii) $C = \{0\}$.

(iv) D = The set of all triangles having four sides.

5. State which pairs of sets are disjoint or overlapping?

(i) $A = \{f, i, a, s\}$ and $B=\{a, n, f, h, s\}$

(ii) $C = \{x : x$ is a prime number, $x >2\}$ and $D =\{x:x$ is an even prime number$\}$

(iii) $E = \{x : x$ is a factor of $24\}$ and $F=\{x : x$ is a multiple of $3, x < 30\}$

6. If S = {square, rectangle, circle, rhombus, triangle}, list the elements of the following subset of S.

(i) The set of shapes which have 4 equal sides.

(ii) The set of shapes which have radius.

(iii) The set of shapes in which the sum of all interior angles is 180°.

(iv) The set of shapes which have 5 sides.

7. If $A = \{a, \{a, b\}\}$, write all the subsets of A.

8. Write down the power set of the following sets:

(i) $A = \{a, b\}$ (ii) $B = \{1, 2, 3\}$ (iii) $D = \{p, q, r, s\}$ (iv) $E = \varnothing$

9. Find the number of subsets and the number of proper subsets of the following sets.

(i) W = {red, blue, yellow} (ii) $X = \{ x^2 : x \in \mathbb{N}, x^2 \leq 100\}$.

10. (i) If $n(A) = 4$, find $n[P(A)]$. (ii) If $n(A)=0$, find $n[P(A)]$.

(iii) If $n[P(A)] = 256$, find $n(A)$.

1.5 Set Operations

We started with numbers and very soon we learned arithmetical operations on them. In algebra we learnt expressions and soon started adding and multiplying them as well, writing (x^2+2) $(x-3)$ etc. Now that we know sets, the natural question is, what can we do with sets, what are natural operations on them ?

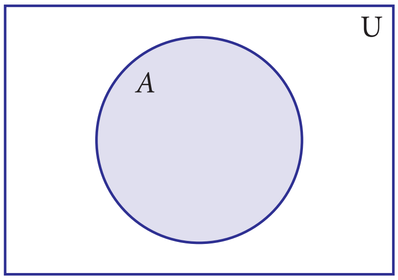
When two or more sets combine together to form one set under the given conditions, then operations on sets can be carried out. We can visualize the relationship between sets and set operations using Venn diagram.

John Venn was an English mathematician. He invented Venn diagrams which pictorially represent the relations between sets. Venn diagrams are used in the field of Set Theory, Probability, Statistics, Logic and Computer Science.

1.5.1 Complement of a Set

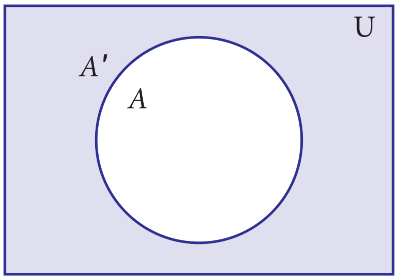
The Complement of a set A is the set of all elements of U (the universal set) that are not in A. It is denoted by A' or A^c. In symbols $A' = \{x : x \in U, x \notin A\}$

Venn diagram for complement of a set



A (shaded region)

Fig. 1.4



A' (shaded region)

Fig. 1.5

For example,

If U = {all boys in a class} and A= {boys who play Cricket}, then complement of the set A is A'= {boys who do not play Cricket}.

Example 1.10

If $U = \{c, d, e, f, g, h, i, j\}$ and $A = \{c, d, g, j\}$, find A'.

Solution

$U = \{c, d, e, f, g, h, i, j\}$, $A = \{c, d, g, j\}$

$A' = \{e, f, h, i\}$

Note

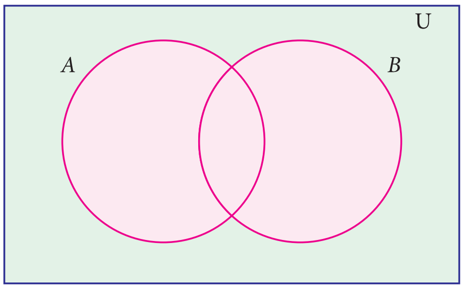
- $(A')' = A$
- $U' = \varnothing$
- $\varnothing' = U$

1.5.2 Union of Two Sets

The union of two sets A and B is the set of all elements which are either in A or in B or in both. It is denoted by $A \cup B$ and read as A union B.

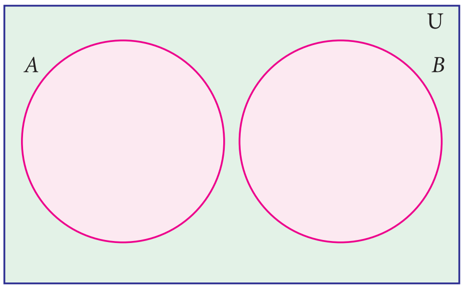
In symbol, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

The union of two sets can be represented by Venn diagram as given below



Sets A and B have common elements

Fig. 1.6



Sets A and B are disjoint

Fig. 1.7

For example,

If P ={Asia,Africa, Antarctica, Australia} and Q = {Europe, North America, South America}, then the union set of P and Q is $P \cup Q$ = {Asia, Africa, Antartica, Australia, Europe, North America, South America}.

Note

- $A \cup A = A$
- $A \cup \varnothing = A$
- $A \cup U = U$ where A is any subset of universal set U
- $A \subseteq A \cup B$ and $B \subseteq A \cup B$
- $A \cup B = B \cup A$ (union of two sets is commutative)

Example 1.11 If P={m, n} and Q= {m, i, j}, then, represent P and Q in Venn diagram and hence find $P \cup Q$.

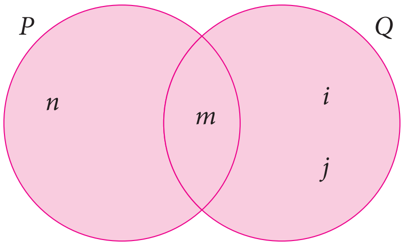


Fig. 1.8

Solution

Given $P=\{m, n\}$ and Q= $\{m, i, j\}$

From the venn diagram, $P \cup Q=\{n, m, i, j\}$.

1.5.3 Intersection of Two Sets

The intersection of two sets A and B is the set of all elements common to both A and B. It is denoted by $A \cap B$ and read as A intersection B.

In symbol , $A \cap B=\{x : x \in A \text{ and } x \in B\}$

Intersection of two sets can be represented by a Venn diagram as given below

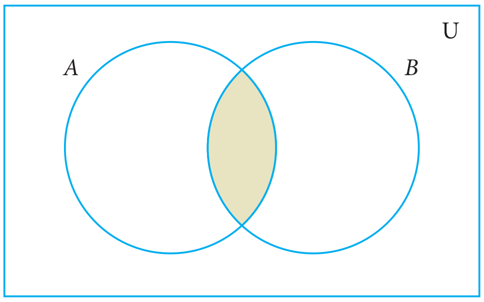


Fig. 1.9

For example,

If $A = \{1, 2, 6\}$; $B = \{2, 3, 4\}$, then $A \cap B = \{2\}$ because 2 is common element of the sets A and B.

Example 1.12

Let $A = \{x : x \text{ is an even natural number and } 1 < x \leq 12\}$ and $B = \{ x : x \text{ is a multiple of } 3, x \in \mathbb{N} \text{ and } x \leq 12\}$ be two sets. Find $A \cap B$.

Solution

Here $A = \{2, 4, 6, 8, 10, 12\}$ and $B = \{3, 6, 9, 12\}$

$A \cap B = \{6, 12\}$

Example 1.13

If $A = \{2, 3\}$ and $C = \{ \}$, find $A \cap C$.

Solution

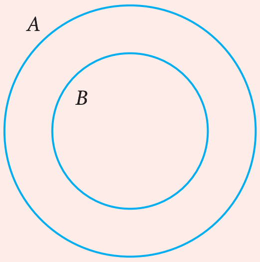
There is no common element and hence $A \cap C = \{ \}$

Note

- $A \cap A = A$ (ii) $A \cap \varnothing = \varnothing$
- $A \cap U = A$ where A is any subset of universal set U
- $A \cap B \subseteq A$ and $A \cap B \subseteq B$
- $A \cap B = B \cap A$ (Intersection of two sets is commutative)

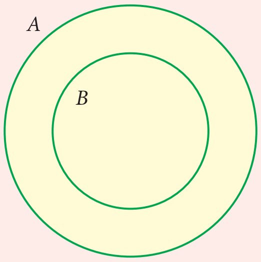
Note

- When $B \subset A$, the union and intersection of A and B are represented in Venn diagram as follows



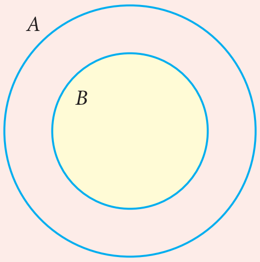
$B \subset A$

Fig. 1.10



shaded region is $A \cup B = A$

Fig. 1.11



shaded region is $A \cap B = B$

Fig. 1.12

- If A and B are any two non empty sets such that $A \cup B = A \cap B$, then $A = B$
- Let $n(A) = p$ and $n(B) = q$ then
  - (a) Minimum of $n(A \cup B) = \max\{p, q\}$
  - (b) Maximum of $n(A \cup B) = p + q$
  - (c) Minimum of $n(A \cap B) = 0$
  - (d) Maximum of $n(A \cap B) = \min\{p, q\}$

1.5.4 Difference of Two Sets

Let A and B be two sets, the difference of sets A and B is the set of all elements which are in A, but not in B. It is denoted by $A-B$ or A\B and read as A difference B.

In symbol, $A-B = \{ x : x \in A \text{ and } x \notin B\}$

$B-A = \{ y : y \in B \text{ and } y \notin A\}$.

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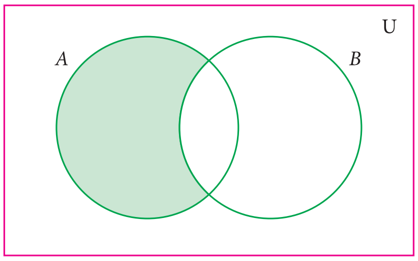
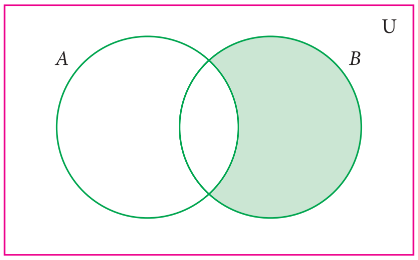
Venn diagram for set difference



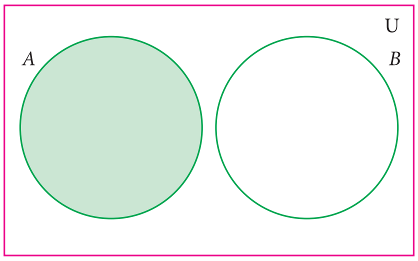
$A-B$

Fig. 1.13



$B-A$

Fig. 1.14



$A-B$

Fig. 1.15

Example 1.14

If $A=\{-3, -2, 1, 4\}$ and $B= \{0, 1, 2, 4\}$, find (i) $A-B$ (ii) $B-A$.

Solution

$A-B = \{-3, -2, 1, 4\} - \{0, 1, 2, 4\} = \{ -3, -2\}$

$B-A = \{0, 1, 2, 4\} - \{-3, -2, 1, 4\} = \{ 0, 2\}$

Note

- $A' = U - A$
- $A-B = A\cap B'$
- $A-A = \varnothing$
- $A-\varnothing = A$
- $A-B = B-A \Leftrightarrow A=B$
- $A-B = A$ if $A\cap B = \varnothing$

1.5.5 Symmetric Difference of Sets

The symmetric difference of two sets A and B is the set $(A-B)\cup(B-A)$. It is denoted by $A\Delta B$.

$A\Delta B=\{ x : x \in A-B \text{ or } x \in B-A\}$

Example 1.15

If $A = \{6, 7, 8, 9\}$ and $B=\{8, 10, 12\}$, find $A\Delta B$.

Solution

$A-B = \{6, 7, 9\}$

$B-A = \{10, 12\}$

$A\Delta B = (A-B)\cup(B-A) = \{6, 7, 9\}\cup\{10,12\}$

$A\Delta B = \{6, 7, 9, 10, 12\}$.

Thinking Corner

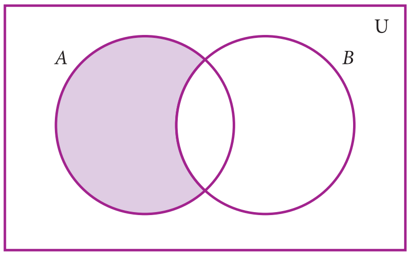
What is $(A-B)\cap(B-A)$?

Example 1.16

Represent $A\Delta B$ through Venn diagram.

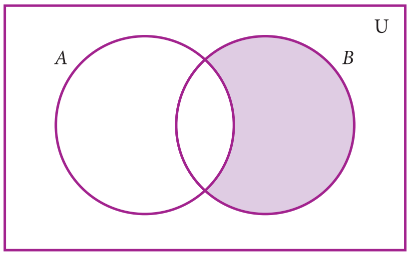
Solution

$A\Delta B = (A-B) \cup (B-A)$



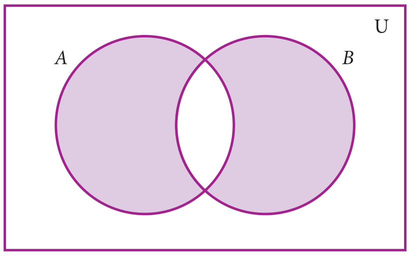
$A-B$

Fig. 1.16



$B-A$

Fig. 1.17



$(A-B) \cup (B-A)$

Fig. 1.18

Note

- $A\Delta A = \varnothing$
- $A\Delta B = B\Delta A$
- $A\Delta B = \{x : x \in A\cup B \text{ and } x \notin A\cap B\}$
- $A\Delta B = (A\cup B) - (A\cap B)$

Example 1.17

From the given Venn diagram, write the elements of

(i) A (ii) B (iii) $A-B$ (iv) $B-A$

(v) A' (vi) B' (vii) U

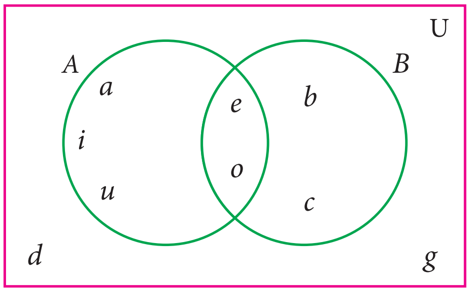


Fig. 1.19

Solution

(i) $A = \{a, e, i, o, u\}$

(ii) $B = \{b, c, e, o\}$

(iii) $A-B = \{a, i, u\}$

(iv) $B-A = \{b, c\}$

(v) $A' = \{b, c, d, g\}$

(vi) $B' = \{a, d, g, i, u\}$

(vii) $U = \{a, b, c, d, e, g, i, o, u\}$

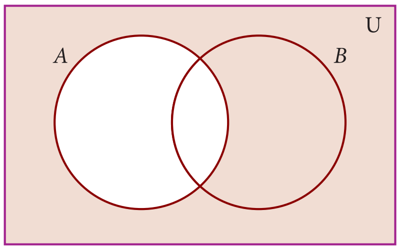
Example 1.18

Draw Venn diagram and shade the region representing the following sets

(i) A' (ii) $(A-B)'$ (iii) $(A\cup B)'$

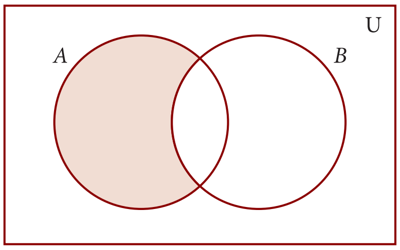
Solution

(i) A' (ii) $(A-B)'$



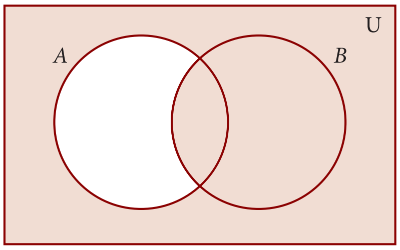
A'

Fig. 1.20



$A-B$

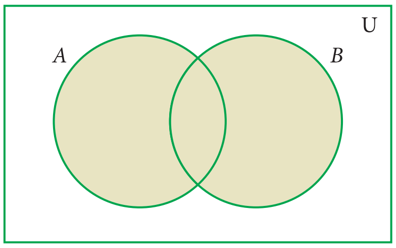
Fig. 1.21



$(A-B)'$

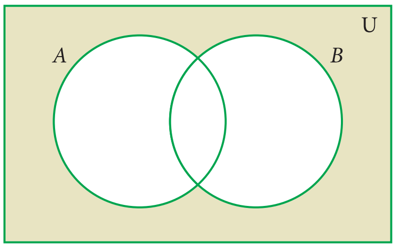
Fig. 1.22

(iii) $(AUB)'$



$A\cup B$

Fig. 1.23



$(A\cup B)'$

Fig. 1.24

Exercise 1.3

1. Using the given Venn diagram, write the elements of

   (i) A (ii) B (iii) $A\cup B$ (iv) $A\cap B$

   (v) $A-B$ (vi) $B-A$ (vii) A' (viii) B'

   (ix) U

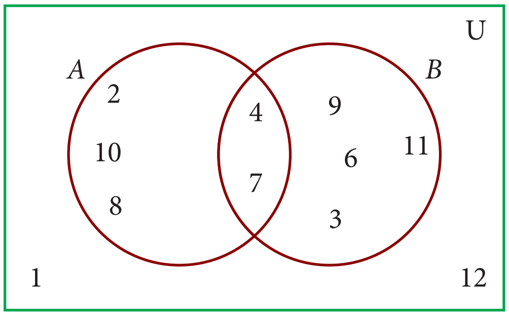


Fig. 1.25

2. Find $A\cup B$, $A\cap B$, $A-B$ and $B-A$ for the following sets.

   (i) $A = \{2, 6, 10, 14\}$ and $B=\{2, 5, 14, 16\}$

   (ii) $A = \{a, b, c, e, u\}$ and $B=\{a, e, i, o, u\}$

   (iii) $A = \{x : x \in N, x \leq 10\}$ and $B=\{x : x \in W, x < 6\}$

   (iv) A = Set of all letters in the word "mathematics" and

   B = Set of all letters in the word "geometry"

3. If U=$\{a, b, c, d, e, f, g, h\}$, $A=\{b, d, f, h\}$ and $B=\{a, d, e, h\}$, find the following sets.

(i) A' (ii) B' (iii) $A'\cup B'$ (iv) $A'\cap B'$ (v) $(A\cup B)'$

(vi) $(A\cap B)'$ (vii) $(A')'$ (viii) $(B')'$

4. Let U=$\{0, 1, 2, 3, 4, 5, 6, 7\}$, $A=\{1, 3, 5, 7\}$ and $B=\{0, 2, 3, 5, 7\}$, find the following sets.

(i) A' (ii) B' (iii) $A'\cup B'$ (iv)$A'\cap B'$ (v)$(A\cup B)'$

(vi) $(A\cap B)'$ (vii) $(A')'$ (viii) $(B')'$

5. Find the symmetric difference between the following sets.

(i) $P = \{2, 3, 5, 7, 11\}$ and $Q=\{1, 3, 5, 11\}$

(ii) $R = \{l, m, n, o, p\}$ and $S = \{j, l, n, q\}$

(iii) $X = \{5, 6, 7\}$ and $Y=\{5, 7, 9, 10\}$

6. Using the set symbols, write down the expressions for the shaded region in the following

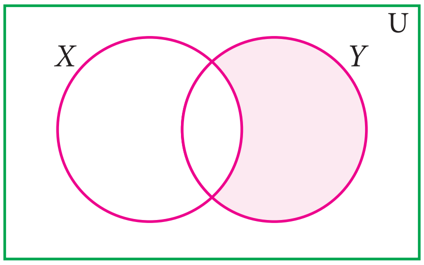


Fig. 1.26

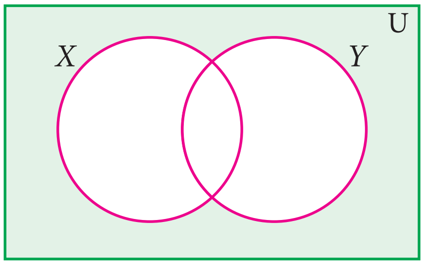


Fig. 1.27

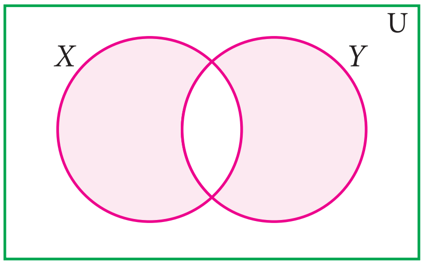


Fig. 1.28

7. Let A and B be two overlapping sets and the universal set be U. Draw appropriate Venn diagram for each of the following,

(i) $A\cup B$ (ii) $A\cap B$ (iii) $(A\cap B)'$ (iv) $(B-A)'$ (v) $A'\cup B'$ (vi) $A'\cap B'$

(vii) What do you observe from the Venn diagram (iii) and (v)?

1.6 Properties of Set Operations

It is an interesting investigation to find out if operations among sets (like union, intersection, etc) follow mathematical properties such as Commutativity, Associativity, etc., We have seen numbers having many of these properties; whether sets also possess these, is to be explored.

We first take up the properties of set operations on union and intersection.

1.6.1 Commutative Property

In set language, commutative situations can be seen when we perform operations. For example, we can look into the Union (and Intersection) of sets to find out if the operation is commutative.

Note

For any set A,

- $A \cup A = A$ and $A \cap A = A$ [Idempotent Laws].
- $A \cup \phi = A$ and $A \cap \mathrm{U} = A$ [Identity Laws].

Let $A = \{2, 3, 8, 10\}$ and $B = \{1, 3, 10, 13\}$ be two sets.

Then, $A \cup B = \{1, 2, 3, 8, 10, 13\}$ and

$B \cup A = \{1, 2, 3, 8, 10, 13\}$

From the above, we see that $A \cup B = B \cup A$.

This is called Commutative property of union of sets.

Now, $A \cap B = \{3, 10\}$ and $B \cap A = \{3, 10\}$. Then, we see that $A \cap B = B \cap A$.

This is called Commutative property of intersection of sets.

Commutative property: For any two sets A and B

(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$

Example 1.19

If $A = \{b, e, f, g\}$ and $B = \{c, e, g, h\}$, then verify the commutative property of (i) union of sets (ii) intersection of sets.

Solution

Given, $A = \{b, e, f, g\}$ and $B = \{c, e, g, h\}$

(i) $A \cup B = \{b, c, e, f, g, h\}$ ... (1)

$B \cup A = \{b, c, e, f, g, h\}$ ... (2)

From (1) and (2) we have $A \cup B = B \cup A$

It is verified that union of sets is commutative.

(ii) $A \cap B = \{e, g\}$ ... (3)

$B \cap A = \{e, g\}$ ... (4)

From (3) and (4) we get, $A \cap B = B \cap A$

It is verified that intersection of sets is commutative.

Thinking Corner

Given, $P = \{l, n, p\}$ and $P \cup Q = \{j, l, m, n, o, p\}$. If P and Q are disjoint sets, then what will be Q and $P \cap Q$?

Note

Recall that subtraction on numbers is not commutative. Is set difference commutative? We expect that the set difference is not commutative as well. For instance, consider $A=\{a,b,c\}$, $B=\{b,c,d\}$. $A-B=\{a\}$, $B-A=\{d\}$; we see that $A-B\neq B-A$.

1.6.2 Associative Property

Now, we perform operations on union and intersection for three sets.

Let $A=\{-1,0,1,2\}$, $B=\{-3,0,2,3\}$ and $C=\{0,1,3,4\}$ be three sets.

Now, $B\cup C = \{-3,0,1,2,3,4\}$

$$A\cup(B\cup C) = \{-1,0,1,2\}\cup\{-3,0,1,2,3,4\}$$

$$= \{-3,-1,0,1,2,3,4\} \quad \text{... (1)}$$

Then, $A\cup B = \{-3,-1,0,1,2,3\}$

$$(A\cup B)\cup C = \{-3,-1,0,1,2,3\}\cup\{0,1,3,4\}$$

$$= \{-3,-1,0,1,2,3,4\} \quad \text{... (2)}$$

From (1) and (2), $A\cup(B\cup C)=(A\cup B)\cup C$.

This is associative property of union among sets A, B, and C.

Now, $B\cap C = \{0,3\}$

$$A\cap(B\cap C) = \{-1,0,1,2\}\cap\{0,3\}$$

$$= \{0\} \quad \text{... (3)}$$

Then, $A\cap B = \{0,2\}$

$$(A\cap B)\cap C = \{0,2\}\cap\{0,1,3,4\}$$

$$= \{0\} \quad \text{... (4)}$$

From (3) and (4), $A\cap(B\cap C)=(A\cap B)\cap C$.

This is associative property of intersection among sets A, B and C.

Associative property: For any three sets A, B and C

(i) $A\cup(B\cup C)=(A\cup B)\cup C$ (ii) $A\cap(B\cap C)=(A\cap B)\cap C$

Example 1.20

If $A=\left\{-\frac{1}{2},0,\frac{1}{4},\frac{3}{4},2\right\}$, $B=\left\{0,\frac{1}{4},\frac{3}{4},2,\frac{5}{2}\right\}$ and $C=\left\{-\frac{1}{2},\frac{1}{4},1,2,\frac{5}{2}\right\}$, then verify $A\cap(B\cap C)=(A\cap B)\cap C$.

Solution

Now, $$(B\cap C) = \left\{\frac{1}{4}, 2, \frac{5}{2}\right\}$$

$$A\cap(B\cap C) = \left\{\frac{1}{4}, 2\right\} \qquad \ldots (1)$$

Then, $$A\cap B = \left\{0, \frac{1}{4}, \frac{3}{4}, 2\right\}$$

$$(A\cap B)\cap C = \left\{\frac{1}{4}, 2\right\} \qquad \ldots (2)$$

From (1) and (2), it is verified that

$$(A\cap B)\cap C = A\cap(B\cap C)$$

Note

The set difference in general is not associative

that is, $(A-B)-C \neq A-(B-C)$.

But, if the sets A, B and C are mutually disjoint then the set difference is associative

that is, $(A-B)-C = A-(B-C)$.

Exercise 1.4

1. If $P=\{1,2,5,7,9\}$, $Q=\{2,3,5,9,11\}$, $R=\{3,4,5,7,9\}$ and $S=\{2,3,4,5,8\}$, then find

   (i) $(P\cup Q)\cup R$ (ii) $(P\cap Q)\cap S$ (iii) $(Q\cap S)\cap R$

2. Test for the commutative property of union and intersection of the sets
   $P = \{ x : x$ is a real number between 2 and 7$\}$ and
   $Q = \{ x : x$ is an irrational number between 2 and 7$\}$.

3. If $A = \{p,q,r,s\}$, $B = \{m,n,q,s,t\}$ and $C = \{m,n,p,q,s\}$, then verify the associative property of union of sets.

4. Verify the associative property of intersection of sets for $A=\{-11, \sqrt{2}, \sqrt{5}, 7\}$, $B=\{\sqrt{3}, \sqrt{5}, 6, 13\}$ and $C=\{\sqrt{2}, \sqrt{3}, \sqrt{5}, 9\}$.

5. If $A=\{x : x = 2^n,\ n\in W$ and $n<4\}$, $B=\{x : x = 2n,\ n\in\mathbb{N}$ and $n\leq 4\}$ and

   $C=\{0,1,2,5,6\}$, then verify the associative property of intersection of sets.

1.6.3 Distributive Property

In lower classes, we have studied distributive property of multiplication over addition on numbers. That is, $a\times(b+c)=(a\times b)+(a\times c)$. In the same way we can define distributive properties on sets.

Distributive property: For any three sets A, B and C

(i) $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$ [Intersection over union]

(ii) $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$ [Union over intersection]

Example 1.21

If $A=\{0,2,4,6,8\}, B=\{x:\ x$ is a prime number and $x<11\}$ and $C=\{x:\ x\in N$ and $5\le x<9\}$ then verify $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$.

Solution

Given $A=\{0,2,4,6,8\},\ B=\{2,3,5,7\}$ and $C=\{5,6,7,8\}$

First, we find $B\cap C=\{5,7\},\ A\cup(B\cap C)=\{0,2,4,5,6,7,8\}$ ... (1)

Next, $A\cup B=\{0,2,3,4,5,6,7,8\},\ A\cup C=\{0,2,4,5,6,7,8\}$

Then, $(A\cup B)\cap(A\cup C)=\{0,2,4,5,6,7,8\}$ ... (2)

From (1) and (2), it is verified that $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$.

Example 1.22

Verify $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$ using Venn diagrams.

Solution

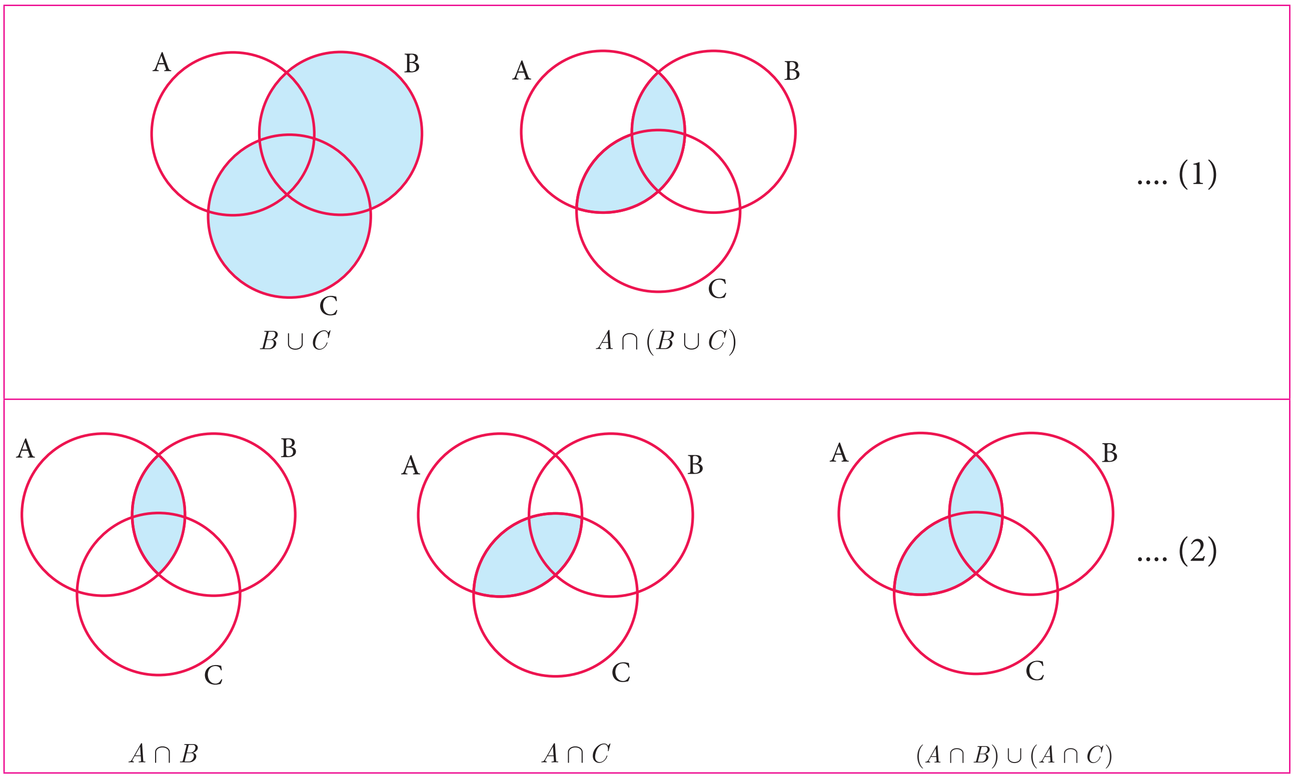


Fig.1.29

From (1) and (2), $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$ is verified.

1.7 De Morgan's Laws

Augustus De Morgan (1806 – 1871) was a British mathematician. He was born on 27th June 1806 in Madurai, Tamilnadu, India. His father was posted in India by the East India Company. When he was seven months old, his family moved back to England. De Morgan was educated at Trinity College, Cambridge, London. He formulated laws for set difference and complementation. These are called De Morgan's laws.

BBGITU

1.7.1 De Morgan's Laws for Set Difference

These laws relate the set operations union, intersection and set difference.

Let us consider three sets A, B and C as $A=\{-5,-2,1,3\}$, $B=\{-3,-2,0,3,5\}$ and $C=\{-2,-1,0,4,5\}$.

Now, $B\cup C = \{-3,-2,-1,0,3,4,5\}$

$$A-(B\cup C) = \{-5,1\} \qquad \ldots (1)$$

Then, $A-B = \{-5,1\}$ and $A-C=\{-5,1,3\}$

$$(A-B)\cup(A-C) = \{-5,1,3\} \qquad \ldots (2)$$

$$(A-B)\cap(A-C) = \{-5,1\} \qquad \ldots (3)$$

From (1) and (2), we see that

$$A-(B\cup C) \neq (A-B)\cup(A-C)$$

But note that from (1) and (3), we see that

$$A-(B\cup C) = (A-B)\cap(A-C)$$

Now, $B\cap C = \{-2,0,5\}$

$$A-(B\cap C) =\{-5,1,3\} \qquad \ldots (4)$$

From (3) and (4) we see that

$$A-(B\cap C) \neq (A-B)\cap(A-C)$$

But note that from (2) and (4), we get $A-(B\cap C)=(A-B)\cup(A-C)$

Thinking Corner

$(A-B)\cup(A-C)\cup(A\cap B)=$____

De Morgan's laws for set difference : For any three sets A, B and C

(i) $A-(B\cup C)=(A-B)\cap(A-C)$ (ii) $A-(B\cap C)=(A-B)\cup(A-C)$

Example 1.23

Verify $A-(B\cup C)=(A-B)\cap(A-C)$ using Venn diagrams.

Solution

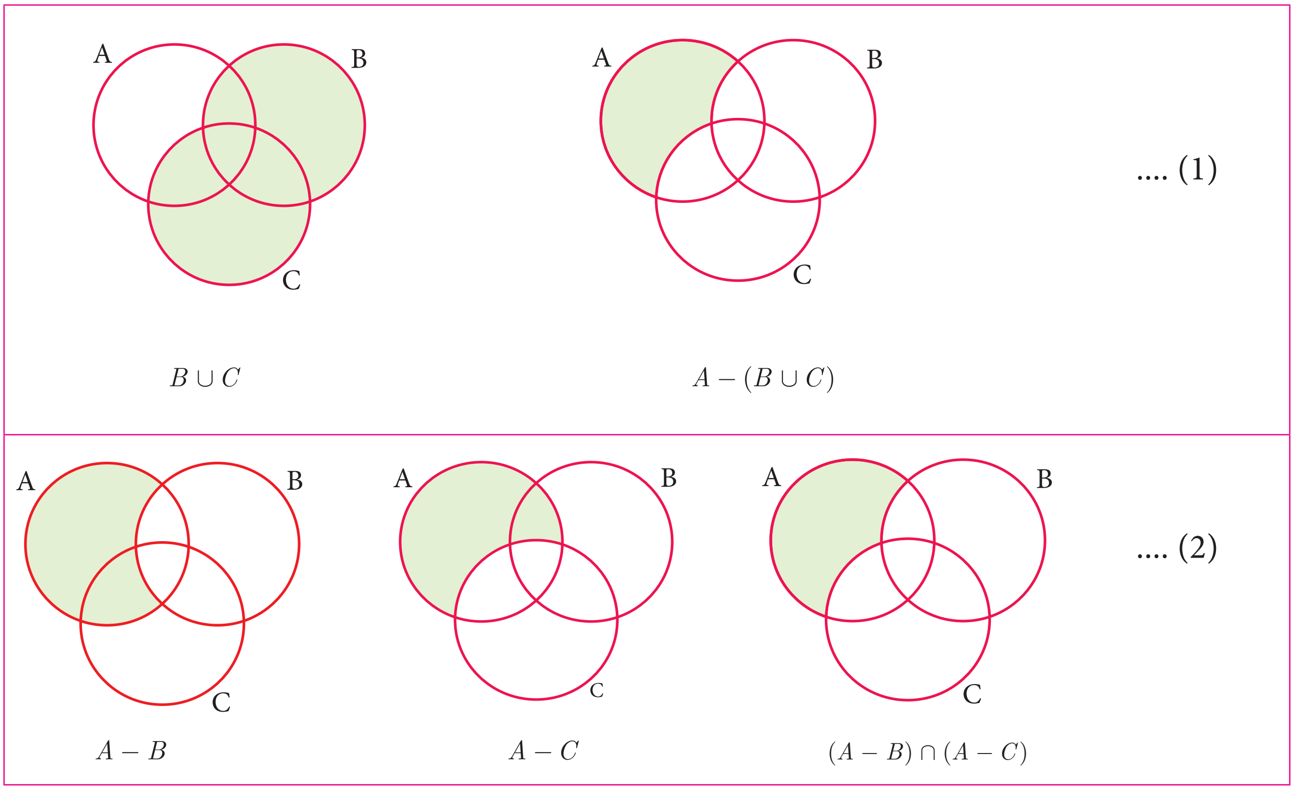


Fig.1.30

From (1) and (2), we get $A-(B\cup C)=(A-B)\cap(A-C)$. Hence it is verified.

Example 1.24

If $P=\{x : x\in \mathbb{W}$ and $0 < x < 10\}$, $Q=\{x : x=2n+1, n\in \mathbb{W}$ and $n<5\}$ and $R=\{2,3,5,7,11,13\}$, then verify $P-(Q\cap R)=(P-Q)\cup(P-R)$

Solution The roster form of sets P, Q and R are

$P=\{1,2,3,4,5,6,7,8,9\}$, $Q=\{1,3,5,7,9\}$

and $R=\{2,3,5,7,11,13\}$

First, we find $(Q\cap R)$ $=\{3,5,7\}$

Then, $P-(Q\cap R)$ $=\{1,2,4,6,8,9\}$ ... (1)

Next, $P–Q=\{2,4,6,8\}$ and

$P–R=\{1,4,6,8,9\}$

and so, $(P-Q)\cup(P-R)$ $=\{1,2,4,6,8,9\}$ ... (2)

Hence from (1) and (2), it is verified that $P-(Q\cap R)=(P-Q)\cup(P-R)$.

Finding the elements of set Q

Given, $x=2n+1$

$n=0\rightarrow x=2(0)+1=0+1=1$

$n=1\rightarrow x=2(1)+1=2+1=3$

$n=2\rightarrow x=2(2)+1=4+1=5$

$n=3\rightarrow x=2(3)+1=6+1=7$

$n=4\rightarrow x=2(4)+1=8+1=9$

Therefore, x takes values such as 1, 3, 5, 7 and 9.

1.7.2 De Morgan's Laws for Complementation

These laws relate the set operations on union, intersection and complementation.

Let us consider universal set $U=\{0,1,2,3,4,5,6\}$, $A=\{1,3,5\}$ and $B=\{0,3,4,5\}$.

Now, $A \cup B = \{0,1,3,4,5\}$

Then, $(A \cup B)' = \{2,6\}$ .....(1)

Next, $A' = \{0,2,4,6\}$ and $B' = \{1,2,6\}$

Then, $A' \cap B' = \{2,6\}$ .....(2)

From (1) and (2), we get $(A \cup B)' = A' \cap B'$

Thinking Corner

Check whether $A - B = A \cap B'$

Also, $A \cap B = \left\{3,5\right\}$,

$(A \cap B)' = \left\{0,1,2,4,6\right\}$ .....(3)

$A' = \left\{0,2,4,6\right\}$ and $B' = \left\{1,2,6\right\}$

$A' \cup B' = \left\{0,1,2,4,6\right\}$ .....(4)

From (3) and (4), we get $(A \cap B)' = A' \cup B'$

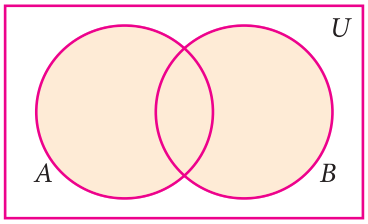
Thinking Corner

$(A - B) \cup (B - A')$=____

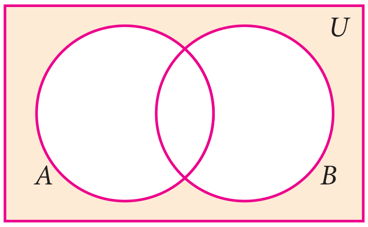
De Morgan's laws for complementation : Let 'U' be the universal set containing finite sets A and B. Then (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

Example 1.25 Verify $(A \cup B)' = A' \cap B'$ using Venn diagrams.

Solution

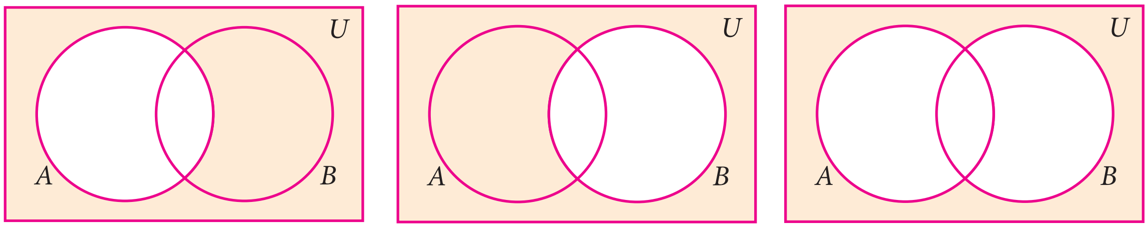


$A \cup B$



$(A \cup B)'$

.... (1)



.... (2)

A' $\qquad$ B' $\qquad$ $A' \cap B'$

Fig.1.31

From (1) and (2), it is verified that $(A \cup B)' = A' \cap B'$

Example 1.26 If $U = \left\{x : x \in \mathbb{Z}, -2 \le x \le 10\right\}$,

$A = \left\{x : x = 2p+1,\ p \in \mathbb{Z}, -1 \le p \le 4\right\}$, $B = \left\{x : x = 3q+1,\ q \in \mathbb{Z}, -1 \le q < 4\right\}$,

verify De Morgan's laws for complementation.

Solution

Given $U = \{-2,-1,0,1,2,3,4,5,6,7,8,9,10\}$,

$A = \{-1,1,3,5,7,9\}$ and $B = \{-2,1,4,7,10\}$

Law (i) $(A \cup B)' = A' \cap B'$

Now, $\quad A \cup B = \{-2,-1,1,3,4,5,7,9,10\}$

$(A \cup B)' = \{0,2,6,8\}$ ..... (1)

Then, $\quad A' = \{-2,0,2,4,6,8,10\}$ and $B' = \{-1,0,2,3,5,6,8,9\}$

$A' \cap B' = \{0,2,6,8\}$ ..... (2)

From (1) and (2), it is verified that $(A \cup B)' = A' \cap B'$

Law (ii) $(A \cap B)' = A' \cup B'$

Now, $\quad A \cap B = \{1,7\}$

$(A \cap B)' = \{-2,-1,0,2,3,4,5,6,8,9,10\}$ ..... (3)

Then, $\quad A' \cup B' = \{-2,-1,0,2,3,4,5,6,8,9,10\}$ ..... (4)

From (3) and (4), it is verified that $(A \cap B)' = A' \cup B'$

Thinking Corner

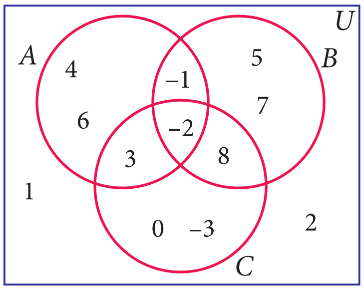
$A \cap (A \cup B)' =$____

Thinking Corner

$(A \cup B)' \cup (A' \cap B) =$___

Exercise 1.5

1. Using the adjacent Venn diagram, find the following sets:



(i) $A-B$ (ii) $B-C$ (iii) $A'\cup B'$

(iv) $A'\cap B'$ (v) $(B\cup C)'$

(vi) $A-(B\cup C)$ (vii) $A-(B\cap C)$

2. If $K=\{a,b,d,e,f\}$, $L=\{b,c,d,g\}$ and $M=\{a,b,c,d,h\}$, then find the following:

(i) $K\cup(L\cap M)$ (ii) $K\cap(L\cup M)$

(iii) $(K\cup L)\cap(K\cup M)$ (iv) $(K\cap L)\cup(K\cap M)$

and verify distributive laws.

3. If $A=\{x : x\in\mathbb{Z}, -2<x\leq 4\}$, $B=\{x : x\in\mathbb{W}, x\leq 5\}$, $C=\{-4,-1,0,2,3,4\}$, then

verify $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$.

4. Verify $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$ using Venn diagrams.

5. If $A=\{b,c,e,g,h\}$, $B=\{a,c,d,g,i\}$ and $C=\{a,d,e,g,h\}$, then show that $A-(B\cap C)=(A-B)\cup(A-C)$.

6. If $A=\{x : x=6n, n\in\mathbb{W}$ and $n<6\}$, $B=\{x : x=2n, n\in\mathbb{N}$ and $2<n\leq 9\}$ and $C=\{x : x=3n, n\in\mathbb{N}$ and $4\leq n<10\}$, then show that $A-(B\cap C)=(A-B)\cup(A-C)$

7. If $A=\{-2,0,1,3,5\}$, $B=\{-1,0,2,5,6\}$ and $C=\{-1,2,5,6,7\}$, then show that $A-(B\cup C)$ $=(A-B)\cap(A-C)$.

8. If $A=\{y : y=\dfrac{a+1}{2}, a\in\mathbb{W}$ and $a\leq 5\}$, $B=\{y : y=\dfrac{2n-1}{2}, n\in\mathbb{W}$ and $n<5\}$ and $C=\left\{-1,-\dfrac{1}{2},1,\dfrac{3}{2},2\right\}$, then show that $A-(B\cup C)=(A-B)\cap(A-C)$.

9. Verify $A-(B\cap C)=(A-B)\cup(A-C)$ using Venn diagrams.

10. If $U=\{4,7,8,10,11,12,15,16\}$, $A=\{7,8,11,12,\}$ and $B=\{4,8,12,15\}$, then verify De Morgan's Laws for complementation.

11. Verify $(A\cap B)'=A'\cup B'$ using Venn diagrams.

1.8 Application on Cardinality of Sets:

We have learnt about the union, intersection, complement and difference of sets. Now we will go through some practical problems on sets related to everyday life.

Results :

If A and B are two finite sets, then

(i) $n(A\cup B) = n(A)+n(B) - n(A\cap B)$

(ii) $n(A-B) = n(A) - n(A\cap B)$

(iii) $n(B-A) = n(B) - n(A\cap B)$

(iv) $n(A') = n(U) - n(A)$

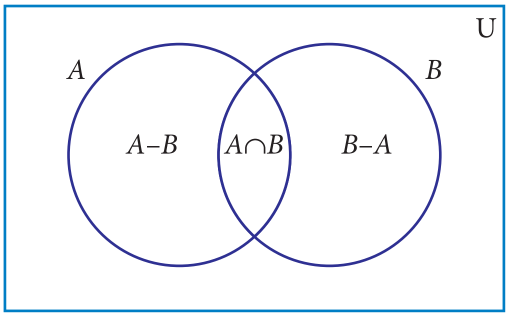


Fig. 1.32

Note

From the above results we may get,

- $n(A\cap B) = n(A)+n(B) - n(A\cup B)$
- $n(U) = n(A)+n(A')$
- If A and B are disjoint sets then, $n(A\cup B) = n(A)+n(B)$.

Example 1.27 From the Venn diagram, verify that $n(A\cup B) = n(A)+n(B) - n(A\cap B)$

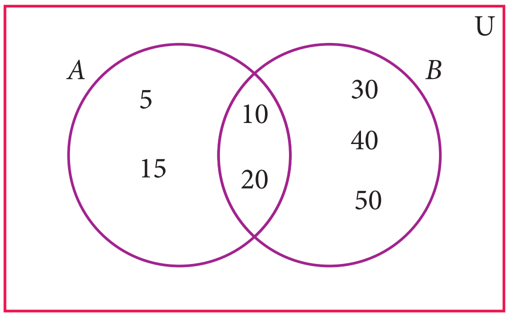


Fig. 1.33

Solution From the venn diagram,

$A = \{5, 10, 15, 20\}$

$B = \{10, 20, 30, 40, 50,\}$

Then $A\cup B = \{5, 10, 15, 20, 30, 40, 50\}$

$A\cap B = \{10, 20\}$

$n(A) = 4,\ n(B) = 5, n(A\cup B) = 7,\ n(A\cap B) = 2$

$n(A\cup B) = 7 \quad \rightarrow (1)$

$n(A)+n(B)-n(A\cap B) = 4+5-2$

$=7 \quad \rightarrow (2)$

From (1) and (2), $n(A\cup B) = n(A)+n(B)-n(A\cap B)$.

Example 1.28 If $n(A) = 36, n(B) = 10, n(A\cup B)=40$, and $n(A')=27$ find $n(U)$ and $n(A\cap B)$.

Solution $n(A) = 36,\ n(B) = 10,\ n(A\cup B)=40,\ n(A')=27$

(i) $n(U) = n(A)+n(A') = 36+27 = 63$

(ii) $n(A\cap B) = n(A)+n(B)-n(A\cup B) = 36+10-40 = 46-40 = 6$

Activity-4

Fill in the blanks with appropriate cardinal numbers.

S.No.	$n(A)$	$n(B)$	$n(A\cup B)$	$n(A\cap B)$	$n(A-B)$	$n(B-A)$
1	30	45	65			
2	20		55	10		
3	50	65		25		
4	30	43	70			

Example 1.29

Let $A=\{b, d, e, g, h\}$ and $B=\{a, e, c, h\}$. Verify that $n(A-B)=n(A)-n(A\cap B)$.

Solution

$$\begin{aligned} A = \{b, d, e, g, h\},\ B &= \{a, e, c, h\} \\ A - B &= \{b, d, g\} \\ n(A-B) &= 3 && \dots (1) \\ A \cap B &= \{e, h\} \\ n(A \cap B) &= 2\ , \quad n(A) = 5 \\ n(A) - n(A\cap B) &= 5\text{-}2 \\ &= 3 && \dots (2) \end{aligned}$$

Form (1) and (2) we get $n(A-B) = n(A)-n(A\cap B)$.

Example 1.30

In a school, all students play either Hockey or Cricket or both. 300 play Hockey, 250 play Cricket and 110 play both games. Find

(i) the number of students who play only Hockey.

(ii) the number of students who play only Cricket.

(iii) the total number of students in the School.

Solution:

Let H be the set of all students who play Hockey and C be the set of all students who play Cricket.

Then $n(H) = 300$, $n(C) = 250$ and $n(H \cap C) = 110$

Using Venn diagram,

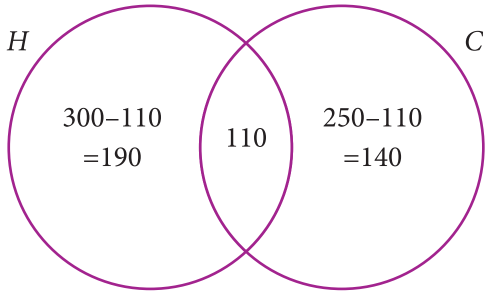


Fig. 1.34

From the Venn diagram,

(i) The number of students who play only Hockey = 190

(ii) The number of students who play only Cricket = 140

(iii) The total number of students in the school = 190+110+140 =440

Aliter

(i) The number of students who play only Hockey

$n(H-C) = n(H) - n(H\cap C)$

$= 300 - 110 = 190$

(ii) The number of students who play only Cricket

$n(C-H) = n(C) - n(H\cap C)$

$= 250 - 110 = 140$

(iii) The total number of students in the school

$n(H\cup C) = n(H) + n(C) - n(H\cap C)$

$= 300+250 - 110 = 440$

E4AHV7

Example 1.31 In a party of 60 people, 35 had Vanilla ice cream, 30 had Chocolate ice cream. All the people had at least one ice cream. Then how many of them had,

(i) both Vanilla and Chocolate ice cream.

(ii) only Vanilla ice cream.

(iii) only Chocolate ice cream.

Solution :

Let V be the set of people who had Vanilla ice cream and C be the set of people who had Chocolate ice cream.

Then $n(V) = 35$, $n(C) = 30$, $n(V\cup C) = 60$,

Let x be the number of people who had both ice creams.

From the Venn diagram

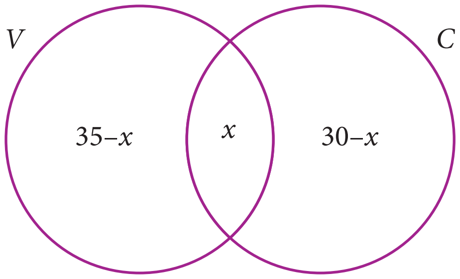


Fig. 1.35

$$35 - x + x + 30 - x = 60$$

$$65 - x = 60$$

$$x = 5$$

Hence 5 people had both ice creams.

(i) Number of people who had only Vanilla ice cream = $35 - x$

$= 35 - 5 = 30$

(ii) Number of people who had only Chocolate ice cream = $30 - x$

$= 30 - 5 = 25$

We have learnt to solve problems involving two sets using the formula $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. Suppose we have three sets, we can apply this formula to get a similar formula for three sets.

For any three finite sets A, B and C

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) \ -n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

Note

Let us consider the following results which will be useful in solving problems using Venn diagram. Let three sets A, B and C represent the students. From the Venn diagram,

Number of students in only set $A = a$, only set $B = b$, only set $C = c$.

- Total number of students in only one set $= (a + b + c)$
- Total number of students in only two sets $= (x + y + z)$
- Number of students exactly in three sets $= r$
- Total number of students in atleast two sets (two or more sets) $= x + y + z + r$
- Total number of students in 3 sets $= (a + b + c + x + y + z + r)$

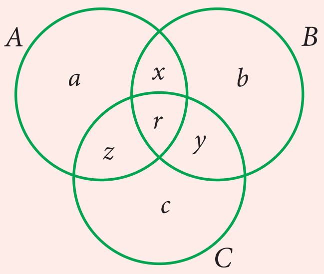


Fig.1.36

Example 1.32 In a college, 240 students play cricket, 180 students play football, 164 students play hockey, 42 play both cricket and football, 38 play both football and hockey, 40 play both cricket and hockey and 16 play all the three games. If each student participate in atleast one game, then find (i) the number of students in the college (ii) the number of students who play only one game.

Solution Let C, F and H represent sets of students who play Cricket, Football and Hockey respectively.

Then , $n(C) = 240$, $n(F) = 180$, $n(H) = 164$, $n(C \cap F) = 42$, $n(F \cap H) = 38$, $n(C \cap H) = 40$, $n(C \cap F \cap H) = 16$.

Let us represent the given data in a Venn diagram.

(i) The number of students in the college

= 174+26+116+22+102+24+16 = 480

(ii) The number of students who play only one game

= 174+116+102 = 392

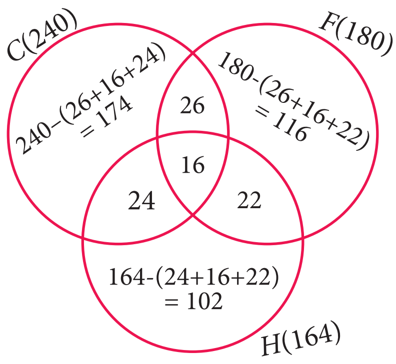


Fig.1.37

Example 1.33 In a residential area with 600 families $\frac{3}{5}$ owned scooter, $\frac{1}{3}$ owned car, $\frac{1}{4}$ owned bicycle, 120 families owned scooter and car, 86 owned car and bicylce while 90 families owned scooter and bicylce. If $\frac{2}{15}$ of families owned all the three types of vehicles, then find (i) the number of families owned atleast two types of vehicle. (ii) the number of families owned no vehicle.

Solution Let S, C and B represent sets of families who owned Scooter, Car and Bicycle respectively.

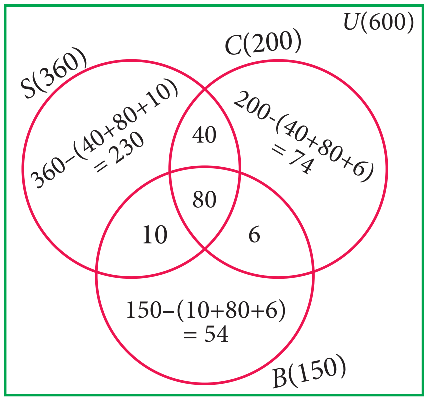


Fig.1.38

Given, $n(\text{U}) = 600 \quad n(S) = \frac{3}{5} \times 600 = 360$

$$n(C) = \frac{1}{3} \times 600 = 200 \quad n(B) = \frac{1}{4} \times 600 = 150$$

$$n(S \cap C \cap B) = \frac{2}{15} \times 600 = 80$$

From Venn diagram,

(i) The number of families owned atleast two types of vehicles = 40+6+10+80 = 136

(ii) The number of families owned no vehicle

= 600 – (owned atleast one vehicle)

$= 600 - (230 + 40 + 74 + 6 + 54 + 10 + 80)$

$= 600 - 494 = 106$

Example 1.34 In a group of 100 students, 85 students speak Tamil, 40 students speak English, 20 students speak French, 32 speak Tamil and English, 13 speak English and French and 10 speak Tamil and French. If each student knows atleast any one of these languages, then find the number of students who speak all these three languages.

Solution Let A, B and C represent sets of students who speak Tamil, English and French respectively.

Given, $n(A \cup B \cup C) = 100,\ n(A) = 85,\ n(\text{B}) = 40,\ n(\text{C}) = 20,$
$n(\text{A}\cap B) = 32,\ n(\text{B}\cap C) = 13,\ n(\text{A}\cap C) = 10\,.$

We know that,

$$n(A \cup B \cup C) = n(A) + n(B) + n(\text{C}) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

$$100 = 85 + 40 + 20 - 32 - 13 - 10 + n(A \cap B \cap C)$$

Then, $n(A \cap B \cap C) = 100 - 90 = 10$

Therefore, 10 students speak all the three languages.

Example 1.35 A survey was conducted among 200 magazine subscribers of three different magazines A, B and C. It was found that 75 members do not subscribe magazine A, 100 members do not subscribe magazine B, 50 members do not subscribe magazine C and 125 subscribe atleast two of the three magazines. Find

(i) Number of members who subscribe exactly two magazines.

(ii) Number of members who subscribe only one magazine.

Solution

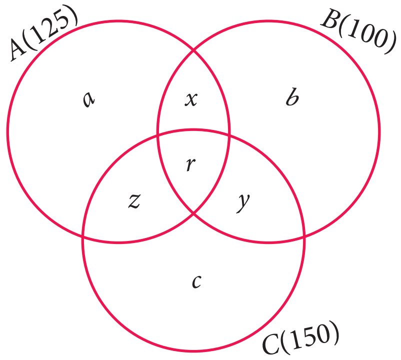


Fig.1.39

Total number of subscribers = 200

Magazine	Do not subscribe	Subscribe
A	75	125
B	100	100
C	50	150

From the Venn diagram,

Number of members who subscribe only one magazine $= a+b+c$

Number of members who subscribe exactly two magazines $= x+y+z$

and 125 members subscribe atleast two magazines.

That is, $x+y+z+r=125$ ... (1)

Now, $n(A\cup B\cup C)=200$, $n(A)=125$, $n(B)=100$, $n(C)=150$, $n(A\cap B)=x+r$

$n(B\cap C)=y+r$, $n(A\cap C)=z+r$, $n(A\cap B\cap C)=r$

We know that,

$$n(A\cup B\cup C)=n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)-n(A\cap C)+n(A\cap B\cap C)$$

$$200=125+100+150-x-r-y-r-z-r+r$$

$$=375-(x+y+z+r)-r$$

$$=375-125-r \quad \left[\because\ x+y+z+r=125\right]$$

$$200=250-r \qquad \Rightarrow r=50$$

From (1) $x+y+z+50=125$

We get, $x+y+z=75$

Therefore, number of members who subscribe exactly two magazines = 75.

From Venn diagram,

$$(a+b+c)+(x+y+z+r)=200 \quad \text{... (2)}$$

substitute (1) in (2),

$$a+b+c+125=200$$

$$a+b+c=75$$

Therefore, number of members who subscribe only one magazine = 75.

Exercise 1.6

1. (i) If $n(A) = 25$, $n(B) = 40$, $n(A \cup B) = 50$ and $n(B') = 25$, find $n(A \cap B)$ and $n(U)$.

   (ii) If $n(A) = 300$, $n(A \cup B) = 500$, $n(A \cap B) = 50$ and $n(B') = 350$, find $n(B)$ and $n(U)$.

2. If $U = \{x : x \in \mathbb{N}, x \leq 10\}$, $A = \{2,3,4,8,10\}$ and $B = \{1,2,5,8,10\}$, then verify that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

3. Verify $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C)$ $+ n(A \cap B \cap C)$ for the following sets.

   (i) $A = \{a, c, e, f, h\}$, $B = \{c, d, e, f\}$ and $C = \{a, b, c, f\}$

   (ii) $A = \{1, 3, 5\}$, $B = \{2, 3, 5, 6\}$ and $C = \{1, 5, 6, 7\}$

4. In a class, all students take part in either music or drama or both. 25 students take part in music, 30 students take part in drama and 8 students take part in both music and drama. Find

   (i) The number of students who take part in only music.

   (ii) The number of students who take part in only drama.

   (iii) The total number of students in the class.

5. In a party of 45 people, each one likes tea or coffee or both. 35 people like tea and 20 people like coffee. Find the number of people who

   (i) like both tea and coffee. (ii) do not like Tea. (iii) do not like coffee.

6. In an examination 50% of the students passed in Mathematics and 70% of students passed in Science while 10% students failed in both subjects. 300 students passed in both the subjects. Find the total number of students who appeared in the examination, if they took examination in only two subjects.

7. A and B are two sets such that $n(A-B) = 32 + x$, $n(B-A) = 5x$ and $n(A \cap B) = x$. Illustrate the information by means of a Venn diagram. Given that $n(A) = n(B)$, calculate the value of x.

8. Out of 500 car owners investigated, 400 owned car A and 200 owned car B, 50 owned both A and B cars. Is this data correct?

9. In a colony, 275 families buy Tamil newspaper, 150 families buy English newspaper, 45 families buy Hindi newspaper, 125 families buy Tamil and English newspapers, 17 families buy English and Hindi newspapers, 5 families buy Tamil and Hindi newspapers and 3 families buy all the three newspapers. If each family buy atleast one of these newspapers then find

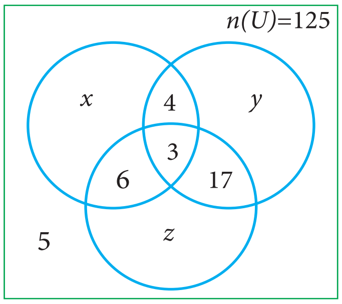
(i) Number of families buy only one newspaper

(ii) Number of families buy atleast two newspapers

(iii) Total number of families in the colony.

10. A survey of 1000 farmers found that 600 grew paddy, 350 grew ragi, 280 grew corn, 120 grew paddy and ragi, 100 grew ragi and corn, 80 grew paddy and corn. If each farmer grew atleast any one of the above three, then find the number of farmers who grew all the three.

11. In the adjacent diagram, if $n(\mathrm{U}) = 125$, y is two times of x and z is 10 more than x, then find the value of x, y and z.



12. Each student in a class of 35 plays atleast one game among chess, carrom and table tennis. 22 play chess, 21 play carrom, 15 play table tennis, 10 play chess and table tennis, 8 play carrom and table tennis and 6 play all the three games. Find the number of students who play (i) chess and carrom but not table tennis (ii) only chess (iii) only carrom (**Hint:** Use Venn diagram)

13. In a class of 50 students, each one come to school by bus or by bicycle or on foot. 25 by bus, 20 by bicycle, 30 on foot and 10 students by all the three. Now how many students come to school exactly by two modes of transport?

Exercise 1.7

Multiple Choice Questions

1. Which of the following is correct?

   (1) $\{7\} \in \{1,2,3,4,5,6,7,8,9,10\}$ (2) $7 \in \{1,2,3,4,5,6,7,8,9,10\}$

   (3) $7 \notin \{1,2,3,4,5,6,7,8,9,10\}$ (4) $\{7\} \not\subseteq \{1,2,3,4,5,6,7,8,9,10\}$

2. The set $P = \{x \mid x \in \mathbb{Z}, -1 < x < 1\}$ is a

   (1) Singleton set (2) Power set (3) Null set (4) Subset

3. If $\mathrm{U} = \{x \mid x \in \mathbb{N}, x < 10\}$ and $A = \{x \mid x \in \mathbb{N}, 2 \le x < 6\}$ then $(A')'$ is

   (1) $\{1, 6, 7, 8, 9\}$ (2) $\{1, 2, 3, 4\}$ (3) $\{2, 3, 4, 5\}$ (4) $\{ \}$

4. If $B \subseteq A$ then $n(A \cap B)$ is

   (1) $n(A-B)$ (2) $n(B)$ (3) $n(B - A)$ (4) $n(A)$

5. If $A = \{x, y, z\}$ then the number of non- empty subsets of A is

   (1) 8 (2) 5 (3) 6 (4) 7

6. Which of the following is correct?

(1) $\varnothing \subseteq \{a, b\}$ (2) $\varnothing \in \{a, b\}$ (3) $\{a\} \in \{a, b\}$ (4) $a \subseteq \{a, b\}$

7. If $A\cup B = A\cap B$, then

(1) $A\neq B$ (2) $A = B$ (3) $A \subset B$ (4) $B \subset A$

8. If $B - A$ is B, then $A\cap B$ is

(1) A (2) B (3) U (4) $\varnothing$

9. From the adjacent diagram $n[P(A\Delta B)]$ is

(1) 8 (2) 16

(3) 32 (4) 64

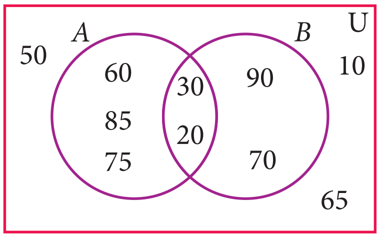


Fig. 1.40

10. If $n(A) = 10$ and $n(B) = 15$, then the minimum and maximum number of elements in $A \cap B$ is

(1) 10,15 (2) 15,10 (3) 10,0 (4) 0,10

11. Let $A = \{\varnothing\}$ and $B = P(A)$, then $A\cap B$ is

(1) $\{ \varnothing, \{\varnothing\} \}$ (2) $\{\varnothing\}$ (3) $\varnothing$ (4) $\{0\}$

12. In a class of 50 boys, 35 boys play Carrom and 20 boys play Chess then the number of boys play both games is

(1) 5 (2) 30 (3) 15 (4) 10.

13. If $\text{U} = \{x : x \in \mathbb{N} \text{ and } x < 10\}$, $A = \{1,2,3,5,8\}$ and $B = \{2,5,6,7,9\}$, then $n\left[(A \cup B)'\right]$ is

(1) 1 (2) 2 (3) 4 (4) 8

14. For any three sets P, Q and R, $P - (Q \cap R)$ is

(1) $P - (Q \cup R)$ (2) $(P \cap Q) - R$

(3) $(P - Q) \cup (P - R)$ (4) $(P - Q) \cap (P - R)$

15. Which of the following is true?

(1) $A - B = A \cap B$ (2) $A - B = B - A$

(3) $(A \cup B)' = A' \cup B'$ (4) $(A \cap B)' = A' \cup B'$

16. If $n(A \cup B \cup C) = 100$, $n(A) = 4x$, $n(B) = 6x$, $n(C) = 5x$, $n(A \cap B) = 20$, $n(B \cap C) = 15$, $n(A \cap C) = 25$ and $n(A \cap B \cap C) = 10$, then the value of x is

(1) 10 (2) 15 (3) 25 (4) 30

17. For any three sets A, B and C, $(A - B) \cap (B - C)$ is equal to

(1) A only (2) B only (3) C only (4) ϕ

18. If J = Set of three sided shapes, K = Set of shapes with two equal sides and L = Set of shapes with right angle, then $J \cap K \cap L$ is

(1) Set of isoceles triangles
(2) Set of equilateral triangles
(3) Set of isoceles right triangles
(4) Set of right angled triangles

19. The shaded region in the Venn diagram is

(1) $Z-(X \cup Y)$
(2) $(X \cup Y) \cap Z$
(3) $Z-(X \cap Y)$
(4) $Z \cup (X \cap Y)$

20. In a city, 40% people like only one fruit, 35% people like only two fruits, 20% people like all the three fruits. How many percentage of people do not like any one of the above three fruits?

(1) 5
(2) 8
(3) 10
(4) 15

Points to Remember

- A set is a well defined collection of objects.
- Sets are represented in three forms (i) Descriptive form (ii) Set – builder form (iii) Roster form.
- If every element of A is also an element of B, then A is called a subset of B.
- If $A \subseteq B$ and $A \neq B$, then A is a proper subset of B.
- The power set of the set A is the set of all the subsets of A and it is denoted by $P(A)$.
- The number of subsets of a set with $\boldsymbol{m}$ elements is $\mathbf{2^m}$.
- The number of proper subsets of a set with $\boldsymbol{m}$ elements is $\mathbf{2^m - 1}$.
- If $A \cap B = \varnothing$ then A and B are disjoint sets. If $A \cap B \neq \varnothing$ then A and B are overlapping.
- The difference of two sets A and B is the set of all elements in A but not in B.
- The symmetric difference of two sets A and B is the union of A-B and B-A.
- **Commutative Property**

  For any two sets A and B,

  $A \cup B = B \cup A$; $A \cap B = B \cap A$

- **Associative Property**

For any three sets A, B and C

$$A \cup (B \cup C) = (A \cup B) \cup C \quad ; \quad A \cap (B \cap C) = (A \cap B) \cap C$$

- **Distributive Property**

For any three sets A, B and C

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \qquad [\text{Intersection over union}]$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \qquad [\text{Union over intersection}]$$

- **De Morgan's Laws for Set Difference**

For any three sets A, B and C

$$A - (B \cup C) = (A - B) \cap (A - C)$$

$$A - (B \cap C) = (A - B) \cup (A - C)$$

- **De Morgan's Laws for Complementation**

Consider an Universal set and A, B are two subsets, then

$$(A \cup B)' = A' \cap B' \quad ; \quad (A \cap B)' = A' \cup B'$$

- **Cardinality of Sets**

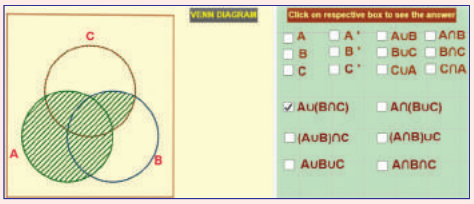
If A and B are any two sets, then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

If A, B and C are three sets, then

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

ICT Corner 1

Expected Result is shown in this picture


VENN DIAGRAM

Click on respective box to see the answer

Step – 1 : Open the Browser, type the URL Link given below (or) Scan the QR Code. GeoGebra work sheet named "Set Language" will open. In the work sheet there are two activities. 1. Venn Diagram for two sets and 2. Venn Diagram for three sets. In the first activity Click on the boxes on the right side to see the respective shading and analyse.

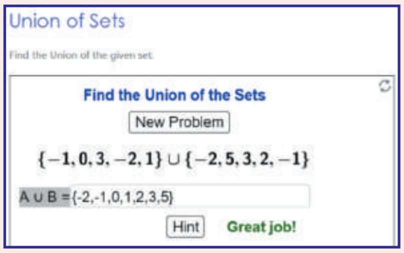
Step - 2 : Do the same for the second activity for three sets.

Scan the QR Code.

B566_9_MAT_EM_T3

ICT Corner 2

Expected Result is shown in this picture


Union of Sets

Find the Union of the given set

Find the Union of the Sets

$\{-1, 0, 3, -2, 1\} \cup \{-2, 5, 3, 2, -1\}$

A ∪ B ={-2,-1,0,1,2,3,5}

Great job!

Step – 1

Open the Browser and copy and paste the Link given below (or) by typing the URL given (or) Scan the QR Code.

Step - 2

GeoGebra worksheet "Union of Sets" will appear. You can create new problems by clicking on the box "NEW PROBLEM"

Step-3

Enter your answer by typing the correct numbers in the Question Box and then hit enter. If you have any doubt, you can hit the "HINT" button

Step-4

If your answer is correct "GREAT JOB" menu will appear. And if your answer is Wrong "Try Again!" menu will appear.

Keep on working new problems until you get 5 consecutive trials as correct.

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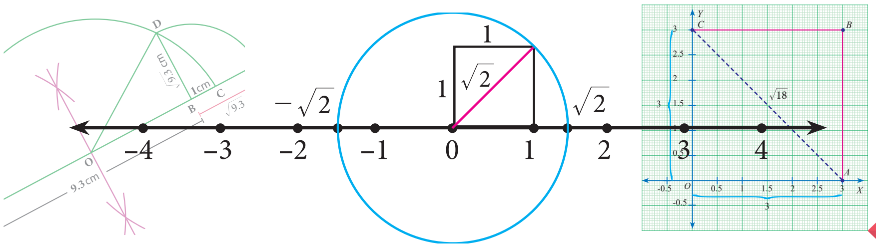
2



REAL NUMBERS

"When I consider what people generally want in calculating, I found that it always is a number"

- Al-Khwarizmi

Al-Khwarizmi
(A D (CE) 780 - 850)

Al-Khwarizmi, a Persian scholar, is credited with identification of surds as something noticeable in mathematics. He referred to the irrational numbers as 'inaudible', which was later translated to the Latin word 'surdus' (meaning 'deaf' or 'mute'). In mathematics, a surd came to mean a root (radical) that cannot be expressed (spoken) as a rational number.

E4JDHU

Learning Outcomes

- To know that there exists infinitely many rational numbers between two given rational numbers.
- To represent rational and irrational numbers on number line and express them in decimal form.
- To visualize the real numbers on the number line.
- To identify surds.
- To carry out basic operations of addition, subtraction, multiplication and division using surds.
- To rationalise denominators of surds.
- To understand the scientific notation.

2.1 Introduction

Numbers, numbers, everywhere!

- Do you have a phone at home? How many digits does its dial have?
- What is the Pin code of your locality? How is it useful?
- When you park a vehicle, do you get a 'token'? What is its purpose?
- Have you handled 24 'carat' gold? How do you decide its purity?
- How high is the 'power' of your spectacles?

- How much water does the overhead tank in your house can hold?
- Does your friend have fever? What is his body temperature?

You have learnt about many types of numbers so far. Now is the time to extend the ideas further.

2.2 Rational Numbers

When you want to count the number of books in your cupboard, you start with 1, 2, 3, … and so on. These counting numbers 1, 2, 3, … , are called Natural numbers. You know to show these numbers on a line (see Fig. 2.1).

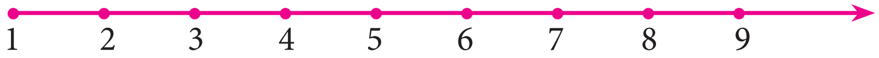


Fig. 2.1

We use $\mathbb{N}$ to denote the set of all **Natural numbers**.

$$\mathbb{N} = \{ 1, 2, 3, \dots \}$$

If the cupboard is empty (since no books are there). To denote such a situation we use the symbol 0. Including zero as a digit you can now consider the numbers 0, 1, 2, 3, … and call them **Whole numbers**. With this additional entity, the number line will look as shown below

Fig. 2.2

We use $\mathbb{W}$ to denote the set of all Whole numbers.

$$\mathbb{W} = \{ 0, 1, 2, 3, \dots \}$$

Certain conventions lead to more varieties of numbers. Let us agree that certain conventions may be thought of as **"positive"** denoted by a '+' sign. A thing that is 'up' or 'forward' or 'more' or 'increasing' is positive; and anything that is 'down' or 'backward' or 'less' or 'decreasing' is **"negative"** denoted by a '–' sign.

You can treat natural numbers as positive numbers and rename them as **positive integers**; thereby you have enabled the entry of negative integers –1, –2, –3, … .

Note that –2 is "more negative" than –1. Therefore, among –1 and –2, you find that –2 is smaller and –1 is bigger. Are –2 and –1 smaller or greater than –3? Think about it.

The number line at this stage may be given as follows:

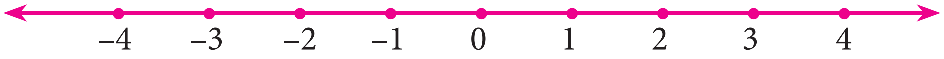


Fig. 2.3

We use $\mathbb{Z}$ to denote the set of all **Integers**.

$$\mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}.$$

When you look at the figures (Fig. 2.2 and 2.3) above, you are sure to get amused by the gap between any pair of consecutive integers. Could there be some numbers in between?

You have come across fractions already. How will you mark the point that shows $\frac{1}{2}$ on $\mathbb{Z}$? It is just midway between 0 and 1. In the same way, you can plot $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, 2\frac{3}{4}$ ... etc. These are all fractions of the form $\frac{a}{b}$ where a and b are integers with one restriction that $b \neq 0$. (Why?) If a fraction is in decimal form, even then the setting is same.

Because of the connection between fractions and ratios of lengths, we name them as **Rational numbers**. Here is a rough picture of the situation:

Fig. 2.4

A **rational number** is a fraction indicating the quotient of two integers, excluding division by zero.

Since a fraction can have many equivalent fractions , there are many possible names for the same rational number. Thus $\frac{1}{3}, \frac{2}{6}, \frac{8}{24}$ all these denote the same rational number.

2.2.1 Denseness Property of Rational Numbers

Consider a, b where $a > b$ and their AM(Arithmetic Mean) given by $\frac{a+b}{2}$. Is this AM a rational number? Let us see.

If $a = \frac{p}{q}$ (p, q integers and $q \neq 0$); $b = \frac{r}{s}$ (r, s integers and $s \neq 0$), then

$$\frac{a+b}{2} = \frac{\frac{p}{q}+\frac{r}{s}}{2} = \frac{ps+qr}{2qs} \text{ which is a rational number.}$$

We have to show that this rational number lies between a and b.

$$a - \left(\frac{a+b}{2}\right) = \frac{2a-a-b}{2} = \frac{a-b}{2} \text{ which is} > 0 \text{ since } a>b.$$

$$\text{Therefore, } a > \left(\frac{a+b}{2}\right) \qquad \ldots (1)$$

$$\left(\frac{a+b}{2}\right) - b = \frac{a+b-2b}{2} = \frac{a-b}{2} \text{ which is} > 0 \text{ since } a>b.$$

$$\text{Therefore, } \left(\frac{a+b}{2}\right) > b \qquad \ldots (2)$$

From (1) and (2) we see that $a > \left(\frac{a+b}{2}\right) > b$, which can be visualized as follows:

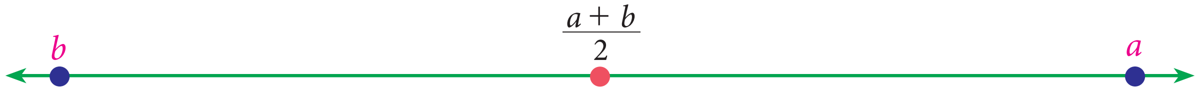


Fig. 2.5

Thus, for any two rational numbers, their average/mid point is rational. Proceeding similarly, we can generate infinitely many rational numbers.

Example 2.1 Find any two rational numbers between $\frac{1}{2}$ and $\frac{2}{3}$.

Solution 1

A rational number between $\frac{1}{2}$ and $\frac{2}{3}$ $= \frac{1}{2}\left(\frac{1}{2}+\frac{2}{3}\right)=\frac{1}{2}\left(\frac{3+4}{6}\right)=\frac{1}{2}\left(\frac{7}{6}\right)=\frac{7}{12}$

A rational number between $\frac{1}{2}$ and $\frac{7}{12}$ $= \frac{1}{2}\left(\frac{1}{2}+\frac{7}{12}\right)=\frac{1}{2}\left(\frac{6+7}{12}\right)=\frac{1}{2}\left(\frac{13}{12}\right)=\frac{13}{24}$

Hence two rational numbers between $\frac{1}{2}$ and $\frac{2}{3}$ are $\frac{7}{12}$ and $\frac{13}{24}$ (of course, there are many more!)

There is an interesting result that could help you to write instantly rational numbers between any two given rational numbers.

Result

If $\frac{p}{q}$ and $\frac{r}{s}$ are any two rational numbers such that $\frac{p}{q}<\frac{r}{s}$, then $\frac{p+r}{q+s}$ is a rational number, such that $\frac{p}{q}<\frac{p+r}{q+s}<\frac{r}{s}$.

Let us take the same example: Find any two rational numbers between $\frac{1}{2}$ and $\frac{2}{3}$

Solution 2

$\frac{1}{2}<\frac{2}{3}$ gives $\frac{1}{2}<\frac{1+2}{2+3}<\frac{2}{3}$ *or* $\frac{1}{2}<\frac{3}{5}<\frac{2}{3}$ gives $\frac{1}{2}<\frac{1+3}{2+5}<\frac{3}{5}<\frac{3+2}{5+3}<\frac{2}{3}$ *or* $\frac{1}{2}<\frac{4}{7}<\frac{3}{5}<\frac{5}{8}<\frac{2}{3}$

Solution 3

Any more new methods to solve? Yes, if decimals are your favourites, then the above example can be given an alternate solution as follows:

$$\frac{1}{2}=0.5 \text{ and } \frac{2}{3}=0.66\ldots$$

Hence rational numbers between $\frac{1}{2}$ and $\frac{2}{3}$ can be listed as 0.51, 0.57,0.58,…

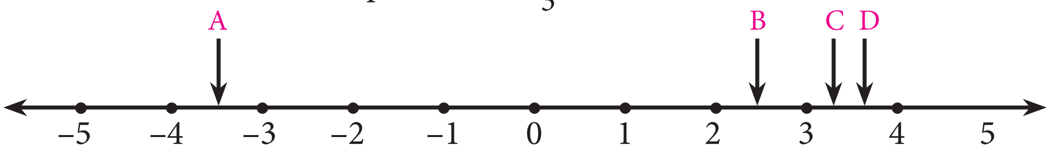
Solution 4

There is one more way to solve some problems. For example, to find four rational numbers between $\frac{4}{9}$ and $\frac{3}{5}$, note that the LCM of 9 and 5 is 45; so we can write $\frac{4}{9}=\frac{20}{45}$ and $\frac{3}{5}=\frac{27}{45}$.

Therefore, four rational numbers between $\frac{4}{9}$ and $\frac{3}{5}$ are $\frac{21}{45}, \frac{22}{45}, \frac{23}{45}, \frac{24}{45}, \ldots$

Exercise 2.1

1. Which arrow best shows the position of $\frac{11}{3}$ on the number line?



2. Find any three rational numbers between $\frac{-7}{11}$ and $\frac{2}{11}$.

3. Find any five rational numbers between (i) $\frac{1}{4}$ and $\frac{1}{5}$ (ii) 0.1 and 0.11 (iii) –1 and –2

2.3 Irrational Numbers

You know that each rational number is assigned to a point on the number line and learnt about the denseness property of the rational numbers. Does it mean that the line is entirely filled with the rational numbers and there are no more numbers on the number line? Let us explore.

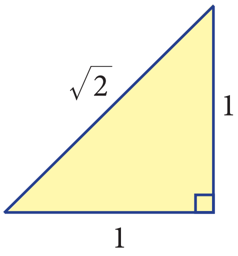


Fig.2.6

Consider an isosceles right-angled triangle whose base and height are each 1 unit long. Using Pythagoras theorem, the hypotenuse can be seen having a length $\sqrt{1^2+1^2}=\sqrt{2}$ (see Fig. 2.6). Greeks found that this $\sqrt{2}$ is neither a whole number nor an ordinary fraction. The belief of relationship between points on the number line and all numbers was shattered! $\sqrt{2}$ was called an irrational number.

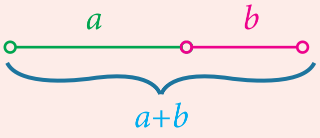
An irrational number is a number that cannot be expressed as an ordinary ratio of two integers.

Examples

1. Apart from $\sqrt{2}$, one can produce a number of examples for such irrational numbers. Here are a few: $\sqrt{5}, \sqrt{7}, 2\sqrt{3}, \ldots$
2. π, the ratio of the circumference of a circle to the diameter of that same circle, is another example for an irrational number.
3. *e*, also known as Euler's number, is another common irrational number.

GOLDEN RATIO

The Golden Ratio has been heralded as the most beautiful ratio in art and architecture.



$a+b : a = a : b$

Take a line segment and divide it into two smaller segments such that the ratio of the whole line segment ($a+b$) to segment a is the same as the ratio of segment a to the segment b.

This gives the proportion $\frac{a+b}{a}=\frac{a}{b}$

Notice that '**a**' is the geometric mean of $a+b$ and b.

4. The golden ratio, also known as golden mean, or golden section, is a number often stumbled upon when taking the ratios of distances in simple geometric figures such as the pentagon, the pentagram, decagon and dodecahedron, etc., it is an irrational number.

2.3.1 Irrational Numbers on the Number Line

Where are the points on the number line that correspond to the irrational numbers? As an example, let us locate $\sqrt{2}$ on the number line. This is easy.

Remember that $\sqrt{2}$ is the length of the diagonal of the square whose side is 1 unit (How?)Simply construct a square and transfer the length of one of its diagonals to our number line. (see Fig.2.7).

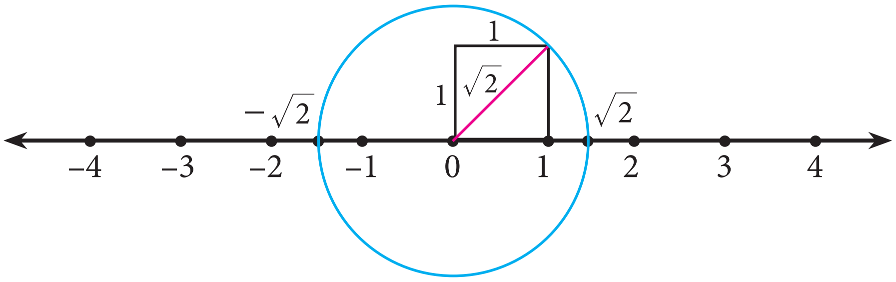
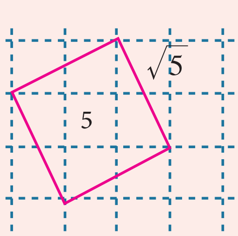
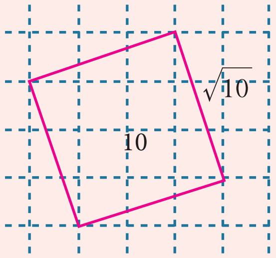


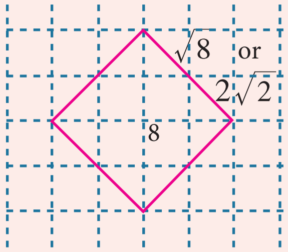
Fig.2.7

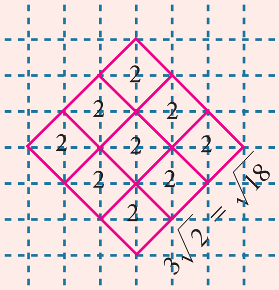
Squares on grid sheets can be used to produce irrational lengths.

Here are a few examples :









We draw a circle with centre at 0 on the number line,with a radius equal to that of diagonal of the square. This circle cuts the number line in two points, locating $\sqrt{2}$ on the right of 0 and $-\sqrt{2}$ on its left. (You wanted to locate $\sqrt{2}$; you have also got a bonus in $-\sqrt{2}$)

You started with Natural numbers and extended it to rational numbers and then irrational numbers. You may wonder if further extension on the number line waits for us. Fortunately it stops and you can learn about it in higher classes.

Representation of a Rational number as terminating and non terminating decimal helps us to understand irrational numbers. Let us see the decimal expansion of rational numbers.

2.3.2 Decimal Representation of a Rational Number

If you have a rational number written as a fraction, you get the decimal representation by long division. Study the following examples where the remainder is always zero.

Consider the examples,

$$\begin{array}{r|r}
\multicolumn{2}{r}{0.875} \\ \cline{2-2}
8 & 7.000 \\
 & 64 \\ \cline{2-2}
 & 60 \\
 & 56 \\ \cline{2-2}
 & 40 \\
 & 40 \\ \cline{2-2}
 & 0 \\ \cline{2-2}
\end{array}$$

$\frac{7}{8} = 0.875$

$$\begin{array}{r|r}
\multicolumn{2}{r}{0.84} \\ \cline{2-2}
25 & 21.00 \\
 & 200 \\ \cline{2-2}
 & 100 \\
 & 100 \\ \cline{2-2}
 & 0 \\ \cline{2-2}
\end{array}$$

$\frac{21}{25} = 0.84$

$$\begin{array}{r|r}
\multicolumn{2}{r}{2.71875} \\ \cline{2-2}
32 & 87.00000 \\
 & 64 \\ \cline{2-2}
 & 230 \\
 & 224 \\ \cline{2-2}
 & 60 \\
 & 32 \\ \cline{2-2}
 & 280 \\
 & 256 \\ \cline{2-2}
 & 240 \\
 & 224 \\ \cline{2-2}
 & 160 \\
 & 160 \\ \cline{2-2}
 & 0 \\ \cline{2-2}
\end{array}$$

$\frac{-87}{32} = -2.71875$

Note

These show that the process could lead to a decimal with finite number of decimal places. They are called terminating decimals.

Can the decimal representation of a rational number lead to forms of decimals that do not terminate? The following examples (with non-zero remainder) throw some light on this point.

Example 2.2

Represent the following as decimal form (i) $\frac{-4}{11}$ (ii) $\frac{11}{75}$

Solution

$$\begin{array}{r|r}
\multicolumn{2}{r}{0.3636....} \\ \cline{2-2}
11 & 4.0000 \\
 & 33 \\ \cline{2-2}
 & 70 \\
 & 66 \\ \cline{2-2}
 & 40 \\
 & 33 \\ \cline{2-2}
 & 70 \\
 & 66 \\ \cline{2-2}
 \multicolumn{2}{r}{4} \\
 \multicolumn{2}{r}{\vdots}
\end{array}$$

$$\begin{array}{r|r}
\multicolumn{2}{r}{0.1466...} \\ \cline{2-2}
75 & 11.0000 \\
 & 7\ 5 \\ \cline{2-2}
 & 350 \\
 & 300 \\ \cline{2-2}
 & 500 \\
 & 450 \\ \cline{2-2}
 & 500 \\
 & 450 \\ \cline{2-2}
 \multicolumn{2}{r}{50} \\
 \multicolumn{2}{r}{\vdots}
\end{array}$$

Thus we see that, $\frac{-4}{11} = -0.\overline{36}$ $\frac{11}{75} = 0.14\overline{6}$

The reciprocals of Natural Numbers are Rational numbers. It is interesting to note their decimal forms. See the first ten.

S.No.	Reciprocal	Decimal Representation
1	$\frac{1}{1} = 1.0$	Terminating
2	$\frac{1}{2} = 0.5$	Terminating
3	$\frac{1}{3} = 0.\overline{3}$	Non-terminating Recurring
4	$\frac{1}{4} = 0.25$	Terminating
5	$\frac{1}{5} = 0.2$	Terminating
6	$\frac{1}{6} = 0.1\overline{6}$	Non-terminating Recurring

7	$\frac{1}{7} = 0.\overline{142857}$	Non-terminating Recurring
8	$\frac{1}{8} = 0.125$	Terminating
9	$\frac{1}{9} = 0.\overline{1}$	Non-terminating Recurring
10	$\frac{1}{10} = 0.1$	Terminating

A rational number can be expressed by

(i) **either** a terminating

(ii) **or** a non-terminating and recurring (repeating) decimal expansion.

The converse of this statement is also true.

That is, if the decimal expansion of a number is terminating or non-terminating and recurring, then the number is a rational number.

Note

In this case the decimal expansion does not terminate!

The remainders repeat again and again! We get non-terminating but recurring block of digits.

2.3.3 Period of Decimal

In the decimal expansion of the rational numbers, the number of repeating decimals is called the length of the period of decimals.

For example,

(i) $\frac{25}{7} = 3.\overline{571428}$ has the length of the period of decimal = 6

(ii) $\frac{27}{110} = 0.2\overline{45}$ has the length of the period of decimal = 2

Example 2.3 Express the rational number $\frac{1}{27}$ in recurring decimal form by using the recurring decimal expansion of $\frac{1}{3}$. Hence write $\frac{59}{27}$ in recurring decimal form.

Solution

We know that $\frac{1}{3} = 0.\overline{3}$

Therefore, $\frac{1}{27} = \frac{1}{9} \times \frac{1}{3} = \frac{1}{9} \times 0.333... = 0.037037... = 0.\overline{037}$

Also, $\frac{59}{27} = 2\frac{5}{27} = 2 + \frac{5}{27}$

$$= 2 + \left(5 \times \frac{1}{27}\right)$$

$$= 2 + (5 \times 0.\overline{037}) = 2 + (5 \times 0.037037037...) = 2 + 0.185185... = 2.185185... = 2.\overline{185}$$

2.3.4 Conversion of Terminating Decimals into Rational Numbers

Let us now try to convert a terminating decimal, say 2.945 as rational number in the fraction form.

$2.945 \quad = 2 + 0.945$

$$\begin{aligned} 2.945 &= 2+\frac{9}{10}+\frac{4}{100}+\frac{5}{1000} \\ &= 2+\frac{900}{1000}+\frac{40}{1000}+\frac{5}{1000} \text{ (making denominators common)} \\ &= 2+\frac{945}{1000} \\ &= \frac{2945}{1000} \text{ or } \frac{589}{200} \text{ which is required} \end{aligned}$$

(In the above, is it possible to write directly $2.945 = \frac{2945}{1000}$?)

Example 2.4 Convert the following decimal numbers in the form of $\frac{p}{q}$, where p and q are integers and $q \neq 0$: (i) 0.35 (ii) 2.176 (iii) – 0.0028

Solution

(i) $0.35 = \frac{35}{100} = \frac{7}{20}$

(ii) $2.176 = \frac{2176}{1000} = \frac{272}{125}$

(iii) $-0.0028 = \frac{-28}{10000} = \frac{-7}{2500}$

2.3.5 Conversion of Non-terminating and recurring decimals into Rational Numbers

It was very easy to handle a terminating decimal. When we come across a decimal such as 2.4, we get rid of the decimal point, by just divide it by 10.

Thus $2.4 = \frac{24}{10}$, which is simplified as $\frac{12}{5}$. But, when we have a decimal such as $2.\overline{4}$, the problem is that we have infinite number of 4s and hence will need infinite number of 0s in the denominator. For example,

$$\begin{aligned} 2.4 &= 2+\frac{4}{10} \\ 2.44 &= 2+\frac{4}{10}+\frac{4}{100} \\ 2.444 &= 2+\frac{4}{10}+\frac{4}{100}+\frac{4}{1000} \end{aligned}$$

How tough it is to have infinite 4's and work with them. We need to get rid of the infinite sequence in some way. The good thing about the infinite sequence is that even if we pull away one , two or more 4 out of it, the sequence still remains infinite.

Let $x = 2.\overline{4}$ ...(1)

Then, $10x = 24.\overline{4}$ ...(2) [When you multiply by 10, the decimal moves one place to the right but you still have infinite 4s left over).

Subtract the first equation from the second to get,

$9x = 24.\overline{4} - 2.\overline{4} = 22$ (Infinite 4s subtract out the infinite 4s and the left out is $24 - 2 = 22$)

$x = \frac{22}{9}$, the required value.

We use the same exact logic to convert any number with a non terminating repeating part into a fraction.

Example 2.5 Convert the following decimal numbers in the form of $\frac{p}{q}\,(p, q \in Z \text{ and } q \neq 0)$.

(i) $0.\overline{3}$ (ii) $2.\overline{124}$ (iii) $0.4\overline{5}$ (iv) $0.5\overline{68}$

Solution

(i) Let $x = 0.\overline{3} = 0.3333\ldots$ (1)

(Here period of decimal is 1, multiply equation (1) by 10)

$10\,x = 3.3333...$ (2)

(2) – (1): $9\,x = 3$ or $x = \frac{1}{3}$

(ii) Let $x = 2.\overline{124} = 2.124124124\ldots$ (1)

(Here period of decimal is 3, multiply equation (1) by 1000.)

$1000\,x = 2124.124124124\ldots$ (2)

(2)–(1): $999\,x = 2122$ $x = \frac{2122}{999}$

(iii) Let $x = 0.4\overline{5} = 0.45555\ldots$ (1)

(Here the repeating decimal digit is 5, which is the second digit after the decimal point, multiply equation (1) by 10)

$10\,x = 4.5555\ldots$ (2)

(Now period of decimal is 1, multiply equation (2) by 10)

$100\,x = 45.5555\ldots$ (3)

(3) – (2): $90\,x = 41$ or $x = \frac{41}{90}$

(iv) Let $x = 0.5\overline{68} = 0.5686868\ldots$ (1)

(Here the repeating decimal digit is 68, which is the second digit after the decimal point, so multiply equation (1) by 10)

$10\,x = 5.686868\ldots$ (2)

(Now period of decimal is 2, multiply equation (2) by 100)

$1000\,x = 568.686868\ldots$ (3)

(3) – (2): $990\,x = 563$ or $x = \frac{563}{990}$.

Note

To determine whether the decimal form of a rational number will terminate or non - terminate, we can make use of the following rule

If a rational number $\frac{p}{q}$, $q \neq 0$ can be expressed in the form $\frac{p}{2^m \times 5^n}$, where $p \in \mathbb{Z}$ and $m, n \in \mathbb{W}$, then rational number will have a terminating decimal expansion. Otherwise, the rational number will have a non- terminating and recurring decimal expansion

Example 2.6 Without actual division, classify the decimal expansion of the following numbers as terminating or non – terminating and recurring.

(i) $\frac{13}{64}$ (ii) $\frac{-71}{125}$ (iii) $\frac{43}{375}$ (iv) $\frac{31}{400}$

Solution

(a) $\frac{13}{64} = \frac{13}{2^6}$, So $\frac{13}{64}$ has a terminating decimal expansion.

(b) $\frac{-71}{125} = \frac{-71}{5^3}$, So $\frac{-71}{125}$ has a terminating decimal expansion.

(c) $\frac{43}{375} = \frac{43}{3^1 \times 5^3}$, So $\frac{43}{375}$ has a non – terminating recurring decimal expansion.

(d) $\frac{31}{400} = \frac{31}{2^4 \times 5^2}$, So $\frac{31}{400}$ has a terminating decimal expansion.

Example 2.7 Verify that $1 = 0.\overline{9}$

Solution

Let $x = 0.\overline{9} = 0.99999\ldots$ (1)

(Multiply equation (1) by 10)

$10\,x = 9.99999\ldots$ (2)

Subtract (1) from (2)

$9x = 9$ or $x = 1$

Thus, $0.\overline{9} = 1$

$1 = 0.9999\ldots$

$7 = 6.9999\ldots$

$3.7 = 3.6999\ldots$

The pattern suggests that any terminating decimal can be represented as a non-terminating and recurring decimal expansion with an endless block of 9's.

Exercise 2.2

1. Express the following rational numbers into decimal and state the kind of decimal expansion

(i) $\frac{2}{7}$ (ii) $-5\frac{3}{11}$ (iii) $\frac{22}{3}$ (iv) $\frac{327}{200}$

2. Express $\frac{1}{13}$ in decimal form. Find the length of the period of decimals.

3. Express the rational number $\frac{1}{33}$ in recurring decimal form by using the recurring decimal expansion of $\frac{1}{11}$. Hence write $\frac{71}{33}$ in recurring decimal form.

4. Express the following decimal expression into rational numbers.

(i) $0.\overline{24}$ (ii) $2.\overline{327}$ (iii) -5.132

(iv) $3.1\overline{7}$ (v) $17.2\overline{15}$ (vi) $-21.213\overline{7}$

5. Without actual division, find which of the following rational numbers have terminating decimal expansion.

(i) $\frac{7}{128}$ (ii) $\frac{21}{15}$ (iii) $4\frac{9}{35}$ (iv) $\frac{219}{2200}$

2.3.6 Decimal Representation to Identify Irrational Numbers

It can be shown that irrational numbers, when expressed as decimal numbers, do not terminate, nor do they repeat. For example, the decimal representation of the number π starts with 3.14159265358979..., but no finite number of digits can represent π exactly, nor does it repeat.

Consider the following decimal expansions:

(i) 0.10110011100011111… (ii) 3.012012120121212…

(iii) 12.230223300222333000… (in) $\sqrt{2}$ = 1.4142135624…

Are the above numbers terminating (or) recurring and non- terminating? No… They are neither terminating, nor non–terminating and recurring. Hence they are not rational numbers. They cannot be written in the form of $\frac{p}{q}$,where p, q ,$\in \mathbb{Z}$ and $q \neq 0$. They are irrational numbers.

A number having non- terminating and non- recurring decimal expansion is an irrational number.

Example 2.8

Find the decimal expansion of $\sqrt{3}$.

Solution

$$\begin{array}{r|l} & 1.7320508\ldots \\ \hline 1 & 3.00,00,00,00,00,\ldots \\ & 1 \\ \hline 27 & 200 \\ & 189 \\ \hline 343 & 1100 \\ & 1029 \\ \hline 3462 & 7100 \\ & 6924 \\ \hline 346405 & 1760000 \\ & 1732025 \\ \hline 34641008 & 279750000 \\ & 277128064 \\ \hline & 2621936 \end{array}$$

We often write $\sqrt{2}$ = 1.414, $\sqrt{3}$ = 1.732, π = 3.14 etc. These are only approximate values and not exact values. In the case of the irrational number π, we take frequently $\frac{22}{7}$ (which gives the value $3.\overline{142857}$) to be its correct value but in reality these are only approximations. This is because, the decimal expansion of an irrational number is non-terminating and non-recurring. None of them gives an exact value!

Thus, by division method, $\sqrt{3} = 1.7320508\ldots$

It is found that the square root of every positive non perfect square number is an irrational number. $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \ldots$ are all irrational numbers.

Example 2.9

Classify the numbers as rational or irrational:

(i) $\sqrt{10}$ (ii) $\sqrt{49}$ (iii) 0.025 (iv) $0.7\overline{6}$ (v) 2.505500555... (vi) $\frac{\sqrt{2}}{2}$

Solution

(i) $\sqrt{10}$ is an irrational number (since 10 is not a perfect square number).

(ii) $\sqrt{49} = 7 = \frac{7}{1}$, a rational number(since 49 is a perfect square number).

(iii) 0.025 is a rational number (since it is a terminating decimal).

(iv) $0.7\overline{6} = 0.7666\ldots$ is a rational number (since it is a non – terminating and recurring decimal expansion).

(v) 2.505500555…. is an irrational number (since it is a non – terminating and non–recurring decimal).

(vi) $\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{\sqrt{2}\times\sqrt{2}} = \frac{1}{\sqrt{2}}$ is an irrational number (since 2 is not a perfect square number).

Note

The above example(vi) it is not to be misunderstood as $\frac{p}{q}$ form, because both p and q must be integers and not an irrational number.

Example 2.10

Find any 3 irrational numbers between 0.12 and 0.13 .

E4T9YH

Solution

Three irrational numbers between 0.12 and 0.13 are 0.12010010001…, 0.12040040004…, 0.12070070007…

Note

We state (without proof) an important result worth remembering.

If 'a' is a rational number and $\sqrt{b}$ is an irrational number then each one of the following is an irrational number:

(i) $a+\sqrt{b}$; (ii) $a-\sqrt{b}$; (iii) $a\sqrt{b}$; (iv) $\frac{a}{\sqrt{b}}$; (v) $\frac{\sqrt{b}}{a}$.

For example, when you consider the rational number 4 and the irrational number $\sqrt{5}$, then $4+\sqrt{5}$, $4-\sqrt{5}$, $4\sqrt{5}$, $\frac{4}{\sqrt{5}}$, $\frac{\sqrt{5}}{4}$ ... are all irrational numbers.

Example 2.11

Give any two rational numbers lying between 0.5151151115…. and 0.5353353335…

Solution Two rational numbers between the given two irrational numbers are 0.5152 and 0.5352

Example 2.12

Find whether x and y are rational or irrational in the following.

(i) $a = 2+\sqrt{3}$, $b = 2-\sqrt{3}$; $x = a+b$, $y = a-b$

(ii) $a = \sqrt{2}+7$, $b = \sqrt{2}-7$; $x = a+b$, $y = a-b$

(iii) $a = \sqrt{75}$, $b = \sqrt{3}$; $x = ab$, $y = \frac{a}{b}$

(iv) $a = \sqrt{18}$, $b = \sqrt{3}$; $x = ab$, $y = \frac{a}{b}$

Note From these examples, it is clear that the sum , difference, product, quotient of any two irrational numbers could be rational or irrational.

Solution

(i) Given that $a = 2+\sqrt{3}, b = 2-\sqrt{3}$

$x = a + b = (2+\sqrt{3}) + (2-\sqrt{3}) = 4$, a rational number.

$y = a - b = (2+\sqrt{3}) - (2-\sqrt{3}) = 2\sqrt{3}$, an irrational number.

(ii) Given that $a = \sqrt{2}+7$, $b = \sqrt{2}-7$

$x = a + b = (\sqrt{2}+7) + (\sqrt{2}-7) = 2\sqrt{2}$, an irrational number.

$y = a - b = (\sqrt{2}+7) - (\sqrt{2}-7) = 14$, a rational number.

(iii) Given that $a = \sqrt{75}$, $b = \sqrt{3}$

$x = ab = \sqrt{75} \times \sqrt{3} = \sqrt{75 \times 3} = \sqrt{5 \times 5 \times 3 \times 3} = 5 \times 3 = 15$, a rational number.

$y = \frac{a}{b} = \frac{\sqrt{75}}{\sqrt{3}} = \sqrt{\frac{75}{3}} = \sqrt{25} = 5$, rational number.

(iv) Given that $a = \sqrt{18}$, $b = \sqrt{3}$

$x = ab = \sqrt{18} \times \sqrt{3} = \sqrt{18 \times 3} = \sqrt{6 \times 3 \times 3} = 3\sqrt{6}$, an irrational number.

$y = \frac{a}{b} = \frac{\sqrt{18}}{\sqrt{3}} = \sqrt{\frac{18}{3}} = \sqrt{6}$, an irrational number.

Example 2.13

Represent $\sqrt{9.3}$ on a number line.

Solution

- Draw a line and mark a point A on it.
- Mark a point B such that $AB = 9.3$ cm.
- Mark a point C on this line such that $BC = 1$ cm.
- Find the midpoint of AC by drawing perpendicular bisector of AC and let it be O
- With O as center and $OC = OA$ as radius, draw a semicircle.

- Draw a line BD, which is perpendicular to AB at B.
- Now $BD = \sqrt{9.3}$, which can be marked in the number line as the value of $BE = BD = \sqrt{9.3}$

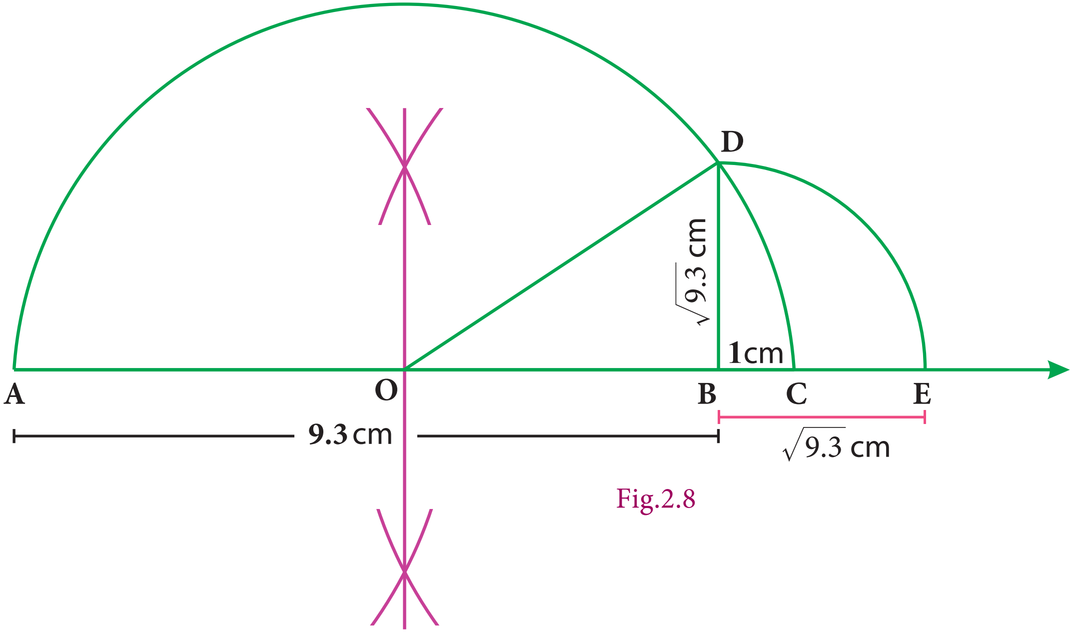


Fig.2.8

Exercise 2.3

1. Represent the following irrational numbers on the number line.

   (i) $\sqrt{3}$ (ii) $\sqrt{4.7}$ (iii) $\sqrt{6.5}$

2. Find any two irrational numbers between

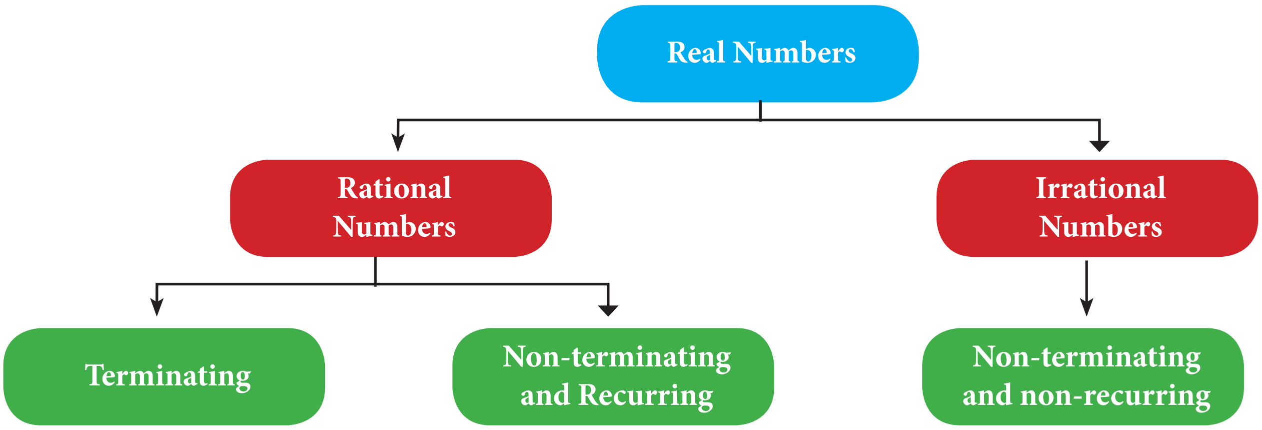
   (i) 0.3010011000111…. and 0.3020020002…. (ii) $\frac{6}{7}$ and $\frac{12}{13}$ (iii) $\sqrt{2}$ and $\sqrt{3}$

3. Find any two rational numbers between 2.2360679….. and 2.236505500….

2.4 Real Numbers

The real numbers consist of all the rational numbers and all the irrational numbers.

Real numbers can be thought of as points on an infinitely long number line called the real line, where the points corresponding to integers are equally spaced.



Any real number can be determined by a possibly infinite decimal representation, (as we have already seen decimal representation of the rational numbers and the irrational numbers).

2.4.1 The Real Number Line

Visualisation through Successive Magnification.

We can visualise the representation of numbers on the number line, as if we glimpse through a magnifying glass.

Example 2.14 Represent 4.863 on the number line.

Solution

4.863 lies between 4 and 5(see Fig. 2.9)

(i) Divide the distance between 4 and 5 into 10 equal intervals.

(ii) Mark the point 4.8 which is second from the left of 5 and eighth from the right of 4

(iii) 4.86 lies between 4.8 and 4.9. Divide the distance into 10 equal intervals.

(iv) Mark the point 4.86 which is fourth from the left of 4.9 and sixth from the right of 4.8

(v) 4.863 lies between 4.86 and 4.87. Divide the distance into 10 equal intervals.

(vi) Mark point 4.863 which is seventh from the left of 4.87 and third from the right of 4.86.

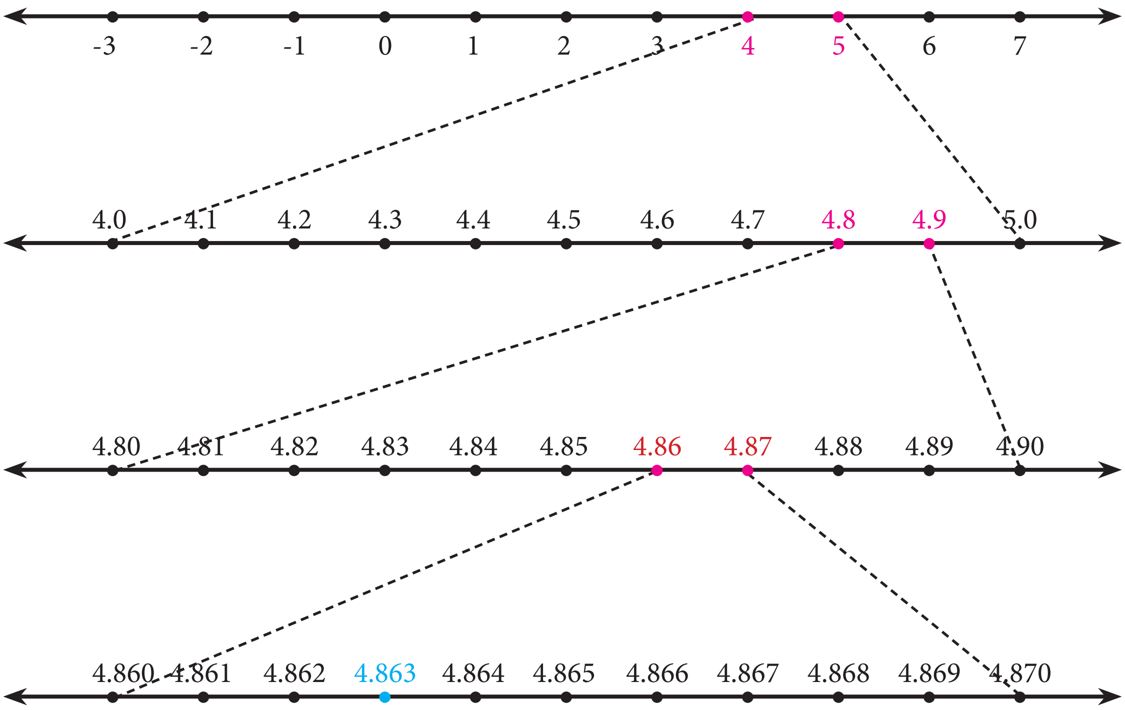


Fig. 2.9

Example 2.15 Represent $3.\overline{45}$ on the number line upto 4 decimal places.

Solution

$3.\overline{45} = 3.45454545\ldots..$

= 3.4545 (correct to 4 decimal places).

The number lies between 3 and 4

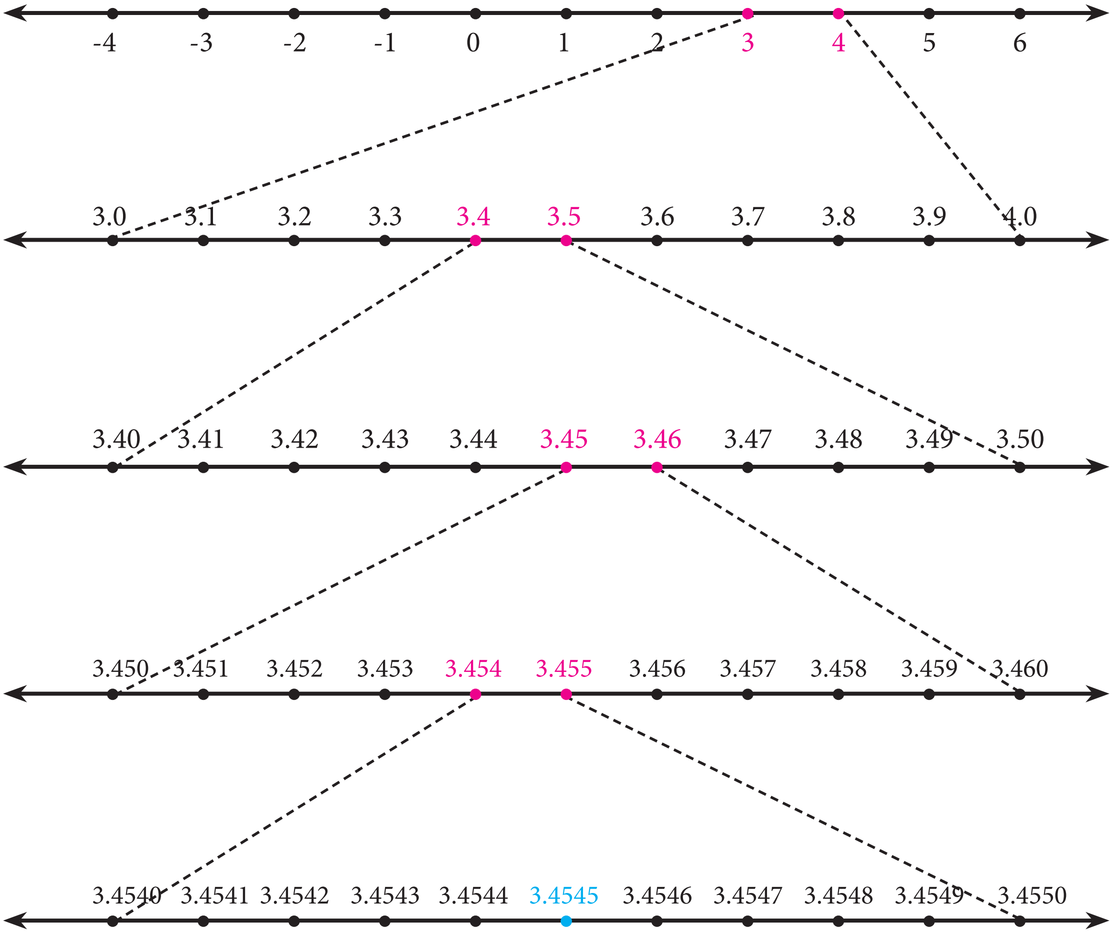


Fig. 2.10

Exercise 2.4

1. Represent the following numbers on the number line.

   (i) 5.348 (ii) $6.\overline{4}$ upto 3 decimal places. (iii) $4.\overline{73}$ upto 4 decimal places.

2.5 Radical Notation

Let n be a positive integer and r be a real number. If $r^n = x$, then r is called the n^{th} root of x and we write

$$\sqrt[n]{x} = r$$

The symbol $\sqrt[n]{}$ (read as n^{th} root) is called a radical; n is the index of the radical (hitherto we named it as exponent); and x is called the radicand.

Note

It is worth spending some time on the concepts of the 'square root' and the 'cube root', for better understanding of surds.

What happens when $n = 2$? Then we get $r^2 = x$, so that r is $\sqrt[2]{x}$, our good old friend, the square root of x. Thus $\sqrt[2]{16}$ is written as $\sqrt{16}$, and when $n = 3$, we get the cube root of x, namely $\sqrt[3]{x}$. For example, $\sqrt[3]{8}$ is cube root of 8, giving 2. (Is not $8 = 2^3$?)

How many square roots are there for 4? Since $(+2)\times(+2) = 4$ and also $(-2)\times(-2) = 4$, we can say that both +2 and −2 are square roots of 4. But it is incorrect to write that $\sqrt{4} = \pm 2$. This is because, when n is even, it is an accepted convention to reserve the symbol $\sqrt[n]{x}$ for the positive n^{th} root and to denote the negative n^{th} root by $-\sqrt[n]{x}$. Therefore we need to write $\sqrt{4} = 2$ and $-\sqrt{4} = -2$.

When n is odd, for any value of x, there is exactly one real n^{th} root. For example, $\sqrt[3]{8} = 2$ and $\sqrt[5]{-32} = -2$.

Thinking Corner

Which one of the following is false?

(1) The square root of 9 is 3 or −3.

(2) $\sqrt{9} = 3$

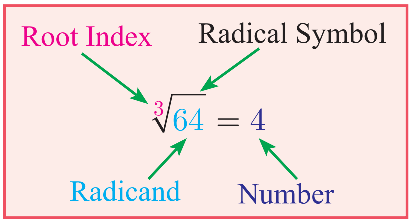
(3) $-\sqrt{9} = -3$

(4) $\sqrt{9} = \pm 3$

2.5.1 Fractional Index

Consider again results of the form $r = \sqrt[n]{x}$.

In the adjacent notation, the index of the radical (namely n which is 3 here) tells you how many times the answer (that is 4) must be multiplied with itself to yield the radicand.



To express the powers and roots, there is one more way of representation. It involves the use of fractional indices.

We write $\sqrt[n]{x}$ as $x^{\frac{1}{n}}$.

With this notation, for example

$$\sqrt[3]{64} \text{ is } 64^{\frac{1}{3}} \text{ and } \sqrt{25} \text{ is } 25^{\frac{1}{2}}.$$

Observe in the following table just some representative patterns arising out of this new acquaintance:

Power	Radical Notation	Index Notation	Read as
$2^6 = 64$	$2 = \sqrt[6]{64}$	$2 = 64^{\frac{1}{6}}$	2 is the 6th root of 64
$2^5 = 32$	$2 = \sqrt[5]{32}$	$2 = 32^{\frac{1}{5}}$	2 is the 5th root of 32
$2^4 = 16$	$2 = \sqrt[4]{16}$	$2 = 16^{\frac{1}{4}}$	2 is the 4th root of 16
$2^3 = 8$	$2 = \sqrt[3]{8}$	$2 = 8^{\frac{1}{3}}$	2 is the cube root of 8 meaning 2 is the 3rd root of 8
$2^2 = 4$	$2 = \sqrt[2]{4}$ or simply $2 = \sqrt{4}$	$2 = 4^{\frac{1}{2}}$	2 is the square root of 4 meaning 2 is the 2nd root of 4

Example 2.16 Express the following in the form 2^n :

(i) 8 (ii) 32 (iii) $\frac{1}{4}$ (iv) $\sqrt{2}$ (v) $\sqrt{8}$.

Solution

(i) $8 = 2\times2\times2;\quad$ therefore $8 = 2^3$

(ii) $32 = 2\times2\times2\times2\times2 = 2^5$

(iii) $\frac{1}{4} = \frac{1}{2\times2} = \frac{1}{2^2} = 2^{-2}$

(iv) $\sqrt{2} = 2^{1/2}$

(v) $\sqrt{8} = \sqrt{2}\times\sqrt{2}\times\sqrt{2} = \left(2^{\frac{1}{2}}\right)^3$ which may be written as $2^{\frac{3}{2}}$

Meaning of $x^{\frac{m}{n}}$, (where m and n are Positive Integers)

We interpret $x^{\frac{m}{n}}$ either as the n^{th} root of the m^{th} power of x or as the m^{th} power of the n^{th} root of x.

In symbols, $x^{\frac{m}{n}} = (x^m)^{\frac{1}{n}}$ or $(x^{\frac{1}{n}})^m = \sqrt[n]{x^m}$ or $(\sqrt[n]{x})^m$

Example 2.17 Find the value of (i) $81^{\frac{5}{4}}$ (ii) $64^{\frac{-2}{3}}$

Solution

(i) $81^{\frac{5}{4}} = \left(\sqrt[4]{81}\right)^5 = \left(\sqrt[4]{3^4}\right)^5 = 3^5 = 3\times3\times3\times3\times3 = 243$

(ii) $64^{\frac{-2}{3}} = \frac{1}{64^{\frac{2}{3}}} = \frac{1}{\left(\sqrt[3]{64}\right)^2} = \frac{1}{4^2}$ (How?) $= \frac{1}{16}$

Exercise 2.5

1. Write the following in the form of 5^n:

(i) 625 (ii) $\frac{1}{5}$ (iii) $\sqrt{5}$ (iv) $\sqrt{125}$

2. Write the following in the form of 4^n:

(i) 16 (ii) 8 (iii) 32

3. Find the value of

(i) $(49)^{\frac{1}{2}}$ (ii) $(243)^{\frac{2}{5}}$ (iii) $9^{\frac{-3}{2}}$ (vi) $\left(\frac{64}{125}\right)^{\frac{-2}{3}}$

4. Use a fractional index to write:

(i) $\sqrt{5}$ (ii) $\sqrt[2]{7}$ (iii) $\left(\sqrt[3]{49}\right)^5$ (iv) $\left(\frac{1}{\sqrt[3]{100}}\right)^7$

5. Find the 5th root of

   (i) 32 (ii) 243 (iii) 100000 (iv) $\frac{1024}{3125}$

2.6 Surds

Having familiarized with the concept of Real numbers, representing them on the number line and manipulating them, we now learn about surds, a distinctive way of representing certain approximate values.

Can you simplify $\sqrt{4}$ and remove the $\sqrt{\ }$ symbol? Yes; one can replace $\sqrt{4}$ by the number 2. How about $\sqrt{\frac{1}{9}}$? It is easy; without $\sqrt{\ }$ symbol, the answer is $\frac{1}{3}$. What about $\sqrt{0.01}$?. This is also easy and the solution is 0.1

In the cases of $\sqrt{4}$, $\sqrt{\frac{1}{9}}$ and $\sqrt{0.01}$, you can resolve to get a solution and make sure that the symbol $\sqrt{\ }$ is not seen in your solution. Is this possible at all times?

Consider $\sqrt{18}$. Can you evaluate it and also remove the radical symbol? Surds are unresolved radicals, such as square root of 2, cube root of 5, etc.

They are irrational roots of equations with rational coefficients.

A surd is an irrational root of a rational number. $\sqrt[n]{a}$ is a surd, provided $n \in \mathbb{N},\ n > 1$, 'a' is rational.

Examples : $\sqrt{2}$ is a surd. It is an irrational root of the equation $x^2 = 2$. (Note that $x^2 - 2 = 0$ is an equation with rational coefficients. $\sqrt{2}$ is irrational and may be shown as 1.4142135… a non-recurring, non-terminating decimal).

$\sqrt[3]{3}$ (which is same as $3^{\frac{1}{3}}$) is a surd since it is an irrational root of the equation $x^3 - 3 = 0$. ($\sqrt{3}$ is irrational and may be shown as 1.7320508… a non-recurring, non-terminating decimal).

You will learn solving (quadratic) equations like $x^2 - 6x + 7 = 0$ in your next class. This is an equation with rational coefficients and one of its roots is $3+\sqrt{2}$, which is a surd.

Is $\sqrt{\frac{1}{25}}$ a surd? No; it can be simplified and written as rational number $\frac{1}{5}$. How about $\sqrt[4]{\frac{16}{81}}$? It is not a surd because it can be simplified as $\frac{2}{3}$.

The famous irrational number π is not a surd! Though it is irrational, it cannot be expressed as a rational number under the $\sqrt{\ }$ symbol. (In other words, it is not a root of any equation with rational co-efficients).

Why surds are important? For calculation purposes we assume approximate value as $\sqrt{2} = 1.414, \sqrt{3} = 1.732$ and so on.

$$\left(\sqrt{2}\right)^2 = (1.414)^2 = 1.99936 \neq 2 \; ; \quad \left(\sqrt{3}\right)^2 = (1.732)^2 = 3.999824 \neq 3$$

Hence, we observe that $\sqrt{2}$ and $\sqrt{3}$ represent the more accurate and precise values than their assumed values. Engineers and scientists need more accurate values while constructing the bridges and for architectural works. Thus it becomes essential to learn surds.

Progress Check

1. Which is the odd one out? Justify your answer.

   (i) $\sqrt{36}, \; \sqrt{\dfrac{50}{98}}, \; \sqrt{1}, \; \sqrt{1.44}, \; \sqrt[5]{32}, \; \sqrt{120}$

   (ii) $\sqrt{7}, \; \sqrt{48}, \; \sqrt[3]{36}, \; \sqrt{5}+\sqrt{3}, \; \sqrt{1.21}, \; \sqrt{\dfrac{1}{10}}$

2. Are all surds irrational numbers? - Discuss with your answer.
3. Are all irrational numbers surds? Verify with some examples.

2.6.1 Order of a Surd

The order of a surd is the index of the root to be extracted. The order of the surd $\sqrt[n]{a}$ is n. What is the order of $\sqrt[5]{99}$? It is 5.

Surds can be classified in different ways:

(i) Surds of same order : Surds of same order are surds for which the index of the root to be extracted is same. (They are also called equiradical surds).

For example, $\sqrt{x}, \; a^{\frac{3}{2}}, \; \sqrt[2]{m}$ are all 2nd order (called quadratic) surds .

$\sqrt[3]{5}, \; \sqrt[3]{(x-2)}, \; (ab)^{\frac{1}{3}}$ are all 3rd order (called cubic) surds.

$\sqrt{3}, \; \sqrt[3]{10}, \; \sqrt[4]{6}$ and $8^{\frac{2}{5}}$ are surds of different order.

(ii) Simplest form of a surd : A surd is said to be in simplest form, when it is expressed as the product of a rational factor and an irrational factor. In this form the surd has

(a) the smallest possible index of the radical sign.

(b) no fraction under the radical sign.

(c) no factor is of the form a^n, where a is a positive integer under index n.

Example 2.18

Can you reduce the following numbers to surds of same order :

(i) $\sqrt{3}$ (ii) $\sqrt[4]{3}$ (iii) $\sqrt[3]{3}$

Solution

$$\begin{array}{lll}
\text{(i) } \sqrt{3} = 3^{\frac{1}{2}} & \text{(ii) } \sqrt[4]{3} = 3^{\frac{1}{4}} & \text{(iii) } \sqrt[3]{3} = 3^{\frac{1}{3}} \\
= 3^{\frac{6}{12}} & = 3^{\frac{3}{12}} & = 3^{\frac{4}{12}} \\
= \sqrt[12]{3^6} & = \sqrt[12]{3^3} & = \sqrt[12]{3^4} \\
= \sqrt[12]{729} & = \sqrt[12]{27} & = \sqrt[12]{81}
\end{array}$$

The last row has surds of same order.

Example 2.19

1. Express the surds in the simplest form: i) $\sqrt{8}$ ii) $\sqrt[3]{192}$
2. Show that $\sqrt[3]{7} > \sqrt[4]{5}$.

Solution

1. (i) $\sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}$

   (ii) $\sqrt[3]{192} = \sqrt[3]{4 \times 4 \times 4 \times 3} = 4\sqrt[3]{3}$

2. $\sqrt[3]{7} = \sqrt[12]{7^4} = \sqrt[12]{2401}$

   $\sqrt[4]{5} = 5^{\frac{1}{4}} = 5^{\frac{3}{12}} = \sqrt[12]{5^3} = \sqrt[12]{125}$

   $\sqrt[12]{2401} > \sqrt[12]{125}$

   Therefore, $\sqrt[3]{7} > \sqrt[4]{5}$.

(iii) Pure and Mixed Surds : A surd is called a pure surd if its coefficient in its simplest form is 1. For example, $\sqrt{3}$, $\sqrt[3]{6}$, $\sqrt[4]{7}$, $\sqrt[5]{49}$ are pure surds. A surd is called a mixed surd if its co-efficient in its simplest form is other than 1. For example, $5\sqrt{3}$, $2\sqrt[4]{5}$, $3\sqrt[4]{6}$ are mixed surds.

(iv) Simple and Compound Surds : A surd with a single term is said to be a simple surd. For example, $\sqrt{3}$, $2\sqrt{5}$ are simple surds. The algebraic sum of two (or more) surds is called a compound surd. For example, $\sqrt{5}+3\sqrt{2}, \sqrt{3}-2\sqrt{7}, \sqrt{5}-7\sqrt{2}+6\sqrt{3}$ are compound surds.

(v) **Binomial Surd :** A binomial surd is an algebraic sum (or difference) of 2 terms both of which could be surds or one could be a rational number and another a surd. For example, $\frac{1}{2}-\sqrt{19}, 5+3\sqrt{2}, \sqrt{3}-2\sqrt{7}$ are binomial surds.

Example 2.20 Arrange in ascending order: $\sqrt[3]{2},\ \sqrt[2]{4},\ \sqrt[4]{3}$

Solution

The order of the surds $\sqrt[3]{2},\ \sqrt[2]{4}$ and $\sqrt[4]{3}$ are 3, 2, 4.

L.C.M. of 3, 2, 4 = 12.

$$\sqrt[3]{2}=\left(2^{\frac{1}{3}}\right)=\left(2^{\frac{4}{12}}\right)=\sqrt[12]{2^4}=\sqrt[12]{16}\ ; \qquad \sqrt[2]{4}=\left(4^{\frac{1}{2}}\right)=\left(4^{\frac{6}{12}}\right)=\sqrt[12]{4^6}=\sqrt[12]{4096}$$

$$\sqrt[4]{3}=\left(3^{\frac{1}{4}}\right)=\left(3^{\frac{3}{12}}\right)=\sqrt[12]{3^3}=\sqrt[12]{27}$$

The ascending order of the surds $\sqrt[3]{2},\ \sqrt[4]{3},\ \sqrt[2]{4}$ is $\sqrt[12]{16}<\sqrt[12]{27}<\sqrt[12]{4096}$

that is, $\sqrt[3]{2}\,,\ \sqrt[4]{3}\,,\ \sqrt[2]{4}$.

2.6.2 Laws of Radicals

For positive integers m, n and positive rational numbers a and b, it is worth remembering the following properties of radicals:

S.No.	Radical Notation	Index Notation
1.	$\left(\sqrt[n]{a}\right)^n=a=\sqrt[n]{a^n}$	$\left(a^{\frac{1}{n}}\right)^n=a=\left(a^n\right)^{\frac{1}{n}}$
2.	$\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab}$	$a^{\frac{1}{n}}\times b^{\frac{1}{n}}=\left(ab\right)^{\frac{1}{n}}$
3.	$\sqrt[m]{\sqrt[n]{a}}=\sqrt[mn]{a}=\sqrt[n]{\sqrt[m]{a}}$	$\left(a^{\frac{1}{n}}\right)^{\frac{1}{m}}=a^{\frac{1}{mn}}=\left(a^{\frac{1}{m}}\right)^{\frac{1}{n}}$
4.	$\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}$	$\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}=\left(\frac{a}{b}\right)^{\frac{1}{n}}$

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We shall now discuss certain problems which require the laws of radicals for simplifying.

Example 2.21 Express each of the following surds in its simplest form (i) $\sqrt[3]{108}$ (ii) $\sqrt[3]{(1024)^{-2}}$ and find its order, radicand and coefficient.

Solution

(i)
$$\sqrt[3]{108} = \sqrt[3]{27\times 4}$$
$$= \sqrt[3]{3^3\times 4}$$
$$= \sqrt[3]{3^3}\times\sqrt[3]{4} \quad \text{(Laws of radicals - ii)}$$
$$= 3\times\sqrt[3]{4} \quad \text{(Laws of radicals- i)}$$

2	108
2	54
3	27
3	9
3	3
	1

order = 3; radicand = 4; coefficient = 3

(ii)
$$\sqrt[3]{(1024)^{-2}} = \left[\sqrt[3]{\left(2^3\times 2^3\times 2^3\times 2\right)^{-2}}\right]$$
$$= \left[\sqrt[3]{\left(2^3\times 2^3\times 2^3\times 2\right)}\right]^{-2} \quad \text{[Laws of radicals - (i)]}$$
$$= \left[\sqrt[3]{2^3}\times\sqrt[3]{2^3}\times\sqrt[3]{2^3}\times\sqrt[3]{2}\right]^{-2} \quad \text{[Laws of radicals – (ii)]}$$
$$= \left[2\times 2\times 2\times\sqrt[3]{2}\right]^{-2} \quad \text{[Laws of radicals – (i)]}$$
$$= \left[8\times\sqrt[3]{2}\right]^{-2} = \left[\frac{1}{8}\right]^{2}\times\left(\frac{1}{\sqrt[3]{2}}\right)^{2}$$
$$= \frac{1}{64}\sqrt[3]{\frac{1}{4}}$$

2	1024
2	512
2	256
2	128
2	64
2	32
2	16
2	8
2	4
2	2
	1

order $= 3$; radicand $= \frac{1}{4}$; coefficient $= \frac{1}{64}$

(These results can also be obtained using index notation).

Note

Consider the numbers 5 and 6. As $5 = \sqrt{25}$ and $6 = \sqrt{36}$

Therefore, $\sqrt{26}$, $\sqrt{27}$, $\sqrt{28}$, $\sqrt{29}$, $\sqrt{30}$, $\sqrt{31}$, $\sqrt{32}$, $\sqrt{33}$, $\sqrt{34}$, and $\sqrt{35}$ are surds between 5 and 6.

Consider $3\sqrt{2} = \sqrt{3^2\times 2} = \sqrt{18}$, $2\sqrt{3} = \sqrt{2^2\times 3} = \sqrt{12}$

Therefore, $\sqrt{17}$, $\sqrt{15}$, $\sqrt{14}$, $\sqrt{13}$ are surds between $2\sqrt{3}$ and $3\sqrt{2}$.

2.6.3 Four Basic Operations on Surds

(i) **Addition and subtraction of surds :** Like surds can be added and subtracted using the following rules:

$$a\sqrt[n]{b} \pm c\sqrt[n]{b} = (a\pm c)\sqrt[n]{b}, \text{ where } b > 0.$$

Example 2.22 (i) Add $3\sqrt{7}$ and $5\sqrt{7}$. Check whether the sum is rational or irrational. (ii) Subtract $4\sqrt{5}$ from $7\sqrt{5}$. Is the answer rational or irrational?

Solution

(i) $3\sqrt{7}+5\sqrt{7} = (3+5)\sqrt{7} = 8\sqrt{7}$. The answer is irrational.

(ii) $7\sqrt{5}-4\sqrt{5} = (7-4)\sqrt{5} = 3\sqrt{5}$. The answer is irrational.

Example 2.23 Simplify the following:

(i) $\sqrt{63}-\sqrt{175}+\sqrt{28}$ (ii) $2\sqrt[3]{40}+3\sqrt[3]{625}-4\sqrt[3]{320}$

Solution

$$\begin{aligned}\text{(i) } \sqrt{63}-\sqrt{175}+\sqrt{28} &= \sqrt{9\times 7}-\sqrt{25\times 7}+\sqrt{4\times 7}\\ &= 3\sqrt{7}-5\sqrt{7}+2\sqrt{7}\\ &= \left(3\sqrt{7}+2\sqrt{7}\right)-5\sqrt{7}\\ &= 5\sqrt{7}-5\sqrt{7}\\ &= 0\end{aligned}$$

$$\begin{aligned}&\text{(ii) } 2\sqrt[3]{40}+3\sqrt[3]{625}-4\sqrt[3]{320}\\ &= 2\sqrt[3]{8\times 5}+3\sqrt[3]{125\times 5}-4\sqrt[3]{64\times 5}\\ &= 2\sqrt[3]{2^3\times 5}+3\sqrt[3]{5^3\times 5}-4\sqrt[3]{4^3\times 5}\\ &= 2\times 2\sqrt[3]{5}+3\times 5\sqrt[3]{5}-4\times 4\sqrt[3]{5}\\ &= 4\sqrt[3]{5}+15\sqrt[3]{5}-16\sqrt[3]{5}\\ &= (4+15-16)\sqrt[3]{5} = 3\sqrt[3]{5}\end{aligned}$$

(ii) Multiplication and division of surds

Like surds can be multiplied or divided by using the following rules:

Multiplication property of surds	Division property of surds
(i) $\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab}$	(iii) $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\dfrac{a}{b}}$
(ii) $a\sqrt[n]{b}\times c\sqrt[n]{d}=ac\sqrt[n]{bd}$ where $b,d>0$	(iv) $\dfrac{a\sqrt[n]{b}}{c\sqrt[n]{d}}=\dfrac{a}{c}\sqrt[n]{\dfrac{b}{d}}$ where $b,d>0$

Example 2.24

Multiply $\sqrt[3]{40}$ and $\sqrt[3]{16}$.

Solution

$$\sqrt[3]{40} \times \sqrt[3]{16} = \left(\sqrt[3]{2\times2\times2\times5}\right) \times \left(\sqrt[3]{2\times2\times2\times2}\right)$$

$$= \left(2\times\sqrt[3]{5}\right)\times\left(2\times\sqrt[3]{2}\right) = 4\times\left(\sqrt[3]{2}\times\sqrt[3]{5}\right) = 4\times\sqrt[3]{2\times5}$$

$$= 4\sqrt[3]{10}$$

Example 2.25

Compute and give the answer in the simplest form: $2\sqrt{72}\times5\sqrt{32}\times3\sqrt{50}$

Solution

$$2\sqrt{72}\times5\sqrt{32}\times3\sqrt{50} = \left(2\times6\sqrt{2}\right)\times\left(5\times4\sqrt{2}\right)\times\left(3\times5\sqrt{2}\right)$$

$$= 2\times5\times3\times6\times4\times5\times\sqrt{2}\times\sqrt{2}\times\sqrt{2}$$

$$= 3600\times2\sqrt{2}$$

$$= 7200\sqrt{2}$$

Let us simplify:

$\sqrt{72} = \sqrt{36\times2} = 6\sqrt{2}$

$\sqrt{32} = \sqrt{16\times2} = 4\sqrt{2}$

$\sqrt{50} = \sqrt{25\times2} = 5\sqrt{2}$

Example 2.26

Divide $\sqrt[9]{8}$ by $\sqrt[6]{6}$.

Solution

$$\frac{\sqrt[9]{8}}{\sqrt[6]{6}} = \frac{8^{\frac{1}{9}}}{6^{\frac{1}{6}}} \text{ (Note that 18 is the LCM of 6 and 9)}$$

$$= \frac{8^{\frac{2}{18}}}{6^{\frac{3}{18}}} \text{ (How?)}$$

$$= \left(\frac{8^2}{6^3}\right)^{\frac{1}{18}} \text{ (How ?) } = \left(\frac{8\times8}{6\times6\times6}\right)^{\frac{1}{18}}$$

$$= \left(\frac{8}{27}\right)^{\frac{1}{18}} = \left[\left(\frac{2}{3}\right)^3\right]^{\frac{1}{18}} = \left(\frac{2}{3}\right)^{\frac{1}{6}} = \sqrt[6]{\frac{2}{3}}$$

Activity - 1

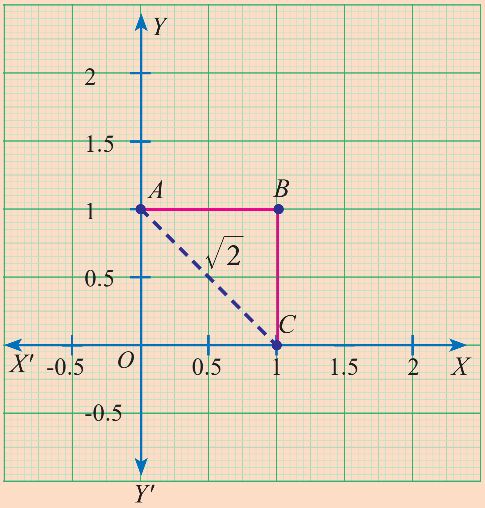
Is it interesting to see this pattern ?

$$\sqrt{4\frac{4}{15}} = 4\sqrt{\frac{4}{15}} \text{ and } \sqrt{5\frac{5}{24}} = 5\sqrt{\frac{5}{24}}$$

Verify it. Can you frame 4 such new surds?

Activity - 2

Take a graph sheet and mark O, A, B, C as follows.



In the square $OABC$,

$$OA = AB = BC = OC = 1\text{unit}$$

Consider right angled ΔOAC

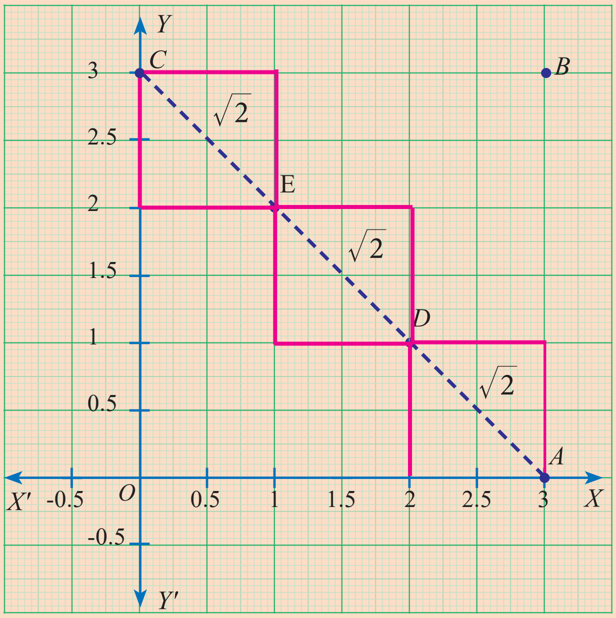
$$AC = \sqrt{1^2 + 1^2}$$

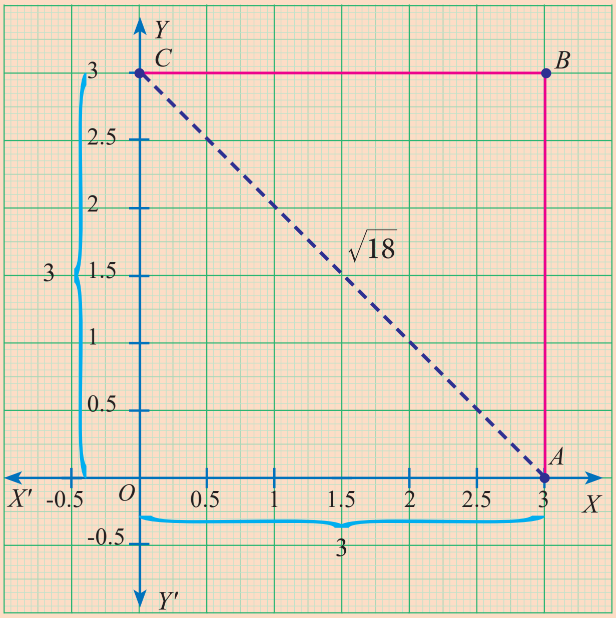
$$= \sqrt{2} \text{ unit} \quad \text{[By Pythagoras theorem]}$$

The length of the diagonal (hypotenuse)

$AC = \sqrt{2}$, which is a surd.

Consider the following graphs:





Let us try to find the length of AC in two different ways :

$$AC = AD + DE + EC$$

(diagonals of units squares)

$$= \sqrt{2} + \sqrt{2} + \sqrt{2}$$

$$AC = 3\sqrt{2} \text{ units}$$

$$AC = \sqrt{OA^2 + OC^2} = \sqrt{3^2 + 3^2}$$

$$= \sqrt{9 + 9}$$

$$AC = \sqrt{18} \text{ units}$$

Are they equal? Discuss. Can you verify the same by taking different squares of different lengths?

Exercise 2.6

1. Simplify the following using addition and subtraction properties of surds:

   (i) $5\sqrt{3}+18\sqrt{3}-2\sqrt{3}$ (ii) $4\sqrt[3]{5}+2\sqrt[3]{5}-3\sqrt[3]{5}$

   (iii) $3\sqrt{75}+5\sqrt{48}-\sqrt{243}$ (iv) $5\sqrt[3]{40}+2\sqrt[3]{625}-3\sqrt[3]{320}$

2. Simplify the following using multiplication and division properties of surds:

   (i) $\sqrt{3} \times \sqrt{5} \times \sqrt{2}$ (ii) $\sqrt{35} \div \sqrt{7}$ (iii) $\sqrt[3]{27} \times \sqrt[3]{8} \times \sqrt[3]{125}$

   (iv) $\left(7\sqrt{a}-5\sqrt{b}\right)\left(7\sqrt{a}+5\sqrt{b}\right)$ (v) $\left[\sqrt{\frac{225}{729}}-\sqrt{\frac{25}{144}}\right] \div \sqrt{\frac{16}{81}}$

3. If $\sqrt{2}=1.414,\ \sqrt{3}=1.732, \sqrt{5}=2.236, \sqrt{10}=3.162$, then find the values of the following correct to 3 places of decimals.

   (i) $\sqrt{40}-\sqrt{20}$ (ii) $\sqrt{300}+\sqrt{90}-\sqrt{8}$

4. Arrange surds in descending order : (i) $\sqrt[3]{5}, \sqrt[9]{4}, \sqrt[6]{3}$ (ii) $\sqrt[2]{\sqrt[3]{5}}, \sqrt[3]{\sqrt[4]{7}}, \sqrt{\sqrt{3}}$

5. Can you get a pure surd when you find

   (i) the sum of two surds (ii) the difference of two surds

   (iii) the product of two surds (iv) the quotient of two surds

   Justify each answer with an example.

6. Can you get a rational number when you compute

   (i) the sum of two surds (ii) the difference of two surds

   (iii) the product of two surds (iv) the quotient of two surds

   Justify each answer with an example.

2.7 Rationalisation of Surds

Rationalising factor is a term with which a term is multiplied or divided to make the whole term rational.

Examples:

(i) $\sqrt{3}$ is a rationalising factor of $\sqrt{3}$ (since $\sqrt{3}\times\sqrt{3}=$ the rational number 3)

(ii) $\sqrt[7]{5^4}$ is a rationalising factor of $\sqrt[7]{5^3}$ (since their product $=\sqrt[7]{5^7}=5$, a rational)

Thinking Corner

1. In the example (i) above, can $\sqrt{12}$ also be a rationalising factor? Can you think of any other number as a rationalising factor for $\sqrt{3}$?
2. Can you think of any other number as a rationalising factor for $\sqrt[7]{5^3}$ in example (ii) ?
3. If there can be many rationalising factors for an expression containing a surd, is there any advantage in choosing the smallest among them for manipulation?

Progress Check

Identify a rationalising factor for each one of the following surds and verify the same in each case:

(i) $\sqrt{18}$ (ii) $5\sqrt{12}$ (iii) $\sqrt[3]{49}$ (iv) $\dfrac{1}{\sqrt{8}}$

2.7.1 Conjugate Surds

Can you guess a rationalising factor for $3+\sqrt{2}$? This surd has one rational part and one radical part. In such cases, the rationalising factor has an interesting form.

A rationalising factor for $3+\sqrt{2}$ is $3-\sqrt{2}$. You can very easily check this.

$$\left(3+\sqrt{2}\right)\left(3-\sqrt{2}\right) = 3^2-\left(\sqrt{2}\right)^2$$
$$= 9-2$$
$$= 7, \text{ a rational.}$$

What is the rationalising factor for $a+\sqrt{b}$ where a and b are rational numbers? Is it $a-\sqrt{b}$? Check it. What could be the rationalising factor for $\sqrt{a}+\sqrt{b}$ where a and b are rational numbers? Is it $\sqrt{a}-\sqrt{b}$? Or, is it $-\sqrt{a}+\sqrt{b}$? Investigate.

Surds like $a+\sqrt{b}$ and $a-\sqrt{b}$ are called conjugate surds. What is the conjugate of $\sqrt{b}+a$? It is $-\sqrt{b}+a$. You would have perhaps noted by now that a conjugate is usually obtained by changing the sign in front of the surd!

Example 2.27 Rationalise the denominator of (i) $\dfrac{7}{\sqrt{14}}$ (ii) $\dfrac{5+\sqrt{3}}{5-\sqrt{3}}$

Solution

(i) Multiply both numerator and denominator by the rationalising factor $\sqrt{14}$.

$$\frac{7}{\sqrt{14}} = \frac{7}{\sqrt{14}}\times\frac{\sqrt{14}}{\sqrt{14}} = \frac{7\sqrt{14}}{14} = \frac{\sqrt{14}}{2}$$

(ii)
$$\frac{5+\sqrt{3}}{5-\sqrt{3}} = \frac{\left(5+\sqrt{3}\right)}{\left(5-\sqrt{3}\right)}\times\frac{\left(5+\sqrt{3}\right)}{\left(5+\sqrt{3}\right)} = \frac{\left(5+\sqrt{3}\right)^2}{5^2-\left(\sqrt{3}\right)^2}$$
$$= \frac{5^2+\left(\sqrt{3}\right)^2+2\times5\times\sqrt{3}}{25-3}$$
$$= \frac{25+3+10\sqrt{3}}{22} = \frac{28+10\sqrt{3}}{22} = \frac{2\times[14+5\sqrt{3}]}{22}$$
$$= \frac{14+5\sqrt{3}}{11}$$

Exercise 2.7

1. Rationalise the denominator

(i) $\frac{1}{\sqrt{50}}$ (ii) $\frac{5}{3\sqrt{5}}$ (iii) $\frac{\sqrt{75}}{\sqrt{18}}$ (iv) $\frac{3\sqrt{5}}{\sqrt{6}}$

2. Rationalise the denominator and simplify

(i) $\frac{\sqrt{48}+\sqrt{32}}{\sqrt{27}-\sqrt{18}}$ (ii) $\frac{5\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

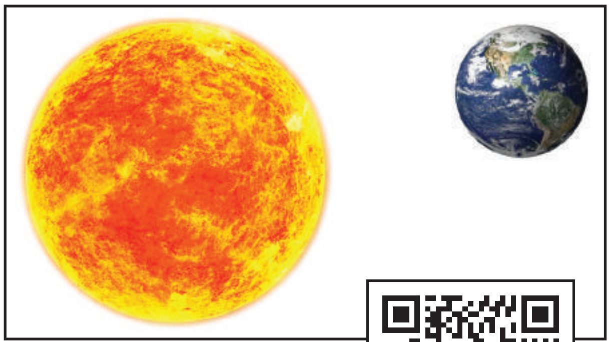
(iii) $\frac{2\sqrt{6}-\sqrt{5}}{3\sqrt{5}-2\sqrt{6}}$ (iv) $\frac{\sqrt{5}}{\sqrt{6}+2}-\frac{\sqrt{5}}{\sqrt{6}-2}$

3. Find the value of a and b if $\frac{\sqrt{7}-2}{\sqrt{7}+2}=a\sqrt{7}+b$

4. If $x=\sqrt{5}+2$, then find the value of $x^2+\frac{1}{x^2}$

5. Given $\sqrt{2}=1.414$, find the value of $\frac{8-5\sqrt{2}}{3-2\sqrt{2}}$ (to 3 places of decimals).

2.8 Scientific Notation



Suppose you are told that the diameter of the Sun is 13,92,000 km and that of the Earth is 12,740 km, it would seem to be a daunting task to compare them. In contrast, if 13,92,000 is written as 1.392×10^6 and 12,740 as 1.274×10^4, one will feel comfortable. This sort of representation is known as scientific notation.

BBZAX5

Since $\frac{1.392 \times 10^6}{1.274 \times 10^4} \approx \frac{14}{13} \times 10^2 \approx 108$.

You can imagine 108 Earths could line up across the face of the sun.

Scientific notation is a way of representing numbers that are too large or too small, to be conveniently written in decimal form. It allows the numbers to be easily recorded and handled.

2.8.1 Writing a Decimal Number in Scientific Notation

Here are steps to help you to represent a number in scientific form:

(i) Move the decimal point so that there is only one non-zero digit to its left.

(ii) Count the number of digits between the old and new decimal point. This gives 'n', the power of 10.

(iii) If the decimal is shifted to the left, the exponent n is positive. If the decimal is shifted to the right, the exponent n is negative.

Expressing a number N in the form of $N = a \times 10^n$ where, $1 \leq a < 10$ and 'n' is an integer is called as Scientific Notation.

The following table of base 10 examples may make things clearer:

Decimal notation	Scientific notation
100	1×10^2
1,000	1×10^3
10,000	1×10^4
1,00,000	1×10^5
10,00,000	1×10^6
1,00,00,000	1×10^7

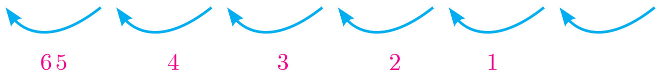
Decimal notation	Scientific notation
0.01	1×10^{-2}
0.001	1×10^{-3}
0.0001	1×10^{-4}
0.00001	1×10^{-5}
0.000001	1×10^{-6}
0.0000001	1×10^{-7}

Let us look into few more examples.

Example 2.28 Express in scientific notation (i) 9768854 (ii) 0.04567891 (iii) 72006865.48

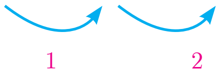
Solution

(i) 9 7 6 8 8 5 4 . 0 $= 9.768854 \times 10^6$



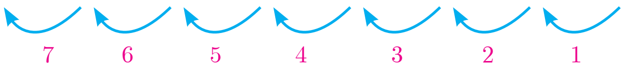
The decimal point is to be moved six places to the left. Therefore $n = 6$.

(ii) 0 . 0 4 5 6 7 8 9 1 $= 4.567891 \times 10^{-2}$



The decimal point is to be moved two places to the right. Therefore $n = -2$.

(iii) 7 2 0 0 6 8 6 5 . 4 8 $= 7.200686548 \times 10^7$



The decimal point is to be moved seven places to the left. Therefore $n = 7$.

2.8.2 Converting Scientific Notation to Decimal Form

The reverse process of converting a number in scientific notation to the decimal form is easily done when the following steps are followed:

(i) Write the decimal number.

(ii) Move the decimal point by the number of places specified by the power of 10, to the right if positive, or to the left if negative. Add zeros if necessary.

(iii) Rewrite the number in decimal form.

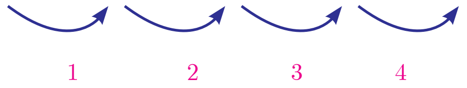
Example 2.29 Write the following numbers in decimal form:

(i) 6.34×10^{4} (ii) 2.00367×10^{-5}

Solution

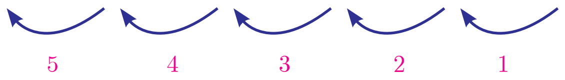
(i)V 6.34×10^{4}

$\Rightarrow$ 6 . 3 4 0 0 = 63400



(ii) 2.00367×10^{-5}

0 0 0 0 0 2 . 0 0 3 6 7



= 0.0000200367

2.8.3 Arithmetic of Numbers in Scientific Notation

(i) If the indices in the scientific notation of two numbers are the same, addition (or subtraction) is easily performed.

Example 2.30 The mass of the Earth is 5.97×10^{24} kg and that of the Moon is 0.073×10^{24} kg. What is their total mass?

Solution

$$\begin{aligned}\text{Total mass} &= 5.97\times10^{24}\text{ kg} + 0.073\times10^{24}\text{ kg}\\ &= (5.97 + 0.073)\times10^{24}\text{ kg}\\ &= 6.043\times10^{24}\text{ kg}\end{aligned}$$

(ii) The product or quotient of numbers in scientific notation can be easily done if we make use of the laws of radicals appropriately.

Example 2.31 Write the following in scientific notation :

(i) $(50000000)^{4}$ (ii) $(0.00000005)^{3}$

(iii) $(300000)^{3} \times (2000)^{4}$ (iv) $(4000000)^{3} \div (0.00002)^{4}$

Solution

(i) $(50000000)^4 = (5.0\times10^7)^4$

$$= (5.0)^4\times(10^7)^4$$

$$= 625.0\times10^{28}$$

$$= 6.25\times10^2\times10^{28}$$

$$= 6.25\times10^{30}$$

(ii) $(0.00000005)^3 = (5.0\times10^{-8})^3$

$$= (5.0)^3\times(10^{-8})^3$$

$$= (125.0)\times(10)^{-24}$$

$$= 1.25\times10^2\times10^{-24}$$

$$= 1.25\times10^{-22}$$

(iii) $(300000)^3\times(2000)^4$

$$= (3.0\times10^5)^3\times(2.0\times10^3)^4$$

$$= (3.0)^3\times(10^5)^3\times(2.0)^4\times(10^3)^4$$

$$= (27.0)\times(10^{15})\times(16.0)\times(10^{12})$$

$$= (2.7\times10^1)\times(10^{15})\times(1.6\times10^1)\times(10^{12})$$

$$= 2.7\times1.6\times10^1\times10^{15}\times10^1\times10^{12}$$

$$= 4.32\times10^{1+15+1+12} \quad = 4.32\times10^{29}$$

(iv) $(4000000)^3\div(0.00002)^4$

$$= (4.0\times10^6)^3\div(2.0\times10^{-5})^4$$

$$= (4.0)^3\times(10^6)^3\div(2.0)^4\times(10^{-5})^4$$

$$= \frac{64.0\times10^{18}}{16.0\times10^{-20}}$$

$$= 4\times10^{18}\times10^{+20}$$

$$= 4.0\times10^{38}$$

Thinking Corner

1. Write two numbers in scientific notation whose product is 2. 83104.
2. Write two numbers in scientific notation whose quotient is 2. 83104.

Exercise 2.8

1. Represent the following numbers in the scientific notation:
   (i) 569430000000 (ii) 2000.57
   (iii) 0.0000006000 (iv) 0.0009000002
2. Write the following numbers in decimal form:
   (i) 3.459×10^6 (ii) 5.678×10^4
   (iii) 1.00005×10^{-5} (iv) 2.530009×10^{-7}
3. Represent the following numbers in scientific notation:
   (i) $(300000)^2\times(20000)^4$ (ii) $(0.000001)^{11}\div(0.005)^3$
   (iii) $\{(0.00003)^6\times(0.00005)^4\}\div\{(0.009)^3\times(0.05)^2\}$

4. Represent the following information in scientific notation:

   (i) The world population is nearly 7000,000,000.

   (ii) One light year means the distance 9460528400000000 km.

   (iii) Mass of an electron is 0.000 000 000 000 000 000 000 000 000 00091093822 kg.

5. Simplify:

   (i) $(2.75 \times 10^{7}) + (1.23 \times 10^{8})$ (ii) $(1.598\times10^{17}) - (4.58 \times10^{15})$

   (iii) $(1.02 \times 10^{10}) \times (1.20 \times 10^{-3})$ (iv) $(8.41 \times 10^{4}) \div (4.3 \times 10^{5})$

Activity - 3

The following list shows the mean distance of the planets of the solar system from the Sun. Complete the following table. Then arrange in order of magnitude starting with the distance of the planet closest to the Sun.

Planet	Decimal form (in Km)	Scientific Notation (in Km)
Jupiter		7.78×10^{8}
Mercury	58000000	
Mars		2.28×10^{8}
Uranus	2870000000	
Venus	108000000	
Neptune	4500000000	
Earth		1.5×10^{8}
Saturn		1.43×10^{8}

Exercise 2.9

BC3XL2

Multiple Choice Questions

1. If n is a natural number then $\sqrt{n}$ is

   (1) always a natural number. (2) always an irrational number.

   (3) always a rational number (4) may be rational or irrational

2. Which of the following is not true?.

   (1) Every rational number is a real number. (2) Every integer is a rational number.

   (3) Every real number is an irrational number. (4) Every natural number is a whole number.

3. Which one of the following, regarding sum of two irrational numbers, is true?

   (1) always an irrational number. (2) may be a rational or irrational number.

   (3) always a rational number. (4) always an integer.

4. Which one of the following has a terminating decimal expansion?.

(1) $\frac{5}{64}$ (2) $\frac{8}{9}$ (3) $\frac{14}{15}$ (4) $\frac{1}{12}$

5. Which one of the following is an irrational number

(1) $\sqrt{25}$ (2) $\sqrt{\frac{9}{4}}$ (3) $\frac{7}{11}$ (4) π

6. An irrational number between 2 and 2.5 is

(1) $\sqrt{11}$ (2) $\sqrt{5}$ (3) $\sqrt{2.5}$ (4) $\sqrt{8}$

7. The smallest rational number by which $\frac{1}{3}$ should be multiplied so that its decimal expansion terminates with one place of decimal is

(1) $\frac{1}{10}$ (2) $\frac{3}{10}$ (3) 3 (4) 30

8. If $\frac{1}{7} = 0.\overline{142857}$ then the value of $\frac{5}{7}$ is

(1) $0.\overline{142857}$ (2) $0.\overline{714285}$ (3) $0.\overline{571428}$ (4) 0.714285

9. Find the odd one out of the following.

(1) $\sqrt{32} \times \sqrt{2}$ (2) $\frac{\sqrt{27}}{\sqrt{3}}$ (3) $\sqrt{72} \times \sqrt{8}$ (4) $\frac{\sqrt{54}}{\sqrt{18}}$

10. $0.\overline{34} + 0.3\overline{4} =$

(1) $0.6\overline{87}$ (2) $0.\overline{68}$ (3) $0.6\overline{8}$ (4) $0.68\overline{7}$

11. Which of the following statement is false?

(1) The square root of 25 is 5 or −5 (3) $\sqrt{25} = 5$

(2) $-\sqrt{25} = -5$ (4) $\sqrt{25} = \pm 5$

12. Which one of the following is not a rational number?

(1) $\sqrt{\frac{8}{18}}$ (2) $\frac{7}{3}$ (3) $\sqrt{0.01}$ (4) $\sqrt{13}$

13. $\sqrt{27} + \sqrt{12} =$

(1) $\sqrt{39}$ (2) $5\sqrt{6}$ (3) $5\sqrt{3}$ (4) $3\sqrt{5}$

14. If $\sqrt{80} = k\sqrt{5}$, then $k =$

(1) 2 (2) 4 (3) 8 (4) 16

15. $4\sqrt{7} \times 2\sqrt{3} =$

(1) $6\sqrt{10}$ (2) $8\sqrt{21}$ (3) $8\sqrt{10}$ (4) $6\sqrt{21}$

16. When written with a rational denominator, the expression $\frac{2\sqrt{3}}{3\sqrt{2}}$ can be simplified as

(1) $\frac{\sqrt{2}}{3}$ (2) $\frac{\sqrt{3}}{2}$ (3) $\frac{\sqrt{6}}{3}$ (4) $\frac{2}{3}$

17. When $\left(2\sqrt{5}-\sqrt{2}\right)^2$ is simplified, we get

(1) $4\sqrt{5}+2\sqrt{2}$ (2) $22-4\sqrt{10}$ (3) $8-4\sqrt{10}$ (4) $2\sqrt{10}-2$

18. $\left(0.000729\right)^{\frac{-3}{4}} \times \left(0.09\right)^{\frac{-3}{4}} =$ ______

(1) $\frac{10^3}{3^3}$ (2) $\frac{10^5}{3^5}$ (3) $\frac{10^2}{3^2}$ (4) $\frac{10^6}{3^6}$

19. If $\sqrt{9^x} = \sqrt[3]{9^2}$, then $x =$ ______

(1) $\frac{2}{3}$ (2) $\frac{4}{3}$ (3) $\frac{1}{3}$ (4) $\frac{5}{3}$

20. The length and breadth of a rectangular plot are 5×10^5 and 4×10^4 metres respectively. Its area is ______.

(1) $9\times10^1\ m^2$ (2) $9\times10^9\ m^2$ (3) $2\times10^{10}\ m^2$ (4) $20\times10^{20}\ m^2$

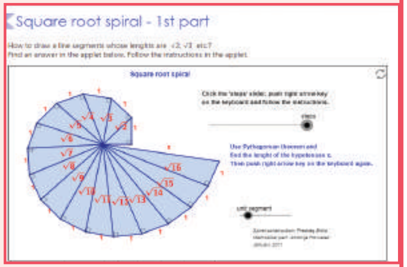
Points to Remember

- When the decimal expansion of $\frac{p}{q}$, q≠0 terminates that is, comes to an end, the decimal is called a terminating decimal.
- In the decimal expansion of $\frac{p}{q}$, q≠0 when the remainder is not zero, we have a repeating (recurring) block of digits in the quotient. In this case, the decimal expansion is called non-terminating and recurring.
- If a rational number $\frac{p}{q}$, q≠ 0 can be expressed in the form $\frac{p}{2^m \times 5^n}$, where $p \in \mathbb{Z}$ and $m, n \in \mathbb{W}$, then the rational number will have a terminating decimals. Otherwise, the rational number will have a non-terminating repeating (recurring) decimal.
- A rational number can be expressed either a terminating or a non- terminating recurring decimal.
- An irrational number is a non-terminating and non-recuring decimal, i.e. it cannot be written in form $\frac{p}{q}$, where p and q are both integers and q≠ 0.
- The union of all rational numbers and all irrational numbers is called the set of real numbers.
- Every real number is either a rational number or an irratonal number.
- If a real number is not rational number, then it must be an irrational number.
- If 'a' is a positive rational number, 'n' is a positive integer and if $\sqrt[n]{a}$ is an irrational number, then $\sqrt[n]{a}$ is called as a surd.
- If 'm', 'n' are positive integers and a, b are positive rational numbers, then

$$\text{(i)}\ \left(\sqrt[n]{a}\right)^n = a = \sqrt[n]{a^n} \quad \text{(ii)}\ \sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab} \quad \text{(iii)}\ \sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} = \sqrt[n]{\sqrt[m]{a}} \quad \text{(iv)}\ \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

- The process of multiplying a surd by another surd to get a rational number is called Rationalisation.
- Expressing a number N in the form of $N = a \times 10^n$ where, $1 \le a < 10$ and 'n' is an integer is called as Scientific Notation.

ICT Corner-1



Square root spiral - 1st part

How to draw a line segments whose lengths are √2, √3 etc?
Find an answer in the applet below. Follow the instructions in the applet.

Square root spiral

Click the 'steps' slider; push right arrow key on the keyboard and follow the instructions.

Use Pythagorean theorem and find the length of the hypotenuse c. Then push right arrow key on the keyboard again.

Expected Result is shown in this picture

Step – 1

Open the Browser and copy and paste the Link given below (or) by typing the URL given (or) Scan the QR Code.

Step - 2

GeoGebra workbook named "Real Numbers" will open. There are several worksheets in the workbook. Open the worksheet named "Square root spiral – 1st part"

Step-3

Drag the slider named "Steps ". The construction of Square root of numbers 2,3,4,5,…. will appear step by step.

Step-4

By dragging the slider named "Unit segment" you can enlarge the diagram for more clarity. Now you can draw the same in a paper and measure the values obtained

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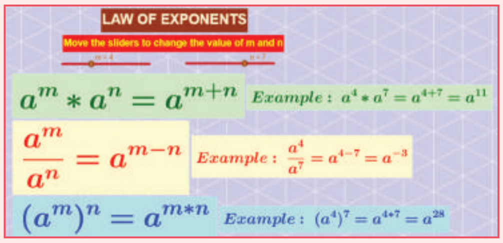
ICT Corner-2

Expected Result is shown in this picture



LAW OF EXPONENTS

Move the sliders to change the value of m and n

$a^m * a^n = a^{m+n}$ Example : $a^4 * a^7 = a^{4+7} = a^{11}$

$\frac{a^m}{a^n} = a^{m-n}$ Example : $\frac{a^4}{a^7} = a^{4-7} = a^{-3}$

$(a^m)^n = a^{m*n}$ Example : $(a^4)^7 = a^{4*7} = a^{28}$

Step – 1

Open the Browser type the URL Link given below (or) Scan the QR Code. GeoGebra work sheet named "Real Numbers" will open. In the work sheet there are two activities. 1. Rationalising the denominator for surds and 2. Law of exponents.

In the first activity procedure for rationalising the denominator is given. Also, example is given under. To change the values of a and b enter the value in the input box given.

Step - 2

In the second activity law of exponents is given. Also, example is given on right side. To change the value of m and n move the sliders and check the answers.

Scan the QR Code.

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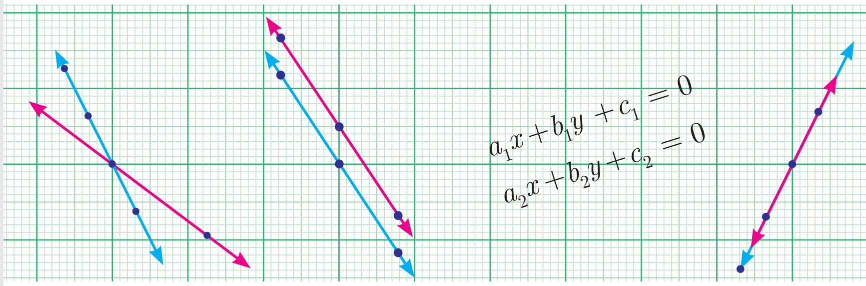
3



ALGEBRA

> *In real life, I assure you, there is no such things as Algebra.*
>
> *- Fran Lebowitz*

Diophantus

of Alexandria an Alexandrian Hellenistic mathematician who lived for about 84 years, was born between A.D(C.E) 201 and A.D(C.E) 215. Diophantus was the author of a series of books called Arithmetica. His texts deal with solving algebraic equations. He is also called as “the father of algebra”.

Learning Outcomes

- To understand the classification of polynomials and to perform basic operations.
- To evaluate the value of a polynomial and understand the zeros of polynomial.
- To understand the remainder and factor theorems.
- To use Algebraic Identities in factorisation.
- To factorise a quadratic and a cubic polynomial.
- To use synthetic division to factorise a polynomial.
- To find GCD of polynomials.
- Able to draw graph for a given linear equation.
- To solve simultaneous linear equations in two variables by Graphical method and Algebraic method
- To understand consistency and inconsistency of linear equations in two variables.

E5C225

3.1 Introduction

Why study polynomials?

This chapter is going to be all about polynomial expressions in algebra. These are your friends, you have already met, without being properly introduced! We will properly introduce them to you, and they are going to be your friends in whatever mathematical journey you undertake from here on.

$(a+1)^{2} = a^2 + 2a + 1$

Now that's a polynomial. That does not look very special, does it? We have seen a lots of algebraic expressions already, so why to bother about these? There are many reasons why polynomials are interesting and important in mathematics.

For now, we will just take one example showing their use. Remember, we studied lots of arithmetic and then came to algebra, thinking of variables as unknown numbers. Actually we can now get back to numbers and try to write them in the language of algebra.

Consider a number like 5418. It is actually 5 thousand 4 hundred and eighteen. Write it as:

$5 \times 1000 + 4 \times 100 + 1 \times 10 + 8$

which again can be written as:

$5 \times 10^3 + 4 \times 10^2 + 1 \times 10^1 + 8$

Now it should be clear what this is about. This is of the form $5x^3 + 4x^2 + x + 8$, which is a polynomial. How does writing in this form help? We always write numbers in decimal system, and hence always $x = 10$. Then what is the fun? Remember divisibility rules? Recall that a number is divisible by 3 only if the sum of its digits is divisible by 3. Now notice that if x divided by 3 gives 1 as remainder, then it is the same for x^2, x^3, etc. They all give remainder 1 when divided by 3. So you get each digit multiplied by 1, added together, which is the sum of digits. If that is divisible by 3, so is the whole number. You can check that the rule for divisibility by 9, or even divisibility by 2 or 5, can be proved similarly with great ease. Our purpose is not to prove divisibility rules but to show that representing numbers as polynomials shows us many new number patterns. In fact, many objects of study, not just numbers, can be represented as polynomials and then we can learn many things about them.

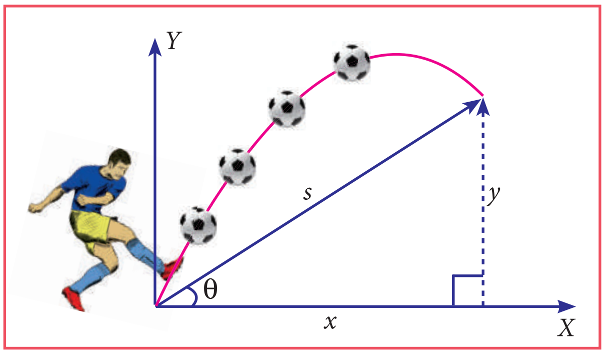


Fig. 3.1

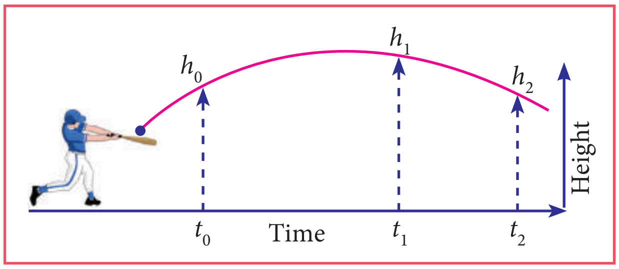


Fig. 3.2

In algebra we think of x^2, $5x^2-3$, $2x+7$ etc as *functions* of x. We draw pictures to see how the function varies as x varies, and this is very helpful to *understand* the function. And now, it turns out that a good number of functions that we encounter in science, engineering, business studies, economics, and of course in mathematics, all can be approximated by polynomials, if not actually be represented as polynomials. In fact, approximating functions using polynomials is a fundamental theme in all of higher mathematics and a large number of people make a living, simply by working on this idea.

Polynomials are extensively used in biology, computer science, communication systems ... the list goes on. The given pictures (Fig. 3.1, 3.2 & 3.3) may be repsresented as a quadratic polynomial. We will not only learn what polynomials are but also how we can use them like in numbers, we add them, multiply them, divide one by another, etc.,

Fig. 3.3

Observe the given figures.

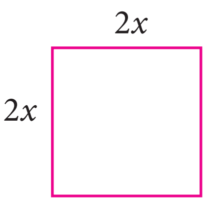


Fig. 3.4

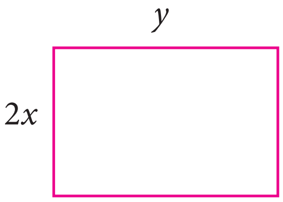


Fig. 3.5

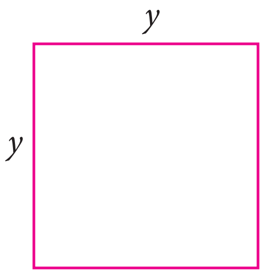


Fig. 3.6

The total area of the above 3 figures is $4x^2 + 2xy + y^2$, we call this expression as an algebraic expression. Here for different values of x and y we get different values of areas. Since the sides x and y can have different values, they are called variables. Thus, a variable is a symbol which can have various numerical values.

Variables are usually denoted by letters such as x, y, z, etc. In the above algebraic expression the numbers 4, 2 are called constants. Hence the constant is a symbol, which has a fixed numeric value.

Constants

Any real number is a constant. We can form numerical expressions using constants and the four arithmetical operations.

Examples of constant are 1, 5, −32, $\frac{3}{7}$, $-\sqrt{2}$, 8.432, 1000000 and so on.

Variables

The use of variables and constants together in expressions give us ways of representing a range of numbers, one for each value of the variable. For instance, we know the expression $2\pi r$, it stands for the circumference of a circle of radius r. As we vary r, say, 1cm, 4cm, 9cm etc, we get larger and larger circles of circumference 2π, 8π, 18π etc.,

The single expression $2\pi r$ is a short and compact description for the circumference of all these circles. We can use arithmetical operations to combine algebraic expressions and get a rich language of functions and numbers. Letters used for representing unknown real numbers called variables are x, y, a, b and so on.

Algebraic Expression

An **algebraic expression** is a combination of constants and variables combined together with the help of the four fundamental signs.

Examples of algebraic expression are

$$x^3 - 4x^2 + 8x - 1,\ 4xy^2 + 3x^2y - \frac{5}{4}xy + 9,\ 5x^2 - 7x + 6$$

Coefficients

Any part of a term that is multiplied by the variable of the term is called the coefficient of the remaining term.

For example,

$x^2 + 5x - 24$ is an algebraic expression containing three terms. The variable of this expression is x, coefficient of x^2 is 1, the coefficient of x is 5 and the constant is –24 (not 24).

Activity-1

Write the Variable, Coefficient and Constant in the given algebraic expression

Expression	$x + 7$	$3y - 2$	$5x^2$	$2xy + 11$	$-\frac{1}{2}p + 7$	$-8 + 3a$
Variable	x			x, y		
Coefficient	1				$-\frac{1}{2}$	
Constant	7					–8

3.2 Polynomials

A polynomial is an arithmetic expression consisting of variables and constants that involves four fundamental arithmetic operations and non-negative integer exponents of variables.

Polynomial in One Variable

An algebraic expression of the form $p(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ is called Polynomial in one variable x of degree 'n' where $a_0, a_1, a_2, \ldots a_n$ are constants $(a_n \neq 0)$ and n is a whole number.

In general polynomials are denoted by $f(x), g(x), p(t), q(z)$ and $r(x)$ and so on.

Note

The coefficient of variables in the algebraic expression may have any real numbers, where as the powers of variables in polynomial must have only non-negative integral powers that is, only whole numbers. Recall that $a^0 = 1$ for all a.

For example,

S.No	Given expression	Polynomial / not a polynomial	Reason
1	$4y^3 + 2y^2 + 3y + 6$	Polynomial	Non- negative integral power
2	$4x^{-4} + 5x^4$	Not a polynomial	One of the powers is negative (–4)
3	$m^2 + \frac{4}{5}m + 8$	Polynomial	Non- negative integral power
4	$\sqrt{5}\,y^2$	Polynomial	Non- negative integral power
5	$2r^2 + 3r - 1 + \frac{1}{r}$	Not a polynomial	One of the power is negative $\left(\frac{1}{r} = r^{-1}\right)$
6	$8 + \sqrt{q}$	Not a polynomial	power of q is fraction $(\sqrt{q} = q^{\frac{1}{2}})$
7	$\sqrt{8}\,p^2 + 5p - 7$	Polynomial	Non- negative integral power
8	$5n^{\frac{4}{5}} + 6n - 1$	Not a polynomial	One of the power of n is a fraction $\frac{4}{5}$

Standard Form of a Polynomial

We can write a polynomial $p(x)$ in the decreasing or increasing order of the powers of x. This way of writing the polynomial is called the standard form of a polynomial.

Activity-2

Write the following polynomials in standard form.

Sl.No.	Polynomial	Standard Form
1	$5m^4 - 3m + 7m^2 + 8$	
2	$\frac{2}{3}y + 8y^3 - 12 + \sqrt{5}\,y^2$	
3	$12p^2 - 8p^5 - 10p^4 - 7$	

For example: (i) $8x^4 + 4x^3 - 7x^2 - 9x + 6$ (ii) $5 - 3y + 6y^2 + 4y^3 - y^4$

Degree of the Polynomial

In a polynomial of one variable, the highest power of the variable is called the **degree of** the polynomial.

In case of a polynomial of more than one variable, the sum of the powers of the variables in each term is considered and the **highest sum** so obtained is called the degree of the polynomial.

This is intended as the most **significant** power of the polynomial. Obviously when we write x^2+5x the value of x^2 becomes much larger than $5x$ for large values of x. So we could think of $x^2 + 5x$ being almost the same as x^2 for large values of x. So the higher the power, the more it dominates. That is why we use the highest power as important information about the polynomial and give it a name.

Example 3.1 Find the degree of each term for the following polynomial and also find the degree of the polynomial $6ab^8 + 5a^2b^3c^2 - 7ab + 4b^2c + 2$

Solution

Given polynomial is $6ab^8 + 5a^2b^3c^2 - 7ab + 4b^2c + 2$

Degree of each of the terms is given below.

$6ab^8$ has degree $(1+8) = 9$

$5a^2b^3c^2$ has degree $(2+3+2) = 7$

$7ab$ has degree $(1+1) = 2$

$4b^2c$ has degree $(2+1) = 3$

The constant term 2 is always regarded as having degree Zero.

The **degree** of the polynomial $6ab^8 + 5a^2b^3c^2 - 7ab + 4b^2c + 2$.

= the largest exponent in the polynomial

$= 9$

A very Special Polynomial

We have said that coefficients can be any real numbers. What if the coefficient is zero? Well that term becomes zero, so we won't write it. What if all the coefficients are zero? We acknowledge that it exists and give it a name.

E5KX4F

It is the polynomial having all its coefficients to be zero.

$$g(t) = 0t^4 + 0t^2 - 0t \ , \quad h(p) = 0p^2 - 0p + 0$$

From the above example we see that we cannot talk of the degree of the zero polynomial, since the above two have different degrees but both are zero polynomial. So we say that the degree of the zero polynomial is not defined.

The degree of the zero polynomial is not defined

Types of Polynomials

(i) Polynomial on the basis of number of terms	
MONOMIAL	A polynomial having one term is called a monomial Examples : 5, 6m, 12ab
BINOMIAL	A polynomial having two terms is called a Binomial Examples : $5x + 3$, $4a - 2$, $10p + 1$
TRINOMIAL	A polynomial having three terms is called a Trinomial Example : $4x^2 + 8x - 12$, $3a^2 + 4a + 10$

(ii) Polynomial based on degree	
CONSTANT	A polynomial of degree zero is called constant polynomial Examples : $5, -7, \frac{2}{3}, \sqrt{5}$
LINEAR	A polynomial of degree one is called linear polynomial Examples : $410x - 7$
QUADRATIC	A polynomial of degree two is called quadratic polynomial Example : $2\sqrt{5}\,x^2 + 8x - 4$
CUBIC	A polynomial of degree three is called cubic polynomial Example : $12y^3$, $6m^3 - 7m + 4$

Example 3.2

Classify the following polynomials based on number of terms.

S.No.	**Polynomial**	**No of Terms**	**Type of polynomial based of terms**
(i)	$5t^3 + 6t + 8t^2$	3 Terms	Trinomial
(ii)	$y–7$	2 Terms	Binomial
(iii)	$\frac{2}{3}r^4$	1 Term	Monomial
(iv)	$6y^5 + 3y - 7$	3 Terms	Trinomial
(v)	$8m^2 + 7m^2$	Like Terms. So, it is $15m^2$ which is 1 term only	Monomial

Example 3.3

Classify the following polynomials based on their degree.

S.No.	**Polynomial**	**Degree**	**Type**
(i)	$\sqrt[3]{4}\,z + 7$	Degree one	Linear polynomial
(ii)	$z^3 - z^2 + 3$	Degree three	Cubic polynomial
(iii)	$\sqrt{7}$	Degree zero	Constant polynomial
(iv)	$y^2 - \sqrt{8}$	Degree two	Quadratic polynomial

3.2.1 Arithmetic of Polynomials

We now have a rich language of polynomials, and we have seen that they can be classified in many ways as well. Now, what can we do with polynomials? Consider a polynomial on x.

We can evaluate the polynomial at a particular value of x. We can ask how the function given by the polynomial changes as x varies. Write the polynomial equation $p(x) = 0$ and solve for x. All this is interesting, and we will be doing plenty of all this as we go along. But there is something else we can do with polynomials, and that is *to treat them like numbers*! We already have a clue to this at the beginning of the chapter when we saw that every positive integer could be represented as a polynomial.

Following arithmetic, we can try to add polynomials, subtract one from another, multiply polynomials, divide one by another. As it turns out, the analogy between numbers and polynomials runs deep, with many interesting properties relating them. For now, it is fun to simply try and define these operations on polynomials and work with them.

Addition of Polynomials

The addition of two polynomials is also a polynomial.

Note

Only like terms can be added. $3x^2 + 5x^2$ gives $8x^2$ but unlike terms such as $3x^2$ and $5x^3$ when added gives $3x^2 + 5x^3$, a new polynomial.

Example 3.4

If $p(x) = 4x^2 - 3x + 2x^3 + 5$ and $q(x) = x^2 + 2x + 4$, then find $p(x) + q(x)$

Solution

Given Polynomial	Standard form
$p(x) = 4x^2 - 3x + 2x^3 + 5$	$2x^3 + 4x^2 - 3x + 5$
$q(x) = x^2 + 2x + 4$	$x^2 + 2x + 4$
$p(x) + q(x) = 2x^3 + 5x^2 - x + 9$	

We see that $p(x) + q(x)$ is also a polynomial. Hence the sum of any two polynomials is also a polynomial.

Subtraction of Polynomials

The subtraction of two polynomials is also a polynomial.

Note Only like terms can be subtracted. $8x^2 - 5x^2$ gives $3x^2$ but when $5x^3$ is subtracted from $3x^2$ we get, $3x^2 - 5x^3$, a new polynomial.

Example 3.5

If $p(x) = 4x^2 - 3x + 2x^3 + 5$ and $q(x) = x^2 + 2x + 4$, then find $p(x) - q(x)$

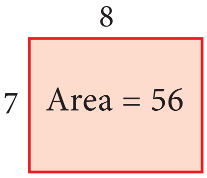
Solution

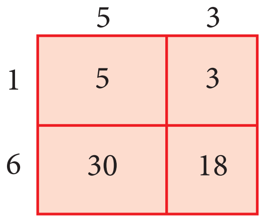
Given Polynomial	Standard form
$p(x) = 4x^2 - 3x + 2x^3 + 5$	$2x^3 + 4x^2 - 3x + 5$
$q(x) = x^2 + 2x + 4$	$x^2 + 2x + 4$
$p(x) - q(x) = 2x^3 + 3x^2 - 5x + 1$	

We see that $p(x) - q(x)$ is also a polynomial. Hence the subtraction of any two polynomials is also a polynomial.

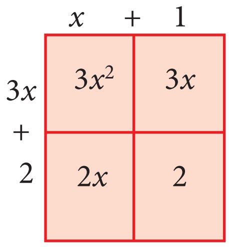
Multiplication of Two Polynomials

Divide a rectangle with 8 units of length and 7 units of breadth into 4 rectangles as shown below, and observe that the area is same, this motivates us to study the multiplication of polynomials.





For example, considering length as $(x+1)$ and breadth as $(3x+2)$ the area of the rectangle can be found by the following way.



Hence the area of the rectangle is

$= 3x^2 + 3x + 2x + 2$

$= 3x^2 + 5x + 2$

If x is a variable and m, n are positive integers then $x^m \times x^n = x^{m+n}$.

When two polynomials are multiplied the product will also be a polynomial.

Example 3.6

Find the product $(4x - 5)$ and $(2x^2 + 3x - 6)$.

Solution

To multiply $(4x - 5)$ and $(2x^2 + 3x - 6)$ distribute each term of the first polynomial to every term of the second polynomial. In this case, we need to distribute the terms $4x$ and -5. Then gather the like terms and combine them:

$$(4x - 5)(2x^2 + 3x - 6) = 4x(2x^2 + 3x - 6) - 5(2x^2 + 3x - 6)$$

$$= 8x^3 + 12x^2 - 24x - 10x^2 - 15x + 30$$

$$= 8x^3 + 2x^2 - 39x + 30$$

Aliter: You may also use the method of detached coefficients:

$$\begin{array}{cccc} & 2 & +3 & -6 \\ & & +4 & -5 \\ \hline & -10 & -15 & +30 \\ 8 & +12 & -24 & \\ \hline 8 & +2 & -39 & +30 \\ \hline \end{array}$$

$\therefore$ $(4x-5)(2x^2+3x-6) = 8x^3+2x^2-39x+30$

Exercise 3.1

1. Which of the following expressions are polynomials. If not give reason:

   (i) $\frac{1}{x^2}+3x-4$ (ii) $x^2(x-1)$

   (iii) $\frac{1}{x}(x+5)$ (iv) $\frac{1}{x^{-2}}+\frac{1}{x^{-1}}+7$

   (v) $\sqrt{5}\,x^2+\sqrt{3}\,x+\sqrt{2}$ (vi) $m^2-\sqrt[3]{m}+7m-10$

2. Write the coefficient of x^2 and x in each of the following polynomials.

   (i) $4+\frac{2}{5}x^2-3x$ (ii) $6-2x^2+3x^3-\sqrt{7}\,x$

   (iii) πx^2-x+2 (iv) $\sqrt{3}\,x^2+\sqrt{2}\,x+0.5$

   (v) $x^2-\frac{7}{2}x+8$

3. Find the degree of the following polynomials.

   (i) $1-\sqrt{2}\,y^2+y^7$ (ii) $\frac{x^3-x^4+6x^6}{x^2}$

   (iii) $x^3(x^2+x)$ (iv) $3x^4+9x^2+27x^6$

   (v) $2\sqrt{5}\,p^4-\frac{8p^3}{\sqrt{3}}+\frac{2p^2}{7}$

4. Rewrite the following polynomial in standard form.

   (i) $x-9+\sqrt{7}\,x^3+6x^2$ (ii) $\sqrt{2}\,x^2-\frac{7}{2}x^4+x-5x^3$

   (iii) $7x^3-\frac{6}{5}x^2+4x-1$ (iv) $y^2+\sqrt{5}\,y^3-11-\frac{7}{3}y+9y^4$

5. Add the following polynomials and find the degree of the resultant polynomial.

   (i) $p(x)=6x^2-7x+2$ $q(x)=6x^3-7x+15$

   (ii) $h(x)=7x^3-6x+1$ $f(x)=7x^2+17x-9$

   (iii) $f(x)=16x^4-5x^2+9$ $g(x)=-6x^3+7x-15$

6. Subtract the second polynomial from the first polynomial and find the degree of the resultant polynomial.

   (i) $p(x)=7x^2+6x-1$ $q(x)=6x-9$

(ii) $f(y) = 6y^2 - 7y + 2$ $g(y) = 7y + y^3$

(iii) $h(z) = z^5 - 6z^4 + z$ $f(z) = 6z^2 + 10z - 7$

7. What should be added to $2x^3 + 6x^2 - 5x + 8$ to get $3x^3 - 2x^2 + 6x + 15$?

8. What must be subtracted from $2x^4 + 4x^2 - 3x + 7$ to get $3x^3 - x^2 + 2x + 1$?

9. Multiply the following polynomials and find the degree of the resultant polynomial:
   (i) $p(x) = x^2 - 9$ $q(x) = 6x^2 + 7x - 2$
   (ii) $f(x) = 7x + 2$ $g(x) = 15x - 9$
   (iii) $h(x) = 6x^2 - 7x + 1$ $f(x) = 5x - 7$

10. The cost of a chocolate is Rs. $(x + y)$ and Amir bought $(x + y)$ chocolates. Find the total amount paid by him in terms of x and y. If x =10, y =5 find the amount paid by him.

11. The length of a rectangle is $(3x+2)$ units and it's breadth is $(3x–2)$ units. Find its area in terms of x. What will be the area if $x = 20$ units.

12. $p(x)$is a polynomial of degree 1 and $q(x)$ is a polynomial of degree 2. What kind of the polynomial $p(x) \times q(x)$ is ?

3.2.2 Value and Zeros of a Polynomial

Consider the two graphs given below. The first is linear, the second is quadratic. The first intersects the X axis at one point ($x = -3$) and the second at two points (x = -1 and x = 2). They both intersect the Y axis only at one point. In general, every polynomial has a graph and the graph is shown as a picture (since we all like pictures more than formulas, don't we?). But also, the graph contains a lot of useful information like whether it is a straight line, what is the shape of the curve, how many places it cuts the x-axis, etc.

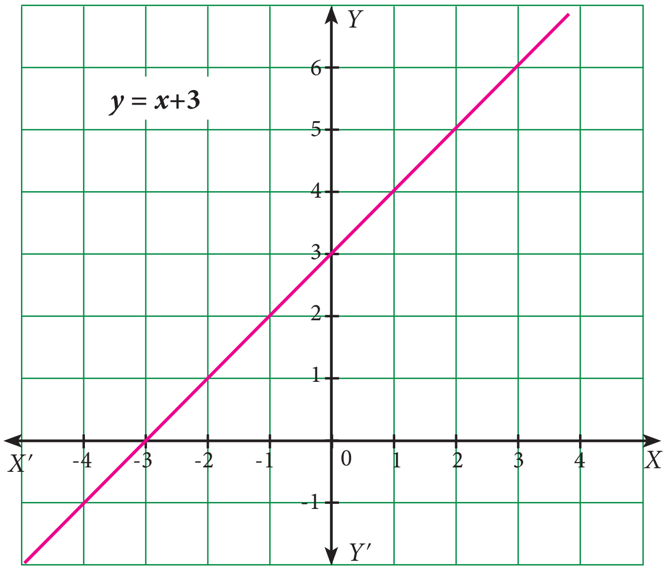


Fig. 3.7

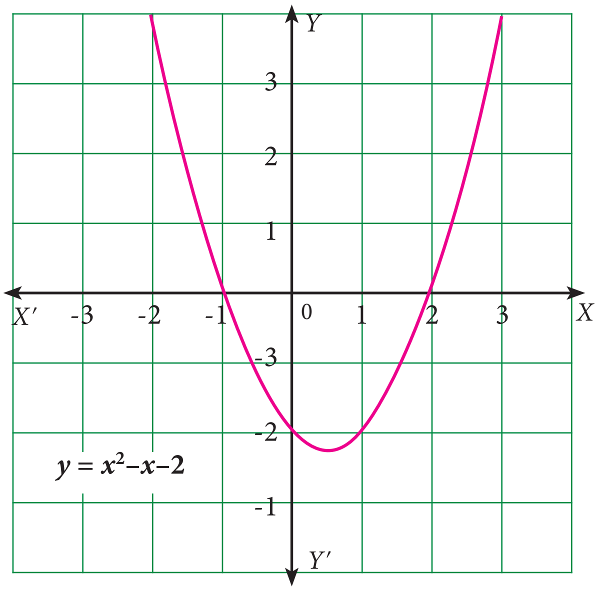


Fig. 3.8

In general, the value of a polynomial $p(x)$ at $x=a$, denoted $p(a)$, is obtained by replacing x by a, where a is any real number.

Notice that the value of $p(x)$ can be zero for many possible values of x as in the second graph 3.8. So it is interesting to ask, for how many values of x, does p(x) become zero, and for which values ? We call these values of x, the zeros of the polynomial $p(x)$.

Once we see that the values of the polynomial are what we plot in the graph of the polynomial, it is also easy to notice that the polynomial becomes zero exactly when the graph intersects the X-axis.

The number of zeros depends on the line or curves intersecting x-axis.

For Fig. 3.7, Number of zeros is equal to 1

For Fig. 3.8, Number of zeros is equal to 2

Value of a Polynomial

Value of a polynomial $p(x)$ at $x = a$ is $p(a)$ obtained on replacing x by a $(a \in R)$

Note

Number of zeros of a polynomial $\leq$ the degree of the polynomial

For example,

Consider $f(x) = x^2 + 3x - 1$.

The value of $f(x)$ at $x = 2$ is

$f(2) = 2^2+3(2)-1 = 4+6-1 = 9$.

Zeros of Polynomial

(i) Consider the polynomial $p(x) = 4x^3 -6x^2 + 3x -14$

The value of $p(x)$ at $x = 1$ is $p(1) = 4(1)^3 - 6(1)^2 + 3(1) - 14$

$= 4 - 6 + 3 - 14$

$= -13$

Then, we say that the value of $p(x)$ at $x = 1$ is -13.

If we replace x by 0, we get $p(0) = 4(0)^3 - 6(0)^2 + 3(0) -14$

$= 0 - 0 + 0 - 14$

$= -14$

we say that the value of $p(x)$ at $x = 0$ is -14.

The value of $p(x)$ at $x = 2$ is $p(2) = 4(2)^3 - 6(2)^2 + 3(2) - 14$

$= 32 - 24 + 6 - 14$

$= 0$

Since the value of $p(x)$ at $x = 2$ is zero, we can say that 2 is one of the zeros of $p(x)$ where $p(x) = 4x^3 -6x^2 + 3x -14$.

Roots of a Polynomial Equation

In general, if $p(a) = 0$ we say that a is zero of polynomial $p(x)$ or a is the root of polynomial equation $p(x) = 0$

Example 3.7 If $f(x) = x^2 - 4x + 3$, then find the values of $f(1)$, $f(-1)$, $f(2)$, $f(3)$. Also find the zeros of the polynomial $f(x)$.

Solution

$$f(x) = x^2 - 4x + 3$$

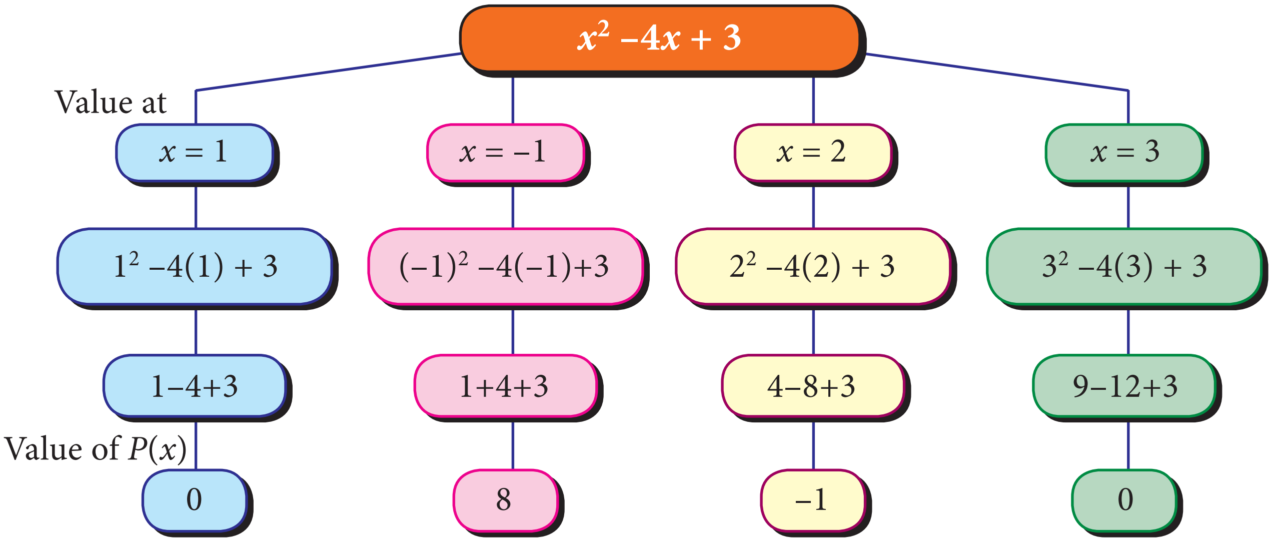


Fig. 3.9

Since the value of the polynomial $f(x)$ at $x = 1$ and $x = 3$ is zero, as the zeros of polynomial $f(x)$ are 1 and 3.

Example 3.8 Find the Zeros of the following polynomials.

(i) $f(x) = 2x + 1$ (ii) $f(x) = 3x - 5$

Solution

(i) Given that

$$f(x) = 2x + 1 = 2\left(x + \frac{1}{2}\right)$$
$$= 2\left(x - \left(-\frac{1}{2}\right)\right)$$
$$f\left(-\frac{1}{2}\right) = 2\left[-\frac{1}{2} - \left(-\frac{1}{2}\right)\right] = 2(0) = 0$$

Since $f\left(-\frac{1}{2}\right) = 0$, $x = -\frac{1}{2}$ is the zero of $f(x)$

(ii) Given that

$$f(x) = 3x - 5 = 3\left(x - \frac{5}{3}\right)$$
$$f\left(\frac{5}{3}\right) = 3\left(\frac{5}{3} - \frac{5}{3}\right) = 3(0) = 0$$

Since $f\left(\frac{5}{3}\right) = 0$, $x = \frac{5}{3}$ is the zero of $f(x)$

Example 3.9

Find the roots of the following polynomial equations.

(i) $5x - 3 = 0$ (ii) $-7 - 4x = 0$

Solution

(i) $5x - 3 = 0$

$$\begin{aligned} \text{(or)} \quad 5x &= 3 \\ \text{Then,} \quad x &= \frac{3}{5} \end{aligned}$$

(ii) $-7 - 4x = 0$

$$\begin{aligned} \text{(or)} \quad 4x &= -7 \\ \text{Then,} \quad x &= \frac{7}{-4} = -\frac{7}{4} \end{aligned}$$

Note

(i) A zero of a polynomial can be any real number not necessarily zero.

(ii) A non zero constant polynomial has no zero.

(iii) By convention, every real number is zero of the zero polynomial

Example 3.10

Check whether -3 and 3 are zeros of the polynomial $x^2 - 9$

Solution

$$\begin{aligned} \text{Let} \quad f(x) &= x^2 - 9 \\ \text{Then,} \quad f(-3) &= (-3)^2 - 9 = 9 - 9 = 0 \\ f(+3) &= 3^2 - 9 = 9 - 9 = 0 \end{aligned}$$

$\therefore$ -3 and 3 are zeros of the polynomial $x^2 - 9$

Exercise 3.2

1. Find the value of the polynomial $f(y) = 6y - 3y^2 + 3$ at

   (i) $y = 1$ (ii) $y = -1$ (iii) $y = 0$

2. If $p(x) = x^2 - 2\sqrt{2}x + 1$, find $p(2\sqrt{2})$.

3. Find the zeros of the polynomial in each of the following :

   (i) $p(x) = x - 3$ (ii) $p(x) = 2x + 5$ (iii) $q(y) = 2y - 3$

   (iv) $f(z) = 8z$ (v) $p(x) = ax$ when $a \neq 0$ (vi) $h(x) = ax + b,\ a \neq 0,\ a, b \in R$

4. Find the roots of the polynomial equations .

   (i) $5x - 6 = 0$ (ii) $x + 3 = 0$

   (iii) $10x + 9 = 0$ (iv) $9x - 4 = 0$

5. Verify whether the following are zeros of the polynomial indicated against them, or not.

   (i) $p(x) = 2x - 1,\ x = \frac{1}{2}$ (ii) $p(x) = x^3 - 1,\ x = 1$

   (iii) $p(x) = ax + b,\ x = \frac{-b}{a}$ (iv) $p(x) = (x + 3)(x - 4),\ x = 4, x = -3$

6. Find the number of zeros of the following polynomials represented by their graphs.

(i)

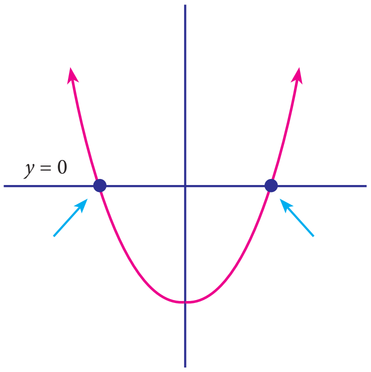


Fig. 3.10

(ii)

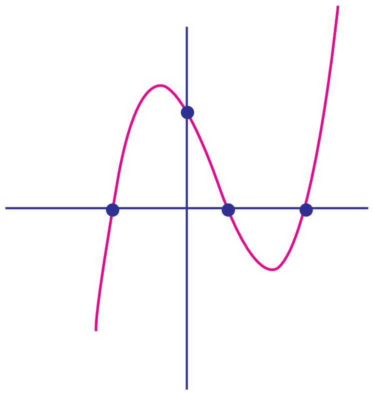


Fig. 3.11

(iii)

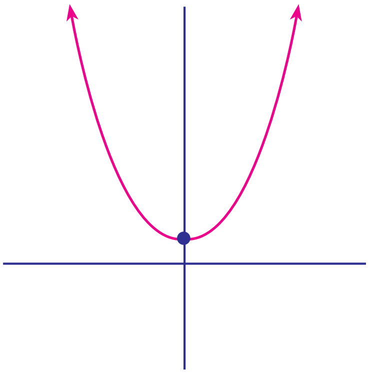


Fig. 3.12

(iv)

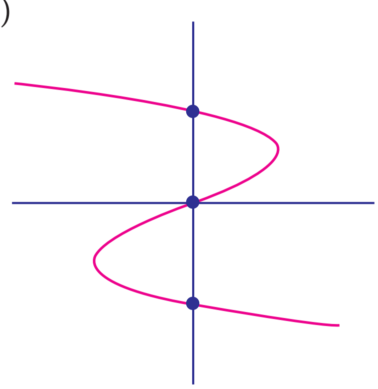


Fig. 3.13

(v)

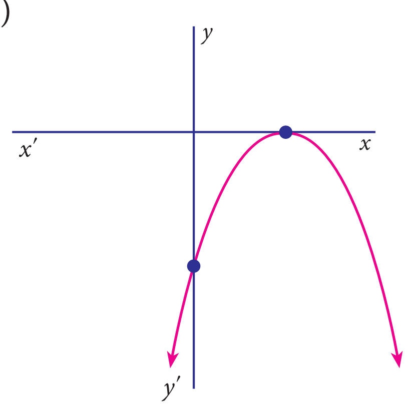


Fig. 3.14

3.3 Remainder Theorem

In this section , we shall study a simple and an elegant method of finding the remainder.

In the case of divisibility of a polynomial by a linear polynomial we use a well known theorem called **Remainder Theorem.**

If a polynomial $p(x)$ of degree greater than or equal to one is divided by a linear polynomial $(x–a)$ then the remainder is $p(a)$, where a is any real number.

Significance of Remainder theorem : It enables us to find the remainder without actually following the cumbersome process of long division.

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Note

(i) If $p(x)$ is divided by $(x+a)$, then the remainder is $p(-a)$

(ii) If $p(x)$ is divided by $(ax–b)$, then the remainder is $p(\frac{b}{a})$

(iii) If $p(x)$ is divided by $(ax+b)$, then the remainder is $p(-\frac{b}{a})$

Example 3.11

S.No.	Question	Solution	Hint
1	Find the remainder when $f(x) = x^3 + 3x^2 + 3x + 1$ is divided by $x+1$.	$f(x) = x^3 + 3x^2 + 3x + 1$ $f(-1) = (-1)^3+3(-1)^2+3(-1)+1$ $= -1+3-3+1 = 0$ Hence, the remainder is 0	$g(x) = x+1$ $g(x) = 0$ $x + 1 = 0$ $x = -1$
2	Check whether $f(x) = x^3 - x + 1$ is a multiple of $g(x) = 2 - 3x$	$\therefore f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^3 - \frac{2}{3} + 1$ $= \frac{8}{27} - \frac{2}{3} + 1$ $= \frac{8 - 18 + 27}{27} = \frac{17}{27} \neq 0$ $\Rightarrow f(x)$ is not multiple of $g(x)$	$g(x) = 2-3x$ $= 0$ gives $x = \frac{2}{3}$
3	Find the remainder when $f(x) = x^3 - ax^2 + 6x - a$ is divided by $(x - a)$	We have $f(x) = x^3 - ax^2 + 6x - a$ $f(a) = a^3 - a(a)^2 + 6a - a$ $= a^3 - a^3 + 5a$ $= 5a$ Hence the required remainder is $5a$	Let $g(x) = x-a$ $g(x) = 0$ $x - a = 0$ $x = a$
4	For what value of k is the polynominal $2x^4 + 3x^3 + 2kx^2 + 3x + 6$ exactly divisible by $(x + 2)$?	Let $f(x) = 2x^4 + 3x^3 + 2kx^2 + 3x + 6$ If $f(x)$ is exactly divisible by $(x+2)$, then the remainder must be zero i.e ., $f(-2) = 0$ i.e ., $2(-2)^4 + 3(-2)^3 + 2k(-2)^2 + 3(-2)+6=0$ $2(16) + 3(-8) + 2k(4) - 6 + 6=0$ $32-24 + 8k = 0$ $8k = -8$, $k = -1$ Hence $f(x)$ is exactly divisible by $(x-2)$ when $k = -1$	Let $g(x)=x+2$ $g(x)=0$ $x+2 = 0$ $x = -2$

Example 3.12 Without actual division , prove that $f(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$ is exactly divisible by $x^2 - 3x + 2$

Solution :

Let

$$\begin{aligned} f(x) &= 2x^4 - 6x^3 + 3x^2 + 3x - 2 \\ g(x) &= x^2 - 3x + 2 \\ &= x^2 - 2x - x + 2 \\ &= x(x-2) - 1(x-2) \\ &= (x-2)(x-1) \end{aligned}$$

we show that $f(x)$ is exactly divisible by $(x-1)$ and $(x-2)$ using remainder theorem

$$\begin{aligned} f(1) &= 2(1)^4 - 6(1)^3 + 3(1)^2 + 3(1) - 2 \\ &= 2 - 6 + 3 + 3 - 2 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(2) &= 2(2)^4 - 6(2)^3 + 3(2)^2 + 3(2) - 2 \\ &= 32 - 48 + 12 + 6 - 2 \\ &= 0 \end{aligned}$$

$\therefore$ $f(x)$ is exactly divisible by $(x - 1)(x - 2)$

i.e., $f(x)$ is exactly divisible by $x^2 - 3x + 2$

If $p(x)$ is divided by $(x-a)$ with the remainder $p(a) = 0$, then $(x-a)$ is a factor of $p(x)$. Remainder Theorem leads to Factor Theorem.

3.3.1 Factor Theorem

If $p(x)$ is a polynomial of degree $n \geq 1$ and 'a' is any real number then

(i) $p(a) = 0$ implies $(x-a)$ is a factor of $p(x)$.

(ii) $(x-a)$ is a factor of $p(x)$ implies $p(a) = 0$.

Proof

If $p(x)$ is the dividend and $(x-a)$ is a divisor, then by division algorithm we write, $p(x) = (x-a)q(x) + p(a)$ where $q(x)$ is the quotient and $p(a)$ is the remainder.

(i) If $p(a) = 0$, we get $p(x) = (x-a)\,q(x)$ which shows that $(x-a)$ is a factor of $p(x)$.

(ii) Since $(x-a)$ is a factor of $p(x)$, $p(x) = (x-a)g(x)$ for some polynomial $g(x)$. In this case

$$\begin{aligned} p(a) &= (a-a)\,g(a) \\ &= 0 \times g(a) \\ &= 0 \end{aligned}$$

Hence, $p(a) = 0$, when $(x-a)$ is a factor of $p(x)$.

Thinking Corner

For any two integers a $(a \neq 0)$ and b, a divides b if $b = ax$, for some integer x.

Note

- $(x-a)$ is a factor of $p(x)$, if $p(a) = 0$ $\quad(\because x-a = 0,\ x = a)$
- $(x+a)$ is a factor of $p(x)$, if $p(-a) = 0$ $\quad(\because x+a = 0,\ x = -a)$
- $(ax+b)$ is a factor of $p(x)$, if $p\left(-\frac{b}{a}\right) = 0$ $\quad\left(\because ax+b = 0,\ ax = -b,\ x = -\frac{b}{a}\right)$
- $(ax-b)$ is a factor of $p(x)$, if $p\left(\frac{b}{a}\right) = 0$ $\quad\left(\because ax-b = 0,\ ax = b,\ x = \frac{b}{a}\right)$
- $(x-a)(x-b)$ is a factor of $p(x)$, if $p(a) = 0$ and $p(b) = 0$ $\quad\left(\because \begin{array}{ccc} x-a=0 & \text{or} & x-b=0 \\ x=a & \text{or} & x=b \end{array}\right)$

Example 3.13 Show that $(x+2)$ is a factor of $x^3-4x^2-2x+20$

Solution

Let $p(x) = x^3-4x^2-2x+20$

By factor theorem, $(x+2)$ is factor of $p(x)$, *if* $p(-2) = 0$

$$p(-2) = (-2)^3-4(-2)^2-2(-2)+20$$

$$= -8-4(4)+4+20$$

$$p(-2) = 0$$

Therefore, $(x+2)$ is a factor of $x^3-4x^2-2x+20$

To find the zero of $x+2$;

put $x+2 = 0$

we get $x = -2$

Example 3.14 Is $(3x-2)$ a factor of $3x^3+x^2-20x+12$?

Solution

Let $p(x) = 3x^3+x^2-20x+12$

By factor theorem, $(3x-2)$ is a factor, if $p\left(\frac{2}{3}\right) = 0$

$$p\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right)^3+\left(\frac{2}{3}\right)^2-20\left(\frac{2}{3}\right)+12$$

$$= 3\left(\frac{8}{27}\right)+\left(\frac{4}{9}\right)-20\left(\frac{2}{3}\right)+12$$

$$= \frac{8}{9}+\frac{4}{9}-\frac{120}{9}+\frac{108}{9}$$

To find the zero of $3x-2$;

put $3x-2 = 0$

$3x = 2$

we get $x = \frac{2}{3}$

Progress Check

1. $(x+3)$ is a factor of $p(x)$, if $p(\_\_) = 0$
2. $(3-x)$ is a factor of $p(x)$, if $p(\_\_) = 0$
3. $(y-3)$ is a factor of $p(y)$, if $p(\_\_) = 0$
4. $(-x-b)$ is a factor of $p(x)$, if $p(\_\_) = 0$
5. $(-x+b)$ is a factor of $p(x)$, if $p(\_\_) = 0$

$$p\left(\frac{2}{3}\right) = \frac{(120-120)}{9}$$

$$= 0$$

Therefore, $(3x-2)$ is a factor of
$3x^3 + x^2 - 20x + 12$

Example 3.15 Find the value of m, if $(x-2)$ is a factor of the polynomial $2x^3 - 6x^2 + mx + 4$.

Solution

Let $p(x) = 2x^3 - 6x^2 + mx + 4$

By factor theorem, $(x-2)$ is a factor of $p(x)$, if $p(2) = 0$

$$\begin{aligned} p(2) &= 0 \\ 2(2)^3 - 6(2)^2 + m(2) + 4 &= 0 \\ 2(8) - 6(4) + 2m + 4 &= 0 \\ -4 + 2m &= 0 \\ m &= 2 \end{aligned}$$

To find the zero of x–2;
put $x - 2 = 0$
we get $x = 2$

Exercise 3.3

1. Check whether $p(x)$ is a multiple of g(x) or not .
$p(x) = x^3 - 5x^2 + 4x - 3$; $g(x) = x - 2$

2. By remainder theorem, find the remainder when, $p(x)$ is divided by $g(x)$ where,
(i) $p(x) = x^3 - 2x^2 - 4x - 1$; $g(x) = x + 1$
(ii) $p(x) = 4x^3 - 12x^2 + 14x - 3$; $g(x) = 2x - 1$
(iii) $p(x) = x^3 - 3x^2 + 4x + 50$; $g(x) = x - 3$

3. Find the remainder when $3x^3 - 4x^2 + 7x - 5$ is divided by $(x+3)$

4. What is the remainder when $x^{2018} + 2018$ is divided by x–1

5. For what value of k is the polynomial
$p(x) = 2x^3 - kx^2 + 3x + 10$ exactly divisible by $(x$–$2)$

6. If two polynomials $2x^3 + ax^2 + 4x - 12$ and $x^3 + x^2 - 2x + a$ leave the same remainder when divided by $(x - 3)$, find the value of a and also find the remainder.

7. Determine whether $(x-1)$ is a factor of the following polynomials:
$i)$ $x^3 + 5x^2 - 10x + 4$ $ii)$ $x^4 + 5x^2 - 5x + 1$

8. Using factor theorem, show that $(x-5)$ is a factor of the polynomial $2x^3-5x^2-28x + 15$

9. Determine the value of m, if $(x+3)$ is a factor of $x^3-3x^2-mx+24$.

10. If both $(x-2)$ and $\left(x-\frac{1}{2}\right)$ are the factors of ax^2+5x+b, then show that $a=b$.

11. If $(x-1)$ divides the polynomial $kx^3-2x^2+25x-26$ without remainder, then find the value of k.

12. Check if $(x+2)$ and $(x-4)$ are the sides of a rectangle whose area is x^2-2x-8 by using factor theorem.

3.4 Algebraic Identities

An identity is an equality that remains true regardless of the values chosen for its variables.

We have already learnt about the following identities:

(1) $(a+b)^2 \equiv a^2+2ab+b^2$ (2) $(a-b)^2 \equiv a^2-2ab+b^2$

(3) $(a+b)(a-b) \equiv a^2-b^2$ (4) $(x+a)(x+b) \equiv x^2+(a+b)x+ab$

Note

(i) $a^2+b^2=(a+b)^2-2ab$ (ii) $a^2+b^2=(a-b)^2+2ab$

(iii) $a^2+\frac{1}{a^2}=\left(a+\frac{1}{a}\right)^2-2$ (iv) $a^2+\frac{1}{a^2}=\left(a-\frac{1}{a}\right)^2+2$

Example 3.16

Expand the following using identities: (i) $(3x+4y)^2$ (ii) $(2a-3b)^2$ (iii) $(5x+4y)(5x-4y)$ (iv) $(m+5)(m-8)$

Solution

(i) $(3x+4y)^2$ $\left[\text{we have } (a+b)^2=a^2+2ab+b^2\right]$

$(3x+4y)^2=(3x)^2+2(3x)(4y)+(4y)^2$ put $[a=3x,\ b=4y]$

$=9x^2+24xy+16y^2$

(ii) $(2a-3b)^2$ $\left[\text{we have } (a-b)^2=a^2-2ab+b^2\right]$

$(2a-3b)^2=(2a)^2-2(2a)(3b)+(3b)^2$ put $[a=2a,\ b=3b]$

$=4a^2-12ab+9b^2$

(iii) $(5x+4y)(5x-4y)$ $\quad$ $\left[\text{we have } (a+b)(a-b) \;\; = a^2-b^2\right]$

$$(5x+4y)(5x-4y) = (5x)^2-(4y)^2 \qquad \text{put } [a=5x \quad b=4y]$$

$$= 25x^2-16y^2$$

(iv) $(m+5)(m-8)$ $\quad$ $\left[\text{we have } (x+a)(x-b) = x^2+(a-b)x-ab\right]$

$$(m+5)(m-8) = m^2+(5-8)m-(5)(8) \qquad \text{put } [x=m,\ a=5,\ b=8]$$

$$= m^2-3m-40$$

3.4.1 Expansion of Trinomial $(a+b+c)^2$

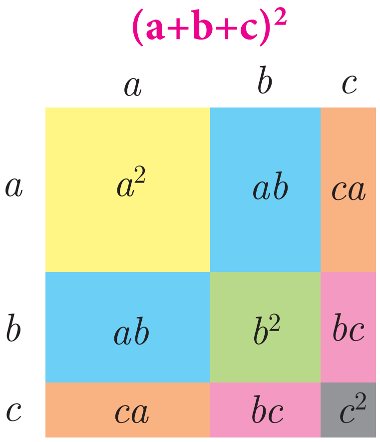
We know that $(x+y)^2 = x^2+2xy+y^2$

Put $x=a+b,\ y=c$

Then, $\quad (a+b+c)^2 = (a+b)^2+2(a+b)(c)+c^2$

$$= a^2+2ab+b^2+2ac+2bc+c^2$$

$$= a^2+b^2+c^2+2ab+2bc+2ca$$



Thus, $\quad (a+b+c)^2 \equiv a^2+b^2+c^2+2ab+2bc+2ca$

Example 3.17

Expand $(a-b+c)^2$

Solution

Replacing 'b' by '$-b$' in the expansion of

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$$

$$(a+(-b)+c)^2 = a^2+(-b)^2+c^2+2a(-b)+2(-b)c+2ca$$

$$= a^2+b^2+c^2-2ab-2bc+2ca$$

Progress Check

Expand the following and verify :

$$(a+b+c)^2 = (-a-b-c)^2$$

$$(-a+b+c)^2 = (a-b-c)^2$$

$$(a-b+c)^2 = (-a+b-c)^2$$

$$(a+b-c)^2 = (-a-b+c)^2$$

Example 3.18

Expand $(2x+3y+4z)^2$

Solution

We know that,

$(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$

Substituting, $a=2x, b=3y$ and $c=4z$

$$(2x+3y+4z)^2 = (2x)^2+(3y)^2+(4z)^2+2(2x)(3y)+2(3y)(4z)+2(4z)(2x)$$

$$= 4x^2+9y^2+16z^2+12xy+24yz+16xz$$

Example 3.19 Find the area of square whose side length is $3m+2n-4l$

Solution

$$\text{Area of square} = \text{side} \times \text{side}$$
$$= (3m+2n-4l)\times(3m+2n-4l)$$
$$= (3m+2n-4l)^2$$

We know that, $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$

substituting
$a = 3m,$
$b = 2n$
$c = -4l$

$$\left[3m+2n+(-4l)\right]^2 = (3m)^2+(2n)^2+(-4l)^2+2(3m)(2n)+2(2n)(-4l)+2(-4l)(3m)$$
$$= 9m^2+4n^2+16l^2+12mn-16ln-24lm$$

Therefore, Area of square $= [9m^2+4n^2+16l^2+12mn-16ln-24lm]$ sq.units.

3.4.2 Identities involving Product of Three Binomials

$$(x+a)(x+b)(x+c) = [(x+a)\,(x+b)]\,(x+c)$$
$$= [x^2+(a+b)x+ab](x+c)$$
$$= x^2(x)+(a+b)(x)(x)+abx+x^2c+(a+b)(x)c+abc$$
$$= x^3+ax^2+bx^2+abx+cx^2+acx+bcx+abc$$
$$= x^3+(a+b+c)x^2+(ab+bc+ca)x+abc$$

Thus, $(x+a)(x+b)(x+c) \equiv x^3+(a+b+c)x^2+(ab+bc+ca)x+abc$

Example 3.20 Expand the following:

(*i*) $(x+5)(x+6)(x+4)$ (*ii*) $(3x-1)(3x+2)(3x-4)$

Solution

We know that $(x+a)(x+b)(x+c) = x^3+(a+b+c)x^2+(ab+bc+ca)x+abc$ --(1)

(i) $(x+5)(x+6)(x+4)$

$$= x^3+(5+6+4)x^2+(30+24+20)x+(5)(6)(4)$$
$$= x^3+15x^2+74x+120$$

Replace
a by 5
b by 6
c by 4 in (1)

(ii) $(3x-1)(3x+2)(3x-4)$

$$= (3x)^3+(-1+2-4)(3x)^2+(-2-8+4)(3x)+(-1)(2)(-4)$$
$$= 27x^3+(-3)9x^2+(-6)(3x)+8$$
$$= 27x^3-27x^2-18x+8$$

Replace
x by 3*x*, *a* by –1,
b by 2, *c* by –4
in (1)

3.4.3 Expansion of $(x+y)^3$ and $(x-y)^3$

$$(x+a)(x+b)(x+c) \equiv x^3 + (a+b+c)x^2 + (ab+bc+ca)x + abc$$

substituting $a = b = c = y$ in the identity

$$\text{we get, } (x+y)(x+y)(x+y) = x^3 + (y+y+y)x^2 + (yy+yy+yy)x + yyy$$

$$= x^3 + (3y)x^2 + (3y^2)x + y^3$$

Thus, $(x+y)^3 \equiv x^3 + 3x^2y + 3xy^2 + y^3$ (or) $(x+y)^3 \equiv x^3 + y^3 + 3xy(x+y)$

by replacing y by $-y$, we get

$$(x-y)^3 \equiv x^3 - 3x^2y + 3xy^2 - y^3 \text{ (or) } (x-y)^3 \equiv x^3 - y^3 - 3xy(x-y)$$

Example 3.21 Expand $(5a-3b)^3$

Solution

We know that, $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

$$(5a-3b)^3 = (5a)^3 - 3(5a)^2(3b) + 3(5a)(3b)^2 - (3b)^3$$

$$= 125a^3 - 3(25a^2)(3b) + 3(5a)(9b^2) - (3b)^3$$

$$= 125a^3 - 225a^2b + 135ab^2 - 27b^3$$

The following identity is also used:

$$x^3 + y^3 + z^3 - 3xyz \equiv (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$$

We can check this by performing the multiplication on the right hand side.

Note

(i) If $(x+y+z) = 0$ then $x^3 + y^3 + z^3 = 3xyz$

Some identities involving sum, difference and product are stated without proof

(i) $x^3 + y^3 \equiv (x+y)^3 - 3xy(x+y)$ (ii) $x^3 - y^3 \equiv (x-y)^3 + 3xy(x-y)$

Example 3.22 Find the product of

$(2x+3y+4z)(4x^2+9y^2+16z^2-6xy-12yz-8zx)$

Solution

We know that, $(a+b+c)(a^2+b^2+c^2-ab-bc-ca) = a^3+b^3+c^3-3abc$

$$(2x+3y+4z)(4x^2+9y^2+16z^2-6xy-12yz-8zx)$$
$$= (2x)^3+(3y)^3+(4z)^3-3(2x)(3y)(4z)$$
$$= 8x^3+27y^3+64z^3-72xyz$$

Example 3.23 Evaluate $10^3-15^3+5^3$

Solution

We know that, if $a+b+c=0$, then $a^3+b^3+c^3=3abc$

Here, $a+b+c = 10-15+5=0$

Replace
a by 10, b by −15,
c by 5

$$\text{Therefore, } 10^3+(-15)^3+5^3 = 3(10)(-15)(5)$$
$$10^3-15^3+5^3 = -2250$$

Exercise 3.4

1. Expand the following:

(i) $(2x+3y+4z)^2$ (ii) $(-p+2q+3r)^2$

(iii) $(2p+3)(2p-4)(2p-5)$ (iv) $(3a+1)(3a-2)(3a+4)$

2. Using algebraic identity, find the coefficients of x^2, x and constant term without actual expansion.

(i) $(x+5)(x+6)(x+7)$ (ii) $(2x+3)(2x-5)(2x-6)$

3. If $(x+a)(x+b)(x+c)=x^3+14x^2+59x+70$, find the value of

(i) $a+b+c$ (ii) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ (iii) $a^2+b^2+c^2$ (iv) $\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}$

4. Expand:

(i) $(3a-4b)^3$ (ii) $\left(x+\frac{1}{y}\right)^3$

5. Evaluate the following by using identities:

(i) 98^3 (ii) 1001^3

6. If $(x+y+z)=9$ and $(xy+yz+zx)=26$, then find the value of $x^2+y^2+z^2$.

7. Find $27a^3+64b^3$, if $3a+4b=10$ and $ab=2$.

8. Find x^3-y^3, if $x-y=5$ and $xy=14$.

9. If $a+\frac{1}{a}=6$, then find the value of $a^3+\frac{1}{a^3}$.

10. If $x^2+\frac{1}{x^2}=23$, then find the value of $x+\frac{1}{x}$ and $x^3+\frac{1}{x^3}$.

11. If $\left(y-\frac{1}{y}\right)^3=27$, then find the value of $y^3-\frac{1}{y^3}$.

12. Simplify: (i) $(2a+3b+4c)(4a^2+9b^2+16c^2-6ab-12bc-8ca)$

(ii) $(x-2y+3z)(x^2+4y^2+9z^2+2xy+6yz-3xz)$

13. By using identity evaluate the following:

(i) $7^3-10^3+3^3$ (ii) $1+\frac{1}{8}-\frac{27}{8}$

14. If $2x-3y-4z=0$, then find $8x^3-27y^3-64z^3$.

3.5 Factorisation

Factorisation is the reverse of multiplication.

For Example : Multiply 3 and 5; we get product 15.

Factorise 15; we get factors 3 and 5.

For Example : Multiply (*x* + 2) and (*x* + 3); we get product x^2+5x+6.

Factorise x^2+5x+6 ; we get factors (*x* + 2) and (*x* + 3).

Multiply ↓	Factorise ↑
$(x+3)(x+2)$	$(x+3)(x+2)$
$x(x+2)+3(x+2)$	$x(x+2)+3(x+2)$
$x^2+2x+3x+6$	$x^2+2x+3x+6$
x^2+5x+6	x^2+5x+6

Thus, the process of converting the given higher degree polynomial as the product of factors of its lower degree, which cannot be further factorised is called factorisation.

Two important ways of factorisation are :

(i) By taking common factor

$ab+ac$
$a\times b\ +a\times c$
$a(b+c)$ factored form

(ii) By grouping them

$a+b-pa-pb$
$(a+b)-p(a+b)$ group in pairs
$(a+b)(1-p)$ factored form

When a polynomial is factored, we "factored out" the common factor.

Example 3.24 Factorise the following:

(i) $am+bm+cm$ (ii) a^3-a^2b (iii) $5a-10b-4bc+2ac$ (iv) $x+y-1-xy$

Solutions

(i) $am+bm+cm$
$am+bm+cm$
$m(a+b+c)$ factored form

(ii) a^3-a^2b
$a^2\cdot a-a^2\cdot b$ group in pairs
$a^2\times(a-b)$ factored form

(iii) $5a - 10b - 4bc + 2ac$

$5a - 10b + 2ac - 4bc$

$5(a - 2b) + 2c(a - 2b)$

$(a - 2b)(5 + 2c)$

(iv) $x + y - 1 - xy$

$x - 1 + y - xy$

$(x - 1) + y(1 - x)$

$(x - 1) - y(x - 1)$

$(x - 1)(1 - y)$

$(a-b) = -(b-a)$

3.5.1 Factorisation using Identity

(i) $a^2 + 2ab + b^2 \equiv (a + b)^2$ (ii) $a^2 - 2ab + b^2 \equiv (a - b)^2$

(iii) $a^2 - b^2 \equiv (a + b)(a - b)$ (iv) $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \equiv (a + b + c)^2$

(v) $a^3 + b^3 \equiv (a + b)(a^2 - ab + b^2)$ (vi) $a^3 - b^3 \equiv (a - b)(a^2 + ab + b^2)$

(vii) $a^3 + b^3 + c^3 - 3abc \equiv (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

Note

$(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$; $a^4 - b^4 = (a^2 + b^2)(a + b)(a - b)$

$(a + b)^2 - (a - b)^2 = 4ab$; $a^6 - b^6 = (a + b)(a - b)(a^2 - ab + b^2)(a^2 + ab + b^2)$

Progress Check

Prove: (i) $\left(a + \frac{1}{a}\right)^2 + \left(a - \frac{1}{a}\right)^2 = 2\left(a^2 + \frac{1}{a^2}\right)$ (ii) $\left(a + \frac{1}{a}\right)^2 - \left(a - \frac{1}{a}\right)^2 = 4$

Example 3.25 Factorise the following:

(i) $9x^2 + 12xy + 4y^2$ (ii) $25a^2 - 10a + 1$ (iii) $36m^2 - 49n^2$

(iv) $x^3 - x$ (v) $x^4 - 16$ (vi) $x^2 + 4y^2 + 9z^2 - 4xy + 12yz - 6xz$

Solution

(i) $9x^2 + 12xy + 4y^2 = (3x)^2 + 2(3x)(2y) + (2y)^2$ $[\because a^2 + 2ab + b^2 = (a + b)^2]$

$= (3x + 2y)^2$

(ii) $25a^2 - 10a + 1 = (5a)^2 - 2(5a)(1) + 1^2$

$= (5a - 1)^2$ $\left[\because a^2 - 2ab + b^2 = (a - b)^2\right]$

(iii) $36m^2 - 49n^2. = (6m)^2 - (7n)^2$

$= (6m + 7n)(6m - 7n)$ $\left[\because a^2 - b^2 = (a + b)(a - b)\right]$

(iv) $x^3 - x = x(x^2 - 1)$

$= x(x^2 - 1^2)$

$= x(x+1)(x-1)$

(v) $x^4 - 16 = x^4 - 2^4$ $\left[\because a^4 - b^4 = (a^2+b^2)(a+b)(a-b)\right]$

$= (x^2 + 2^2)(x^2 - 2^2)$

$= (x^2 + 4)(x+2)(x-2)$

(vi) $x^2 + 4y^2 + 9z^2 - 4xy + 12yz - 6xz$

$= (-x)^2 + (2y)^2 + (3z)^2 + 2(-x)(2y) + 2(2y)(3z) + 2(3z)(-x)$

$= (-x + 2y + 3z)^2$ (or) $(x - 2y - 3z)^2$

Example 3.26

Factorise the following:

(i) $27x^3 + 125y^3$ (ii) $216m^3 - 343n^3$

(iii) $2x^4 - 16xy^3$ (iv) $8x^3 + 27y^3 + 64z^3 - 72xyz$

Solution

(i) $27x^3 + 125y^3 = (3x)^3 + (5y)^3$ $\left[\because (a^3 + b^3) = (a+b)(a^2 - ab + b^2)\right]$

$= (3x + 5y)\left((3x)^2 - (3x)(5y) + (5y)^2\right)$

$= (3x + 5y)(9x^2 - 15xy + 25y^2)$

(ii) $216m^3 - 343n^3 = (6m)^3 - (7n)^3$ $\left[\because (a^3 - b^3) = (a-b)(a^2 + ab + b^2)\right]$

$= (6m - 7n)\left((6m)^2 + (6m)(7n) + (7n)^2\right)$

$= (6m - 7n)(36m^2 + 42mn + 49n^2)$

(iii) $2x^4 - 16xy^3 = 2x(x^3 - 8y^3)$

$= 2x\left(x^3 - (2y)^3\right)$ $\left[\because (a^3 - b^3) = (a-b)(a^2 + ab + b^2)\right]$

$= 2x\left((x - 2y)(x^2 + (x)(2y) + (2y)^2)\right)$

$= 2x(x - 2y)(x^2 + 2xy + 4y^2)$

(iv) $8x^3 + 27y^3 + 64z^3 - 72xyz$

$= (2x)^3 + (3y)^3 + (4z)^3 - 3(2x)(3y)(4z)$

$= (2x + 3y + 4z)(4x^2 + 9y^2 + 16z^2 - 6xy - 12yz - 8xz)$

Thinking Corner

Check 15 divides the following

(i) $2017^3 + 2018^3$

(ii) $2018^3 - 1973^3$

Exercise 3.5

1. Factorise the following expressions:

   (i) $2a^2 + 4a^2b + 8a^2c$ (ii) $ab - ac - mb + mc$

2. Factorise the following:

   (i) $x^2 + 4x + 4$ (ii) $3a^2 - 24ab + 48b^2$ (iii) $x^5 - 16x$

   (iv) $m^2 + \dfrac{1}{m^2} - 23$ (v) $6 - 216x^2$ (vi) $a^2 + \dfrac{1}{a^2} - 18$

3. Factorise the following:

   (i) $4x^2 + 9y^2 + 25z^2 + 12xy + 30yz + 20xz$

   (ii) $25x^2 + 4y^2 + 9z^2 - 20xy + 12yz - 30xz$

4. Factorise the following:

   (i) $8x^3 + 125y^3$ (ii) $27x^3 - 8y^3$ (iii) $a^6 - 64$

5. Factorise the following:

   (i) $x^3 + 8y^3 + 6xy - 1$ (ii) $l^3 - 8m^3 - 27n^3 - 18lmn$

3.5.2 Factorising the Quadratic Polynomial (Trinomial) of the type $ax^2 + bx + c,\ a \neq 0$

The linear factors of $ax^2 + bx + c$ will be in the form $(kx + m)$ and $(lx + n)$

$$\text{Thus, } ax^2 + bx + c = (kx + m)(lx + n) = klx^2 + (lm + kn)x + mn$$

Comparing coefficients of x^2, x and constant term c on both sides.
We have, $a = kl$, $b = (lm + kn)$ and $c = mn$, where ac is the product of kl and mn that is, equal to the product of lm and kn which are the coefficient of x. Therefore $(kl \times mn) = (lm \times kn)$.

Steps to be followed to factorise $ax^2 + bx + c$:

Step 1 : Multiply the coefficient of x^2 and constant term, that is ac.

Step 2 : Split ac into two factors whose sum and product is equal to b and ac respectively.

Step 3 : The terms are grouped into two pairs and factorise.

Example 3.27

Factorise $2x^2 + 15x + 27$

Solution

Compare with $ax^2 + bx + c$

we get, $a = 2,\ b = 15,\ c = 27$

product $ac = 2 \times 27 = 54$ and sum $b = 15$

We find the pair 6, 9 only satisfies "$b = 15$"

Product of factors $ac = 54$	Sum of factors $b = 15$	Product of factors $ac = 54$	Sum of factors $b = 15$
1×54	55	-1×-54	-55
2×27	29	-2×-27	-29
3×18	21	-3×-18	-21
6×9	**15**	-6×-9	-15
The required factors are **6** and **9**			

and also “$ac = 54$”.

$\therefore$ we split the middle term as $6x$ and $9x$

$$\begin{aligned} 2x^2+15x+27 &= 2x^2+6x+9x+27 \\ &= 2x(x+3)+9(x+3) \\ &= (x+3)(2x+9) \end{aligned}$$

Therefore, $(x+3)$ and $(2x+9)$ are the factors of $2x^2+15x+27$.

Example 3.28

Factorise $2x^2-15x+27$

Product of factors $ac = 54$	Sum of factors $b = -15$	Product of factors $ac = 54$	Sum of factors $b =-15$
1×54	55	-1×-54	-55
2×27	29	-2×-27	-29
3×18	21	-3×-18	-21
6×9	15	**-6×-9**	**-15**
The required factors are **-6** and **-9**			

Solution

Compare with ax^2+bx+c

$a=2,\ b=-15,\ c=27$

product $ac = 2\times 27 = 54$, sum b=-15

$\therefore$ we split the middle term as $-6x$ and $-9x$

$$\begin{aligned} 2x^2-15x+27 &= 2x^2-6x-9x+27 \\ &= 2x(x-3)-9(x-3) \\ &= (x-3)(2x-9) \end{aligned}$$

Therefore, $(x-3)$ and $(2x-9)$ are the factors of $2x^2-15x+27$.

Example 3.29

Factorise $2x^2+15x-27$

Product of factors $ac =-54$	Sum of factors $b = 15$	Product of factors $ac =-54$	Sum of factors $b = 15$
-1×54	53	1×-54	-53
-2×27	25	2×-27	-25
-3×18	**15**	3×-18	-15
-6×9	3	6×-9	-3
The required factors are **-3** and **18**			

Solution

Compare with ax^2+bx+c

Here, $a=2,\ b=15,\ c=-27$

product $ac = 2\times -27 = -54$, sum b=15

$\therefore$ we split the middle term as $18x$ and $-3x$

$$\begin{aligned} 2x^2+15x-27 &= 2x^2+18x-3x-27 \\ &= 2x(x+9)-3(x+9) \\ &= (x+9)(2x-3) \end{aligned}$$

Therefore, $(x+9)$ and $(2x-3)$ are the factors of $2x^2+15x-27$.

Example 3.30

Factorise $2x^2-15x-27$

Product of factors $ac =-54$	Sum of factors $b =-15$	Product of factors $ac =-54$	Sum of factors $b =-15$
-1×54	53	1×-54	-53
-2×27	25	2×-27	-25
-3×18	15	**3×-18**	**-15**
-6×9	3	6×-9	-3
The required factors are **3** and **-18**			

Solution

Compare with ax^2+bx+c

Here, $a=2,\ b=-15,\ c=-27$

product $ac = 2\times -27$=-54, sum b=-15

$\therefore$ we split the middle term as $-18x$ and $3x$

$$\begin{aligned} 2x^2-15x-27 &= 2x^2-18x+3x-27 \\ &= 2x(x-9)+3(x-9) \\ &= (x-9)(2x+3) \end{aligned}$$

Therefore, $(x-9)$ and $(2x+3)$ are the factors of $2x^2-15x-27$

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Example 3.31 Factorise $(x+y)^2+9(x+y)+20$

Solution

Let $x+y=p$, we get $p^2+9p+20$

Compare with ax^2+bx+c,

We get $a=1,\ b=9,\ c=20$

product $ac = 1\times 20 = 20$, sum b=9

$\therefore$ we split the middle term as $4p$ and $5p$

Product of factors ac = 20	Sum of factors b = 9	Product of factors ac = 20	Sum of factors b = 9
1 × 20	21	−1 × −20	−21
2 × 10	12	−2 × −10	−12
4 × 5	**9**	−4 × −5	−9
The required factors are **4** and **5**			

$$\begin{aligned} p^2+9p+20 &= p^2+4p+5p+20 \\ &= p(p+4)+5(p+4) \\ &= (p+4)(p+5) \end{aligned}$$

Put, $p=x+y$ we get, $(x+y)^2+9(x+y)+20=(x+y+4)(x+y+5)$

Exercise 3.6

1. Factorise the following:

   (i) $x^2+10x+24$ (ii) $z^2+4z-12$ (iii) $p^2-6p-16$

   (iv) $t^2+72-17t$ (v) $y^2-16y-80$ (vi) $a^2+10a-600$

2. Factorise the following:

   (i) $2a^2+9a+10$ (ii) $5x^2-29xy-42y^2$ (iii) $9-18x+8x^2$

   (iv) $6x^2+16xy+8y^2$ (v) $12x^2+36x^2y+27y^2x^2$

   (vi) $\left(a+b\right)^2+9\left(a+b\right)+18$

3. Factorise the following:

   (i) $(p-q)^2-6(p-q)-16$ (ii) $m^2+2mn-24n^2$ (iii) $\sqrt{5}a^2+2a-3\sqrt{5}$

   (iv) a^4-3a^2+2 (v) $8m^3-2m^2n-15mn^2$ (vi) $\frac{1}{x^2}+\frac{1}{y^2}+\frac{2}{xy}$

3.6 Division of Polynomials

Let us consider the numbers 13 and 5. When 13 is divided by 5 what is the quotient and remainder.?

Yes, of course, the quotient is 2 and the remainder is 3. We write 13 = (5×2)+3

Let us try.

Divide	Expressed as	Remainder	Divisor
11 by 4	(4×2)+3	3	4
22 by 11	(11×2)+0	0	11

Dividend = (Divisor × Quotient) + Remainder.

From the above examples, we observe that the remainder is less than the divisor.

3.6.1 Division Algorithm for Polynomials

Let $p(x)$ and $g(x)$ be two polynomials such that degree of $p(x) \geq$ degree of $g(x)$ and $g(x) \neq 0$. Then there exists unique polynomials $q(x)$ and $r(x)$ such that

$$p(x) = g(x) \times q(x) + r(x) \qquad \ldots (1)$$

where $r(x) = 0$ or degree of $r(x) <$ degree of $g(x)$.

The polynomial $p(x)$ is the Dividend, $g(x)$ is the Divisor, $q(x)$ is the Quotient and $r(x)$ is the Remainder. Now (1) can be written as

Dividend = (Divisor × Quotient) + Remainder.

If $r(x)$ is zero, then we say $p(x)$ is a multiple of $g(x)$. In other words, $g(x)$ divides $p(x)$.

If it looks complicated, don't worry! it is important to know how to divide polynomials, and that comes easily with practice. The examples below will help you.

Example 3.32

Divide $x^3 - 4x^2 + 6x$ by x, where , $x \neq 0$

Solution

We have

$$\frac{x^3 - 4x^2 + 6x}{x} = \frac{x^3}{x} - \frac{4x^2}{x} + \frac{6x}{x}, x \neq 0$$

$$= x^2 - 4x + 6$$

Example 3.33

Find the quotient and the remainder when $(5x^2 - 7x + 2) \div (x - 1)$

Solution

$(5x^2 - 7x + 2) \div (x - 1)$

$$\begin{array}{r|l} & 5x-2 \\ \hline x-1 & 5x^2-7x+2 \\ & 5x^2-5x \\ & (-)\ \ (+) \\ \hline & \quad -2x+2 \\ & \quad -2x+2 \\ & \quad (+)\ \ (-) \\ \hline & \qquad 0 \\ \hline \end{array}$$

(i) $\frac{5x^2}{x} = 5x$

(ii) $5x(x-1) = 5x^2-5x$

(iii) $-\frac{2x}{x} = -2$

(iv) $-2(x-1) = (-2x+2)$

$\therefore$ Quotient $= 5x-2$ Remainder $= 0$

Example 3.34 Find quotient and the remainder when $f(x)$ is divided by $g(x)$

(i) $f(x) = (8x^3-6x^2+15x-7)$, $g(x) = 2x+1$. (ii) $f(x) = x^4-3x^3+5x^2-7$, $g(x) = x^2+x+1$

Solution

(i) $f(x) = (8x^3-6x^2+15x-7)$, $g(x) = 2x+1$

$$\begin{array}{r|l} & 4x^2-5x+10 \\ \hline 2x+1 & 8x^3-6x^2+15x-7 \\ & 8x^3+4x^2 \\ & (-)\ \ (-) \\ \hline & \quad -10x^2+15x \\ & \quad -10x^2-5x \\ & \quad\ \ (+)\quad\ (+) \\ \hline & \qquad\quad 20x-7 \\ & \qquad\quad 20x+10 \\ & \qquad\quad (-)\quad (-) \\ \hline & \qquad\qquad -17 \\ \hline \end{array}$$

$\therefore$ Quotient $= 4x^2-5x+10$ and

Remainder $= -17$

(ii) $f(x) = x^4-3x^3+5x^2-7$, $g(x) = x^2+x+1$

$$\begin{array}{r|l} & x^2-4x+8 \\ \hline x^2+x+1 & x^4-3x^3+5x^2+0x-7 \\ & x^4+x^3+x^2 \\ & (-)\ \ (-)\ \ (-) \\ \hline & \quad -4x^3+4x^2+0x \\ & \quad -4x^3-4x^2-4x \\ & \quad (+)\quad (+)\quad (+) \\ \hline & \qquad\quad 8x^2+4x-7 \\ & \qquad\quad 8x^2+8x+8 \\ & \qquad\quad (-)\ \ (-)\ \ (-) \\ \hline & \qquad\qquad -4x-15 \\ \hline \end{array}$$

$\therefore$ Quotient $= x^2-4x+8$ and

Remainder $= -4x-15$

3.6.2 Synthetic Division

Synthetic Division is a shortcut method of polynomial division. The advantage of synthetic division is that it allows one to calculate without writing variables, than long division.

Example 3.35 Find the quotient and remainder when $p(x) = (3x^3 - 2x^2 - 5 + 7x)$ is divided by $d(x) = x + 3$ using synthetic division.

Solution

Step 1 Arrange dividend and the divisor in standard form.

$3x^3 - 2x^2 + 7x - 5$ (standard form of dividend)

$x + 3$ (standard form of divisor)

Write the coefficients of dividend in the first row. Put '0' for missing term(s).

3 −2 7 −5 (first row)

Step 2 Find out the zero of the divisor.

$x + 3 = 0$ implies $x = -3$

Step 3 Write the zero of divisor in front of dividend in the first row. Put '0' in the first column of second row.

$$\begin{array}{r|cccc|l} -3 & 3 & -2 & 7 & -5 & \text{(first row)} \\ & 0 & & & & \text{(second row)} \\ \hline \end{array}$$

Step 4 Complete the second row and third row as shown below.

$$\begin{array}{r|ccccccc|l} -3 & 3 & & -2 & & 7 & & -5 & \text{(first row)} \\ & 0 & \xrightarrow{-3\times 3} & -9 & \xrightarrow{-3\times -11} & 33 & \xrightarrow{-3\times 40} & -120 & \text{(second row)} \\ \hline & 3 & & -11 & & 40 & & \boxed{-125} & \text{(third row)} \end{array}$$

All the entries except the last one in the third row are the coefficients of the quotient.

Then quotient is $3x^2 - 11x + 40$ and remainder is -125.

Example 3.36 Find the quotient and remainder when $(3x^3 - 4x^2 - 5)$ is divided by $(3x+1)$ using synthetic division.

Solution

Let $p(x) = 3x^3 - 4x^2 - 5$, $d(x) = (3x + 1)$

Standard form: $p(x) = 3x^3 - 4x^2 + 0x - 5$ and $d(x) = 3x + 1$

$$\begin{array}{r|cccc|l} -\frac{1}{3} & 3 & -4 & 0 & -5 & \\ & 0 & -1 & \frac{5}{3} & -\frac{5}{9} & \\ \hline & 3 & -5 & \frac{5}{3} & \boxed{-\frac{50}{9}} & \text{(remainder)} \end{array}$$

To find the zero of $3x$+1;

put $3x + 1 = 0$

we get $3x = -1$

$x = -\frac{1}{3}$

$$3x^3 - 4x^2 - 5 = \left(x + \frac{1}{3}\right)\left(3x^2 - 5x + \frac{5}{3}\right) - \frac{50}{9}$$

$$3x^3-4x^2-5 = \frac{(3x+1)}{3}\times 3\left(x^2-\frac{5}{3}x+\frac{5}{9}\right)-\frac{50}{9}$$

$$= (3x+1)\left(x^2-\frac{5}{3}x+\frac{5}{9}\right)-\left(\frac{50}{9}\right) \qquad \text{(since, } p(x)=d(x)q(x)+r\text{)}$$

Hence the quotient is $\left(x^2-\frac{5}{3}x+\frac{5}{9}\right)$ and remainder is $\frac{-50}{9}$

Example 3.37

If the quotient on dividing $x^4+10x^3+35x^2+50x+29$ by $(x+4)$ is x^3-ax^2+bx+6, then find the value of a, b and also remainder.

To find the zero of x+4;

put $x+4=0$

we get $x=-4$

Solution

Let $p(x)=x^4+10x^3+35x^2+50x+29$

Standard form $=x^4+10x^3+35x^2+50x+29$

Coefficient are 1 10 35 50 29

–4	1	10	35	50	29	
	0	–4	–24	–44	–24	
	1	6	11	6	5	(remainder)

quotient $x^3+6x^2+11x+6$ is compared with given quotient x^3-ax^2+bx+6

coefficient of x^2 is $6=-a$ and coefficient of x is $11=b$

Therefore, $a=-6$, $b=11$ and remainder$=5$.

Exercise 3.7

1. Find the quotient and remainder of the following.

   (i) $(4x^3+6x^2-23x+18)\div(x+3)$ (ii) $(8y^3-16y^2+16y-15)\div(2y-1)$

   (iii) $(8x^3-1)\div(2x-1)$ (iv) $(-18z+14z^2+24z^3+18)\div(3z+4)$

2. The area of a rectangle is $x^2+7x+12$. If its breadth is $(x+3)$, then find its length.

3. The base of a parallelogram is $(5x+4)$. Find its height, if the area is $25x^2-16$.

4. The sum of $(x+5)$ observations is (x^3+125). Find the mean of the observations.

5. Find the quotient and remainder for the following using synthetic division:

   (i) $(x^3+x^2-7x-3)\div(x-3)$ (ii) $(x^3+2x^2-x-4)\div(x+2)$

   (iii) $(3x^3-2x^2+7x-5)\div(x+3)$ (iv) $(8x^4-2x^2+6x+5)\div(4x+1)$

6. If the quotient obtained on dividing $(8x^4-2x^2+6x-7)$ by $(2x+1)$ is $(4x^3+px^2-qx+3)$, then find p, q and also the remainder.

7. If the quotient obtained on dividing $3x^3+11x^2+34x+106$ by $x-3$ is $3x^2+ax+b$, then find a, b and also the remainder.

3.6.3 Factorisation using Synthetic Division

In this section, we use the synthetic division method that helps to factorise a cubic polynomial into linear factors. If we identify one linear factor of cubic polynomial $p(x)$ then using synthetic division we can get the quadratic factor of $p(x)$. Further if possible one can factorise the quadratic factor into linear factors.

Note

- For any non constant polynomial $p(x)$, $x = a$ is zero if and only if $p(a) = 0$
- $x-a$ is a factor for $p(x)$ if and only if $p(a) = 0$ (Factor theorem)

To identify $(x-1)$ and $(x+1)$ are the factors of a polynomial

- $(x-1)$ is a factor of $p(x)$ if and only if the sum of coefficients of $p(x)$ is 0.
- $(x+1)$ is a factor of $p(x)$ if and only if the sum of the coefficients of even power of x, including constant is equal to the sum of the coefficients of odd powers of x

Example 3.38

(i) Prove that $(x-1)$ is a factor of $x^3-7x^2+13x-7$

(ii) Prove that $(x+1)$ is a factor of $x^3+7x^2+13x+7$

Solution

(i) Let $p(x)=x^3-7x^2+13x-7$

Sum of coefficients $=1-7+13-7=0$

Thus $(x-1)$ is a factor of $p(x)$

(ii) Let $q(x)=x^3+7x^2+13x+7$

Sum of coefficients of even powers of x and constant term = 7 + 7 = 14

Sum of coefficients of odd powers of x = 1 + 13 = 14

Hence, $(x+1)$ is a factor of $q(x)$

Example 3.39

Factorise $x^3+13x^2+32x+20$ into linear factors.

Solution

Let, $p(x)=x^3+13x^2+32x+20$

Sum of all the coefficients $=1+13+32+20=66\neq 0$

Hence, $(x-1)$ is not a factor.

Sum of coefficients of even powers and constant term = 13 + 20 = 33

Sum of coefficients of odd powers = 1 + 32 = 33

Hence, $(x+1)$ is a factor of $p(x)$

Now we use synthetic division to find the other factors

Method I

−1	1	13	32	20
	0	−1	−12	−20
−2	1	12	20	0 (remainder)
	0	−2	−20	
	1	10	0 (remainder)	

$$p(x) = (x+1)(x+2)(x+10)$$

Hence,

$$x^3 + 13x^2 + 32x + 20$$
$$= (x+1)(x+2)(x+10)$$

Method II

−1	1	13	32	20
	0	−1	−12	−20
	1	12	20	0 (remainder)

Then $p(x) = (x+1)(x^2 + 12x + 20)$

Now $x^2 + 12x + 20 = x^2 + 10x + 2x + 20$

$= x(x+10) + 2(x+10)$

$= (x+2)(x+10)$

Hence, $x^3 + 13x^2 + 32x + 20$
$= (x+1)(x+2)(x+10)$

Example 3.40

Factorise $x^3 - 5x^2 - 2x + 24$

Solution

Let $p(x) = x^3 - 5x^2 - 2x + 24$

When $x = 1, p(1) = 1 - 5 - 2 + 24 = 18 \neq 0$ $(x-1)$ is not a factor.

When $x = -1, p(-1) = -1 - 5 + 2 + 24 = 20 \neq 0$ $(x+1)$ is not a factor.

Therefore, we have to search for different values of x by trial and error method.

When $x = 2$

$p(2) = 2^3 - 5(2)^2 - 2(2) + 24$

$= 8 - 20 - 4 + 24$

$= 8 \neq 0$ Hence, $(x-2)$ is not a factor

When $x = -2$

$p(-2) = \left(-2\right)^3 - 5(-2)^2 - 2(-2) + 24$

$= -8 - 20 + 4 + 24$

$p(-2) = 0$

Hence, $(x+2)$ is a factor

Note

Check whether 3 is a zero of $x^2 - 7x + 12$. If it is not, then check for −3 or 4 or −4 and so on.

−2	1	−5	−2	24
	0	−2	+14	−24
3	1	−7	12	0 (remainder)
	0	3	−12	
	1	−4	0 (remainder)	

Thus, $(x+2)(x-3)(x-4)$ are the factors.

Therefore, $x^3 - 5x^2 - 2x^2 + 24 = (x+2)(x-3)(x-4)$

Exercise 3.8

1. Factorise each of the following polynomials using synthetic division:

(i) $x^3 - 3x^2 - 10x + 24$ (ii) $2x^3 - 3x^2 - 3x + 2$

(iii) $-7x + 3 + 4x^3$ (iv) $x^3 + x^2 - 14x - 24$

(v) $x^3 - 7x + 6$ (vi) $x^3 - 10x^2 - x + 10$

3.7 Greatest Common Divisor (GCD)

The Greatest Common Divisor, abbreviated as GCD, of two or more polynomials is a polynomial, of the highest common possible degree, that is a factor of the given two or more polynomials. It is also known as the Highest Common Factor (HCF).

This concept is similar to the greatest common divisor of two integers.

For example, Consider the expressions $14xy^2$ and $42xy$. The common divisors of 14 and 42 are 2, 7 and 14. Their GCD is thus 14. The only common divisors of xy^2 and xy *are* x, y and xy; their GCD is thus xy.

$$14xy^2 = 1 \times 2 \times 7 \times x \times y \times y$$

$$42xy = 1 \times 2 \times 3 \times 7 \times x \times y$$

Therefore the requried GCD of $14xy^2$ and $42xy$ is $14xy$.

To find the GCD by Factorisation

(i) Each expression is to be resolved into factors first.

(ii) The product of factors having the highest common powers in those factors will be the GCD.

(iii) If the expression have numerical coefficient, find their GCD separately and then prefix it as a coefficient to the GCD for the given expressions.

Example 3.41 Find GCD of the following:

(i) $16x^3y^2$, $24xy^3z$

(ii) $(y^3 + 1)$ and $(y^2 - 1)$

(iii) $2x^2 - 18$ and $x^2 - 2x - 3$

(iv) $(a-b)^2$, $(b-c)^3$, $(c-a)^4$

Solutions

(i) $16x^3y^2 = 2 \times 2 \times 2 \times 2 \times x^3y^2 = 2^4 \times x^3 \times y^2 = 2^3 \times 2 \times x^2 \times x \times y^2$

$$24xy^3z = 2\times2\times2\times3\times x\times y^3\times z = 2^3\times3\times x\times y^3\times z = 2^3\times3\times x\times y\times y^2\times z$$

Therefore, $GCD = 2^3xy^2$

(ii) $y^3+1 = y^3+1^3 = (y+1)(y^2-y+1)$

$y^2-1 = y^2-1^2 = (y+1)(y-1)$

Therefore, $GCD = (y+1)$

(iii) $2x^2-18 = 2(x^2-9) = 2(x^2-3^2) = 2(x+3)(x-3)$

$$x^2-2x-3 = x^2-3x+x-3$$

$$= x(x-3)+1(x-3)$$

$$= (x-3)(x+1)$$

Therefore, $GCD = (x-3)$

(iv) $(a-b)^2,\ (b-c)^3,\ (c-a)^4$

There is no common factor other than one.

Therefore, $GCD = 1$

Exercise 3.9

1. Find the GCD for the following:

(i) $p^5,\ p^{11},\ p^9$

(ii) $4x^3,\ y^3,\ z^3$

(iii) $9a^2b^2c^3,\ 15a^3b^2c^4$

(iv) $64x^8,\ 240x^6$

(v) $ab^2c^3,\ a^2b^3c,\ a^3bc^2$

(vi) $35x^5y^3z^4,\ 49x^2yz^3,\ 14xy^2z^2$

(vii) $25ab^3c,\ 100a^2bc,\ 125ab$

(viii) $3abc, 5xyz, 7pqr$

2. Find the GCD of the following:

(i) $(2x+5),\ (5x+2)$

(ii) $a^{m+1},\ a^{m+2},\ a^{m+3}$

(iii) $2a^2+a,\ 4a^2-1$

(iv) $3a^2,\ 5b^3,\ 7c^4$

(v) $x^4-1,\ x^2-1$

(vi) $a^3-9ax^2,\ (a-3x)^2$

3.8 Linear Equation in Two Variables

A linear equation in two variables is of the form $ax + by + c = 0$ where a, b and c are real numbers, both a and b are not zero (The two variables are denoted here by x and y and c is a constant).

Examples

Linear equation in two variables	Not a linear equation in two variables.
$2x + y = 4$	$xy + 2x = 5$ (Why?)
$-5x + \frac{1}{2} = y$	$\sqrt{x} + \sqrt{y} = 25$ (Why?)
$5x = 35y$	$x(x+1) = y$ (Why?)

If an equation has two variables each of which is in first degree such that the variables are not multiplied with each other, then it is a linear equation in two variables (If the degree of an equation in two variables is 1, then it is called a linear equation in two variables).

An understanding of linear equation in two variables will be easy if it is done along with a geometrical visualization (through graphs). We will make use of this resource.

Why do we classify, for example, the equation $2x + y = 4$ is a linear equation? You are right; because its graph will be a line. Shall we check it up?

We try to draw its graph. To draw the graph of $2x + y = 4$, we need some points on the line so that we can join them. (These are the ordered pairs satisfying the equation).

To prepare table giving ordered pairs for $2x + y = 4$. It is better, to take it as

$y = 4 - 2x$. (Why? How?)

When $x = -4$, $y = 4 - 2(-4) = 4 + 8 = 12$

When $x = -2$, $y = 4 - 2(-2) = 4 + 4 = 8$

When $x = 0$, $y = 4 - 2(0) = 4 + 0 = 4$

When $x = +1$, $y = 4 - 2(+1) = 4 - 2 = 2$

When $x = +3$, $y = 4 - 2(+3) = 4 - 6 = -2$

Thus the values are tabulated as follows:

x-value	-4	-2	0	1	3
y-value	12	8	4	2	-2

(To fix a line, do we need so many points? It is enough if we have two and probably one more for verification.)

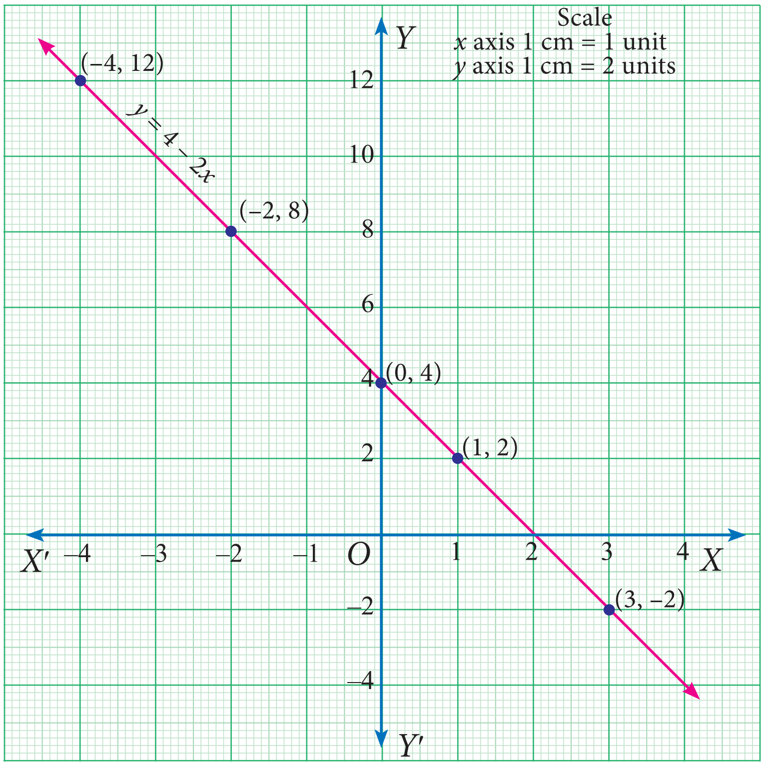


Fig. 3.15

When you plot the points (−4,12), (−2,8), (0,4), (1,2) and (3,–2), you find that they all lie on a line.

This clearly shows that the equation $2x + y = 4$ represents a line (and hence said to be linear).

All the points on the line satisfy this equation and hence the ordered pairs of all the points on the line are the solutions of the equation.

Example 3.42 Draw the graph for the following:

(i) $y = 3x - 1$ (ii) $y = \left(\frac{2}{3}\right)x + 3$

Solution

(i) Let us prepare a table to find the ordered pairs of points for the line $y = 3x - 1$.

We shall assume any value for x, for our convenience let us take −1, 0 and 1.

When $x = -1$, $y = 3(-1) - 1 = -4$

When $x = 0$, $y = 3(0) - 1 = -1$

When $x = 1$, $y = 3(1) - 1 = 2$

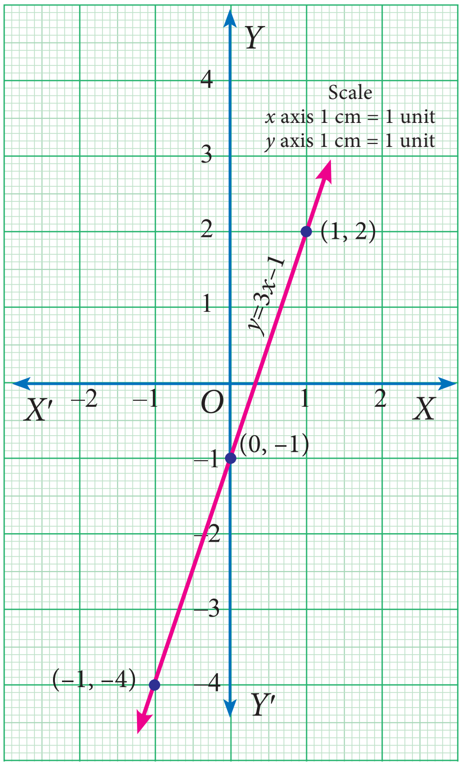


Fig. 3.16

x	-1	0	1
y	-4	-1	2

The points (x,y) to be plotted :

$(-1, -4)$, $(0, -1)$ and $(1, 2)$.

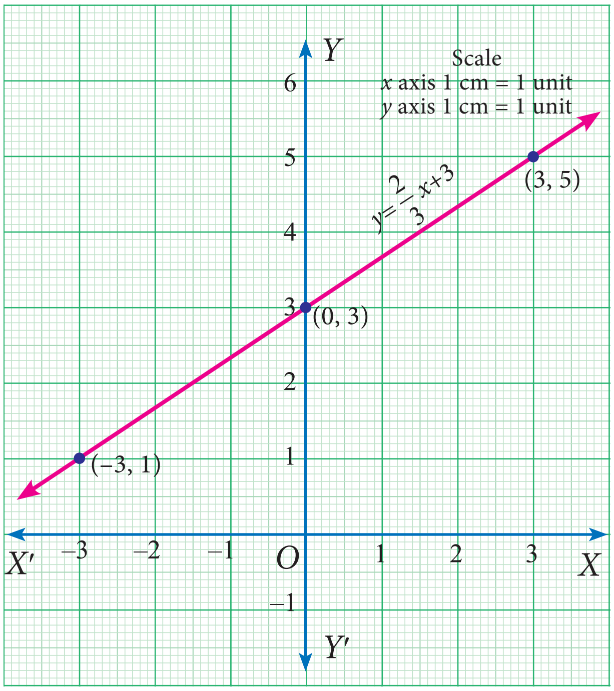


Fig. 3.17

(ii) Let us prepare a table to find the ordered pairs of points

for the line $y = \left(\frac{2}{3}\right)x + 3$.

Let us assume −3, 0, 3 as x values. (why?)

When $x = -3$, $y = \frac{2}{3}(-3) + 3 = 1$

When $x = 0$, $y = \frac{2}{3}(0) + 3 = 3$

When $x = 3$, $y = \frac{2}{3}(3) + 3 = 5$

x	-3	0	3
y	1	3	5

The points (x,y) to be plotted :

$(-3,1)$, $(0,3)$ and $(3,5)$.

3.8.1 Simultaneous Linear Equations

With sufficient background of graphing an equation, now we are set to study about system of equations, particularly pairs of simultaneous equations.

What are simultaneous linear equations? These consists of two or more linear equations with the same variables.

Why do we need them? A single equation like $2x+y=10$ has an unlimited number of solutions. The points (1,8), (2,6), (3,4) and many more lie on the graph of the equation, which means these are some of its endless list of solutions. To be able to solve an equation like this, another equation needs to be used alongside it; then it is possible to find a single ordered pair that solves both equations at the same time.

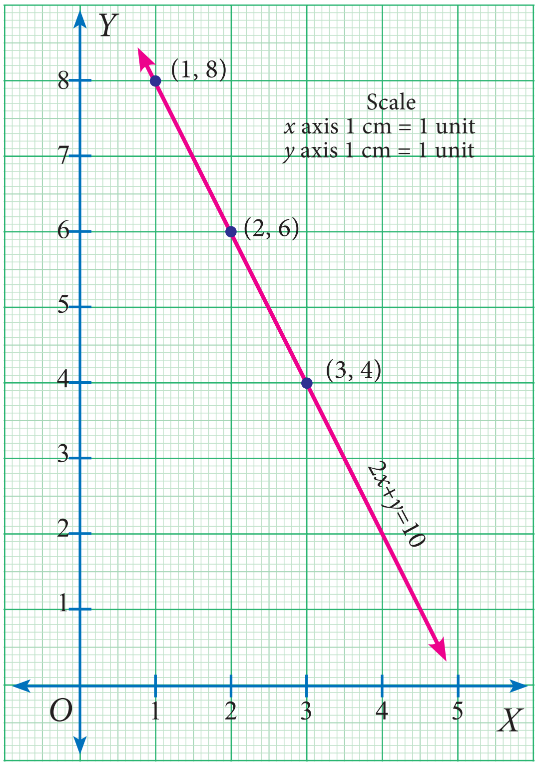


Fig. 3.18

The equations we consider together in such settings make a meaningful situation and are known as simultaneous linear equations.

Real life Situation to understand the simultaneous linear equations

Consider the situation, Anitha bought two erasers and a pencil for ₹10. She does not know the individual cost of each. We shall form an equation by considering the cost of eraser as 'x' and that of pencil as 'y'.

That is $2x + y = 10$ ... (1)

Now, Anitha wants to know the individual cost of an eraser and a pencil. She tries to solve the first equation, assuming various values of x and y.

G2T8JA

2 × cost of eraser + 1 × cost of pencil = 10

$2(1)+8 = 10$

$2(1.5)+7 = 10$

$2(2)+6 = 10$

$2(2.5)+5 = 10$

$2(3)+4 = 10$

⋮ ⋮

Points to be plotted :

x	1	1.5	2	2.5	3	...
y	8	7	6	5	4	...

She gets infinite number of answers. So she tries to find the cost with the second equation.

Again, Anitha needs some more pencils and erasers. This time, she bought 3 erasers and 4 pencils and the shopkeeper received ₹30 as the total cost from her. We shall form an equation like the previous one.

The equation is $3x + 4y = 30$ ...(2)

Even then she arrives at an infinite number of answers.

3 × cost of eraser + 4 × cost of pencil = 30

$3(2)+4\,(6) = 30$

$3(4)+4\,(4.5) = 30$

$3(6)+4\,(3) = 30$

$3(8)+4\,(1.5) = 30$

⋮ ⋮

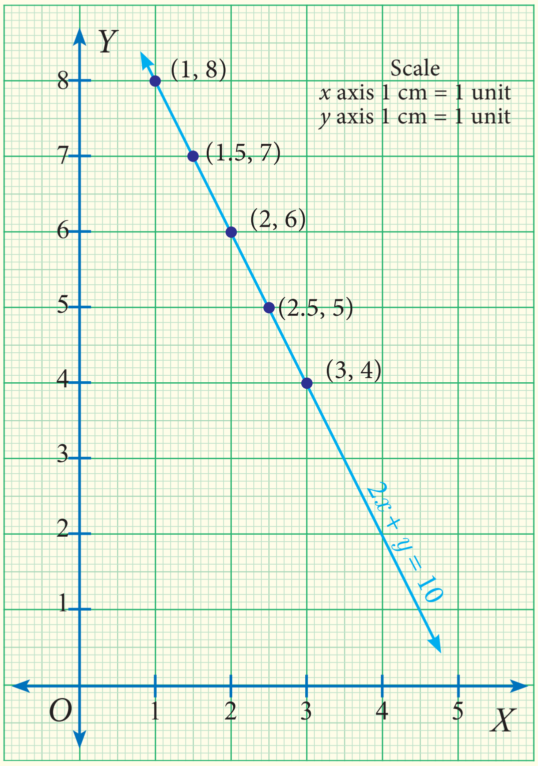


Fig. 3.19

Points to be plotted :

x	2	4	6	8	...
y	6	4.5	3	1.5	...

While discussing this with her teacher, the teacher suggested that she can get a unique answer if she solves both the equations together.

By solving equations (1) and (2) we have the cost of an eraser as ₹2 and cost of a pencil as ₹6. It can be visualised in the graph.

The equations we consider together in such settings make a meaningful situation and are known as simultaneous linear equations.

Thus a system of linear equations consists of two or more linear equations with the same variables. Then such equations are called Simultaneous linear equations or System of linear equations or a Pair of linear equations.

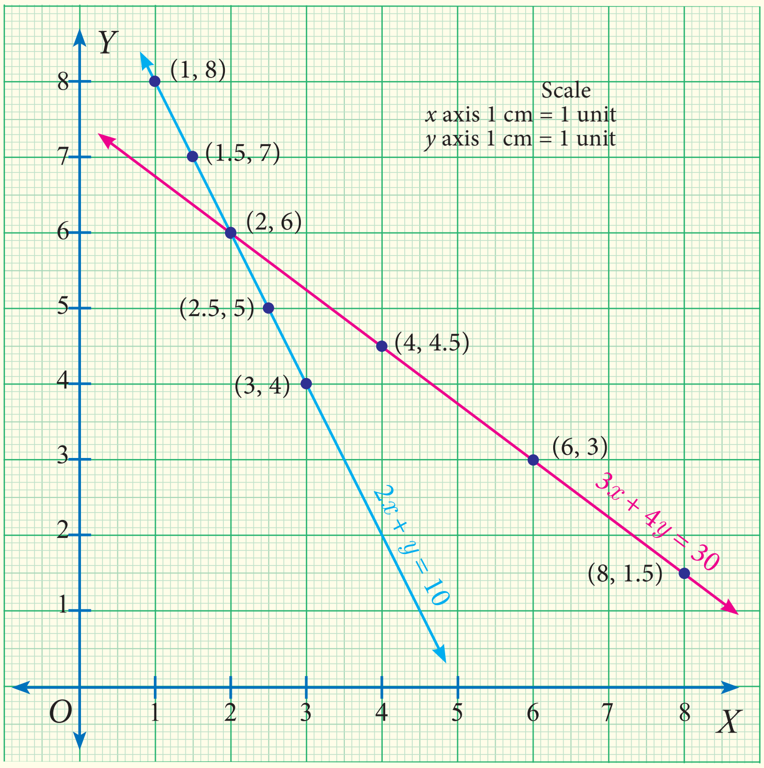


Fig. 3.20

Example 3.43 Check whether (5, −1) is a solution of the simultaneous equations $x - 2y = 7$ and $2x + 3y = 7$.

Solution

Given $x - 2y = 7$ ...(1)

$2x + 3y = 7$ ...(2)

When $x = 5,\ y = -1$ we get

From (1) $x - 2y = 5 - 2(-1) = 5 + 2 = 7$ which is RHS of (1)

From (2) $2x + 3y = 2(5) + 3(-1) = 10 - 3 = 7$ which is RHS of (2)

Thus the values x = 5, y = −1 satisfy both (1) and (2) simultaneously. Therefore (5,–1) is a solution of the given equations.

Progress Check

Examine if (3,3) will be a solution for the simultaneous linear equations $2x - 5y - 2 = 0$ and $x + y - 6 = 0$ by drawing a graph.

3.8.2 Methods of solving simultaneous linear equations

There are different methods to find the solution of a pair of simultaneous linear equations. It can be broadly classified as geometric way and algebraic ways.

Geometric way	Algebraic ways
1. Graphical method	1. Substitution method 2. Elimination method 3. Cross multiplication method

Solving by Graphical Method

Already we have seen graphical representation of linear equation in two variables. Here we shall learn, how we are graphically representing a pair of linear equations in two variables and find the solution of simultaneous linear equations.

Example 3.44 Use graphical method to solve the following system of equations: $x + y = 5$; $2x - y = 4$.

Solution

Given $\quad x + y = 5 \quad$ ...(1)

$\quad 2x - y = 4 \quad$ ...(2)

To draw the graph (1) is very easy. We can find the x and y values and thus two of the points on the line (1).

When $x = 0$, (1) gives $y = 5$.

Thus $A(0,5)$ is a point on the line.

When $y = 0$, (1) gives $x = 5$.

Thus $B(5,0)$ is another point on the line.

Plot A and B ; join them to produce the line (1).

To draw the graph of (2), we can adopt the same procedure.

When $x = 0$, (2) gives $y = -4$.

Thus P(0,–4) is a point on the line.

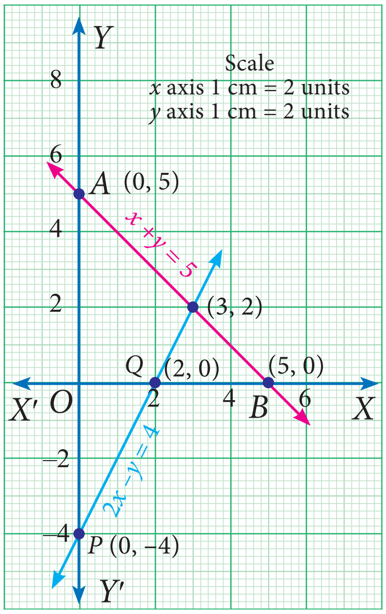


Fig. 3.21

When $y = 0$, (2) gives $x = 2$.

Thus $Q(2,0)$ is another point on the line.

Plot P and Q ; join them to produce the line (2).

The point of intersection (3, 2) of lines (1) and (2) is a solution.

The solution is the point that is common to both the lines. Here we find it to be (3,2). We can give the solution as $x = 3$ and $y = 2$.

Note

It is always good to verify if the answer obtained is correct and satisfies both the given equations.

Example 3.45 Use graphical method to solve the following system of equations: $3x + 2y = 6; 6x + 4y = 8$

Solution

Let us form table of values for each line and then fix the ordered pairs to be plotted.

Graph of $3x + 2y = 6$

x	−2	0	2
y	6	3	0

Points to be plotted :

(−2,6), (0,3), (2,0)

Graph of $6x + 4y = 8$

x	−2	0	2
y	5	2	−1

Points to be plotted :

(−2,5), (0,2), (2,−1)

When we draw the graphs of these two equations, we find that they are parallel and they fail to meet to give a point of intersection. As a result there is no ordered pair that can be common to both the equations. In this case there is no solution to the system.

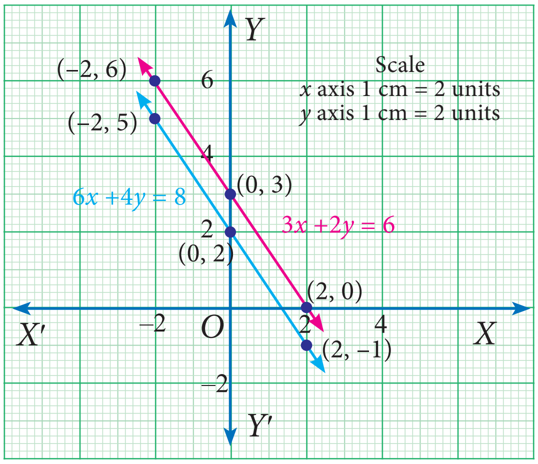


Fig. 3.22

Example 3.46 Use graphical method to solve the following system of equations: $y = 2x + 1; -4x + 2y = 2$

Solution

Let us form table of values for each line and then fix the ordered pairs to be plotted.

Graph of $y = 2x + 1$

Graph of $-4x + 2y = 2$

$$2y = 4x + 2$$

$$y = 2x + 1$$

x	-2	-1	0	1	2
$2x$	-4	-2	0	2	4
1	1	1	1	1	1
$y = 2x+1$	-3	-1	1	3	5

Points to be plotted :

$(-2, -3), (-1, -1), (0,1), (1,3), (2, 5)$

x	-2	-1	0	1	2
$2x$	-4	-2	0	2	4
1	1	1	1	1	1
$y = 2x+1$	-3	-1	1	3	5

Points to be plotted :

$(-2, -3), (-1, -1), (0, 1), (1, 3), (2, 5)$

Here both the equations are identical; they were only represented in different forms. Since they are identical, their solutions are same. All the points on one line are also on the other!

This means we have an infinite number of solutions which are the ordered pairs of all the points on the line.

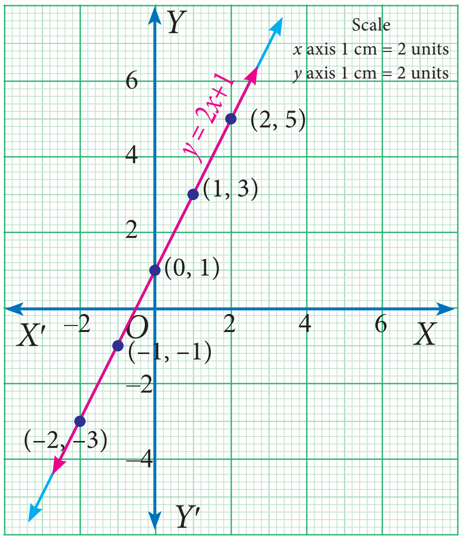


Fig. 3.23

Example 3.47

The perimeter of a rectangle is 36 metres and the length is 2 metres more than three times the width. Find the dimension of rectangle by using the method of graph.

Solution

Let us form equations for the given statement.

Let us consider l and b as the length and breadth of the rectangle respectively.

Now let us frame the equation for the first statement

Perimeter of rectangle = 36

$$2(l+b) = 36$$

$$l+b = \frac{36}{2}$$

$$l = 18 - b \qquad \dots (1)$$

b	2	4	5	8
18	18	18	18	18
−b	−2	−4	−5	−8
l = 18−b	16	14	13	10

Points: (2,16), (4,14), (5,13), (8,10)

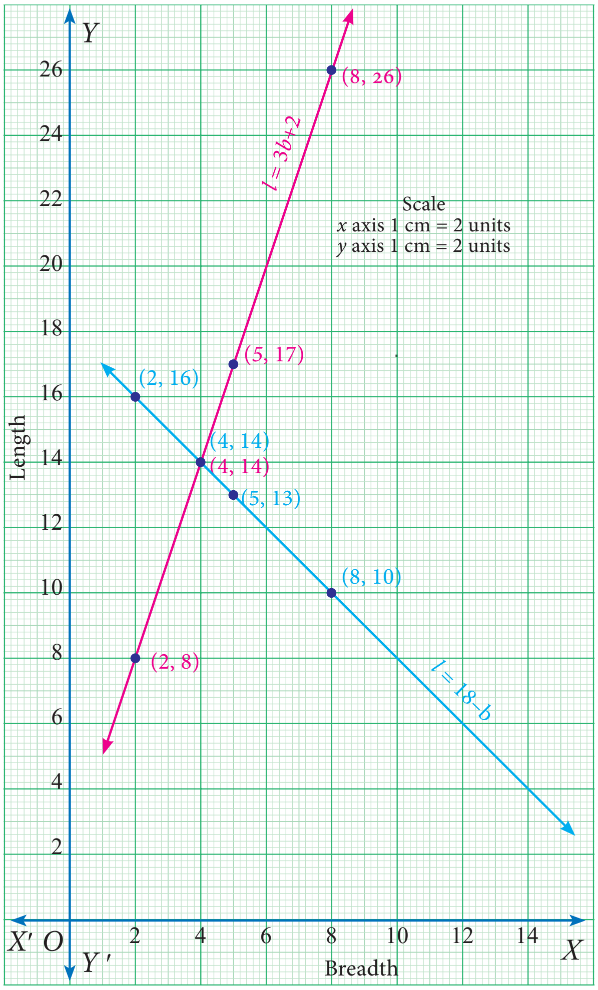


Fig. 3.24

The second statement states that the length is 2 metres more than three times the width which is a straight line written as $l = 3b + 2$ ... (2)

Now we shall form table for the above equation (2).

b	2	4	5	8
$3b$	6	12	15	24
2	2	2	2	2
$l = 3b + 2$	8	14	17	26

Points: (2,8), (4,14), (5,17), (8,26)

The solution is the point that is common to both the lines. Here we find it to be (4,14). We can give the solution to be $b = 4$, $l = 14$.

Verification :

$$\begin{aligned} 2(l+b) &= 36 \quad \text{...(1)} \\ 2(14+4) &= 36 \\ 2 \times 18 &= 36 \\ 36 &= 36 \quad \text{true} \end{aligned} \qquad \begin{aligned} l &= 3b + 2 \quad \text{...(2)} \\ 14 &= 3(4) + 2 \\ 14 &= 12 + 2 \\ 14 &= 14 \quad \text{true} \end{aligned}$$

Exercise 3.10

1. Draw the graph for the following

   (i) $y = 2x$ (ii) $y = 4x - 1$ (iii) $y = \left(\frac{3}{2}\right)x + 3$ (iv) $3x + 2y = 14$

2. Solve graphically

   (i) $x + y = 7;\ x - y = 3$ (ii) $3x + 2y = 4;\ 9x + 6y - 12 = 0$

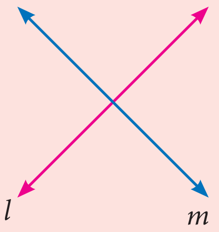
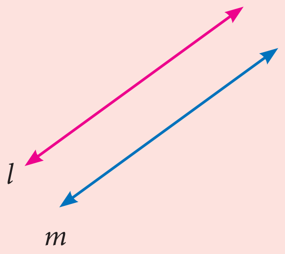
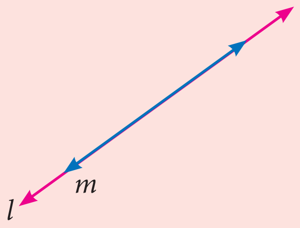
   (iii) $\frac{x}{2} + \frac{y}{4} = 1;\ \frac{x}{2} + \frac{y}{4} = 2$ (iv) $x - y = 0;\ \ y + 3 = 0$

   (v) $y = 2x + 1;\ y + 3x - 6 = 0$ (vi) $x = -3;\ y = 3$

3. Two cars are 100 miles apart. If they drive towards each other they will meet in 1 hour. If they drive in the same direction they will meet in 2 hours. Find their speed by using graphical method.

Some special terminology

We found that the graphs of equations within a system tell us how many solutions are there for that system. Here is a visual summary.

Intersecting lines	Parallel lines	Coinciding lines
		
One single solution	*No solution*	*Infinite number of solutions*

- When a system of linear equation has one solution (the graphs of the equations intersect once), the system is said to be a consistent system.
- When a system of linear equation has no solution (the graphs of the equations don't intersect at all), the system is said to be an inconsistent system.
- When a system of linear equation has infinitely many solutions, the lines are the same (the graph of lines are identical at all points), the system is consistent.

Solving by Substitution Method

In this method we substitute the value of one variable, by expressing it in terms of the other variable to reduce the given equation of two variables into equation of one variable (in order to solve the pair of linear equations). Since we are substituting the value of one variable in terms of the other variable, this method is called substitution method.

The procedure may be put shortly as follows:

Step 1: From any of the given two equations, find the value of one variable in terms of the other.

Step 2: Substitute the value of the variable, obtained in step 1 in the other equation and solve it.

Step 3: Substitute the value of the variable obtained in step 2 in the result of step 1 and get the value of the remaining unknown variable.

Example 3.48

Solve the system of linear equations $x+3y=16$ and $2x-y=4$ by substitution method.

Solution

Given $x+3y=16$ ... (1)

$2x-y=4$ ... (2)

Step 1	Step 2	Step 3	Solution
From equation (2) $2x-y=4$ $-y=4-2x$ $y=2x-4$ ...(3)	Substitute (3) in (1) $x+3y=16$ $x+3(2x-4)=16$ $x+6x-12=16$ $7x=28$ $x=4$	Substitute $x=4$ in (3) $y=2x-4$ $y=2(4)-4$ $y=4$	$x=4$ and $y=4$

Example 3.49

The sum of the digits of a given two digit number is 5. If the digits are reversed, the new number is reduced by 27. Find the given number.

Solution

Let x be the digit at ten's place and y be the digit at unit place.

Given that $x+y=5$ …… (1)

	Tens	**Ones**	**Value**
Given Number	x	y	$10x + y$
New Number (after reversal)	y	x	$10y+x$

Given, Original number − reversing number $= 27$

$$(10x + y) - (10y + x) = 27$$

$$10x - x + y - 10y = 27$$

$$9x - 9y = 27$$

$$\Rightarrow \qquad x - y = 3 \qquad \text{... (2)}$$

Also from (1), $y = 5 - x$ ... (3)

Substitute (3) in (2) to get $x - (5 - x) = 3$

$$x - 5 + x = 3$$

$$2x = 8$$

$$x = 4$$

Substituting $x = 4$ in (3), we get $y = 5 - x = 5 - 4$

$$y = 1$$

Thus, $10x + y = 10 \times 4 + 1 = 40 + 1 = 41$.

Therefore, the given two-digit number is 41.

Verification :

sum of the digits = 5

$x + y = 5$

$4 + 1 = 5$

$5 = 5$ true

Original number − reversed number = 27

$41 - 14 = 27$

$27 = 27$ true

Exercise 3.11

1. Solve, using the method of substitution

   (i) $2x - 3y = 7;\ 5x + y = 9$

   (ii) $1.5x + 0.1y = 6.2;\ 3x - 0.4y = 11.2$

   (iii) $10\%\ of\ x + 20\%\ of\ y = 24;\ 3x - y = 20$

   (iv) $\sqrt{2}\,x - \sqrt{3}\,y = 1;\ \sqrt{3}\,x - \sqrt{8}\,y = 0$

2. Raman's age is three times the sum of the ages of his two sons. After 5 years his age will be twice the sum of the ages of his two sons. Find the age of Raman.

3. The middle digit of a number between 100 and 1000 is zero and the sum of the other digit is 13. If the digits are reversed, the number so formed exceeds the original number by 495. Find the number.

Solving by Elimination Method

This is another algebraic method for solving a pair of linear equations. This method is more convenient than the substitution method. Here we eliminate (i.e. remove) one of the two variables in a pair of linear equations, so as to get a linear equation in one variable which can be solved easily.

The various steps involved in the technique are given below:

Step 1: Multiply one or both of the equations by a suitable number(s) so that either the coefficients of first variable or the coefficients of second variable in both the equations become numerically equal.

Step 2: Add both the equations or subtract one equation from the other, as obtained in step 1, so that the terms with equal numerical coefficients cancel mutually.

Step 3: Solve the resulting equation to find the value of one of the unknowns.

Step 4: Substitute this value in any of the two given equations and find the value of the other unknown.

Example 3.50

Given $4a+3b=65$ and $a+2b=35$ solve by elimination method.

Solution

Given, $4a+3b=65$ .....(1)

$a+2b=35$ .....(2)

$$\begin{array}{lrcrcr} (2)\times 4 \text{ gives} & 4a & + & 8b & = & 140 \\ & (-) & & (-) & & (-) \\ \text{Already (1) is} & \underline{4a} & \underline{+} & \underline{3b} & \underline{=} & \underline{65} \\ & & & 5b & = & 75 \end{array}$$

which gives $b=15$

Put $b=15$ in (2):

$a+2(15)=35$ which simplifies to $a=5$

Thus the solution is $a=5,\ b=15$.

Verification :

$$\begin{aligned} 4a+3b &= 65 \quad ...(1) \\ 4(5)+3(15) &= 65 \\ 20+45 &= 65 \\ 65 &= 65 \quad \text{True} \end{aligned}$$

$$\begin{aligned} a+2b &= 35 \quad ...(2) \\ 5+2(15) &= 35 \\ 5+30 &= 35 \\ 35 &= 35 \quad \text{True} \end{aligned}$$

Example 3.51

Solve for x and y: $8x-3y=5xy,\ \ 6x-5y=-2xy$ by the method of elimination.

Solution

The given system of equations are $8x-3y=5xy$ ...(1)

$6x-5y=-2xy$ ...(2)

Observe that the given system is not linear because of the occurrence of xy term. Also note that if $x=0$, then $y=0$ and vice versa. So, (0,0) is a solution for the system and any other solution would have both $x\neq 0$ and $y\neq 0$.

Let us take up the case where $x \neq 0,\ y \neq 0$.

Dividing both sides of each equation by xy,

$$\frac{8x}{xy} - \frac{3y}{xy} = \frac{5xy}{xy} \qquad \text{we get,} \qquad \frac{8}{y} - \frac{3}{x} = 5 \qquad \text{...(3)}$$

$$\frac{6x}{xy} - \frac{5y}{xy} = \frac{-2xy}{xy} \qquad \frac{6}{y} - \frac{5}{x} = -2 \qquad \text{...(4)}$$

Let $a = \frac{1}{x},\ b = \frac{1}{y}$.

(3)&(4) respectively become, $8b - 3a = 5$ ...(5)

$$6b - 5a = -2 \qquad \text{...(6)}$$

which are linear equations in a and b.

To eliminate a, we have, $(5) \times 5 \Rightarrow \quad 40b - 15a = 25$ .....(7)

$$(6) \times 3 \Rightarrow \quad 18b - 15a = -6 \qquad \text{.....(8)}$$

Now proceed as in the previous example to get the solution $\left(\frac{11}{23}, \frac{22}{31}\right)$.

Thus, the system have two solutions $\left(\frac{11}{23}, \frac{22}{31}\right)$ and $(0,0)$.

Exercise 3.12

1. Solve by the method of elimination

   (i) $2x - y = 3;\quad 3x + y = 7$

   (ii) $x - y = 5;\quad 3x + 2y = 25$

   (iii) $\frac{x}{10} + \frac{y}{5} = 14;\quad \frac{x}{8} + \frac{y}{6} = 15$

   (iv) $3(2x + y) = 7xy;\quad 3(x + 3y) = 11xy$

   (v) $\frac{4}{x} + 5y = 7;\ \frac{3}{x} + 4y = 5$

   (vi) $13x + 11y = 70;\ 11x + 13y = 74$

2. The monthly income of A and B are in the ratio 3:4 and their monthly expenditures are in the ratio 5:7. If each saves ₹ 5,000 per month, find the monthly income of each.

3. Five years ago, a man was seven times as old as his son, while five year hence, the man will be four times as old as his son. Find their present age.

Solving by Cross Multiplication Method

The substitution and elimination methods involves many arithmetic operations, whereas the cross multiplication method utilize the coefficients effectively, which simplifies the procedure to get the solution. This method of cross multiplication is so called because we draw cross ways between the numbers in the denominators and cross multiply the coefficients along the arrows ahead. Now let us discuss this method as follows:

Suppose we are given a pair of linear simultaneous equations such as

$$a_1x+b_1y+c_1=0 \qquad ...(1)$$

$$a_2x+b_2y+c_2=0 \qquad ...(2)$$

such that $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$. We can solve them as follows :

$(1) \times b_2 - (2) \times b_1$ gives $b_2(a_1x+b_1y+c_1) - b_1(a_2x+b_2y+c_2) = 0$

$$\Rightarrow \quad x(a_1b_2 - a_2b_1) = (b_1c_2 - b_2c_1)$$

$$\Rightarrow \quad x = \frac{(b_1c_2 - b_2c_1)}{(a_1b_2 - a_2b_1)}$$

$(1) \times a_2 - (2) \times a_1$ similarly can be considered and that will simplify to

$$y = \frac{(c_1a_2 - c_2a_1)}{(a_1b_2 - a_2b_1)}$$

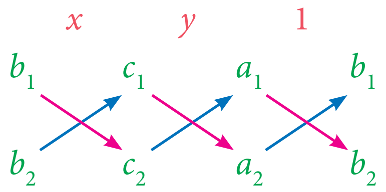
Hence the solution for the system is

$$\boxed{x = \frac{(b_1c_2 - b_2c_1)}{(a_1b_2 - a_2b_1)}, \quad y = \frac{(c_1a_2 - c_2a_1)}{(a_1b_2 - a_2b_1)}}$$

This can also be written as

$$\boxed{\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}}$$

which can be remembered as



Example 3.52 Solve $3x-4y=10$ and $4x+3y=5$ by the method of cross multiplication.

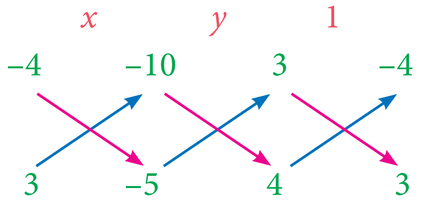
Solution

The given system of equations are

$$3x-4y=10 \qquad \Rightarrow 3x-4y-10=0 \quad .....(1)$$

$$4x+3y=5 \qquad \Rightarrow 4x+3y-5=0 \quad .....(2)$$

For the cross multiplication method, we write the co-efficients as



$$\frac{x}{(-4)(-5)-(3)(-10)}=\frac{y}{(-10)(4)-(-5)(3)}=\frac{1}{(3)(3)-(4)(-4)}$$

$$\frac{x}{(20)-(-30)}=\frac{y}{(-40)-(-15)}=\frac{1}{(9)-(-16)}$$

$$\frac{x}{20+30}=\frac{y}{-40+15}=\frac{1}{9+16}$$

$$\frac{x}{50}=\frac{y}{-25}=\frac{1}{25}$$

Therefore, we get $x=\frac{50}{25}$; $y=\frac{-25}{25}$

$x=2$; $y=-1$

Thus the solution is $x = 2$, $y = -1$.

Verification :

$$\begin{aligned} 3x-4y &= 10 \quad ...(1) \\ 3(2)-4(-1) &= 10 \\ 6+4 &= 10 \\ 10 &= 10 \quad \text{True} \end{aligned}$$

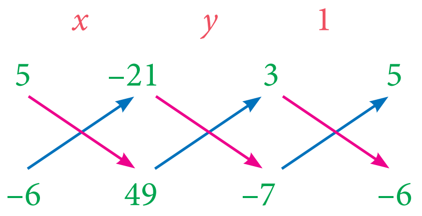
$$\begin{aligned} 4x+3y &= 5 \quad ...(2) \\ 4(2)+3(-1) &= 5 \\ 8-3 &= 5 \\ 5 &= 5 \quad \text{True} \end{aligned}$$

Example 3.53 Solve by cross multiplication method : $3x+5y=21$; $-7x-6y=-49$

Solution

The given system of equations are $3x+5y-21=0$; $-7x-6y+49=0$

Now using the coefficients for cross multiplication, we get,



$$\Rightarrow \frac{x}{(5)(49)-(-6)(-21)}=\frac{y}{(-21)(-7)-(49)(3)}=\frac{1}{(3)(-6)-(-7)(5)}$$

$$\frac{x}{119}=\frac{y}{0}=\frac{1}{17}$$

$$\Rightarrow \quad \frac{x}{119}=\frac{1}{17}, \quad \frac{y}{0}=\frac{1}{17}$$

$$\Rightarrow \quad x=\frac{119}{17}, \quad y=\frac{0}{17}$$

$$\Rightarrow \quad x=7, \quad y=0$$

Verification :

$$\begin{aligned} 3x+5y &= 21 \quad ...(1) \\ 3(7)+5(0) &= 21 \\ 21+0 &= 21 \\ 21 &= 21 \quad \text{True} \end{aligned}$$

$$\begin{aligned} -7x-6y &= -49 \ ...(2) \\ -7(7)-6(0) &= -49 \\ -49 &= -49 \\ -49 &= -49 \ \text{True} \end{aligned}$$

Note

Here $\frac{y}{0}=\frac{1}{17}$ is to mean $y=\frac{0}{17}$. Thus, $\frac{y}{0}$ is only a notation and it is not division by zero. It is always true that division by zero is not defined.

Exercise 3.13

1. Solve by cross-multiplication method

   (i) $8x-3y=12$; $5x=2y+7$ (ii) $6x+7y-11=0$; $5x+2y=13$

   (iii) $\frac{2}{x}+\frac{3}{y}=5$; $\frac{3}{x}-\frac{1}{y}+9=0$

2. Akshaya has 2 rupee coins and 5 rupee coins in her purse. If in all she has 80 coins totalling ₹ 220, how many coins of each kind does she have.

3. It takes 24 hours to fill a swimming pool using two pipes. If the pipe of larger diameter is used for 8 hours and the pipe of the smaller diameter is used for 18 hours. Only half of the pool is filled. How long would each pipe take to fill the swimming pool.

3.8.3 Consistency and Inconsistency of Linear Equations in Two Variables

Consider linear equations in two variables say

$a_1x+b_1y+c_1=0$ ...(1)

$a_2x+b_2y+c_2=0$ ...(2) where a_1, a_2, b_1, b_2, c_1 and c_2 are real numbers.

Then the system has :

(i) a unique solution if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ (Consistent)

(ii) an Infinite number of solutions if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ (Consistent)

(iii) no solution if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ (Inconsistent)

Example 3.54 Check whether the following system of equation is consistent or inconsistent and say how many solutions we can have if it is consistent.

(i) $2x-4y=7$ (ii) $4x+y=3$ (iii) $4x+7=2y$

$x-3y=-2$ $8x+2y=6$ $2x+9=y$

Solution

Sl. No	Pair of lines	$\frac{a_1}{a_2}$	$\frac{b_1}{b_2}$	$\frac{c_1}{c_2}$	Compare the ratios	Graphical represen-tation	Algebraic inter-pretation
(i)	$2x-4y=7$ $x-3y=-2$	$\frac{2}{1}=2$	$\frac{-4}{-3}=\frac{4}{3}$	$\frac{7}{-2}=\frac{-7}{2}$	$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	Intersecting lines	Unique solution
(ii)	$4x+y=3$ $8x+2y=6$	$\frac{4}{8}=\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{6}=\frac{1}{2}$	$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$	Coinciding lines	Infinite many solutions
(iii)	$4x+7=2y$ $2x+9=y$	$\frac{4}{2}=2$	$\frac{2}{1}=2$	$\frac{7}{9}$	$\frac{a_1}{a_2}=\frac{b_1}{b_2} \neq \frac{c_1}{c_2}$	Parallel lines	No solution

Example 3.55

Check the value of k for which the given system of equations $kx+2y=3;\ 2x-3y=1$ has a unique solution.

Solution Given linear equations are

$$\begin{aligned} kx+2y&=3 \text{ .....(1)} \\ 2x-3y&=1 \text{ .....(2)} \end{aligned} \quad \begin{bmatrix} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \end{bmatrix}$$

Here $a_1=k,\ b_1=2,\ a_2=2,\ b_2=-3;$

For unique solution we take $\dfrac{a_1}{a_2}\neq\dfrac{b_1}{b_2}$; therefore $\dfrac{k}{2}\neq\dfrac{2}{-3}$; $k\neq\dfrac{4}{-3}$, that is $k\neq-\dfrac{4}{3}$.

Example 3.56

Find the value of k, for the following system of equation has infinitely many solutions. $2x-3y=7;\ (k+2)x-(2k+1)y=3(2k-1)$

Solution Given two linear equations are

$$\begin{aligned} 2x-3y&=7 \\ (k+2)x-(2k+1)y&=3(2k-1) \end{aligned} \quad \begin{bmatrix} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \end{bmatrix}$$

Here $a_1=2,\ b_1=-3,\ a_2=(k+2),\ b_2=-(2k+1),\ c_1=7,\ c_2=3(2k-1)$

For infinite number of solution we consider $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{c_1}{c_2}$

$$\frac{2}{k+2}=\frac{-3}{-(2k+1)}=\frac{7}{3(2k-1)}$$

$$\begin{aligned} \frac{2}{k+2}&=\frac{-3}{-(2k+1)} \\ 2(2k+1)&=3(k+2) \\ 4k+2&=3k+6 \\ k&=4 \end{aligned}$$

$$\begin{aligned} \frac{-3}{-(2k+1)}&=\frac{7}{3(2k-1)} \\ 9(2k-1)&=7(2k+1) \\ 18k-9&=14k+7 \\ 4k&=16 \\ k&=4 \end{aligned}$$

Example 3.57

Find the value of k for which the system of linear equations $8x+5y=9;$ $kx+10y=15$ has no solution.

Solution Given linear equations are

$$\begin{aligned} 8x+5y&=9 \\ kx+10y&=15 \end{aligned} \quad \begin{bmatrix} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \end{bmatrix}$$

Here $a_1=8,\ b_1=5,\ c_1=9,\ a_2=k, b_2=10, c_2=15$

For no solution, we know that $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}\neq\dfrac{c_1}{c_2}$ and so, $\dfrac{8}{k}=\dfrac{5}{10}\neq\dfrac{9}{15}$

$$\begin{aligned} 80&=5k \\ k&=16 \end{aligned}$$

Activity - 3

1. Find the value of k for the given system of linear equations satisfying the condition below:
   (i) $2x+ky=1;\ 3x-5y=7$ has a unique solution
   (ii) $kx+3y=3;\ 12x+ky=6$ has no solution
   (iii) $(k-3)x+3y=k;\ kx+ky=12$ has infinite number of solution
2. Find the value of a and b for which the given system of linear equation has infinite number of solutions $3x-(a+1)y=2b-1, \quad 5x+(1-2a)y=3b$

Activity - 4

For the given linear equations, find another linear equation satisfying each of the given condition

Given linear equation	**Another linear equation**		
	Unique Solution	**Infinite many solutions**	**No solution**
$2x+3y = 7$	$3x+4y = 8$	$4x+6y = 14$	$6x+9y = 15$
$3x-4y = 5$			
$y-4x = 2$			
$5y-2x = 8$			

Exercise 3.14

Solve by any one of the methods

1. The sum of a two digit number and the number formed by interchanging the digits is 110. If 10 is subtracted from the first number, the new number is 4 more than 5 times the sums of the digits of the first number. Find the first number.
2. The sum of the numerator and denominator of a fraction is 12. If the denominator is increased by 3, the fraction becomes $\frac{1}{2}$. Find the fraction.
3. ABCD is a cyclic quadrilateral such that $\angle A = (4y + 20)^\circ$, $\angle B = (3y - 5)^\circ$, $\angle C = (4x)^\circ$ and $\angle D = (7x + 5)^\circ$. Find the four angles.
4. On selling a T.V. at 5% gain and a fridge at 10% gain, a shopkeeper gains ₹2000. But if he sells the T.V. at 10% gain and the fridge at 5% loss, he gains Rs.1500 on the transaction. Find the actual price of the T.V. and the fridge.
5. Two numbers are in the ratio 5 : 6. If 8 is subtracted from each of the numbers, the ratio becomes 4 : 5. Find the numbers.
6. 4 Indians and 4 Chinese can do a piece of work in 3 days. While 2 Indians and 5 Chinese can finish it in 4 days. How long would it take for 1 Indian to do it? How long would it take for 1 Chinese to do it?

Exercise 3.15

Multiple choice questions

1. If $x^3 + 6x^2 + kx + 6$ is exactly divisible by $(x + 2)$, then k= ?
(1) –6 (2) –7 (3) –8 (4) 11

2. The root of the polynomial equation $2x + 3 = 0$ is
(1) $\frac{1}{3}$ (2) $-\frac{1}{3}$ (3) $-\frac{3}{2}$ (4) $-\frac{2}{3}$

3. The type of the polynomial $4–3x^3$ is
(1) constant polynomial (2) linear polynomial
(3) quadratic polynomial (4) cubic polynomial.

4. If $x^{51} + 51$ is divided by $x + 1$, then the remainder is
(1) 0 (2) 1 (3) 49 (4) 50

5. The zero of the polynomial $2x+5$ is
(1) $\frac{5}{2}$ (2) $-\frac{5}{2}$ (3) $\frac{2}{5}$ (4) $-\frac{2}{5}$

6. The sum of the polynomials $p(x) = x^3 – x^2 – 2$, $q(x) = x^2–3x+ 1$
(1) $x^3 – 3x – 1$ (2) $x^3 + 2x^2 – 1$ (3) $x^3 – 2x^2 – 3x$ (4) $x^3 – 2x^2 + 3x –1$

7. Degree of the polynomial $(y^3–2)(y^3 + 1)$ is
(1) 9 (2) 2 (3) 3 (4) 6

8. Let the polynomials be
(A) $–13q^5 + 4q^2 + 12q$ (B) $(x^2 +4)(x^2 + 9)$
(C) $4q^8 – q^6 + q^2$ (D) $-\frac{5}{7}y^{12} + y^3 + y^5$
Then ascending order of their degree is
(1) A,B,D,C (2) A,B,C,D (3) B,C,D,A (4) B,A,C,D

9. If $p(a) = 0$ then $(x - a)$ is a ___________ of $p(x)$
(1) divisor (2) quotient (3) remainder (4) factor

10. Zeros of $(2 - 3x)$ is ___________
(1) 3 (2) 2 (3) $\frac{2}{3}$ (4) $\frac{3}{2}$

11. Which of the following has $x - 1$ as a factor?
(1) $2x - 1$ (2) $3x - 3$ (3) $4x - 3$ (4) $3x - 4$

12. If $x - 3$ is a factor of $p(x)$, then the remainder is
(1) 3 (2) –3 (3) $p(3)$ (4) $p(–3)$

13. $(x + y)(x^2 - xy + y^2)$ is equal to
(1) $(x + y)^3$ (2) $(x - y)^3$ (3) $x^3 + y^3$ (4) $x^3 - y^3$

14. $(a+b-c)^2$ is equal to __________

(1) $(a-b+c)^2$ (2) $(-a-b+c)^2$ (3) $(a+b+c)^2$ (4) $(a-b-c)^2$

15. In an expression ax^2+bx+c the sum and product of the factors respectively,

(1) a, bc (2) b, ac (3) ac, b (4) bc, a

16. If $(x+5)$ and $(x-3)$ are the factors of ax^2+bx+c, then values of a, b and c are

(1) 1,2,3 (2) 1,2,15 (3) 1,2, −15 (4) 1, −2,15

17. Cubic polynomial may have maximum of ___________ linear factors

(1) 1 (2) 2 (3) 3 (4) 4

18. Degree of the constant polynomial is __________

(1) 3 (2) 2 (3) 1 (4) 0

19. Find the value of m from the equation $2x+3y=m$. If its one solution is $x=2$ and $y=-2$.

(1) 2 (2) −2 (3) 10 (4) 0

20. Which of the following is a linear equation

(1) $x+\frac{1}{x}=2$ (2) $x(x-1)=2$ (3) $3x+5=\frac{2}{3}$ (4) $x^3-x=5$

21. Which of the following is a solution of the equation $2x-y=6$

(1) (2,4) (2) (4,2) (3) (3, −1) (4) (0,6)

22. If (2,3) is a solution of linear equation $2x+3y=k$ then, the value of k is

(1) 12 (2) 6 (3) 0 (4) 13

23. Which condition does not satisfy the linear equation $ax+by+c=0$

(1) $a\neq 0$, $b=0$ (2) $a=0$, $b\neq 0$

(3) $a=0$, $b=0, c\neq 0$ (4) $a\neq 0$, $b\neq 0$

24. Which of the following is not a linear equation in two variable

(1) $ax+by+c=0$ (2) $0x+0y+c=0$

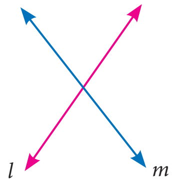
(3) $0x+by+c=0$ (4) $ax+0y+c=0$

25. The value of k for which the pair of linear equations $4x+6y-1=0$ and $2x+ky-7=0$ represents parallel lines is

(1) $k=3$ (2) $k=2$ (3) $k=4$ (4) $k=-3$

26. A pair of linear equations has no solution then the graphical representation is

(1)



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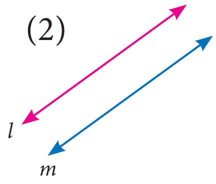

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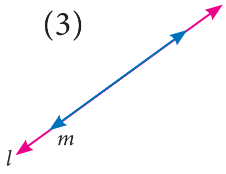

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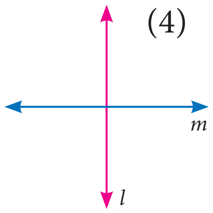


27. If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ where $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ then the given pair of linear equation has __________ solution(s)

(1) no solution (2) two solutions (3) unique (4) infinite

28. If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ where $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ then the given pair of linear equation has __________ solution(s)

(1) no solution (2) two solutions (3) infinite (4) unique

29. GCD of any two prime numbers is __________

(1) –1 (2) 0 (3) 1 (4) 2

30. The GCD of $x^4 - y^4$ and $x^2 - y^2$ is

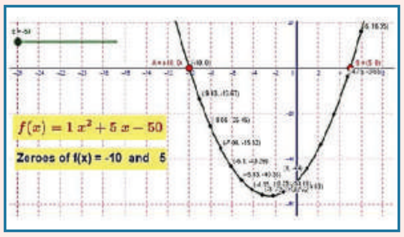
(1) $x^4 - y^4$ (2) $x^2 - y^2$ (3) $(x+y)^2$ (4) $(x+y)^4$

Points to Remember

- An algebraic expression of the form $p(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ is called Polynomial in one variable x of degree 'n' where $a_0, a_1, a_2, \ldots a_n$ are constants $(a_n \neq 0)$ and n is a whole number.
- Let $p(x)$ be a polynominal. If $p(a) = 0$ then we say that 'a' is a zero of the polynominal $p(x)$
- If $x = a$ statisfies the polynominal $p(x) = 0$ then $x = a$ is called a root of the polynominal equation $p(x) = 0$.
- **Remainder Theorem:** If a polynomial $p(x)$ of degree greater than or equal to one is divided by a linear polynomial $(x–a)$, then the remainder is $p(a)$, where a is any real number.
- **Factor Theorem**

  If $p(x)$ is divided by $(x-a)$ and the remainder $p(a)=0$, then $(x-a)$ is a factor of the polynomial $p(x)$
- Solution of an equation is the set of all values that when substituted for unknowns make an equation true.
- An equation with two variable each with exponent as 1 and not multiplied with each other is called a linear equation with two variables.
- Linear equation in two variables has infinite number of solutions.
- The graph of a linear equation in two variables is a straight line.
- Simultaneous linear equations consists of two or more linear equations with the same variables.

ICT Corner-1



Expected Result is shown in this picture

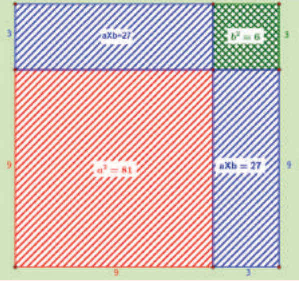
Step – 1 Open the Browser and copy and paste the Link given below (or) by typing the URL given (or) Scan the QR Code.

Step - 2 GeoGebra Work Book called "Polynomials and Quadratic Equations" will appear. There are several work sheets in this work Book. Open the worksheet named "Zeroes: Quadratic Polynomial".

Step-3 Drag the sliders a, b and c to change the quadratic co-efficient. Follow the changes in points *A* and *B* where the curve cuts the *x*-axis. These points are called Zeroes of a Polynomial.

B566_9_MAT_EM_T3

ICT Corner-2



Expected Result is shown in this picture

Step – 1 Open the Browser, type the URL Link given below (or) Scan the QR Code. GeoGebra work sheet named "Algebraic Identities" will open. In the work sheet, there are many activities on Algebraic Identities.

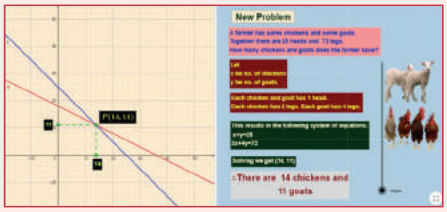
In the first activity diagrammatic approach for $(a+b)^2$ is given. Move the sliders a and b and compare the areas with the Identity given.

Step - 2 Similarly move the sliders a and b and compare the areas with the remaining Identities.

Scan the QR Code.

B566_9_MAT_EM_T3

ICT Corner-3



Expected Result is shown in this picture

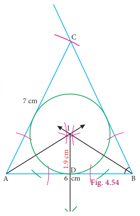
Step – 1 Open the Browser by typing the URL Link given below (or) Scan the QR Code. GeoGebra work sheet named "Algebra" will open. There are three worksheets under the title Solving by rule of cross multiplication, Graphical method and Chick-Goat puzzle.

Step - 2 Move the sliders or type the respective values in the respective boxes to change the equations. Work out the solution and check the solutions. In third title click on new problem and solve. Move the slider to see the steps.

Scan the QR Code.

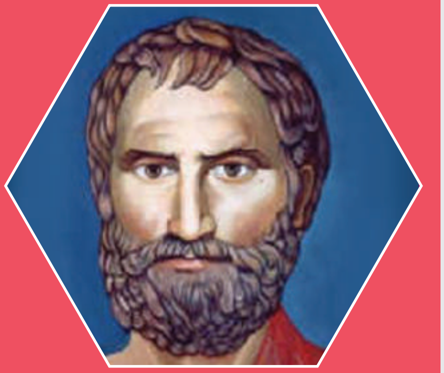
B566_9_MAT_EM_T3

4



GEOMETRY

Inspiration is needed in geometry just as much as in poetry.
- Alexander Pushkin



Thales
(BC (BCE) 624 – 546)

Thales (Pronounced THAYLEES) was born in the Greek city of Miletus. He was known for theoretical and practical understanding of geometry, especially triangles. He used geometry to solve many problems such as calculating the height of pyramids and the distance of ships from the sea shore. He was one of the so-called Seven Sagas or Seven Wise Men of Greece and many regarded him as the first philosopher in the western tradition.

E64P7Q

Learning Outcomes

- To understand theorems on linear pairs and vertically opposite angles.
- To understand the angle sum property of triangle.
- To understand the properties of quadrilaterals and use them in problem solving.
- To understand, interpret and apply theorems on the chords and the angles subtended by arcs of a circle
- To understand, interpret and apply theorems on the cyclic quadrilaterals.
- To construct and locate centroid, orthocentre, circumcentre and incentre of a triangle.

4.1 Introduction

In geometry, we study **shapes**. But what is there to *study* in shapes, you may ask. Think first, what are all the things we do with shapes? We draw shapes, we compare shapes, we *measure* shapes. *What* do we measure in shapes?

Take some shapes like this:

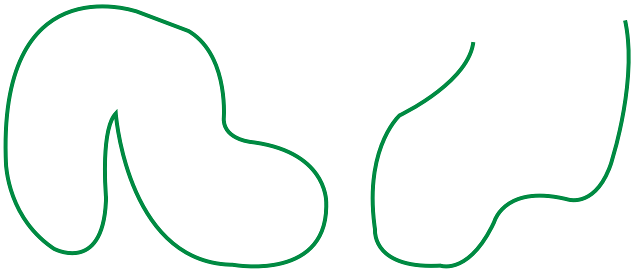


Fig. 4.1

In both of them, there is a *curve* forming the shape: one is a closed curve, enclosing a region, and the other is an open curve. We can use a rope (or a thick string) to measure the **length** of the open curve and the length of the boundary of the region in the case of the closed curve.

Curves are tricky, aren't they? It is so much easier to measure length of straight lines using the scale, isn't it? Consider the two shapes below.

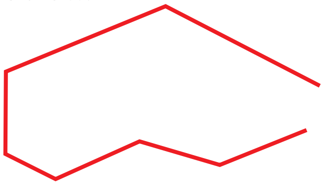


Fig. 4.2

Now we are going to focus our attention only on shapes made up of straight lines and on closed figures. As you will see, there is plenty of interesting things to do. Fig.4.2 shows an open figure.

We not only want to draw such shapes, we want to compare them, measure them and do much more. For doing so, we want to **describe** them. How would you describe these closed shapes? (See Fig 4.3) They are all made up of straight lines and are closed.

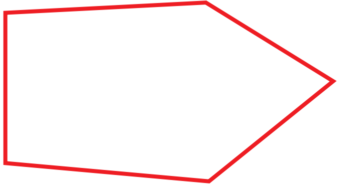
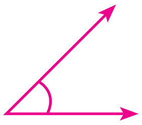


Fig. 4.3

4.2 Types of Angles-Recall

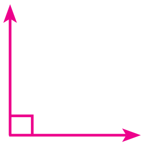
Plumbers measure the angle between connecting pipes to make a good fitting. Wood workers adjust their saw blades to cut wood at the correct angle. Air Traffic Controllers (ATC) use angles to direct planes. Carom and billiards players must know their angles to plan their shots. An angle is formed by two rays that share a common end point provided that the two rays non-collinear.



acute

an angle that is less than 90°

Fig. 4.4



right

an angle that is exactly 90°

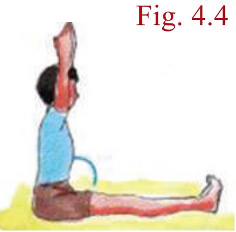
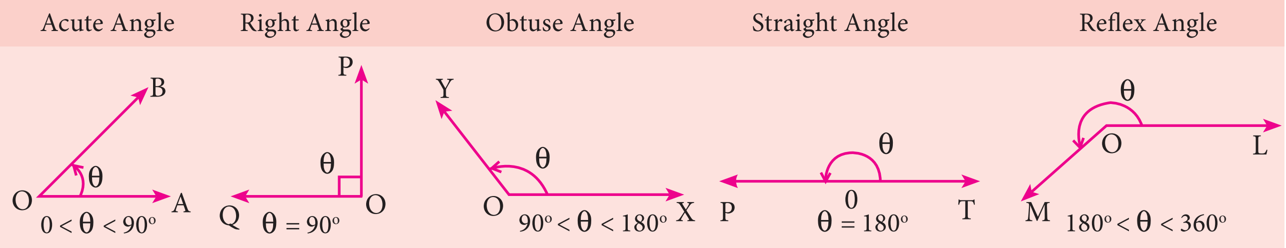


Fig. 4.5



Complementary Angles

Two angles are Complementary if their sum is 90°. For example, if $\angle ABC=64^\circ$ and $\angle DEF=26^\circ$, then angles $\angle ABC$ and $\angle DEF$ are complementary to each other because $\angle ABC + \angle DEF = 90^\circ$

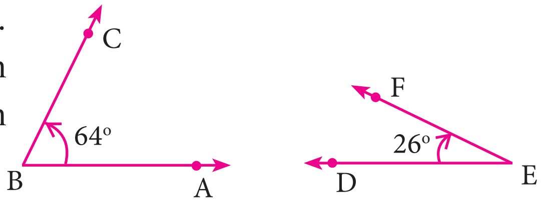


Fig. 4.6

Supplementary Angles

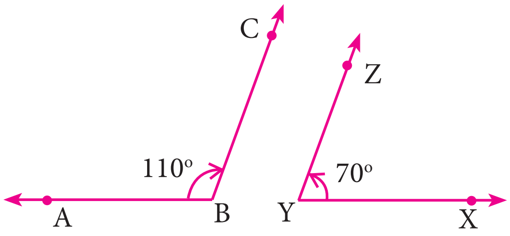


Fig. 4.7

Two angles are Supplementary if their sum is 180°. For example if $\angle ABC=110^\circ$ and $\angle XYZ=70^\circ$

Here $\angle ABC + \angle XYZ = 180^\circ$

$\therefore \angle ABC$ and $\angle XYZ$ are supplementary to each other

Adjacent Angles

Two angles are called adjacent angles if

(i) They have a common vertex.

(ii) They have a common arm.

(iii) The common arm lies between the two non-common arms.

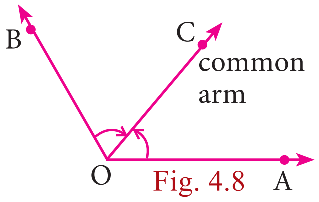


Fig. 4.8

Linear Pair of Angles

If a ray stands on a straight line then the sum of two adjacent angle is 180°. We then say that the angles so formed is a linear pair.

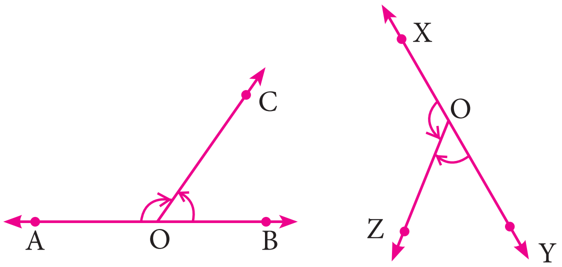


Fig. 4.9

$\angle AOC + \angle BOC=180^\circ$

$\therefore \angle AOC$ and $\angle BOC$ form a linear pair

$\angle XOZ + \angle YOZ = 180^\circ$

$\angle XOZ$ and $\angle YOZ$ form a linear pair

Vertically Opposite Angles

If two lines intersect each other, then vertically opposite angles are equal.

In this figure $\angle POQ = \angle SOR$

$\angle POS = \angle QOR$

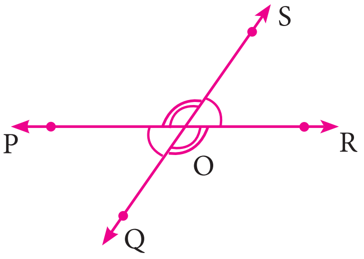


Fig. 4.10

4.2.1 Transversal

A line which intersects two or more lines at distinct points is called a transversal of those lines.

E6DK9D

Case (i) When a transversal intersect two lines, we get eight angles.

In the figure the line *l* is the transversal for the lines *m* and *n*

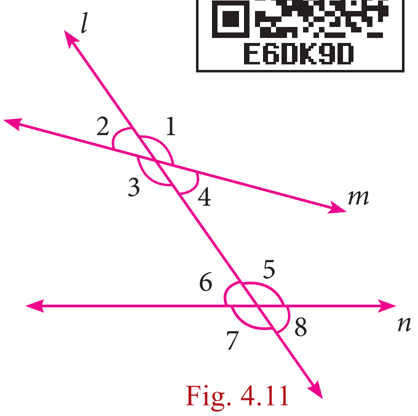


Fig. 4.11

(i) Corresponding Angles: ∠1 and ∠5, ∠2 and ∠6, ∠3 and ∠7, ∠4 and ∠8

(ii) Alternate Interior Angles: ∠4 and ∠6, ∠3 and ∠5

(iii) Alternate Exterior Angles: ∠1 and ∠7, ∠2 and ∠8

(iv) ∠4 and ∠5,∠3 and ∠6 are interior angles on the same side of the transversal.

(v) ∠1 and ∠8, ∠2 and ∠7 are exterior angles on the same side of the transversal.

Case (ii) If a transversal intersects two parallel lines. The transversal forms different pairs of angles.

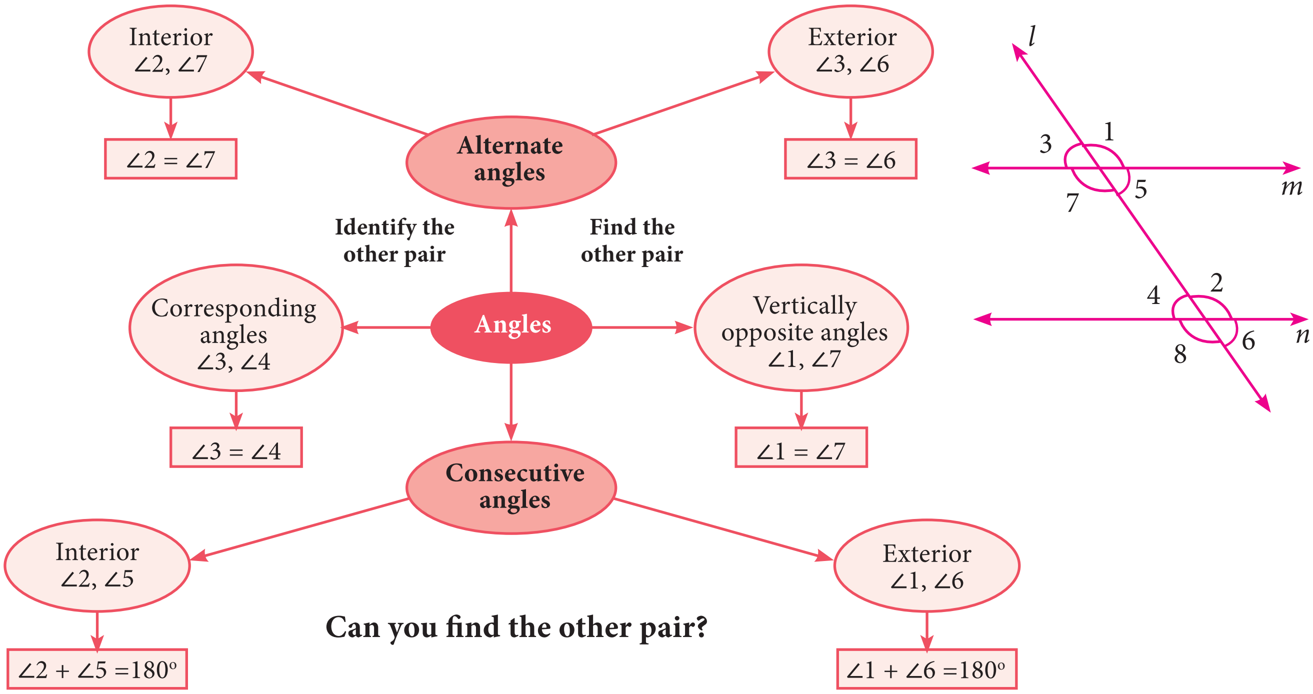


Fig. 4.12

4.2.2 Triangles

Activity 1

1. Take three different colour sheets; place one over the other and draw a triangle on the top sheet. Cut the sheets to get triangles of different colour which are identical. Mark the vertices and the angles as shown. Place the interior angles ∠1, ∠2 and ∠3 on a straight line, adjacent to each other, without leaving any gap. What can you say about the total measure of the three angles ∠1, ∠2 and ∠3?

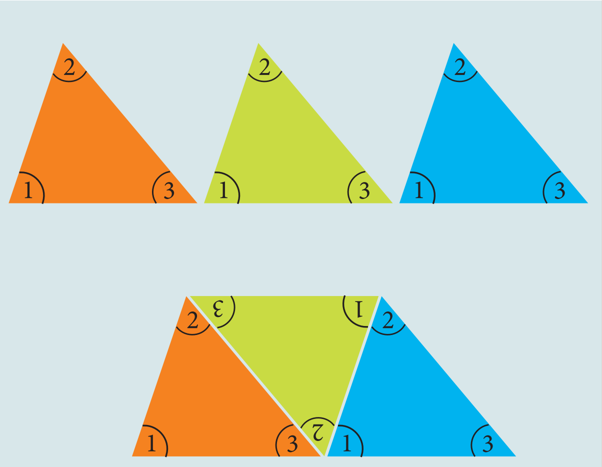


Fig. 4.13

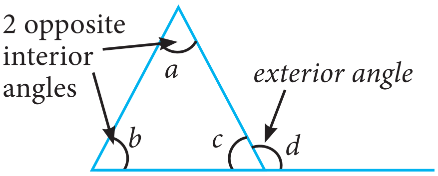


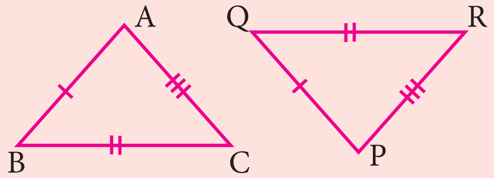
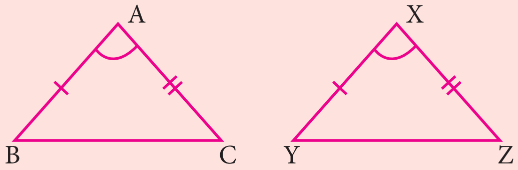
Fig. 4.14

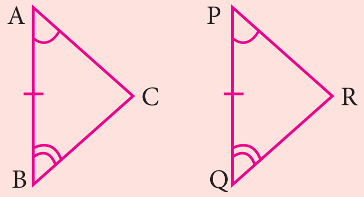
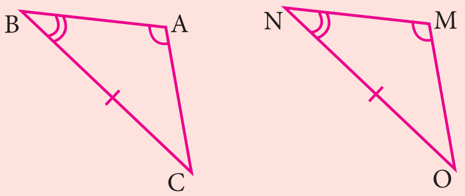
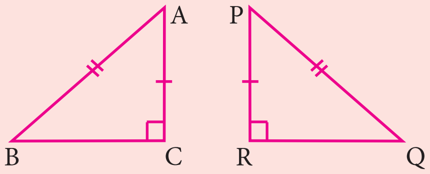
Can you use the same figure to explain the "**Exterior angle property**" of a triangle?

If a side of a triangle is stretched, the exterior angle so formed is equal to the sum of the two interior opposite angles. That is $d=a+b$ (see Fig 4.14)

4.2.3 Congruent Triangles

Two triangles are congruent if the sides and angles of one triangle are equal to the corresponding sides and angles of another triangle.

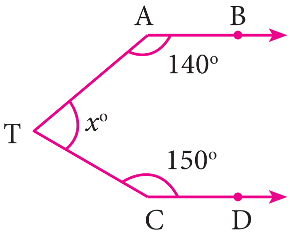
Rule	Diagrams	Reason
SSS		$AB = PQ$ $BC = QR$ $AC = PR$ $\Delta ABC \cong \Delta PQR$
SAS		$AB = XY$ $\angle BAC = \angle YXZ$ $AC = XZ$ $\Delta ABC \cong \Delta XYZ$

ASA		$\angle A = \angle P$ $AB = PQ$ $\angle B = \angle Q$ $\Delta ABC \cong \Delta PQR$
AAS		$\angle A = \angle M$ $\angle B = \angle N$ $BC = NO$ $\Delta ABC \cong \Delta MNO$
RHS		$\angle ACB = \angle PRQ = 90°(R)$ AB = PQ hypotenuse (H) $AC = PR$ (S) $\Delta ABC \cong \Delta PQR$

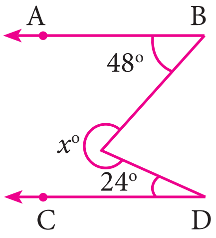
Exercise 4.1

1. In the figure, AB is parallel to CD, find x

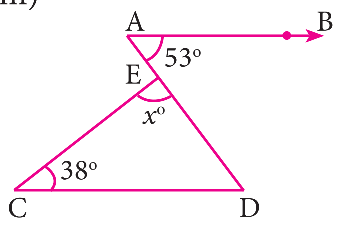
(i)



(ii)



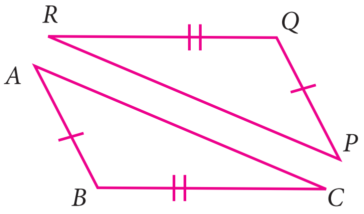
(iii)



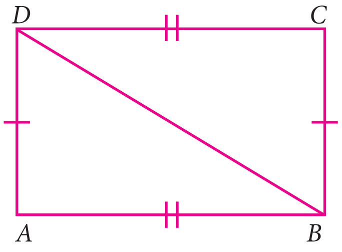
2. The angles of a triangle are in the ratio 1: 2 : 3, find the measure of each angle of the triangle.

3. Consider the given pairs of triangles and say whether each pair is that of congruent triangles. If the triangles are congruent, say 'how'; if they are not congruent say 'why' and also say if a small modification would make them congruent:

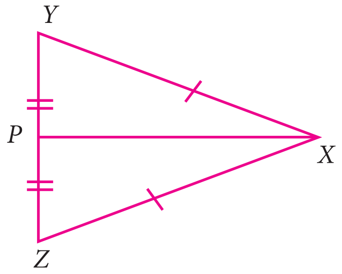
(i)



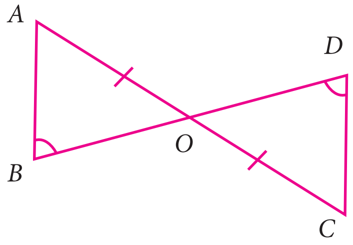
(ii)



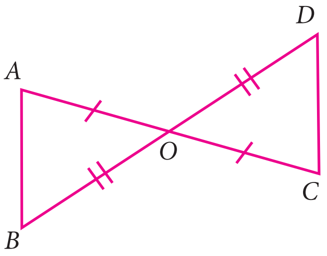
(iii)



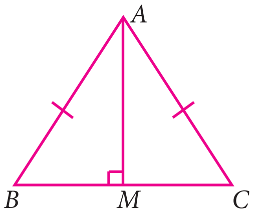
(iv)



(v)



(vi)

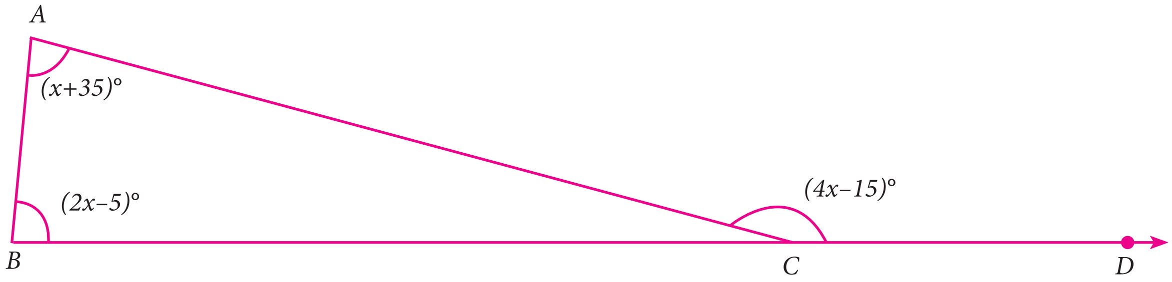


4. ΔABC and ΔDEF are two triangles in which

   $AB=DF,\ \angle ACB=70°,\ \angle ABC=60°;\ \angle DEF=70°$ and $\angle EDF=60°$.

   Prove that the triangles are congruent.

5. Find all the three angles of the ΔABC



4.3 Quadrilaterals

Activity 2

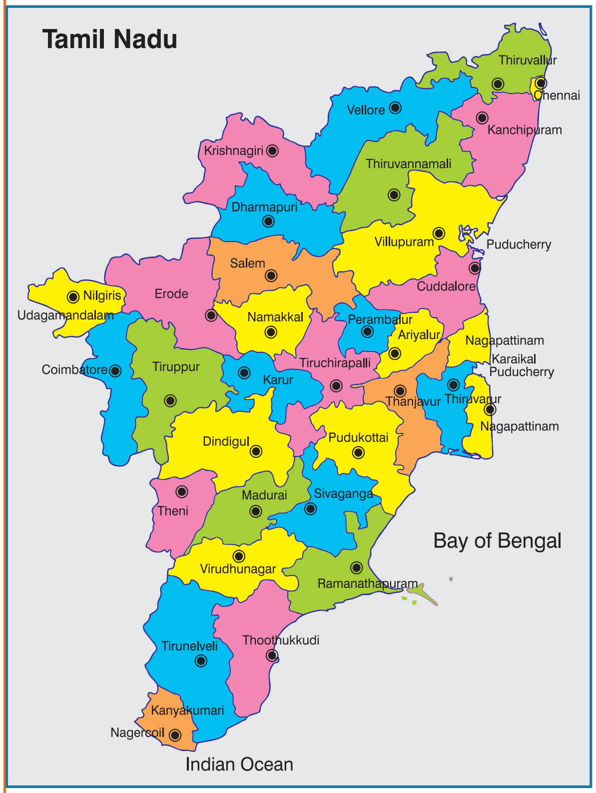


Fig. 4.15

Four Tamil Nadu State Transport buses take the following routes. The first is a one-way journey, and the rest are round trips. Find the places on the map, put points on them and connect them by lines to draw the routes. The places connecting four different routes are given as follows.

(i) Nagercoil, Tirunelveli, Virudhunagar, Madurai

(ii) Sivagangai, Puthukottai, Thanjavur, Dindigul, Sivagangai

(iii) Erode, Coimbatore, Dharmapuri, Karur, Erode

(iv) Chennai, Cuddalore, Krishnagiri, Vellore, Chennai

You will get the following shapes.

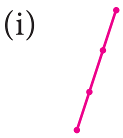


Fig. 4.16

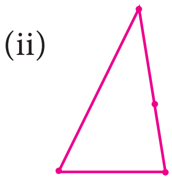


Fig. 4.17

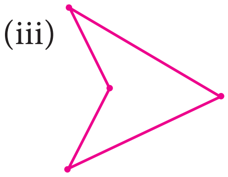


Fig. 4.18

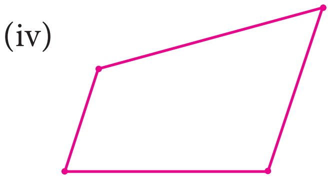


Fig. 4.19

Label the vertices with city names, draw the shapes exactly as they are shown on the map without rotations.

We observe that the first is a single line, the four points are collinear. The other three are closed shapes made of straight lines, of the kind we have seen before. We need names to call such closed shapes, we will call them **polygons** from now on.

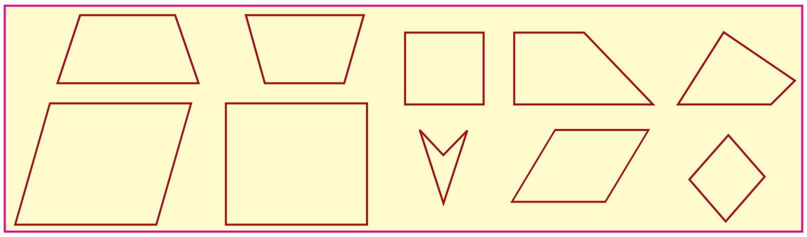


Fig. 4.20

How do polygons look? They have sides, with points at either end. We call these points as **vertices** of the polygon. The sides are line segments joining the vertices. The word *poly* stands for many, and a polygon is a many-sided figure.

Note

Concave polygon: Polygon having any one of the interior angle greater than 180^{o}
Convex Polygon: Polygon having each interior angle less than 180^{o}
(Diagonals should be inside the polygon)

How many sides can a polygon have? One? But that is just a line segment. Two ? But how can you get a closed shape with two sides? Three? Yes, and this is what we know as a triangle. Four sides?

Squares and rectangles are examples of polygons with 4 sides but they are not the only ones. Here (Fig. 4.20) are some examples of 4-sided polygons. We call them ***quadrilaterals.***

4.3.1 Special Names for Some Quadrilaterals

1. A **parallelogram** is a quadrilateral in which opposite sides are parallel and equal.
2. A **rhombus** is a quadrilateral in which opposite sides are parallel and all sides are equal.
3. A **trapezium** is a quadrilateral in which *one pair of* opposite sides are parallel.

Draw a few parallellograms, a few rhombuses (correctly called rhombii, like cactus and cactii) and a few trapeziums (correctly written trapezia).

Fig. 4.21

The great advantage of knowing properties of quadrilateral is that we can see the relationships among them immediately.

- Every parallelogram is a trapezium, but not necessarily the other way.
- Every rhombus is a parallelogram, but not necessarily the other way.
- Every rectangle is a parallelogram, but not necessarily the other way.
- Every square is a rhombus and hence every square is a parallelogram as well.

For "not necessarily the other way" mathematicians usually say "the converse is not true". A smart question then is: just *when* is the other way also true? For instance, when is a parallelogram also a rectangle? Any parallelogram in which all angles are also equal is a rectangle. (Do you see why?) Now we can observe many more interesting properties. For instance, we see that a rhombus is a parallelogram in which all **sides** are also equal.

Note

You know *bi*-cycles and *tri*-cycles? When we attach *bi or tri* to the front of any word, they stand for 2 (bi) or 3 (tri) of them. Similarly *quadri* stands for 4 of them. We should really speak of quadri-cycles also, but we don't. *Lateral* stands for sideways, thus quadrilateral means a 4-sided figure. You know *trilaterals*; they are also called triangles !
After 4 ? We have: 5 – *penta*, 6 – *hexa*, 7 – *hepta*, 8 – *octa*, 9 – *nano*, 10 – *deca*. Conventions are made by history. Trigons are called triangles, quadrigons are called quadrilaterals. Continuing in th same way we get pentagons, hexagons, heptagons, octagons, nanogons and decagons. Beyond these, we have 11-gons, 12-gons etc. Perhaps you can draw a 23-gon !

4.3.2 More Special Names

When all sides of a quadrilateral are equal, we call it **equilateral**. When all angles of a quadrilateral are equal, we call it **equiangular**. In triangles, we talked of equilateral triangles as those with all sides equal. Now we can call them equiangular triangles as well!

We thus have:

A rhombus is an equilateral parallelogram.
A rectangle is an equiangular parallelogram.
A square is an equilateral and equiangular parallelogram.

Here are two more special quadrilaterals, called **kite** and **isosceles trapezium**.

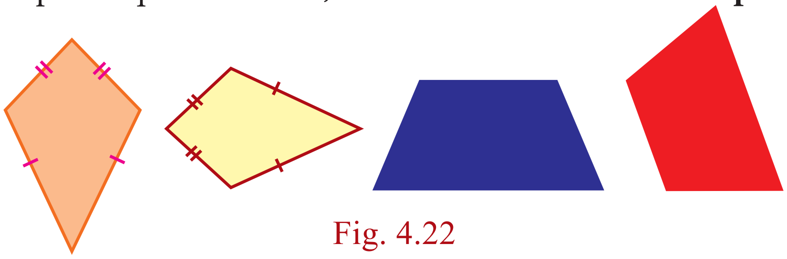


Fig. 4.22

4.3.3 Types of Quadrilaterals

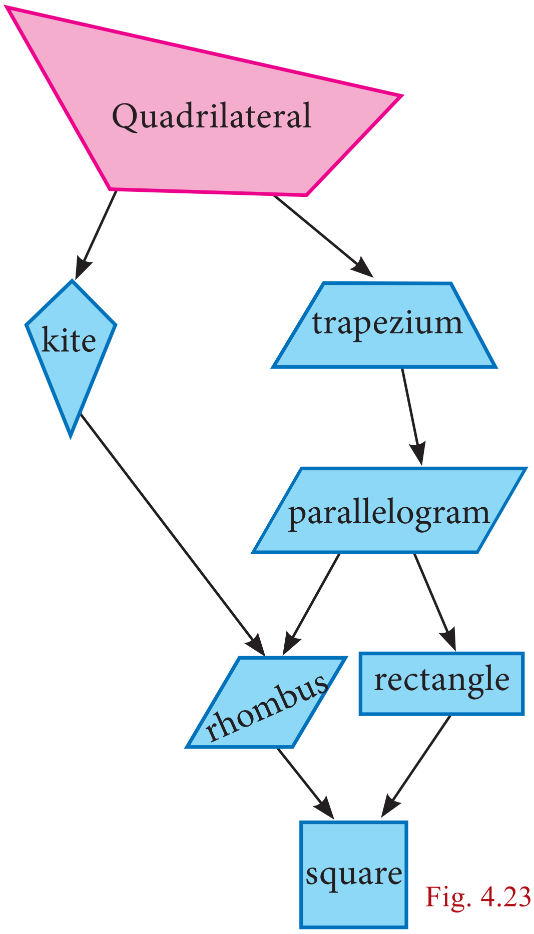


Fig. 4.23

Progress Check

Answer the following question.

(i) Are the opposite angles of a rhombus equal?

(ii) A quadrilateral is a ______________ if a pair of opposite sides are equal and parallel.

(iii) Are the opposite sides of a kite equal?

(iv) Which is an equiangular but not an equilateral parallelogram?

(v) Which is an equilateral but not an equiangular parallelogram?

(vi) Which is an equilateral and equiangular parallelogram?

(vii) _________ is a rectangle, a rhombus and a parallelogram.

Activity 3

Step – 1: Cut out four different quadrilaterals from coloured glazed papers.

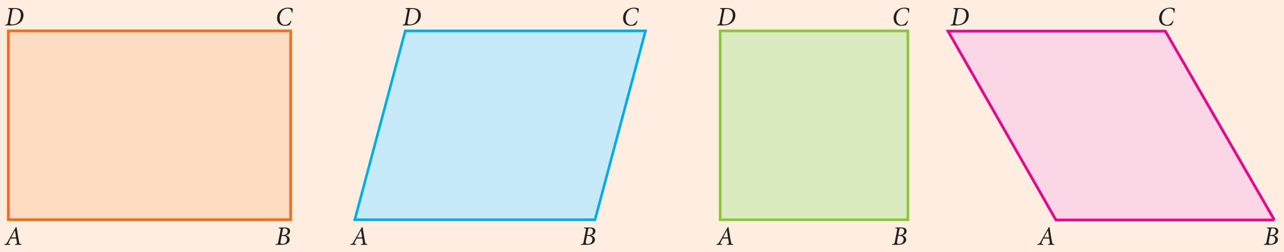


Fig. 4.24

Step – 2: Fold the quadrilaterals along their respective diagonals. Press to make creases. Here, dotted line represent the creases.

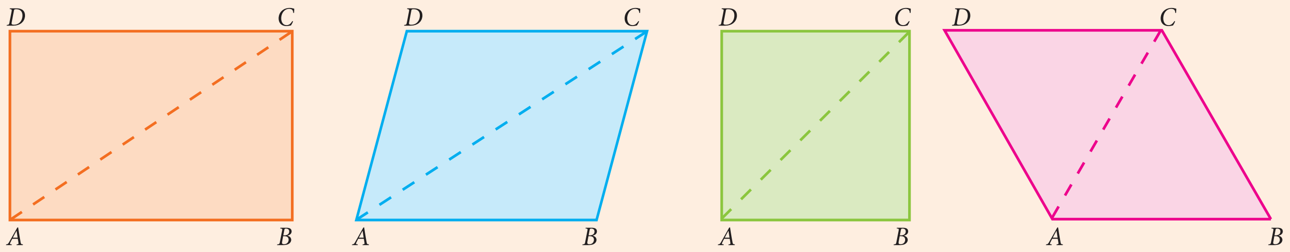


Fig. 4.25

Step – 3: Fold the quadrilaterals along both of their diagonals. Press to make creases.

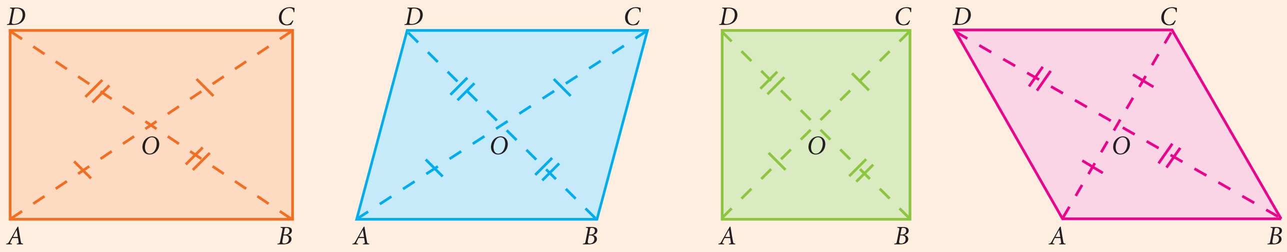


Fig. 4.26

We observe that two imposed triangles are congruent to each other. Measure the lengths of portions of diagonals and angles between the diagonals.

Also do the same for the quadrilaterals such as Trapezium, Isosceles Trapezium and Kite.

From the above activity, measure the lengths of diagonals and angles between the diagonals and record them in the table below:

S. No.	**Name of the quadrilateral**	**Length along diagonals**						**Measure of angles**			
		AC	BD	OA	OB	OC	OD	$\angle AOB$	$\angle BOC$	$\angle COD$	$\angle DOA$
1	Trapezium										
2	Isosceles Trapezium										
3	Parallelogram										
4	Rectangle										
5	Rhombus										
6	Square										
7	Kite										

Activity 4

Angle sum for a polygon

Draw any quadrilateral $ABCD$.

Mark a point P in its interior.

Join the segments PA, PB, PC and PD.

You have 4 triangles now.

How much is the sum of all the angles of the 4 triangles?

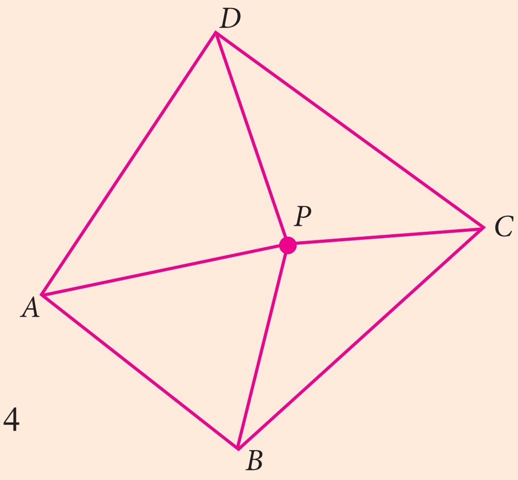


Fig. 4.27

How much is the sum of the angles at P?

Can you now find the ‘angle sum’ of the quadrilateral $ABCD$?

Can you extend this idea to any polygon?

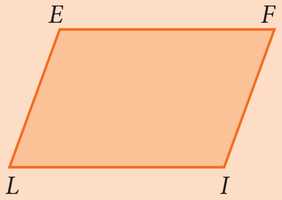
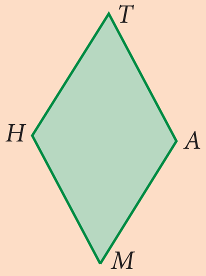
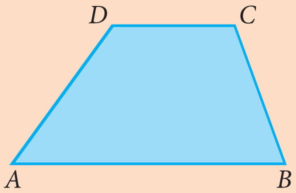
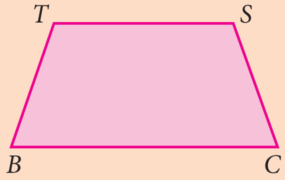
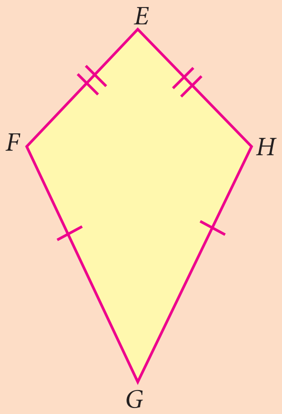
Thinking Corner

1. If there is a polygon of n sides ($n \geq 3$), then the sum of all interior angles is $(n-2) \times 180°$

2. For the regular polygon (All the sides of a polygon are equal in size)

   - Each interior angle is $\frac{(n-2)}{n} \times 180°$
   - Each exterior angle is $\frac{360°}{n}$
   - The sum of all the exterior angles formed by producing the sides of a convex polygon in the order is 360° .
   - If a polygon has ‘n’ sides , then the number of diagonals of the polygon is $\frac{n(n-3)}{2}$

4.3.4 Properties of Quadrilaterals

Name	Diagram	Sides	Angles	Diagonals
Parallelogram		Opposite sides are parallel and equal	Opposite angles are equal and sum of any two adjacent angles is 180°	Diagonals bisect each other.
Rhombus		All sides are equal and opposite sides are parallel	Opposite angles are equal and sum of any two adjacent angles is 180°	Diagonals bisect each other at right angle.
Trapezium		One pair of opposite sides are parallel	The angles at the ends of each non-parallel sides are supplementary	Diagonals need not be equal
Isosceles Trapezium		One pair of opposite sides are parallel and non-parallel sides are equal in length.	The angles at the ends of each parallel sides are equal.	Diagonals are of equal length.
Kite		Two pairs of adjacent sides are equal	One pair of opposite angles are equal	1. Diagonals intersect at right angle. 2. Shorter diagonal bisected by longer diagonal 3. Longer diagonal divides the kite into two congruent triangles

Note

(i) A rectangle is an equiangular parallelogram.
(ii) A rhombus is an equilateral parallelogram.
(iii) A square is an equilateral and equiangular parallelogram.
(iv) A square is a rectangle , a rhombus and a parallelogram.

Progress Check

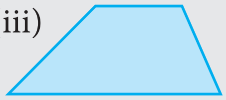
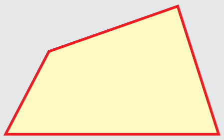
1. State the reasons for the following.
   (i) A square is a special kind of a rectangle.
   (ii) A rhombus is a special kind of a parallelogram.
   (iii) A rhombus and a kite have one common property.
   (iv) A square and a rhombus have one common property.

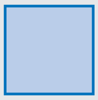
2. What type of quadrilateral is formed when the following pairs of congruent triangles are joined together?
   (i) Equilateral triangle.
   (ii) Right angled triangle.
   (iii) Isosceles triangle.

3. Identify which ones are parallelograms and which are not.

(i)

(ii)

(iii)

(iv)

(v)

(vi)


4. Which ones are not quadrilaterals?

(i)

(ii)

(iii)

(iv)

(v)

(vi)

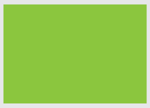
(vii)

(viii)


5. Identify which ones are trapeziums and which are not.

(i)

(ii)

(iii)

(iv)

(v)

(vi)


4.3.5 Properties of Parallelogram

We can now embark on an interesting journey. We can tour among lots of quadrilaterals, noting down interesting properties. What properties do we look for, and how do we know they are true?

For instance, opposite sides of a parallelogram are parallel, but are they also **equal**? We could draw any number of parallelograms and verify whether this is true or not. In fact, we see that opposite sides are equal in **all of** them. Can we then conclude that opposite sides are equal in *all* parallelograms? No, because we might later find a parallelogram, one which we had not thought of until then, in which opposite sides are unequal. So, we need an argument, a **proof**.

Consider the parallelogram $ABCD$ in the given Fig. 4.28. We believe that $AB = CD$ and $AD = BC$, but how can we be sure? We know triangles and their properties. So we can try and see if we can use that knowledge. But we don't have any triangles in the parallelogram $ABCD$.

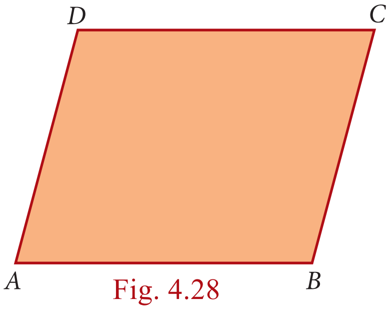


Fig. 4.28

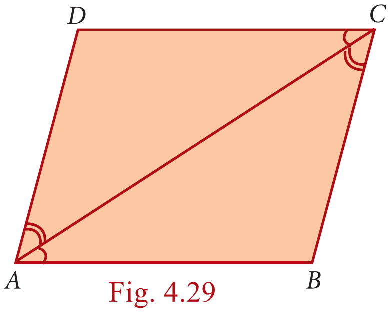


Fig. 4.29

This is easily taken care of by joining AC. (We could equally well have joined BD, but let it be AC for now.) We now have 2 triangles ADC and ABC with a common side AC. If we could somehow prove that these two triangles are congruent, we would get $AB = CD$ and $AD = BC$, which is what we want!

Is there any hope of proving that ΔADC and ΔABC are congruent? There are many criteria for congruence, it is not clear which one is relevant here.

So far we have not used the fact that $ABCD$ is a parallelogram at all. So we need to use the facts that $AB \| DC$ and $AD \| BC$ to show that ΔADC and ΔABC are congruent. From sides being parallel we have to get into some angles being equal. Do we know any such properties? we do, and that is all about **transversals!**

Now we can see it clearly. $AD \| BC$ and AC is a transversal, hence $\angle DAC = \angle BCA$. Similarly, $AB \| DC$, AC is a transversal, hence $\angle BAC = \angle DCA$. With AC as common side, the ASA criterion tells us that ΔADC and ΔABC are congruent, just what we needed. From this we can conclude that $AB = CD$ and $AD = BC$.

Thus opposite sides are indeed equal in a parallelogram.

The argument we now constructed is written down as a ***formal proof*** in the following manner.

Theorem 1

In a parallelogram, opposite sides are equal

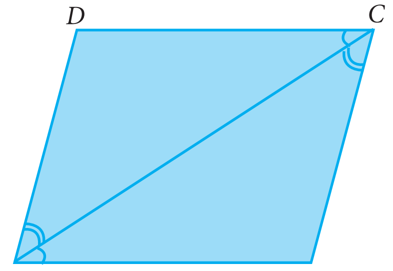


Fig. 4.30

Given $ABCD$ is a parallelogram

To Prove $AB=CD$ and $DA=BC$

Construction Join AC

Proof

Since $ABCD$ is a parallelogram

$AD \parallel BC$ and AC is the transversal

$\angle DAC = \angle BCA$ →(1) (alternate angles are equal)

$AB \parallel DC$ and AC is the transversal

$\angle BAC = \angle DCA$ →(2) (alternate angles are equal)

In ΔADC and ΔCBA

$\angle DAC = \angle BCA$ from (1)

AC is common

$\angle DCA = \angle BAC$ from (2)

$\Delta ADC \cong \Delta CBA$ (By ASA)

Hence $AD = CB$ and $DC = BA$ (Corresponding sides are equal)

Along the way in the proof above, we have proved another property that is worth recording as a theorem.

Theorem 2

A diagonal of a parallelogram divides it into two congruent triangles.

Notice that the proof above established that $\angle DAC = \angle BCA$ and $\angle BAC = \angle DCA$. Hence we also have, in the figure above,

$$\angle BCA + \angle BAC = \angle DCA + \angle DAC$$

But we know that:

$$\angle B + \angle BCA + \angle BAC = 180$$

and $\angle D + \angle DCA + \angle DAC = 180$

Therefore we must have that $\angle B = \angle D$.

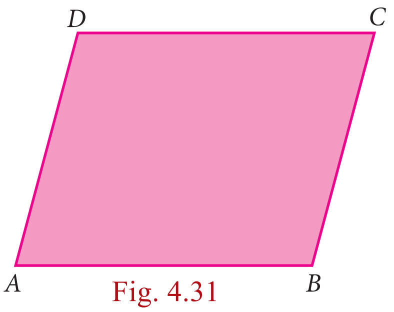


Fig. 4.31

With a little bit of work, proceeding similarly, we could have shown that $\angle A = \angle C$ as well. Thus we have managed to prove the following theorem:

Theorem 3

The opposite angles of a parallelogram are equal.

Now that we see congruence of triangles as a good "strategy", we can look for more triangles. Consider both diagonals AC and DB. We already know that ΔADC and ΔCBA are congruent. By a similar argument we can show that ΔDAB and ΔBCD are congruent as well. Are there more congruent triangles to be found in this figure ?

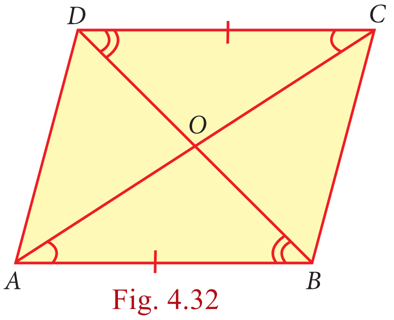


Fig. 4.32

Yes. The two diagonals intersect at point O. We now see 4 new ΔAOB, ΔBOC, ΔCOD and ΔDOA. Can you see any congruent pairs among them?

Since AB and CD are parallel and equal, one good guess is that ΔAOB and ΔCOD are congruent. We could again try the ASA crierion, in which case we want $\angle OAB = \angle OCD$ and $\angle ABO = \angle CDO$. But the first of these follows from the fact that $\angle CAB = \angle ACD$ (which we already established) and observing that $\angle CAB$ and $\angle OAB$ are the same (and so also $\angle OCD$ and $\angle ACD$). We now use the fact that BD is a transversal to get that $\angle ABD = \angle CDB$, but then $\angle ABD$ is the same as $\angle ABO$, $\angle CDB$ is the same as $\angle CDO$, and we are done.

Again, we need to write down the formal proof, and we have another theorem.

Theorem 4

The diagonals of a parallelogram bisect each other.

It is time now to reinforce our concepts on parallelograms. Consider each of the given statements, in the adjacent box, one by one. Identifiy the type of parallelogram which satisfies each of the statements. Support your answer with reason.

- Each pair of its opposite sides are parallel.
- Each pair of opposite sides is equal.
- All of its angles are right angles.
- Its diagonals bisect each other.
- The diagonals are equal.
- The diagonals are perpendicular and equal.
- The diagonals are perpendicular bisectors of each other.
- Each pair of its consecutive angles is supplementary.

Now we begin with lots of interesting properties of parallelograms. Can we try and prove some property relating to two or more parallelograms ? A simple case to try is when two parallelograms share the same base, as in Fig.4.33

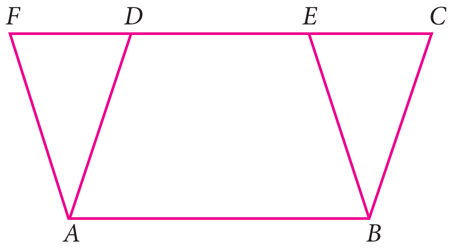


Fig. 4.33

We see parallelograms $ABCD$ and $ABEF$ are on the common base AB. At once we can see a pair of triangles for being congruent ΔADF and ΔBCE. We already have that $AD = BC$ and $AF = BE$. But then since $AD \parallel BC$ and $AF \parallel BE$, the angle formed by AD and AF must be the same as the angle formed by BC tand BE. Therefore $\angle DAF = \angle CBE$. Thus ΔADF and ΔBCE are congruent.

That is an interesting observation; can we infer anything more from this ? Yes, we know that congruent triangles have the *same area*. This makes us think about the areas of the parallelograms $ABCD$ and $ABEF$.

$$\begin{aligned}\text{Area of } ABCD &= \text{area of quadrilateral } ABED + \text{area of } \Delta BCE\\ &= \text{area of quadrilateral } ABED + \text{area of } \Delta ADF\\ &= \text{area of } ABEF\end{aligned}$$

Thus we have proved another interesting theorem:

Theorem 5:

Parallelograms on the same base and between the same parallels are equal in area.

In this process, we have also proved other interesting statements. These are called *Corollaries*, which do not need separate detailed proofs.

Corollary 1: Triangles on the same base and between the same parallels are equal in area.

Corollary 2: A rectangle and a parallelogram on the same base and between the same parallels are equal in area.

These statements that we called Theorems and Corollaries, hold for all parallelograms, however large or small, with whatever be the lengths of sides and angles at vertices.

Example 4.1

In a parallelogram $ABCD$, the bisectors of the consecutive angles $\angle A$ and $\angle B$ insersect at P. Show that $\angle APB = 90°$

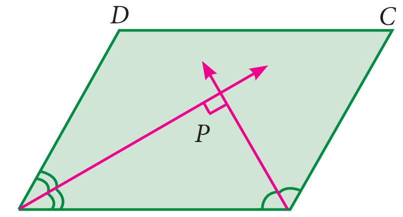


Fig. 4.34

Solution

$ABCD$ is a parallelogram AP and BP are bisectors of consecutive angles $\angle A$ and $\angle B$.

Since the consecutive angles of a parallelogram are supplementary

$$\angle A + \angle B = 180°$$

$$\frac{1}{2}\angle A + \frac{1}{2}\angle B = \frac{180^{\circ}}{2}$$

$$\Longrightarrow \angle PAB + \angle PBA = 90°$$

In ΔAPB,

$$\angle PAB + \angle APB + \angle PBA = 180° \text{ (angle sum property of triangle)}$$

$$\angle APB = 180° - [\angle PAB + \angle PBA]$$

$$= 180° - 90° = 90°$$

Hence Proved.

Example 4.2

In the Fig.4.35 $ABCD$ is a parallelogram, P and Q are the mid-points of sides AB and DC respectively. Show that $APCQ$ is a parallelogram.

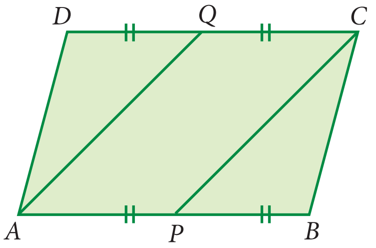


Fig. 4.35

Solution

Since P and Q are the mid points of

AB and DC respectively

Therefore $AP = \frac{1}{2} AB$ and

$QC = \frac{1}{2} DC$ (1)

But $AB = DC$ (Opposite sides of a parallelogram are equal)

$\Longrightarrow \frac{1}{2} AB = \frac{1}{2} DC$

$\Longrightarrow AP = QC$ (2)

Also , $AB \parallel DC$

$\Longrightarrow AP \parallel QC$ (3) [$\because$ $ABCD$ is a parallelogram]

Thus , in quadrilateral $APCQ$ we have $AP = QC$ and $AP \parallel QC$ [from (2) and (3)]

Hence , quadrilateral $APCQ$ is a parallelogram.

Example 4.3

$ABCD$ is a parallelogram Fig.4.36 such that $\angle BAD = 120^{o}$ and AC bisects $\angle BAD$ show that $ABCD$ is a rhombus.

Solution

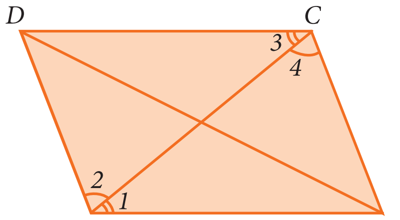


Fig. 4.36

Given $\angle BAD = 120°$ and AC bisects $\angle BAD$

$\angle BAC = \frac{1}{2} \times 120° = 60°$

$\angle 1 = \angle 2 = 60°$

$AD \parallel BC$ and AC is the traversal

$\angle 2 = \angle 4 = 60°$

ΔABC is isosceles triangle [$\because \angle 1 = \angle 4 = 60°$]

$\Longrightarrow AB = BC$

Parallelogram $ABCD$ is a rhombus.

Example 4.4 In a parallelogram $ABCD$, P and Q are the points on line DB such that $PD = BQ$ show that $APCQ$ is a parallelogram

Solution

$ABCD$ is a parallelogram.

$$OA = OC \text{ and}$$

$$OB = OD \ (\because \text{Diagonals bisect each other})$$

now $$OB + BQ = OD + DP$$

$$OQ = OP \text{ and } OA = OC$$

$APCQ$ is a parallelogram.

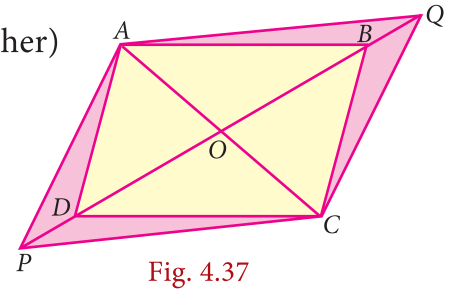


Fig. 4.37

Exercise 4.2

1. The angles of a quadrilateral are in the ratio 2 : 4 : 5 : 7. Find all the angles.
2. In a quadrilateral $ABCD$, $\angle A = 72°$ and $\angle C$ is the supplementary of $\angle A$. The other two angles are $2x–10$ and $x + 4$. Find the value of x and the measure of all the angles.
3. $ABCD$ is a rectangle whose diagonals AC and BD intersect at O. If $\angle OAB = 46°$, find $\angle OBC$
4. The lengths of the diagonals of a Rhombus are 12 cm and 16 cm . Find the side of the rhombus.
5. Show that the bisectors of angles of a parallelogram form a rectangle .
6. If a triangle and a parallelogram lie on the same base and between the same parallels, then prove that the area of the triangle is equal to half of the area of parallelogram.
7. Iron rods a, b, c, d, e, and f are making a design in a bridge as shown in the figure. If $a \parallel b$, $c \parallel d$, $e \parallel f$, find the marked angles between
   (i) b and c
   (ii) d and e
   (iii d and f
   (iv) c and f

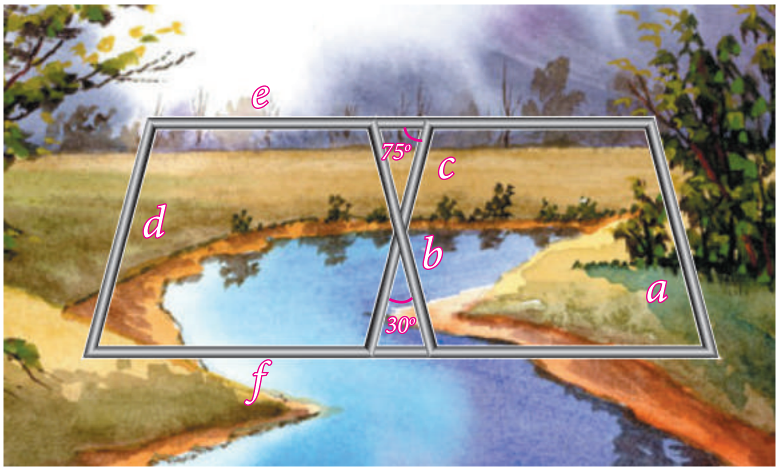


Fig. 4.38

8. In the given Fig. 4.39, $\angle A = 64°$, $\angle ABC = 58°$. If BO and CO are the bisectors of $\angle ABC$ and $\angle ACB$ respectively of ΔABC, find $x°$ and $y°$

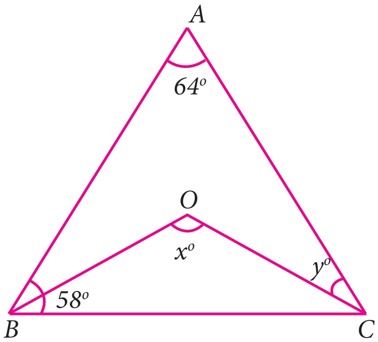


Fig. 4.39

9. In the given Fig. 4.40, if $AB = 2$, $BC = 6$, $AE = 6$, $BF = 8$, $CE = 7$, and $CF = 7$, compute the ratio of the area of quadrilateral $ABDE$ to the area of ΔCDF.

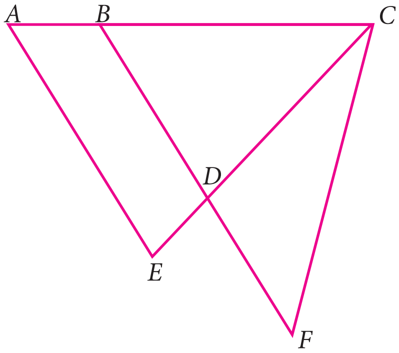


Fig. 4.40

10. In the Fig. 4.41 , $ABCD$ is a rectangle and $EFGH$ is a parallelogram. Using the measurements given in the figure, what is the length d of the segment that is perpendicular to $\overline{HE}$ and $\overline{FG}$?

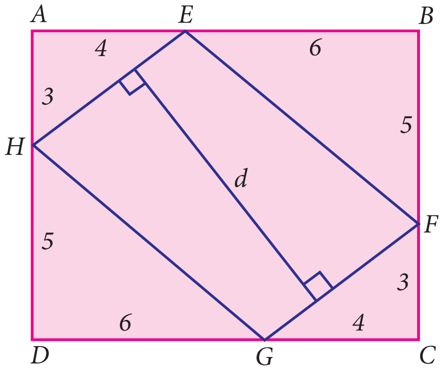


Fig. 4.41

11. In parallelogram $ABCD$ of the accompanying diagram, line DP is drawn bisecting BC at N and meeting AB (extended) at P. From vertex C, line CQ is drawn bisecting side AD at M and meeting AB (extended) at Q. Lines DP and CQ meet at O. Show that the area of triangle QPO is $\frac{9}{8}$ of the area of the parallelogram $ABCD$.

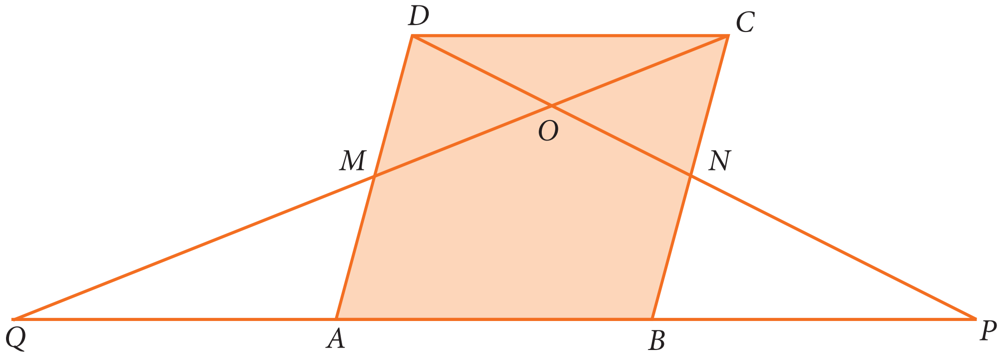


Fig. 4.42

4.4 Parts of a Circle

Circles are geometric shapes you can see all around you. The significance of the concept of a circle can be well understood from the fact that the wheel is one of the ground-breaking inventions in the history of mankind.

Fig. 4.43

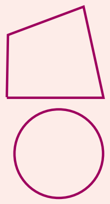
A circle, you can describe, is the set of all points in a plane at a constant distance from a fixed point. The fixed point is the centre of the circle; the constant distance corresponds to a radius of the circle.

A line that cuts the circle in two points is called a secant of the circle.

A line segment whose end points lie on the circle is called a chord of the circle.

Note

A circle notably differs from a polygon. A polygon (for example, a quadrilateral) has edges and corners while, a circle is a 'smooth' curve.



A chord of a circle that has the centre is called a diameter of the circle. The circumference of a circle is its boundary. (We use the term perimeter in the case of polygons).

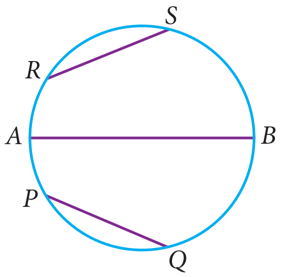


Fig. 4.44

In Fig.4.44, we see that all the line segments meet at two points on the circle. These line segments are called the chords of the circle. So, a line segment joining any two points on the circle is called a chord of the circle. In this figure AB, PQ and RS are the chords of the circle.

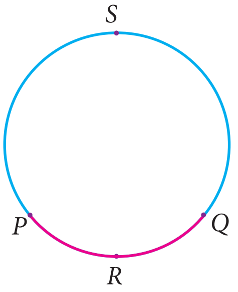


Fig. 4.45

Now place four points P, R, Q and S on the same circle (Fig.4.45), then PRQ and QSP are the continuous parts (sections) of the circle. These parts (sections) are to be denoted by $\overset{\frown}{PRQ}$ and $\overset{\frown}{QSP}$ or simply by $\overset{\frown}{PQ}$ and $\overset{\frown}{QP}$. This continuous part of a circle is called an arc of the circle. Usually the arcs are denoted in anti-clockwise direction.

Now consider the points P and Q in the circle (Fig.4.45). It divides the whole circle into two parts. One is longer and another is shorter. The longer one is called major arc $\overset{\frown}{QP}$ and shorter one is called minor arc $\overset{\frown}{PQ}$.

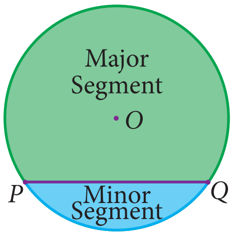


Fig. 4.46

Now in (Fig.4.46), consider the region which is surrounded by the chord PQ and major arc $\overset{\frown}{QP}$. This is called the major segment of the circle. In the same way, the segment containing the minor arc and the same chord is called the minor segment.

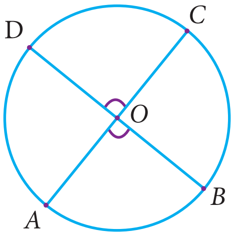


Fig. 4.47

In (Fig.4.47), if two arcs $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ of a circle subtend the same angle at the centre, they are said to be congruent arcs and we write,

$$\overset{\frown}{AB} \equiv \overset{\frown}{CD} \text{ implies } m\overset{\frown}{AB} = m\overset{\frown}{CD}$$

$$\text{implies } \angle AOB = \angle COD$$

Note

A diameter of a circle is:

- the line segment which bisects the circle.
- the largest chord of a circle.
- a line of symmetry for the circle.
- twice in length of a radius in a circle.

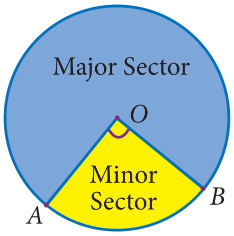


Fig. 4.48

Now, let us observe (Fig.4.48). Is there any special name for the region surrounded by two radii and arc? Yes, its name is sector. Like segment, we find that the minor arc corresponds to the minor sector and the major arc corresponds to the major sector.

Concentric Circles

Circles with the same centre but different radii are said to be concentric.

Here are some real-life examples:

An Archery target

A carom board coin

Water ripples

Fig. 4.49

Congruent Circles

Two circles are congruent if they are copies of one another or identical. That is, they have the same size. Here are some real life examples:

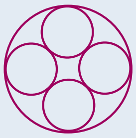

The two wheels of a bullock cart

The Olympic rings

Fig. 4.50

Thinking Corner

Draw four congruent circles as shown. What do you infer?



Position of a Point with respect to a Circle

Consider a circle in a plane (Fig.4.51). Consider any point *P* on the circle. If the distance from the centre *O* to the point *P* is *OP*, then

(i) OP = radius (If the point *P* lies on the circle)

(ii) $OP <$ radius (If the point *P* Point lies inside the circle)

(iii) $OP >$ radius(If the point P lies outside the circle)

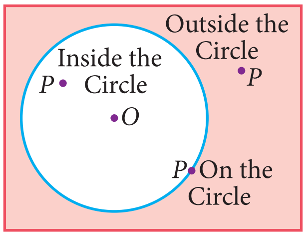


Fig. 4.51

So, a circle divides the plane on which it lies into three parts.

Progress Check

Say True or False

1. Every chord of a circle contains exactly two points of the circle.
2. All radii of a circle are of same length.
3. Every radius of a circle is a chord.
4. Every chord of a circle is a diameter.
5. Every diameter of a circle is a chord.
6. There can be any number of diameters for a circle.
7. Two diameters cannot have the same end-point.
8. A circle divides the plane into three disjoint parts.
9. A circle can be partitioned into a major arc and a minor arc.
10. The distance from the centre of a circle to the circumference is that of a diameter

Thinking Corner

1. How many sides does a circle have ?
2. Is circle, a polygon?

4.4.1 Circle Through Three Points

We have already learnt that there is one and only one line passing through two points. In the same way, we are going to see how many circles can be drawn through a given point, and through two given points. We see that in both cases there can be infinite number of circles passing through a given point P (Fig.4.52) , and through two given points A and B (Fig.4.53).

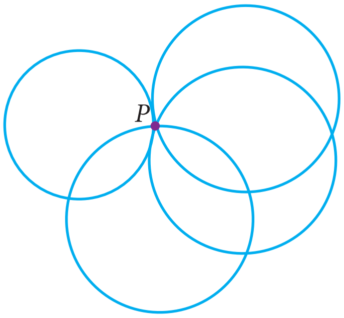


Fig. 4.52

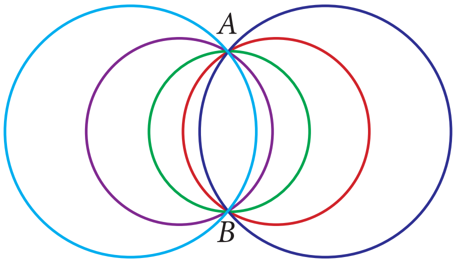


Fig. 4.53

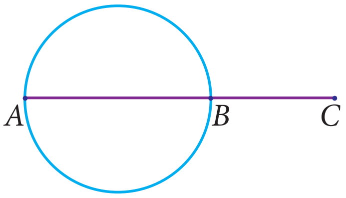


Fig. 4.54

Now consider three collinear points A, B and C (Fig.4.14). Can we draw a circle passing through these three points? Think over it. If the points are collinear, we can't?

If the three points are non collinear, they form a triangle (Fig.4.55). Recall the construction of the circumcentre. The intersecting point of the perpendicular bisector of the sides is the circumcentre and the circle is circumcircle.

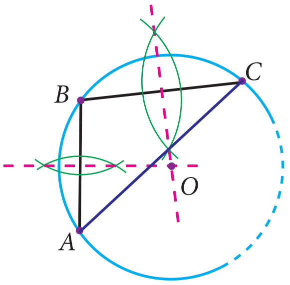


Fig. 4.55

Therefore from this we know that, there is a unique circle which passes through A, B and C. Now, the above statement leads to a result as follows.

Theorem 6 There is one and only one circle passing through three non-collinear points.

4.5 Properties of Chords of a Circle

In this chapter, already we come across lines, angles, triangles and quadrilaterals. Recently we have seen a new member circle. Using all the properties of these, we get some standard results one by one. Now, we are going to discuss some properties based on chords of the circle.

Considering a chord and a perpendicular line from the centre to a chord, we are going to see an interesting property.

4.5.1 Perpendicular from the Centre to a Chord

Consider a chord AB of the circle with centre O. Draw $OC \perp AB$ and join the points OA, OB. Here, easily we get two triangles ΔAOC and ΔBOC (Fig.4.56).

Can we prove these triangles are congruent? Now we try to prove this using the congruence of triangle rule which we have already learnt. $\angle OCA = \angle OCB = 90^\circ (OC \perp AB)$ and $OA = OB$ is the radius of the circle. The side OC is common. RHS criterion tells us that ΔAOC and ΔBOC are congruent. From this we can conclude that $AC = BC$. This argument leads to the result as follows.

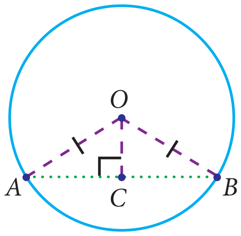


Fig. 4.56

Theorem 7 The perpendicular from the centre of a circle to a chord bisects the chord.

Converse of Theorem 7 The line joining the centre of the circle and the midpoint of a chord is perpendicular to the chord.

Example 4.5 Find the length of a chord which is at a distance of $2\sqrt{11}$ cm from the centre of a circle of radius 12cm.

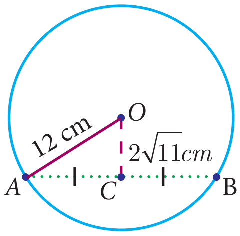


Fig. 4.57

Solution

Let AB be the chord and C be the mid point of AB

Therefore, $OC \perp AB$

Join OA and OC.

OA is the radius

Given $OC = 2\sqrt{11}\text{cm}$ and OA = 12cm

In a right ΔOAC,

using Pythagoras Theorem, we get,

$$AC^2 = OA^2 - OC^2$$

$$= 12^2 - (2\sqrt{11})^2$$

$$= 144 - 44$$

$$= 100\text{cm}$$

$$AC^{\ 2} = 100\text{cm}$$

$$AC = 10\text{cm}$$

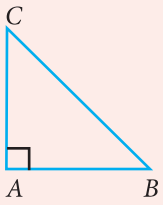
Therefore, length of the chord $AB = 2AC$

$$= 2 \times 10\text{cm} = 20\text{cm}$$

Note

Pythagoras theorem

One of the most important and well known results in geometry is Pythagoras Theorem. "In a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides".



In right Δ $ABC, BC^2 = AB^2 + AC^2$.

Application of this theorem is most useful in this unit.

Example 4.6 In the concentric circles, chord AB of the outer circle cuts the inner circle at C and D as shown in the diagram. Prove that, $AB - CD = 2AC$

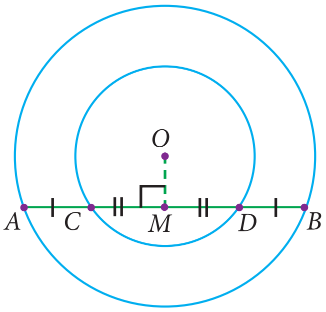


Fig. 4.58

Solution

Given : Chord AB of the outer circle cuts the inner circle at C and D.

To prove : $AB - CD = 2AC$

Construction : Draw $OM \perp AB$

Proof : Since, $OM \perp AB$ (By construction)

Also, $OM \perp CD$

Therefore, $AM = MB$ ... (1) (Perpendicular drawn from centre to chord bisect it)

$CM = MD$ ... (2)

Now, $AB - CD = 2AM - 2CM$

$= 2(AM - CM)$ from (1) and (2)

$AB - CD = 2AC$

Progress Check

1. The radius of the circle is 25 cm and the length of one of its chord is 40cm. Find the distance of the chord from the centre.
2. Draw three circles passing through the points P and Q, where PQ = 4cm.

4.5.2 Angle Subtended by Chord at the Centre

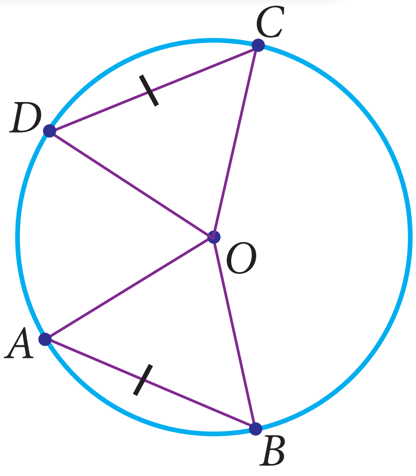


Fig. 4.59

Instead of a single chord we consider two equal chords. Now we are going to discuss another property.

Let us consider two equal chords in the circle with centre O. Join the end points of the chords with the centre to get the triangles ΔAOB and ΔOCD, chord AB = chord CD (because the given chords are equal). The other sides are radii, therefore $OA=OC$ and $OB=OD$. By SSS rule, the triangles are congruent, that is $\Delta OAB \equiv \Delta OCD$. This gives $m\angle AOB = m\angle COD$. Now this leads to the following result.

BDTH9N

Theorem 8 Equal chords of a circle subtend equal angles at the centre.

Activity - 5

Procedure

1. Draw a circle with centre O and with suitable radius.
2. Make it a semi-circle through folding. Consider the point A, B on it.
3. Make crease along AB in the semi circles and open it.
4. We get one more crease line on the another part of semi circle, name it as CD ($observe\ AB = CD$)
5. Join the radius to get the ΔOAB and ΔOCD.
6. Using trace paper, take the replicas of triangle ΔOAB and ΔOCD.
7. Place these triangles ΔOAB and ΔOCD one on the other.

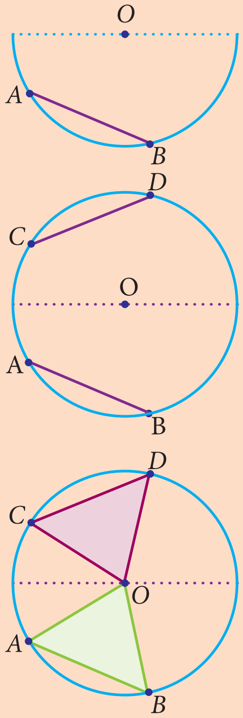


Fig. 4.60

Observation

1. What do you observe? Is $\Delta OAB \equiv \Delta OCD$?
2. Construct perpendicular line to the chords AB and CD passing through the centre O. Measure the distance from O to the chords.

Now we are going to find out the length of the chords AB and CD, given the angles subtended by two chords at the centre of the circle are equal. That is, $\angle AOB = \angle COD$ and the two sides which include these angles of the ΔAOB and ΔCOD are radii and are equal.

By SAS rule, $\Delta AOB \equiv \Delta COD$. This gives chord AB = chord CD. Now let us write the converse result as follows:

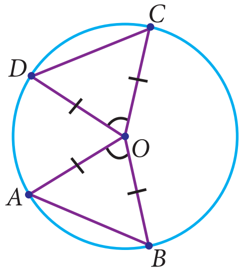


Fig. 4.61

Converse of theorem 8

If the angles subtended by two chords at the centre of a circle are equal, then the chords are equal.

In the same way we are going to discuss about the distance from the centre, when the equal chords are given. Draw the perpendicular $OL \perp AB$ and $OM \perp CD$. From theorem 7, these perpendicular divides the chords equally. So $AL = CM$. By comparing the ΔOAL and ΔOCM, the angles $\angle OLA = \angle OMC = 90°$ and $OA = OC$ are radii. By RHS rule, the $\Delta OAL \equiv \Delta OCM$. It gives the distance from the centre $OL = OM$ and write the conclusion as follows.

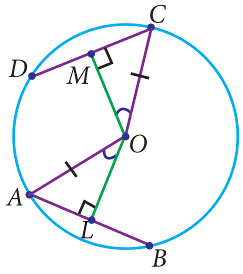


Fig. 4.62

Theorem 9 Equal chords of a circle are equidistant from the centre.

Let us know the converse of theorem 9, which is very useful in solving problems.

Converse of theorem 9

The chords of a circle which are equidistant from the centre are equal.

4.5.3 Angle Subtended by an Arc of a Circle

Activity - 6

Procedure :

1. Draw three circles of any radius with centre O on a chart paper.
2. From these circles, cut a semi-circle, a minor segment and a major segment.
3. Consider three points on these segment and name them as A, B and C.

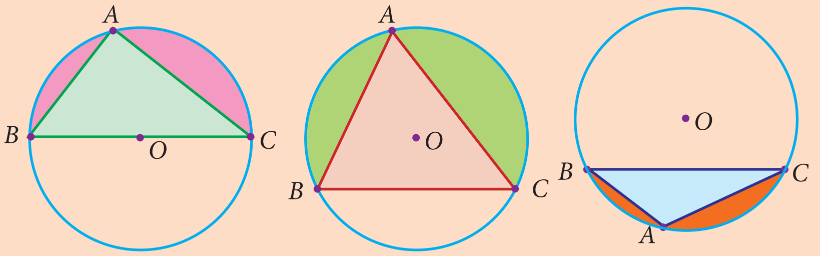


Fig. 4.63

4. (iv) Cut the triangles and paste it on the graph sheet so that the point A coincides with the origin as shown in the figure.

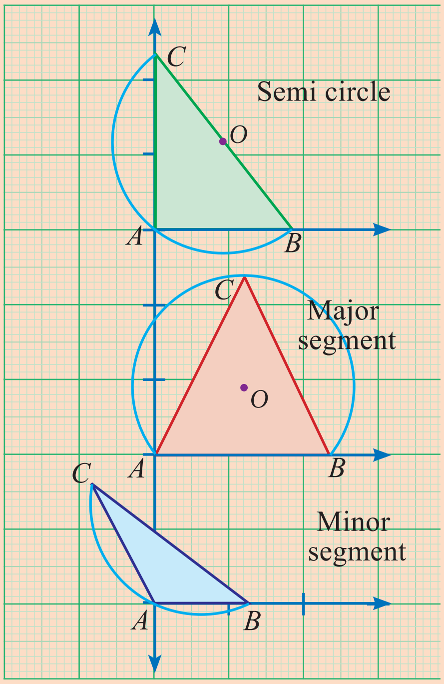


Fig. 4.64

Observation :

(i) Angle in a Semi-Circle is _______ angle.

(ii) Angle in a major segment is _____ angle.

(iii) Angle in a minor segment is _____ angle.

Now we are going to verify the relationship between the angle subtended by an arc at the centre and the angle subtended on the circumference.

4.5.4 Angle at the Centre and the Circumference

Let us consider any circle with centre O. Now place the points A, B and C on the circumference.

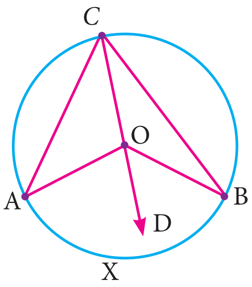


Fig. 4.65

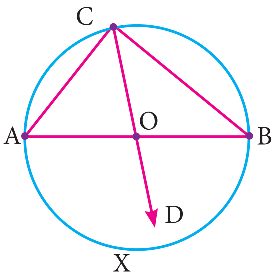


Fig. 4.66

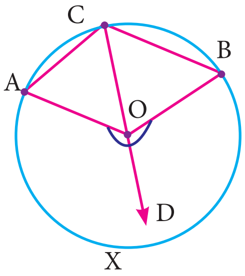


Fig. 4.67

Here $\overset{\frown}{AB}$ is a minor arc in Fig.4.65, a semi circle in Fig.4.66 and a major arc in Fig.4.67. The point C makes different types of angles in different positions (Fig. 4.65 to 4.67). In all these circles, $\overset{\frown}{AXB}$ subtends $\angle AOB$ at the centre and $\angle ACB$ at a point on the circumference of the circle.

We want to prove $\angle AOB = 2\angle ACB$. For this purpose extend CO to D and join CD.

$\angle OCA = \angle OAC$ since $OA = OC$ (radii)

Exterior angle = sum of two interior opposite angles.

$$\begin{aligned} \angle AOD &= \angle OAC + \angle OCA \\ &= 2\angle OCA \quad \text{.... (1)} \end{aligned}$$

Similarly,

$$\begin{aligned} \angle BOD &= \angle OBC + \angle OCB \\ &= 2\angle OCB \quad \text{.... (2)} \end{aligned}$$

From (1) and (2),

$$\angle AOD + \angle BOD = 2(\angle OCA + \angle OCB)$$

Finally we reach our result $\angle AOB = 2\angle ACB$.

From this we get the result as follows :

Progress Check

i. Draw the outline of different size of bangles and try to find out the centre of each using set square.

ii. Trace the given cresent and complete as full moon using ruler and compass.

Fig. 4.68

Theorem 10

The angle subtended by an arc of the circle at the centre is double the angle subtended by it at any point on the remaining part of the circle.

Note

- Angle inscribed in a semicircle is a right angle.
- Equal arcs of a circle subtend equal angles at the centre.

Example 4.7 Find the value of $x°$ in the following figures:

(i)

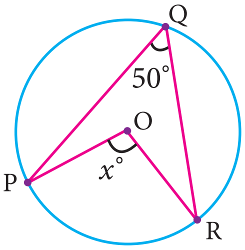


Fig. 4.69

(ii)

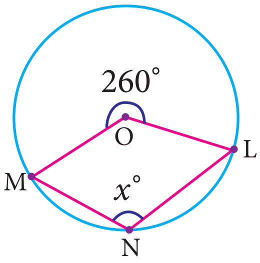


Fig. 4.70

(iii)

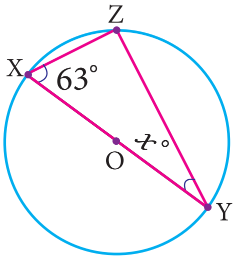


Fig. 4.71

(iv)

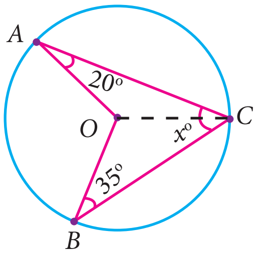


Fig. 4.72

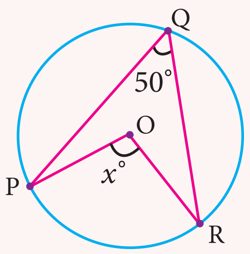
Solution

Using the theorem the angle subtended by an arc of a circle at the centre is double the angle subtended by it at any point on the remaining part of a circle.

(i) $\angle POR = 2\angle PQR$

$x^\circ = 2 \times 50^\circ$

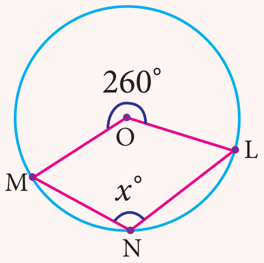
$x^\circ = 100^\circ$



(ii) $\angle MNL = \frac{1}{2}$ Reflex $\angle MOL$

$= \frac{1}{2} \times 260^\circ$

$x^\circ = 130^\circ$



(iii) XY is the diameter of the circle.

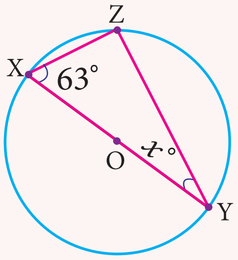
Therefore $\angle XZY = 90^\circ$

(Angle on a semi – circle)

In ΔXYZ

$x^\circ + 63^\circ + 90^\circ = 180^\circ$

$x^\circ = 27^\circ$

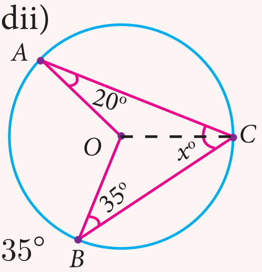


(iv) $OA = OB = OC$ (Radii)

In ΔOAC,

$\angle OAC = \angle OCA = 20^\circ$

In ΔOBC,



$\angle OBC = \angle OCB = 35^\circ$

(angles opposite to equal sides are equal)

$\angle ACB = \angle OCA + \angle OCB$

$x^\circ = 20^\circ + 35^\circ$

$x^\circ = 55^\circ$

Example 4.8 If O is the centre of the circle and $\angle ABC = 30^\circ$ then find $\angle AOC$. (see Fig. 4.73)

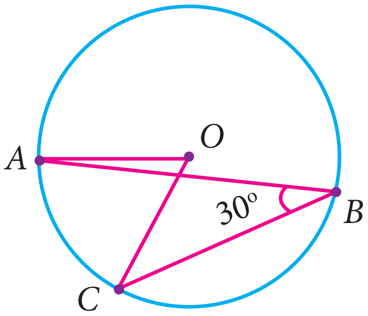


Fig. 4.73

Solution

Given $\angle ABC = 30^\circ$

$\angle AOC = 2\angle ABC$ (The angle subtended by an arc at the centre is double the angle at any point on the circle)

$= 2 \times 30^\circ$

$= 60^\circ$

Now we shall see, another interesting theorem. We have learnt that minor arc subtends obtuse angle, major arc subtends acute angle and semi circle subtends right angle on the circumference. If a chord AB is given and C and D are two different points on the circumference of the circle, then find $\angle ACB$ and $\angle ADB$. Is there any difference in these angles?

4.5.5 Angles in the same segment of a circle

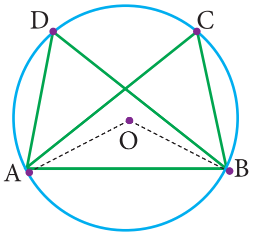


FIG. 4.74

Consider the circle with centre O and chord AB. C and D are the points on the circumference of the circle in the same segment. Join the radius OA and OB.

$$\frac{1}{2}\angle AOB = \angle ACB \quad \text{(by theorem 10)}$$

$$\text{and} \quad \frac{1}{2}\angle AOB = \angle ADB \quad \text{(by theorem 10)}$$

$$\angle ACB = \angle ADB$$

This conclusion leads to the new result.

Theorem 11 Angles in the same segment of a circle are equal.

Example 4.9 In the given figure, O is the center of the circle. If the measure of $\angle OQR = 48^\circ$, what is the measure of $\angle P$?

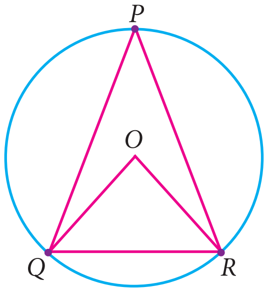


Fig. 4.75

Solution

Given $\angle OQR = 48^\circ$.

Therefore, $\angle ORQ$ also is 48°. (Why?__________)

$\angle QOR = 180^\circ - (2 \times 48^\circ) = 84^\circ$.

The central angle made by chord QR is twice the inscribed angle at P.

Thus, measure of $\angle QPR = \frac{1}{2} \times 84^\circ = 42^\circ$.

Exercise 4.3

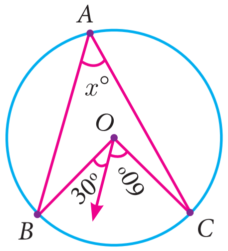
1. The diameter of the circle is 52cm and the length of one of its chord is 20cm. Find the distance of the chord from the centre.
2. The chord of length 30 cm is drawn at the distance of 8cm from the centre of the circle. Find the radius of the circle
3. Find the length of the chord AC where AB and CD are the two diameters perpendicular to each other of a circle with radius $4\sqrt{2}$ cm and also find $\angle OAC$ and $\angle OCA$.
4. A chord is 12cm away from the centre of the circle of radius 15cm. Find the length of the chord.

5. In a circle, AB and CD are two parallel chords with centre O and radius 10 cm such that $AB = 16$ cm and $CD = 12$ cm determine the distance between the two chords?

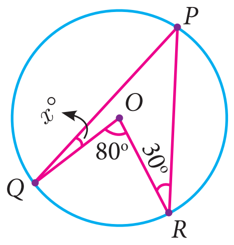
6. Two circles of radii 5 cm and 3 cm intersect at two points and the distance between their centres is 4 cm. Find the length of the common chord.

7. Find the value of x° in the following figures:

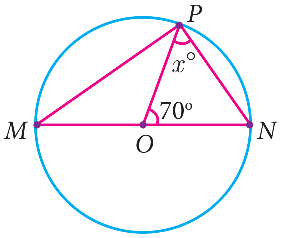
(i)



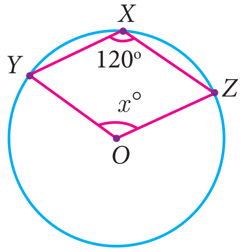
(ii)



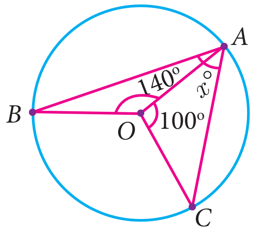
(iii)



(iv)

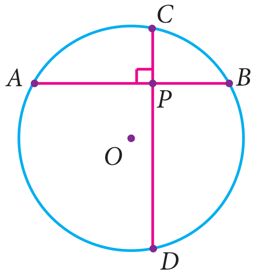


(v)



8. In the given figure, $\angle CAB = 25^\circ$,

find $\angle BDC$, $\angle DBA$ and $\angle COB$



4.6 Cyclic Quadrilaterals

Now, let us see a special quadrilateral with its properties called "Cyclic Quadrilateral". A quadrilateral is called cyclic quadrilateral if all its four vertices lie on the circumference of the circle. Now we are going to learn the special property of cyclic quadrilateral.

Consider the quadrilateral $ABCD$ whose vertices lie on a circle. We want to show that its opposite angles are supplementary. Connect the centre O of the circle with each vertex. You now see four radii OA, OB, OC and OD giving rise to four isosceles triangles OAB, OBC, OCD and ODA. The sum of the angles around the centre of the circle is 360°. The angle sum of each isosceles triangle is 180°

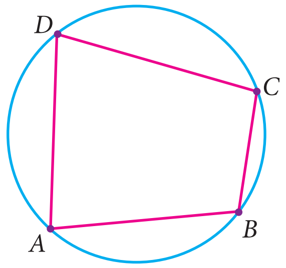


Fig. 4.76

Thus, we get from the figure,

$$2\times(\angle 1+\angle 2+\angle 3+\angle 4) + \text{Angle at centre } O = 4\times 180^\circ$$

$$2\times(\angle 1+\angle 2+\angle 3+\angle 4) + 360^\circ = 720^\circ$$

Simplifying this, $(\angle 1+\angle 2+\angle 3+\angle 4) = 180^\circ$.

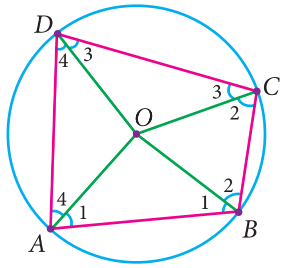


Fig. 4.77

You now interpret this as

(i) $(\angle 1+\angle 2)+(\angle 3+\angle 4)=180^\circ$ (Sum of opposite angles B and D)

(ii) $(\angle 1+\angle 4)+(\angle 2+\angle 3)=180^\circ$ (Sum of opposite angles A and C)

Now the result is given as follows.

BDAQ6D

Theorem 12 Opposite angles of a cyclic quadrilateral are supplementary.

Let us see the converse of theorem 12, which is very useful in solving problems

Converse of Theorem 12 If a pair of opposite angles of a quadrilateral is supplementary, then the quadrilateral is cyclic.

Activity - 7

Procedure

1. Draw a circle of any radius with centre O.
2. Mark any four points A, B, C and D on the boundary. Make a cyclic quadrilateral $ABCD$ and name the angles as in Fig. 4.78
3. Make a replica of the cyclic quadrilateral $ABCD$ with the help of tracing paper.
4. Make the cutout of the angles A, B, C and D as in Fig. 4.79
5. Paste the angle cutout $\angle 1, \angle 2, \angle 3$ and $\angle 4$ adjacent to the angles opposite to A, B, C and D as in Fig. 4.80
6. Measure the angles $\angle 1+\angle 3$, and $\angle 2+\angle 4$.

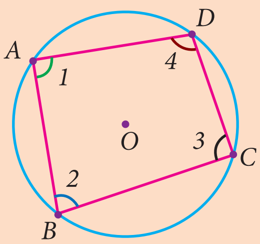


Fig. 4.78

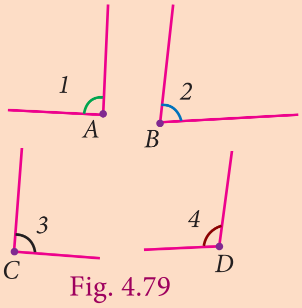


Fig. 4.79

Observe and complete the following:

1. (i) $\angle A+\angle C=$ ______ (ii) $\angle B+\angle D=$ ______

   (iii) $\angle C+\angle A=$ ______ (iv) $\angle D+\angle B=$ ______

2. Sum of opposite angles of a cyclic quadrilateral is __________.

3. The opposite angles of a cyclic quadrilateral is ________.

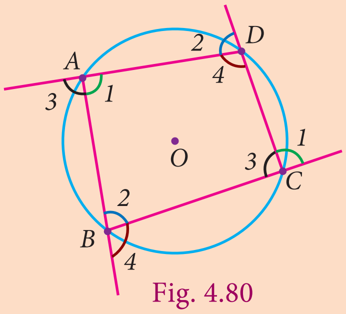


Fig. 4.80

Example 4.10

If $PQRS$ is a cyclic quadrilateral in which $\angle PSR=70^\circ$ and $\angle QPR=40^\circ$, then find $\angle PRQ$ (see Fig. 4.81).

Solution

$PQRS$ is a cyclic quadrilateral

Given $\angle PSR=70^\circ$

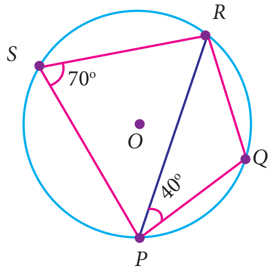


Fig. 4.81

$$\begin{aligned} \angle PSR + \angle PQR &= 180^\circ \quad \text{(state reason\_\_\_\_\_\_\_\_)} \\ 70^\circ + \angle PQR &= 180^\circ \\ \angle PQR &= 180^\circ - 70^\circ \\ \angle PQR &= 110^\circ \end{aligned}$$

In ΔPQR we have,

$$\begin{aligned} \angle PQR + \angle PRQ + \angle QPR &= 180^\circ \quad \text{(state reason\_\_\_\_\_\_\_\_\_)} \\ 110^\circ + \angle PRQ + 40^\circ &= 180^\circ \\ \angle PRQ &= 180^\circ - 150^\circ \\ \angle PRQ &= 30^\circ \end{aligned}$$

Exterior Angle of a Cyclic Quadrilateral

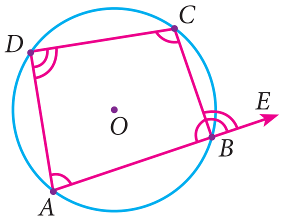


Fig. 4.82

An exterior angle of a quadrilateral is an angle in its exterior formed by one of its sides and the extension of an adjacent side.

Let the side AB of the cyclic quadrilateral $ABCD$ be extended to E. Here $\angle ABC$ and $\angle CBE$ are linear pair, their sum is 180° and the angles $\angle ABC$ and $\angle ADC$ are the opposite angles of a cyclic quadrilateral, and their sum is also 180°. From this, $\angle ABC + \angle CBE = \angle ABC + \angle ADC$ and finally we get $\angle CBE = \angle ADC$. Similarly it can be proved for other angles.

Theorem 13 If one side of a cyclic quadrilateral is produced then the exterior angle is equal to the interior opposite angle.

Progress Check

1. If a pair of opposite angles of a quadrilateral is supplementary, then the quadrilateral is ______________.
2. As the length of the chord decreases, the distance from the centre ______________.
3. If one side of a cyclic quadrilateral is produced then the exterior angle is ______________ to the interior opposite angle.
4. Opposite angles of a cyclic quadrilateral are ______________.

Example 4.11 In the figure given, find the value of x° and y°.

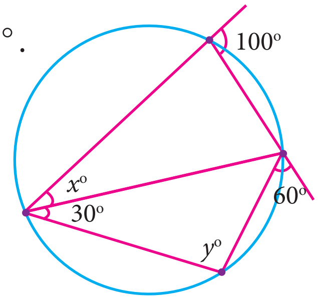


Fig. 4.83

Solution

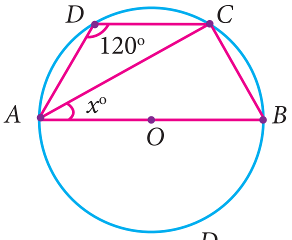
By the exterior angle property of a cyclic quadrilateral,

we get, $y^\circ = 100^\circ$ and

$x^\circ + 30^\circ = 60^\circ$ and so $x^\circ = 30^\circ$

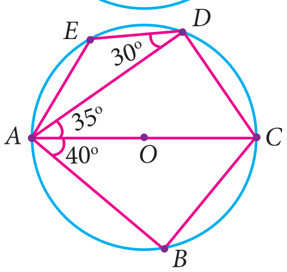
Exercise 4.4

1. Find the value of x in the given figure.

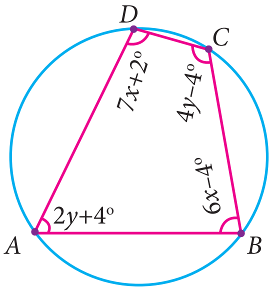


2. In the given figure, AC is the diameter of the circle with centre O. If $\angle ADE = 30°$; $\angle DAC = 35°$ and $\angle CAB = 40°$.

   Find (i) $\angle ACD$ (ii) $\angle ACB$ (iii) $\angle DAE$

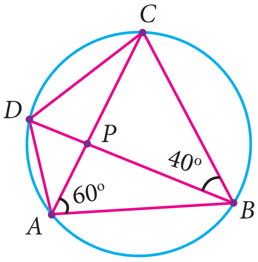


3. Find all the angles of the given cyclic quadrilateral $ABCD$ in the figure.

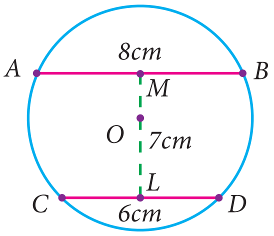


4. In the given figure, $ABCD$ is a cyclic quadrilateral where diagonals intersect at P such that $\angle DBC = 40°$ and $\angle BAC = 60°$ find

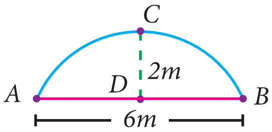
   (i) $\angle CAD$ (ii) $\angle BCD$



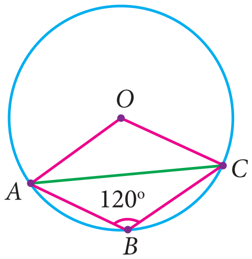
5. In the given figure, AB and CD are the parallel chords of a circle with centre O. Such that AB = 8cm and CD = 6cm. If $OM \perp AB$ and $OL \perp CD$ distance between LM is 7cm. Find the radius of the circle?



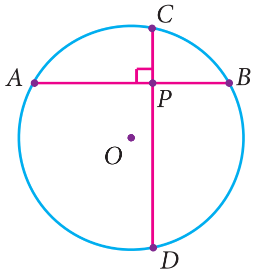
6. The arch of a bridge has dimensions as shown, where the arch measure 2m at its highest point and its width is 6m. What is the radius of the circle that contains the arch?



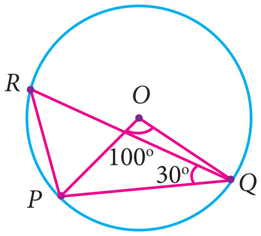
7. In figure, $\angle ABC = 120^0$, where A, B and C are points on the circle with centre O. Find $\angle OAC$?



8. A school wants to conduct tree plantation programme. For this a teacher allotted a circle of radius 6m ground to nineth



standard students for planting sapplings. Four students plant trees at the points A, B, C and D as shown in figure. Here AB = 8m, CD = 10m and $AB \perp CD$. If another student places a flower pot at the point P, the intersection of AB and CD, then find the distance from the centre to P.



9. In the given figure, $\angle POQ = 100^0$ and $\angle PQR = 30^0$, then find $\angle RPO$.

4.7 Practical Geometry

Practical geometry is the method of applying the rules of geometry dealt with the properties of points, lines and other figures to construct geometrical figures. "Construction" in Geometry means to draw shapes, angles or lines accurately. The geometric constructions have been discussed in detail in Euclid's book 'Elements'. Hence these constructions are also known as Euclidean constructions. These constructions use only compass and straightedge (i.e. ruler). The compass establishes equidistance and the straightedge establishes collinearity. All geometric constructions are based on those two concepts.

It is possible to construct rational and irrational numbers using straightedge and a compass as seen in chapter II. In 1913 the Indian mathematical Genius, Ramanujam gave a geometrical construction for 355/113 =π. Today with all our accumulated skill in exact measurements. it is a noteworthy feature that lines driven through a mountain meet and make a tunnel.In the earlier classes, we have learnt the construction of angles and triangles with the given measurements.

In this chapter we are going to learn to construct Centroid, Orthocentre, Circumcentre and Incentre of a triangle by using concurrent lines.

4.7.1 Construction of the Centroid of a Triangle

Centroid

The point of concurrency of the medians of a triangle is called the centroid of the triangle and is usually denoted by *G*.

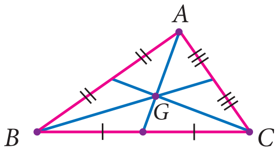


Fig. 4.84

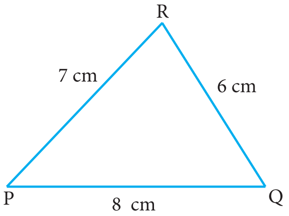
Activity 8

Objective To find the mid-point of a line segment using paper folding

Procedure Make a line segment on a paper by folding it and name it *PQ*. Fold the line segment *PQ* in such a way that *P* falls on *Q* and mark the point of intersection of the line segment and the crease formed by folding the paper as *M*. *M* is the midpoint of *PQ*.

Example 4.12 Construct the centroid of ΔPQR whose sides are $PQ = 8\text{cm}$; $QR = 6\text{cm}$; $RP = 7\text{cm}$.

Rough Diagram



Solution

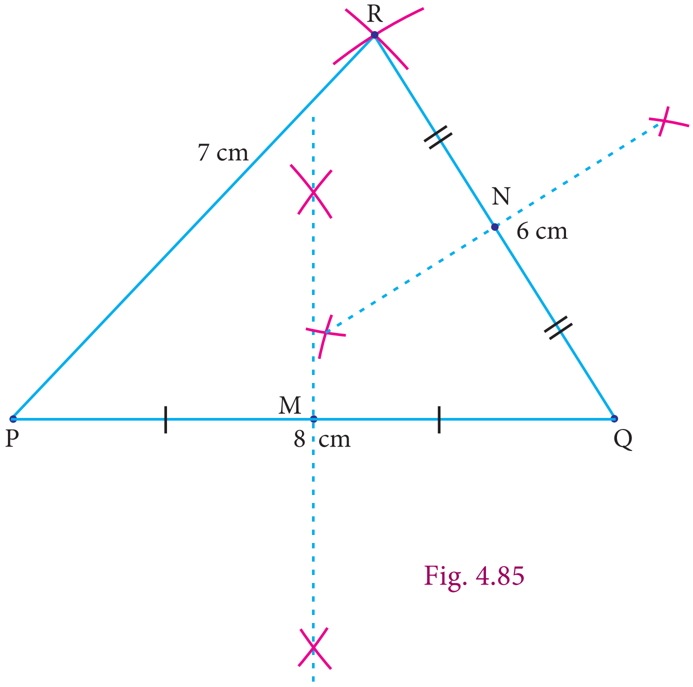


Fig. 4.85

Step 1 : Draw ΔPQR using the given measurements $PQ = 8\text{cm}$ $QR = 6\text{cm}$ and $RP = 7\text{cm}$ and construct the perpendicular bisector of any two sides (PQ and QR) to find the mid-points M and N of PQ and QR respectively.

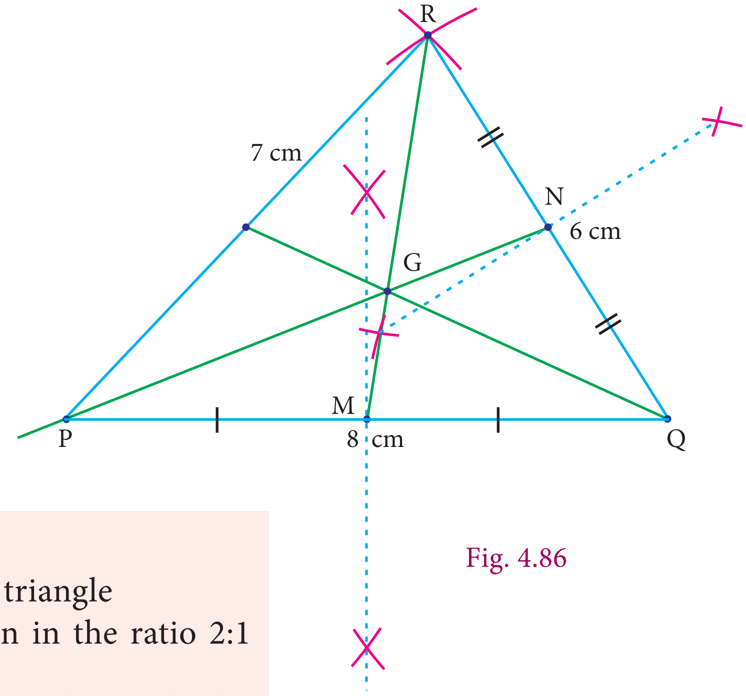


Fig. 4.86

Step 2 : Draw the medians PN and RM and let them meet at G. The point G is the centroid of the given ΔPQR.

Note

- Three medians can be drawn in a triangle
- The centroid divides each median in the ratio 2:1 from the vertex.
- The centroid of any triangle always lie inside the triangle.
- Centroid is often described as the triangle's centre of gravity (where the triangle balances evenly) and also as the barycentre.

4.7.2 Construction of Orthocentre of a Triangle

Orthocentre

The orthocentre is the point of concurrency of the altitudes of a triangle. Usually it is denoted by H.

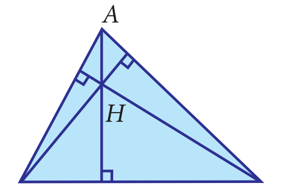


Fig. 4.87

Activity 9

Objective To construct a perpendicular to a line segment from an external point using paper folding.

Procedure Draw a line segment AB and mark an external point P. Move B along BA till the fold passes through P and crease it along that line. The crease thus formed is the perpendicular to AB through the external point P.

Activity 10

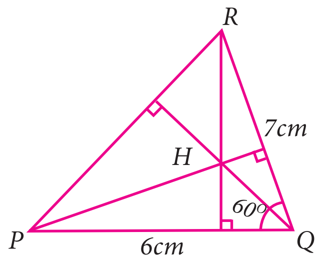
Objective To locate the Orthocentre of a triangle using paper folding.

Procedure Using the above Activity with any two vertices of the triangle as external points, construct the perpendiculars to opposite sides. The point of intersection of the perpendiculars is the Orthocentre of the given triangle.

Example 4.13 Construct ΔPQR whose sides are $PQ = 6$ cm $\angle Q = 60^0$ and $QR = 7$ cm and locate its Orthocentre.

Solution

Step 1 Draw the ΔPQR with the given measurements.



Rough Diagram

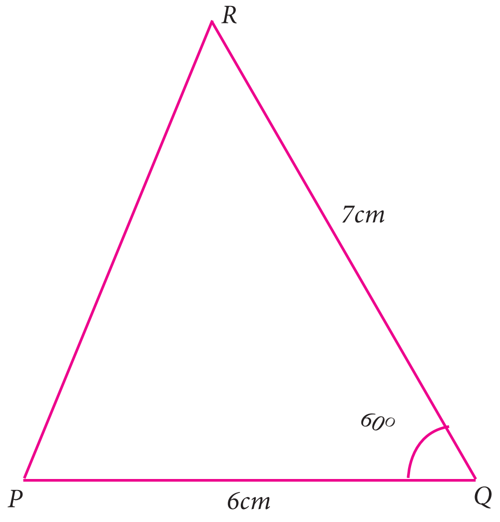


Fig. 4.88

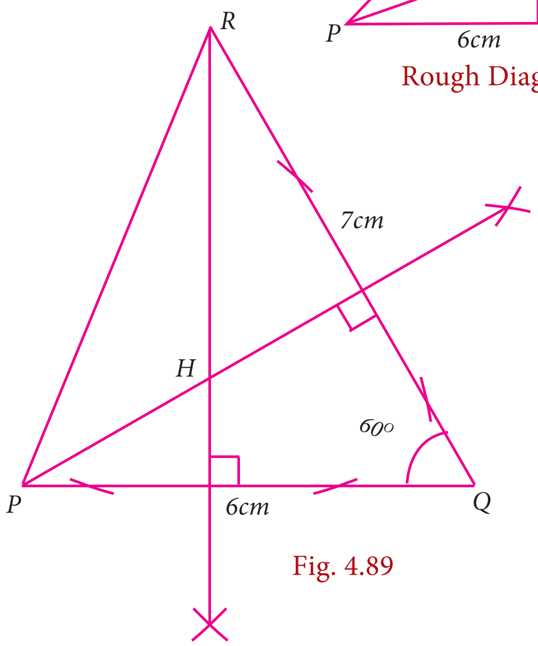


Fig. 4.89

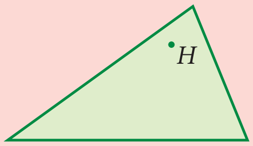
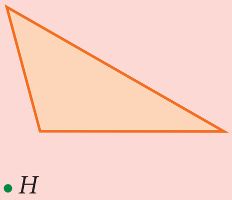
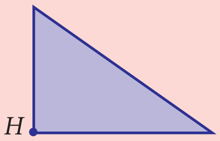
Step 2:

Construct altitudes from any two vertices (say) R and P, to their opposite sides PQ and QR respectively.

The point of intersection of the altitude H is the Orthocentre of the given ΔPQR.

Note

Where do the Orthocentre lie in the given triangles.

	Acute Triangle	**Obtuse Triangle**	**Right Triangle**
Orthocentre	Inside of Triangle  	Outside of Triangle  	Vertex at Right Angle 

Exercise 4.5

1. Construct the ΔLMN such that LM=7.5cm, MN=5cm and LN=8cm. Locate its centroid.
2. Draw and locate the centroid of the triangle ABC where right angle at A, AB = 4cm and AC = 3cm.
3. Draw the ΔABC, where AB = 6cm, $\angle B = 110°$ and AC = 9cm and construct the centroid.
4. Construct the ΔPQR such that PQ = 5cm, PR= 6cm and $\angle QPR = 60°$ and locate its centroid.
5. Draw ΔPQR with sides PQ = 7 cm, QR = 8 cm and PR = 5 cm and construct its Orthocentre.
6. Draw an equilateral triangle of sides 6.5 cm and locate its Orthocentre.
7. Draw ΔABC, where AB = 6 cm, $\angle B = 110^0$ and BC = 5 cm and construct its Orthocentre.
8. Draw and locate the Orthocentre of a right triangle PQR where PQ = 4.5 cm, QR = 6 cm and PR = 7.5 cm.

4.7.3 Construction of the Circumcentre of a Triangle

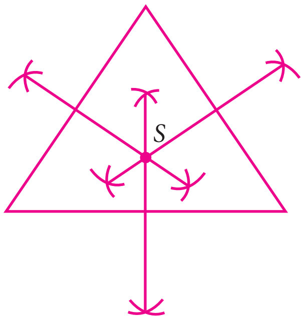


Fig. 4.90

Circumcentre

The Circumcentre is the point of concurrency of the Perpendicular bisectors of the sides of a triangle.
It is usually denoted by S.

Circumcircle

The circle passing through all the three vertices of the triangle with circumcentre (S) as centre is called circumcircle.

Circumradius

The line segment from any vertex of a triangle to the Circumcentre of a given triangle is called circumradius of the circumcircle.

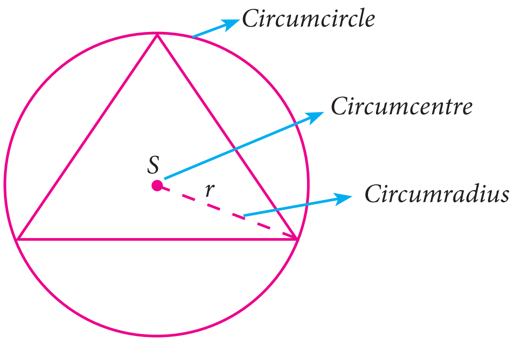


Fig. 4.91

Activity 11

Objective To construct a perpendicular bisector of a line segment using paper folding.

Procedure Make a line segment on a paper by folding it and name it as *PQ*. Fold *PQ* in such a way that *P* falls on *Q* and thereby creating a crease RS. This line *RS* is the perpendicular bisector of *PQ*.

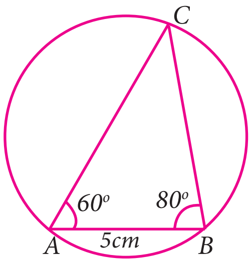
Activity 12

Objective To locate the circumcentre of a triangle using paper folding.

Procedure Using Activity 12, find the perpendicular bisectors for any two sides of the given triangle. The meeting point of these is the circumcentre of the given triangle.

Example 4.14

Construct the circumcentre of the ΔABC with $AB = 5$ cm, $\angle A = 60^0$ and $\angle B = 80^0$. Also draw the circumcircle and find the circumradius of the ΔABC.



Rough Diagram

Solution

Step 1 Draw the ΔABC with the given measurements

Step 2

Construct the perpendicular bisector of any two sides (AC and BC) and let them meet at S which is the circumcentre.

Step 3

S as centre and $SA = SB = SC$ as radius,

draw the Circumcircle to passes through A,B and C.

Circumradius = 3.9 cm.

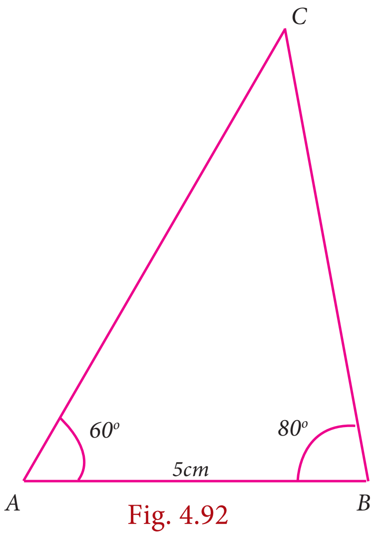


Fig. 4.92

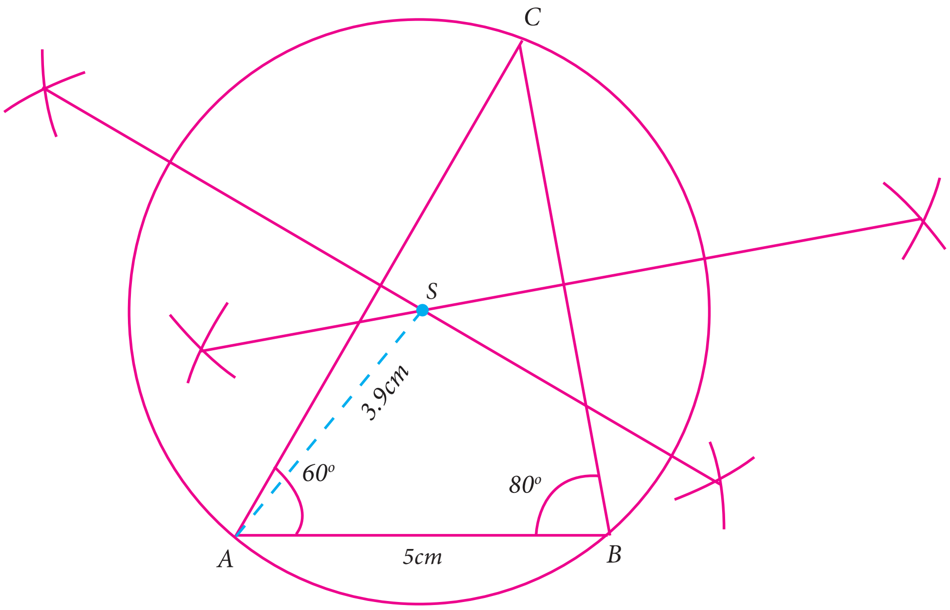
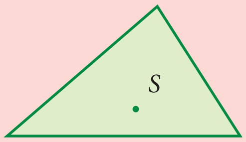
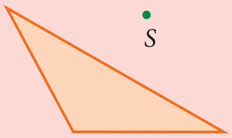
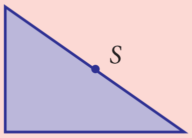


Fig. 4.93

Note

Where do the Circumcentre lie in the given triangles.

	Acute Triangle	Obtuse Triangle	Right Triangle
Circumcentre	Inside of Triangle 	Outside of Triangle 	Midpoint of Hypotenuse 

4.7.4 Construction of the Incircle of a Triangle

Incentre

The incentre is (one of the triangle's points of concurrency formed by) the intersection of the triangle's three angle bisectors.

The incentre is the centre of the incircle ; It is usually denoted by I; it is the one point in the triangle whose distances to the sides are equal.

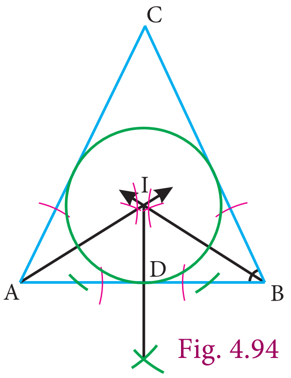
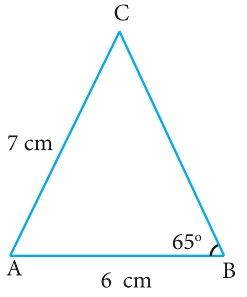


Fig. 4.94

Example 4.15 Construct the incentre of ΔABC with $AB = 6$ cm, $\angle B = 65°$ and $AC = 7$ cm Also draw the incircle and measure its radius.

Solution

Rough Diagram



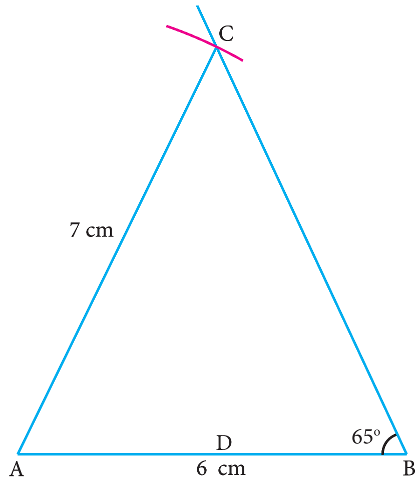


Fig. 4.95

Step 1 : Draw the ΔABC with $AB = 6\text{cm}$, $\angle B = 65°$ and $AC = 7\text{cm}$

Step 2 : Construct the angle bisectors of any two angles (A and B) and let them meet at I. Then I is the incentre of ΔABC. Draw perpendicular from I to any one of the side (AB) to meet AB at D.

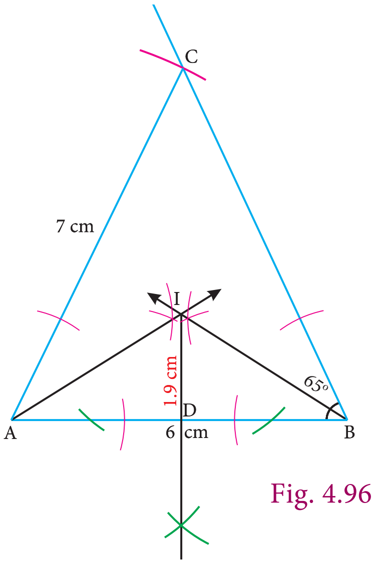


Fig. 4.96

Step 3: With I as centre and ID as radius draw the circle. This circle touches all the sides of the triangle internally.

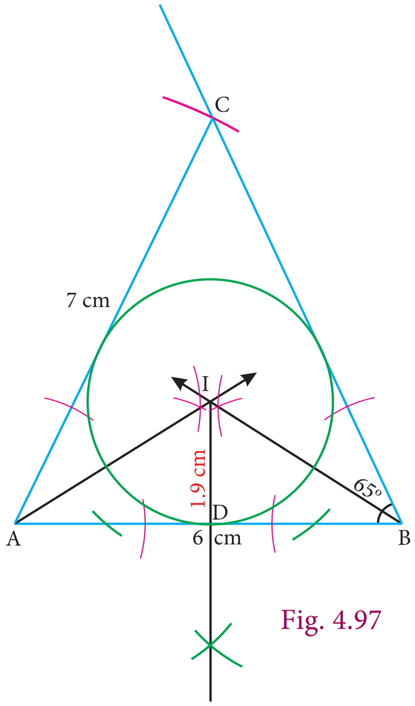


Fig. 4.97

Step 4: Measure inradius

In radius = 1.9 cm.

Exercise 4.6

1 Draw a triangle ABC, where AB = 8 cm, BC = 6 cm and $\angle B = 70^0$ and locate its circumcentre and draw the circumcircle.

2 Construct the right triangle PQR whose perpendicular sides are 4.5 cm and 6 cm. Also locate its circumcentre and draw the circumcircle.

3. Construct ΔABC with $AB = 5$ cm $\angle B = 100^0$ and $BC = 6$ cm. Also locate its circumcentre draw circumcircle.

4. Construct an isosceles triangle PQR where $PQ = PR$ and $\angle Q = 50^0$, $QR = 7$cm. Also draw its circumcircle.

5. Draw an equilateral triangle of side 6.5 cm and locate its incentre. Also draw the incircle.

6. Draw a right triangle whose hypotenuse is 10 cm and one of the legs is 8 cm. Locate its incentre and also draw the incircle.

7. Draw ΔABC given $AB = 9$ cm, $\angle CAB = 115°$ and $\Delta ABC = 40°$. Locatge its incentre and also draw the incircle. (Note: You can check from the above examples that the incentre of any triangle is always in its interior).

8. Construct ΔABC in which $AB = BC$ = 6cm and $\angle B = 80°$. Locate its incentre and draw the incircle.

Exercise 4.7

BE3DBB

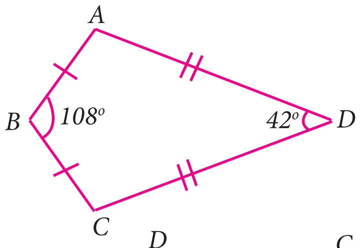
Multiple Choice Questions

1. The exterior angle of a triangle is equal to the sum of two

(1) Exterior angles (2) Interior opposite angles

(3) Alternate angles (4) Interior angles

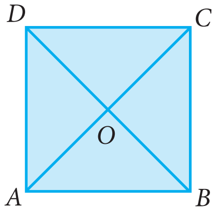
2. In the quadrilateral *ABCD*, *AB* = *BC* and *AD* = *DC* Measure of $\angle BCD$ is



(1) 150° (2) 30°

(3) 105° (4) 72°

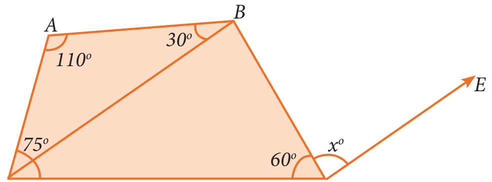
3. *ABCD* is a square, diagonals *AC* and *BD* meet at *O*. The number of pairs of congruent triangles with vertex O are



(1) 6 (2) 8

(3) 4 (4) 12

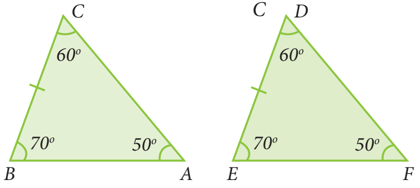
4. In the given figure *CE* || *DB* then the value of x^o is



(1) 45° (2) 30°

(3) 75° (4) 85°

5. The correct statement out of the following is



(1) $\Delta ABC \cong \Delta DEF$ (2) $\Delta ABC \cong \Delta DEF$

(3) $\Delta ABC \cong \Delta FDE$ (4) $\Delta ABC \cong \Delta FED$

6. If the diagonal of a rhombus are equal, then the rhombus is a

(1) Parallelogram but not a rectangle
(2) Rectangle but not a square
(3) Square
(4) Parallelogram but not a square

7. If bisectors of $\angle A$ and $\angle B$ of a quadrilateral $ABCD$ meet at O, then $\angle AOB$ is

(1) $\angle C + \angle D$ (2) $\frac{1}{2}(\angle C + \angle D)$

(3) $\frac{1}{2}\angle C + \frac{1}{3}\angle D$ (4) $\frac{1}{3}\angle C + \frac{1}{2}\angle D$

8. The interior angle made by the side in a parallelogram is 90° then the parallelogram is a

(1) rhombus (2) rectangle (3) trapezium (4) kite

9. Which of the following statement is correct?

(1) Opposite angles of a parallelogram are not equal.
(2) Adjacent angles of a parallelogram are complementary.
(3) Diagonals of a parallelogram are always equal.
(4) Both pairs of opposite sides of a parallelogram are always equal.

10. The angles of the triangle are $3x-40$, $x+20$ and $2x-10$ then the value of x is

(1) 40° (2) 35° (3) 50° (4) 45°

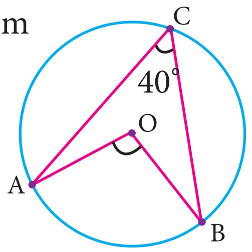
11. PQ and RS are two equal chords of a circle with centre O such that $\angle POQ = 70°$, then $\angle ORS =$

(1) 60° (2) 70° (3) 55° (4) 80°

12. A chord is at a distance of 15cm from the centre of the circle of radius 25cm. The length of the chord is

(1) 25cm (2) 20cm (3) 40cm (4) 18cm

13. In the figure, O is the centre of the circle and $\angle ACB = 40°$ then $\angle AOB =$

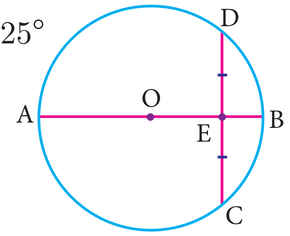


(1) 80° (2) 85° (3) 70° (4) 65°

14. In a cyclic quadrilaterals $ABCD$, $\angle A = 4x$, $\angle C = 2x$ the value of x is

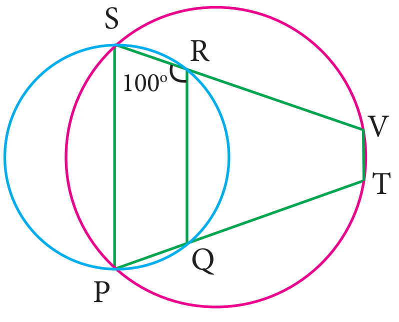
(1) 30° (2) 20° (3) 15° (4) 25°

15. In the figure, O is the centre of a circle and diameter AB bisects the chord CD at a point E such that $CE=ED=8$ cm and $EB=4$cm. The radius of the circle is



(1) 8cm (2) 4cm (3) 6cm (4)10cm

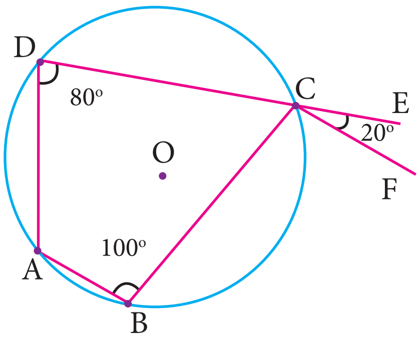
16. In the figure, $PQRS$ and $PTVS$ are two cyclic quadrilaterals, If $\angle QRS = 100°$, then $\angle TVS =$



(1) 80° (2) 100° (3) 70° (4) 90°

17. If one angle of a cyclic quadrilateral is 75°, then the opposite angle is

(1) 100° (2) 105° (3) 85° (4) 90°

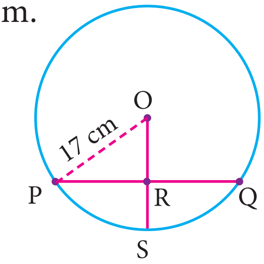


18. In the figure, $ABCD$ is a cyclic quadrilateral in which DC produced to E and CF is drawn parallel to AB such that $\angle ADC = 80°$ and $\angle ECF = 20°$, then $\angle BAD = ?$

(1) 100° (2) 20° (3) 120° (4) 110°

19. AD is a diameter of a circle and AB is a chord If $AD = 30$ cm and $AB = 24$ cm then the distance of AB from the centre of the circle is

(1) 10cm (2) 9cm (3) 8cm (4) 6cm.



20. In the given figure, If $OP = 17$cm, $PQ = 30$cm and OS is perpendicular to PQ, then RS is

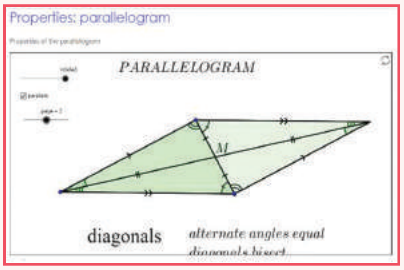
(1) 10cm (2) 6cm

(3) 7cm (4) 9cm.

Points to Remember

- In a parallelogram the opposite sides and opposite angles are equal
- The diagonals of a parallelogram bisect each other.
- The diagonals of a parallelogram divides it into two congruent triangles
- A quadrilateral is a parallelogram if its opposite sides are equal .
- Parallelogram on the same base and between same parallel are equal in area.
- Triangles on the same base and between same parallel are equal in area.
- Parallelogram is a rhombus if its diagonals are perpendicular.
- There is one and only one circle passing through three non-collinear points.
- Equal chords of a circle subtend equal angles at the centre.
- Perpendicular from the centre of a circle to a chord bisects the chord.
- Equal chords of a circle are equidistant from the centre.
- The angle substended by an arc of a circle at the centre is double the angle subtended by it at any point on the remaining part of the circle.
- The angle in a semi circle is a right angle.
- Angles in the same segment of a circle are equal.
- The sum of either pair of opposite angle of a cyclic quadrilateral is 180°.
- If one side of a cyclic quadrilateral is produced then the exterior angle is equal to the interior opposite angle.

ICT Corner-1



Expected Result is shown in this picture

Step – 1

Open the Browser and copy and paste the Link given below (or) by typing the URL given (or) Scan the QR Code.

Step - 2

GeoGebra worksheet "Properties: Parallelogram" will appear. There are two sliders named "Rotate" and "Page"

Step-3

Drag the slider named "Rotate "and see that the triangle is doubled as parallelogram.

Step-4

Drag the slider named "Page" and you will get three pages in which the Properties are explained.

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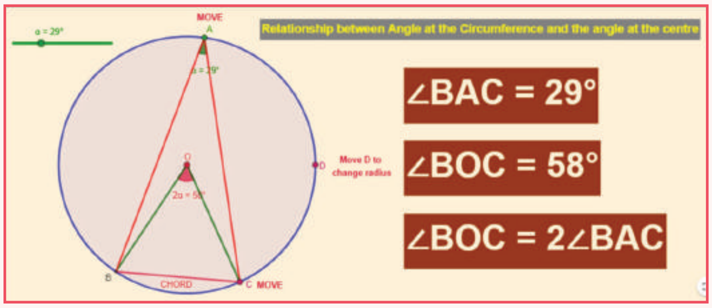
ICT Corner-2



Expected Result is shown in this picture

Step – 1

Open the Browser type the URL Link given below (or) Scan the QR Code. GeoGebra work sheet named "Angles in a circle" will open. In the work sheet there are two activities on Circles.

The first activity is the relation between Angle at the circumference and the angle at the centre. You can change the angle by moving the slider. Also, you can drag on the point A, C and D to change the position and the radius. Compare the angles at A and O.

Step - 2

The second activity is "Angles in the segment of a circle". Drag the points B and D and check the angles. Also drag "Move" to change the radius and chord length of the circle.

Scan the QR Code.

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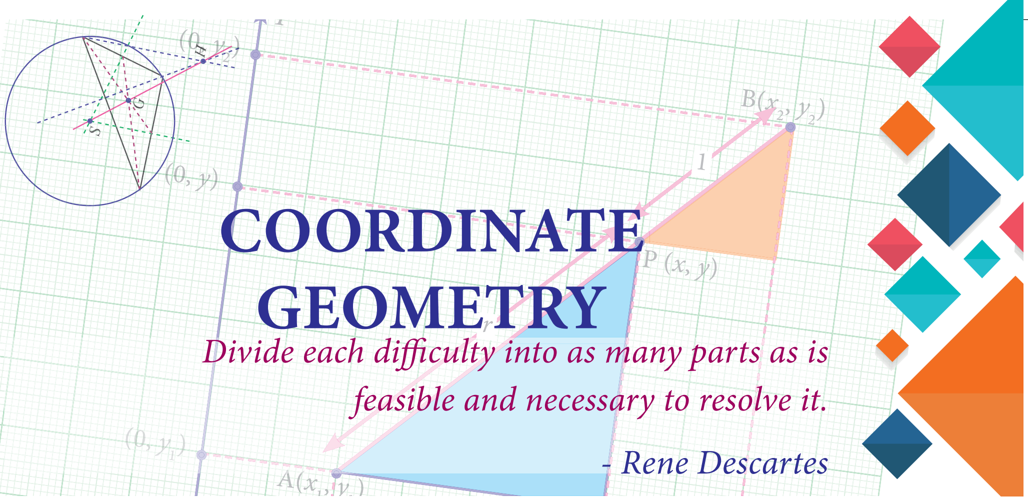
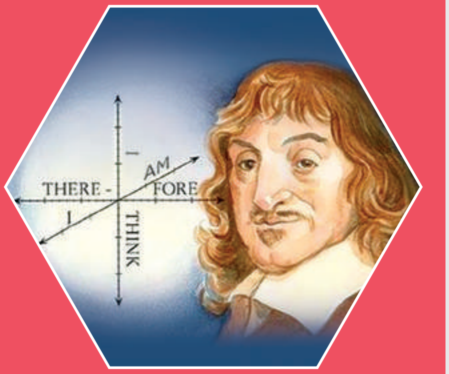
5



COORDINATE GEOMETRY

Divide each difficulty into as many parts as is feasible and necessary to resolve it.

- Rene Descartes



Rene Descartes
(AD(CE)1596-1650)

The French Mathematician Rene Descartes (pronounced "DAY- CART") developed a new branch of Mathematics known as Analytical Geometry or Coordinate Geometry which combined all arithmetic, algebra and geometry of the past ages in a single technique of visualising as points on a graph and equations as geometrical shapes. The fixing of a point position in the plane by assigning two numbers, coordinates, giving its distance from two lines perpendicular to each other, was entirely Descartes' invention.

Learning Outcomes

- To understand the Cartesian coordinate system.
- To identify the abscissa, ordinate and coordinates of any given point.
- To find the distance between any two points in the Cartesian plane using formula.
- To understand the mid-point formula and use it in problem solving.
- To derive the section formula and apply this in problem solving.
- To understand the centroid formula and to know its applications.

5.1 Mapping the Plane

How do you write your address? Here is one.

Sarakkalvilai Primary School
135, Sarakkalvilai,
Sarakkalvilai Housing Board Road,
Keezha Sarakkalvilai,
Nagercoil 629002, Kanyakumari Dist.
Tamil Nadu, India.

E6HCCN

Fig. 5.1

Somehow, this information is enough for anyone in the world from anywhere to locate the school one studied. Just consider there are crores and crores of buildings on the Earth. But yet, we can use an address system to locate a particular person's place of study, however interior it is.

How is this possible? Let us work out the procedure of locating a particular address. We know the World is divided into countries. One among them is India. Subsequently India is divided into States. Among these States we can locate our State Tamilnadu.

Further going deeper, we find our State is divided into Districts. Districts into taluks, taluks into villages proceeding further in this way, one could easily locate "Sarakkalvilai" among the villages in that Taluk. Further among the roads in that village, 'Housing Board Road' is the specific road which we are interested to explore. Finally we end up the search by locating Primary School building bearing the door number 135 to enable us precisely among the buildings in that road.

In New York city of USA, there is an area called Manhattan. The map shows Avenues run in the North – South direction and the Streets run in the East – West direction. So, if you know that the place you are looking for is on 57th street between 9th and 10th Avenues, you can find it immediately on the map. Similarly it is easy to find a place on 2nd Avenue between 34th and 35th streets. In fact, New Yorkers make it even simpler. From the door number on a street, you can actually calculate which avenues it lies between, and from the door number on an avenue, you can calculate which streets it stands.

Fig. 5.2

All maps do just this for us. They help us in finding our way and locate a place easily by using information of any landmark which is nearer to our search to make us understand whether we are near or far, how far are we, or what is in between etc. We use latitudes (east – west, like streets in Manhattan) and longitudes (north – south, like avenues in Manhattan) to pin point places on Earth. It is interesting to see how using numbers in maps helps us so much.

This idea, of using numbers to map places, comes from geometry. Mathematicians wanted to build maps of planes, solids and shapes of all kinds. Why would they want such maps? When we work with a geometric figure, we want to observe wheather a point lies inside the region or outside or on the boundary. Given two points on the boundary and a point outside, we would like to examine which of the two points on the boundary is closer to the one outside, and how much closer and so on. With solids like cubes, you can imagine how interesting and complicated such questions can be.

Mathematicians asked such questions and answered them only to develop their own understanding of circles, polygons and spheres. But the mathematical tools and techniques were used to find immense applications in day to day life. Mapping the world using latitudes and longitudes would not have been developed at all in 18th century, if the co-ordinate system had not been developed mathematically in the 17th century.

You already know the map of the real number system is the number line. It extends infinitely on both directions. In between any two points, on a number line, there lies infinite number of points. We are now going to build a map of the plane so that we can discuss about the points on the plane, of the distance between the points etc. We can then draw on the plane all the geometrical shapes we have discussed so far, precisely.

Arithmetic introduced us to the world of numbers and operations on them. Algebra taught us how to work with unknown values and find them using equations. Geometry taught us to describe shapes by their properties. Co-ordinate geometry will teach us how to use numbers and algebraic equations for studying geometry and beautiful integration of many techniques in one place. In a way, that is also great fun as an activity. Can't wait? Let us plunge in.

5.2 Devising a Coordinate System

You ask your friend to draw a rectangle on a blank sheet of paper, 5 cm by 3 cm. He says, "Sure, but where on this sheet?" How would you answer him?

Now look at the picture. How will you describe it to another person?

Fig. 5.3

Let us analyse the given picture (Fig. 5.3). Just like that a particular house is to be pointed out, it is going to be a difficult task. Instead, if any place or an object is fixed for identification then it is easy to identify any other place or object relative to it. For example, you fix the flag and talk about the house to the left of it, the hotel below it, the antenna on the house to right of it etc.

As shown in Fig. 5.4 draw two perpendicular lines in such a way that the flag is pointed out at near the intersection point. Now if you tell your friend the total length and width of the picture frame, keeping the flagpole as landmark, you can also say 2 cm to the right, 3 cm above etc. Since you know directions, you can also say 2 cm east, 3 cm north.

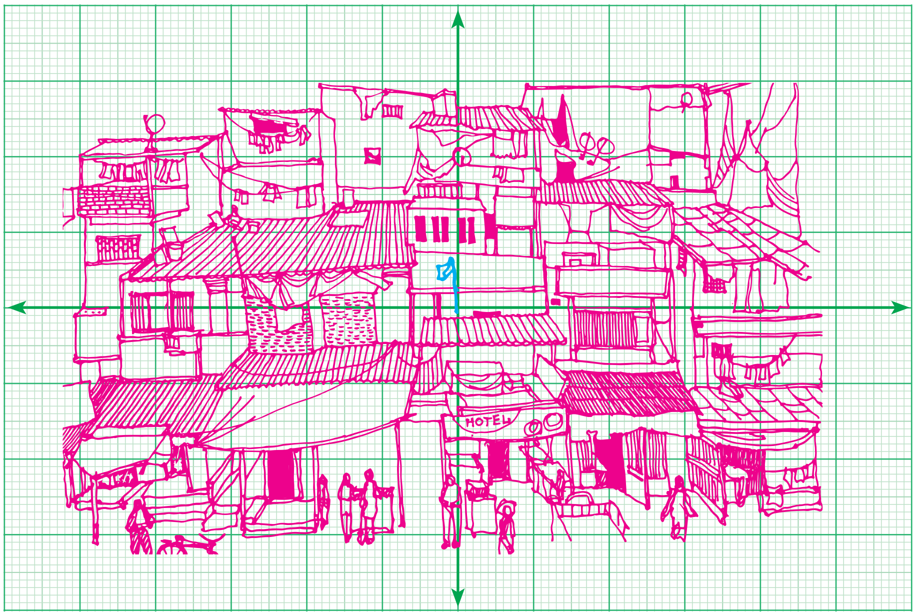


Fig. 5.4

This is what we are going to do. A number line is usually represented as horizontal line on which the positive numbers always lie on the right side of zero, negative on the left side of zero. Now consider another copy of the number line, but drawn vertically: the positive integers are represented above zero and the negative integers are below zero (fig 5.5).

Where do the two number lines meet? Obviously at zero for both lines. That will be our location where the "flag" is fixed. We can talk of other numbers relative to it, on both the lines. But now you see that we talk not only of numbers on the two number lines but lots more !

Suppose we go 2 to the right and then 3 to the top. We would call this place (→ 2, ↑ 3).

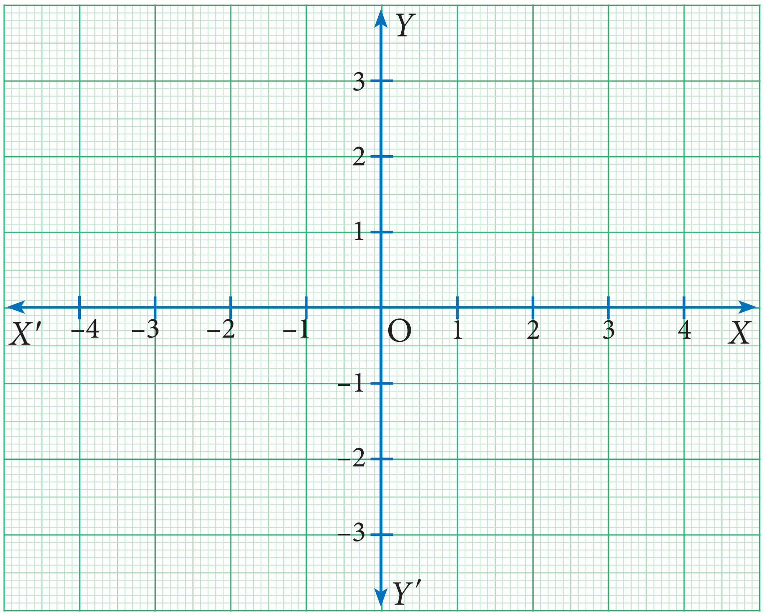


Fig. 5.5

All this vertical, horizontal, up, down etc is all very cumbersome. We simply say (2,3) and understand this as 2 to the right and then 3 up. Notice that we would reach the same place if we first went 3 up and then 2 right, so for us, the instruction (2,3) is not the same as the instruction (3,2). What about (–2,3)? It would mean 2 left and 3 up. From where? Always from (0,0). What about the instruction (2,–3)? It would mean 2 right and 3 down. We need names for horizontal and vertical number lines too. We call the horizontal number line the x-axis and the vertical number line the y-axis. To the right we mark it as X, to the left as X', to the top as Y, to the bottom as Y'.

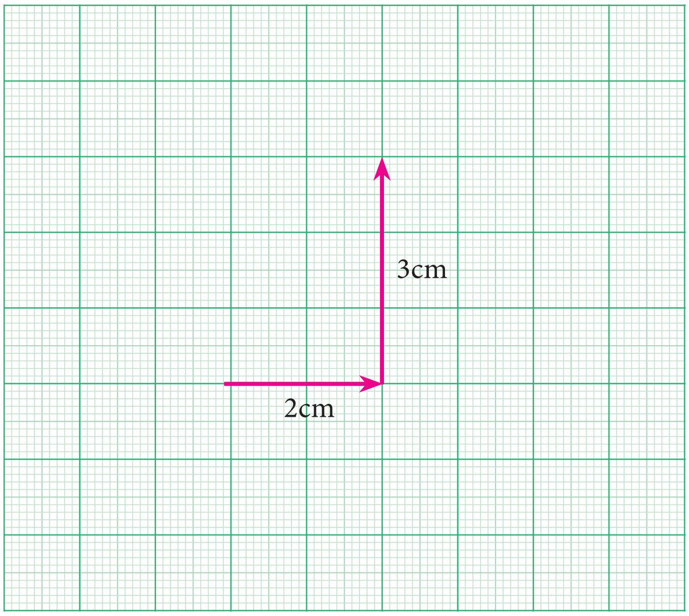


Fig. 5.6

The x-co-ordinate is called the *abscissa* and the y-co-ordinate is called the *ordinate*. We call the meeting point of the axes (0,0) the origin.

Note

Whether we place (0,0) at the centre of the sheet, or somewhere else does not matter, (0,0) is always the origin for us, and all "instructions" are relative to that point. We usually denote the origin by the letter 'O' .

Now we can describe any point on a sheet of paper by a pair (x,y). However, what do (1, 2) etc mean on our paper? We need to choose some convenient unit and represent these numbers. For instance we can choose 1 unit to be 1 cm. Thus (2,3) is the instruction to move 2 cm to the right of (0,0) and then to move 3 cm up. Please remember that the choice of units is arbitrary: if we fix 1 unit to be 2 cm, our figures will be larger, but the relative distances will remain the same.

In fact, we now have a language to describe all the infinitely many points on the plane, not just our sheet of paper !

Since the x-axis and the y-axis divide the plane into four regions, we call them quadrants. (Remember, quadrilateral has 4 sides, quadrants are 4 regions.) They are usually numbered as I, II, III and IV, with I for upper east side, II for upper west side, III for lower west side and IV for lower east side, thus making an anti-clockwise tour of them all.

Note

Why this way (anti-clockwise) and not clockwise, or not starting from any of the other quadrants? It does not matter at all, but it is good to follow some convention, and this is what we have been doing for a few centuries now.

Region	Quadrant	Nature of x,y	Signs of the coordinates
XOY	I	$x>0,\ y>0$	(+,+)
$X'OY$	II	$x<0,\ y>0$	(–,+)
$X'OY'$	III	$x<0,\ y<0$	(–,–)
XOY'	IV	$x>0,\ y<0$	(+,–)

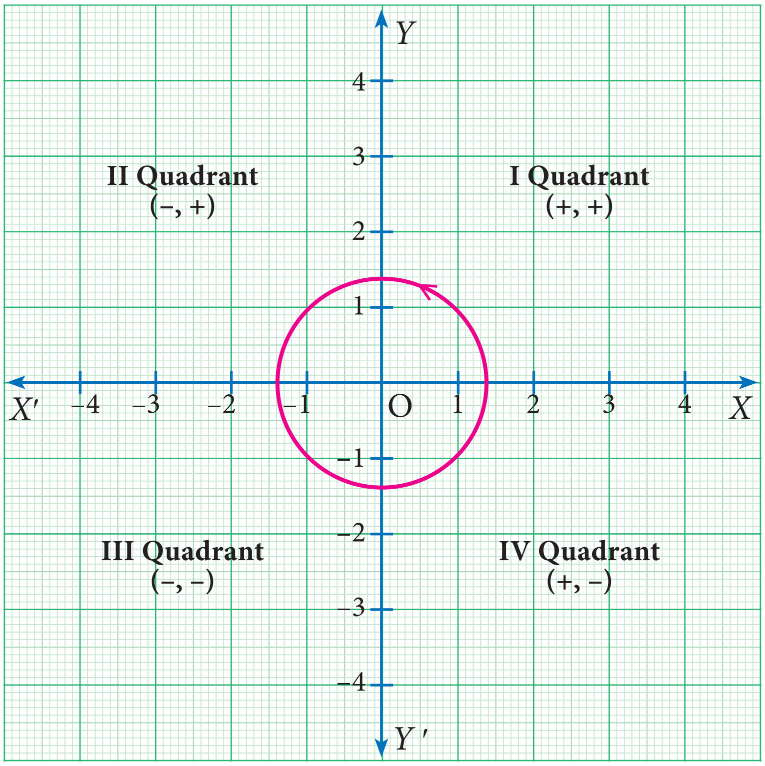


Fig. 5.7

Note

- For any point P on the x axis,the value of y coordinate (ordinate) is zero that is, P (x,0).
- For any point Q on the y axis, the value of x coordinate (abscissa) is zero. that is, Q (0, y)
- (x, y) ≠ (y, x) unless x= y
- A plane with the rectangular coordinate system is called the Cartesian plane.

5.2.1 Plotting Points in Cartesian Coordinate Plane

To plot the points (4, 5) in the Cartesian coordinate plane. We follow the x – axis until we reach 4 and draw a vertical line at $x = 4$.

Similarly, we follow the y – axis until we reach 5 and draw a horizontal line at $y = 5$.

The intersection of these two lines is the position of (4, 5) in the Cartesian plane.

This point is at a distance of 4 units from the y-axis and 5 units from the x-axis. Thus the position of (4, 5) is located in the Cartesian plane.

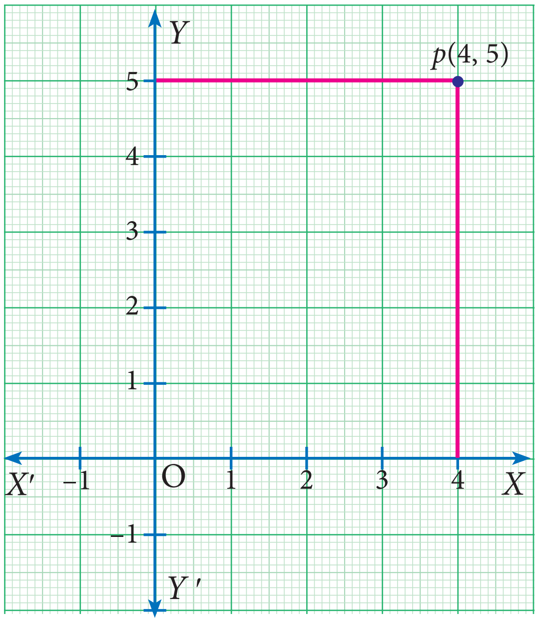


Fig. 5.8

Example 5.1 In which quadrant does the following points lie?

(a) (3,–8) (b) (–1,–3) (c) (2, 5) (d) (–7, 3)

Solution

(a) The x- coordinate is positive and y – coordinate is negative. So, point(3,–8) lies in the IV quadrant.

(b) The x-coordinate is negative and y – coordinate is negative. So, point(–1,–3) lies in the III quadrant.

(c) The x-coordinate is positive and y – coordinate is positive. So point(2,5) lies in the I quadrant.

(d) The x-coordinate is negative and y – coordinate is positive. So, point(–7,3) lies in the II quadrant

Example 5.2 Plot the points $A(2,4)$, $B(-3,5)$, $C(-4,-5)$, $D(4,-2)$ in the Cartesian plane.

Solution

(i) To plot (2, 4), draw a vertical line at $x = 2$ and draw a horizontal line at $y = 4$. The intersection of these two lines is the position of (2, 4) in the Cartesian plane. Thus, the Point A (2, 4) is located in the I quadrant of Cartesian plane.

(ii) To plot (–3, 5), draw a vertical line at $x = -3$ and draw a horizontal line at $y = 5$. The intersection of these two lines is the position of (–3, 5) in the Cartesian plane. Thus, the Point B (–3, 5) is located in the II quadrant of Cartesian plane.

(iii) To plot (–4, –5), draw a vertical line at x =–4 and draw a horizontal line at y = –5. The intersection of these two lines is the position of (-4, 5) in the Cartesian plane. Thus, the Point C (–4, –5) is located in the III quadrant of Cartesian plane.

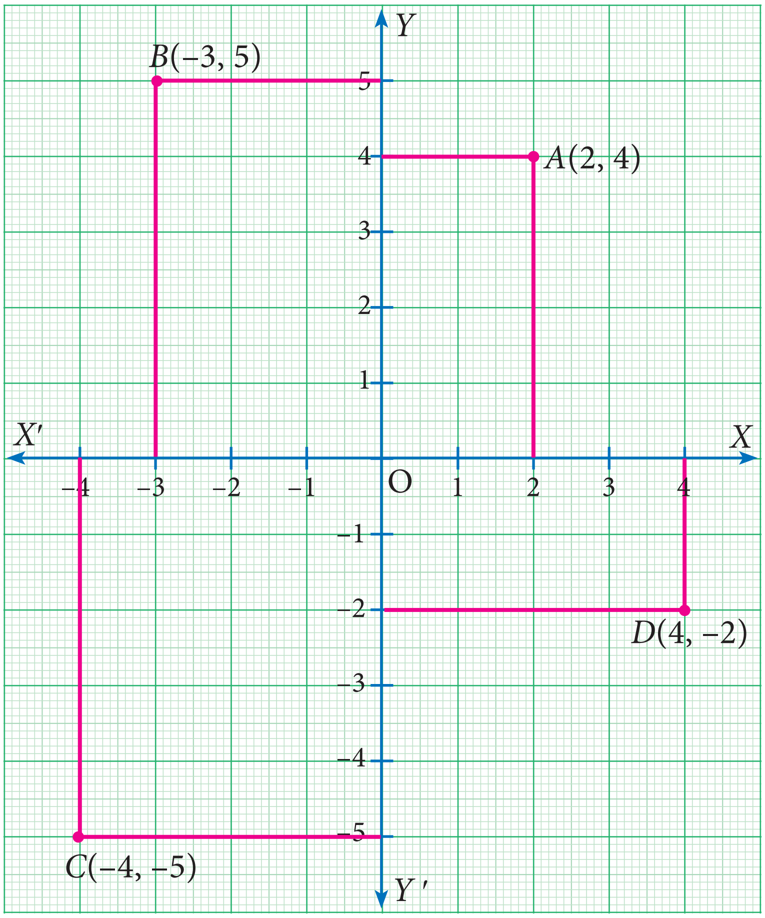


Fig. 5.9

(iv) To plot (4, −2), draw a vertical line at $x = 4$ and draw a horizontal line at $y = -2$. The Intersection of these two lines is the position of (4,−2) in the Cartesian plane. Thus, the Point D (4,−2) is located in the IV quadrant of Cartesian plane.

Example 5.3

Plot the following points $A(2,2)$, $B(-2,2)$, $C(-2,-1)$, $D(2,-1)$ in the Cartesian plane . Discuss the type of the diagram by joining all the points taken in order.

Solution

Point	*A*	*B*	*C*	*D*
Quadrant	I	II	III	IV

ABCD is a rectangle.

Can you find the length, breath and area of the rectangle?

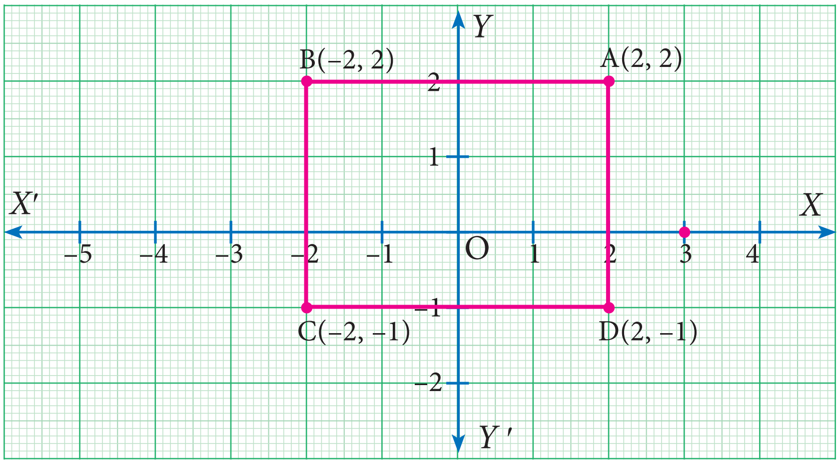


Fig. 5.10

Exercise 5.1

1. Plot the following points in the coordinate system and identify the quadrants $P(-7,6)$, $Q(7,-2)$, $R(-6,-7)$, $S(3,5)$ and $T(3,9)$

2. Write down the abscissa and ordinate of the following from fig 5.11.

(i) P (ii) Q

(iii) R (iv) S

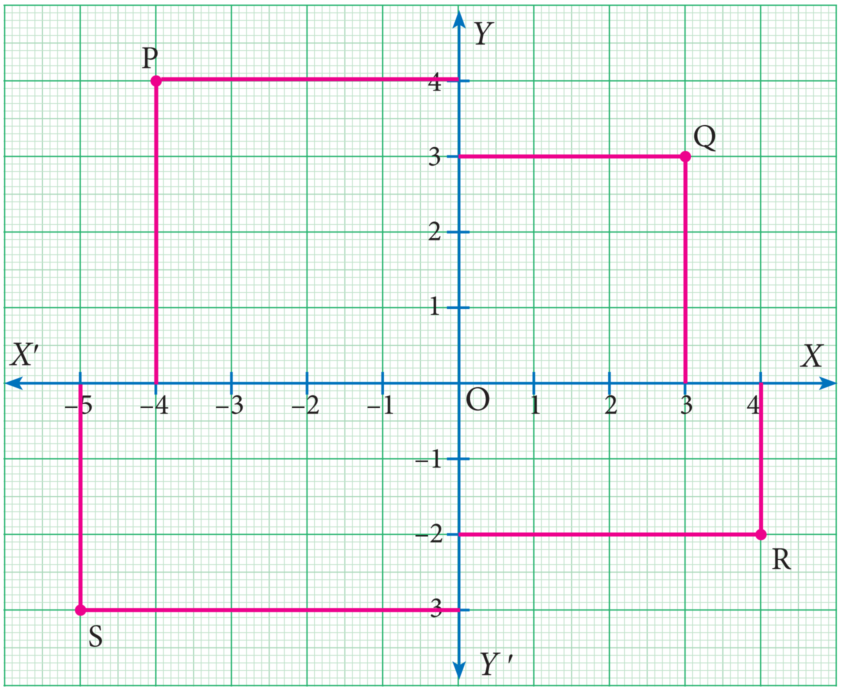


Fig. 5.11

3. Plot the following points in the coordinate plane and join them. What is your conclusion about the resulting figure?
   (i) (–5,3) (–1,3) (0,3) (5,3)   (ii) (0,–4) (0,–2) (0,4) (0,5)

4. Plot the following points in the coordinate plane. Join them in order. What type of geometrical shape is formed?
   (i) (0,0) (–4,0) (–4,–4) (0,–4)   (ii) (–3,3) (2,3) (–6,–1) (5,–1)

Activity - 1

Plot the following points on a graph sheet by taking the scale as 1cm = 1 unit.

Find how far the points are from each other?

A (1, 0) and *D* (4, 0). Find *AD* and also DA.

Is *AD* = *DA*?

You plot another set of points and verify your Result.

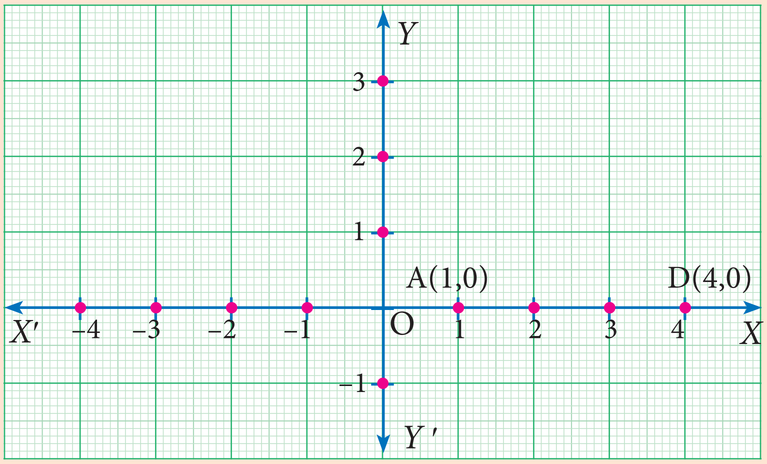


Fig. 5.12

5.3 Distance between any Two Points

Akila and Shanmugam are friends living on the same street in Sathyamangalam. Shanmugam's house is at the intersection of one street with another street on which there is a library. They both study in the same school, and that is not far from Shanmugam's house. Try to draw a picture of their houses, library and school by yourself before looking at the map below. Consider the school as the origin. (We can do this ! That is the whole point about the co-ordinate language we are using.)

Now fix the scale as 1 unit = 50 metres. Here are some questions for you to answer by studying the given figure (Fig 5.13).

1. How far is Akila's house from Shanmugam's house?
2. How far is the library from Shanmugam's house?
3. How far is the school from Shanmugam's and Akila's house?
4. How far is the library from Akila's house?
5. How far is Shanmugam's house from Akila's house?

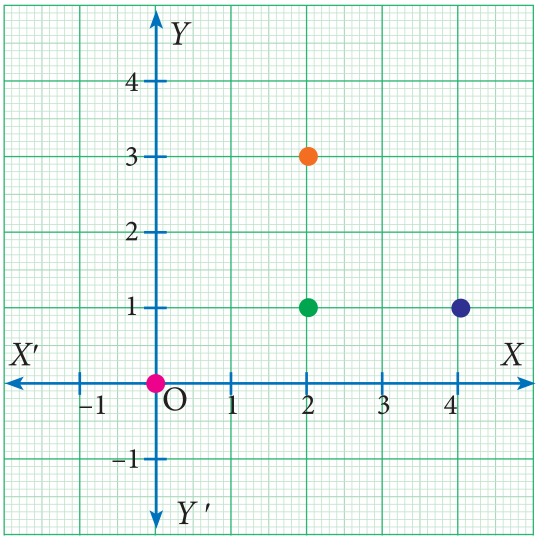


Fig. 5.13

Question 5 is not needed after answering question 1. Obviously, the distance from point A to B is the same as the distance from point B to A, and we usually call it the distance between points A and B. But as mathematicians we are supposed to note down properties as and when we see them, so it is better to note this too: distance (A,B) = distance (B,A). This is true for all points A and B on the plane, so of course question 5 is same as question 1.

Note

The equation distance(A,B)=distance(B,A) is not always obvious. Suppose that the road from A to B is a one-way street on which you cannot go the other way? Then the distance from B to A might be longer ! But we will avoid all these complications and assume that we can go both ways.

What about the other questions? They are not the same. Since we know that the two houses are on the same street which is running north – south, the y-distance tells us the answer to question 1. Similarly, we know that the library and Shanmugam's house are on the same street running east – west, we can take the x-distance to answer question 2.

Questions 3 and 4 depend on what kind of routes are available. If we assume that the only streets available are parallel to the x and y axes at the points marked 1, 2, 3 etc then we answer these questions by adding the x and y distances. But consider the large field east of Akila's house.

If she can walk across the field, of course she would prefer it. Now there are many ways of going from one place to another, so when we talk of the distance between them, it is not precise. We need some way to fix what we mean. When there are many routes between A and B, we will use distance(A,B) to denote the distance on the shortest route between A and B.

Once we think of distance(A,B) as the "straight line distance" between A and B, there is an elegant way of understanding it for any points A and B on the plane. This is the important reason for using the co-ordinate system at all ! Before that, 2 more questions from our example.

1. With the school as origin, define the coordinates of the two houses, the school and the library.
2. Use the coordinates to give the distance between any one of these and another.

The "straight line distance" is usually called "as the crow flies". This is to mean that we don't worry about any obstacles and routes on the ground, but how we would get from A to B if we could fly. No bird ever flies on straight lines, though.

We can give a systematic answer to this: given any two points $A = (x,y)$ and $B = (x',y')$ on the plane, find distance(A,B). It is easy to derive a formula in terms of the four numbers x, y, x' and y'. This is what we set out to do now

5.3.1 Distance Between Two Points on the Coordinate Axes

Points on x – axis: If two points lie on the x- axis, then the distance between them is equal to the difference between the x- coordinates.

Consider two points $A\ (x_1,0)$ and $B\ (x_2,0)$ on the x-axis .

The distance of B from A is

$$AB = OB - OA = x_2 - x_1 \text{ if } x_2 > x_1 \text{ or}$$

$$= x_1 - x_2 \text{ if } x_1 > x_2$$

$$AB = |x_2 - x_1|$$

(Read as modulus or absolute value of $x_2 - x_1$)

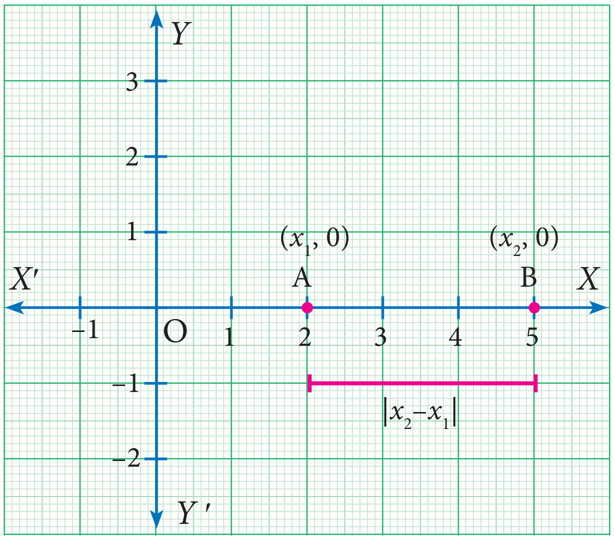


Fig. 5.14

Points on y – axis: If two points lie on y-axis then the distance between them is equal to the difference between the y-coordinates.

Consider two points $P(0,y_1)$ and $Q(0,y_2)$

The distance Q from P is

$$PQ = OQ - OP.$$

$$= y_2 - y_1 \text{ if } y_2 > y_1 \text{ or}$$

$$= y_1 - y_2 \text{ if } y_1 > y_2$$

$$PQ = |y_2 - y_1|$$

(Read as modulus or absolute value of $y_2 - y_1$)

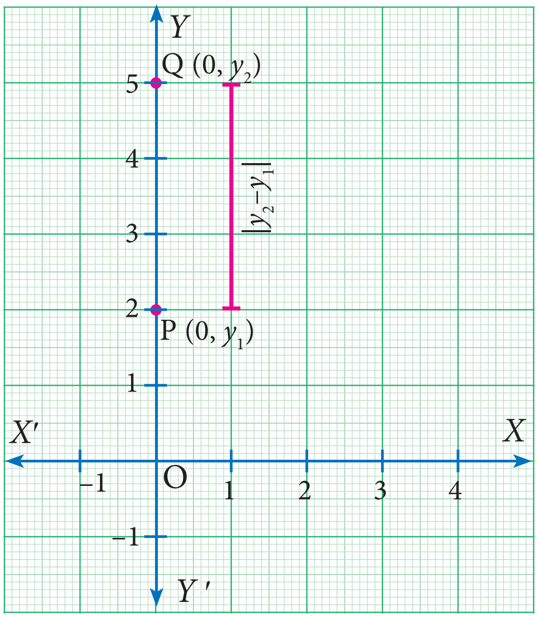


Fig. 5.15

5.3.2 Distance Between Two Points Lying on a Line Parallel to Coordinate Axes

Consider the points $A(x_1, y_1)$ and $B(x_2, y_1)$. Since the y - coordinates are equal the points lie on a line parallel to x- axis. From A and B draw AP and BQ perpendicular to x- axis respectively. Observe the given figure (Fig. 5.16), it is obvious that the distance AB is same as the distance PQ

Distance AB = Distance between $PQ = |x_2 - x_1|$
[The difference between x coordinates]
Similarly consider the line joining the two points

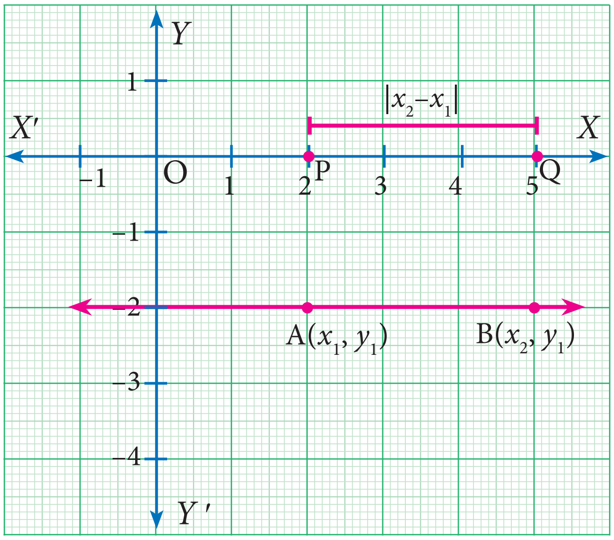


Fig. 5.16

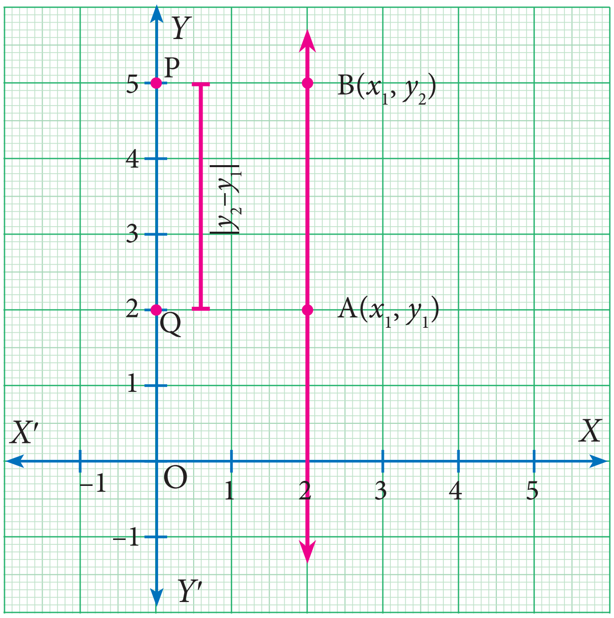


Fig. 5.17

$A(x_1, y_1)$ and $B(x_1, y_2)$, parallel to y – axis.

Then the distance between these two points is

$$|y_2 - y_1|$$

[The difference between y coordinates]

5.3.3 Distance Between Two Points on a Plane.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the Cartesian plane (or xy – plane), at a distance 'd' apart that is $d = PQ$.

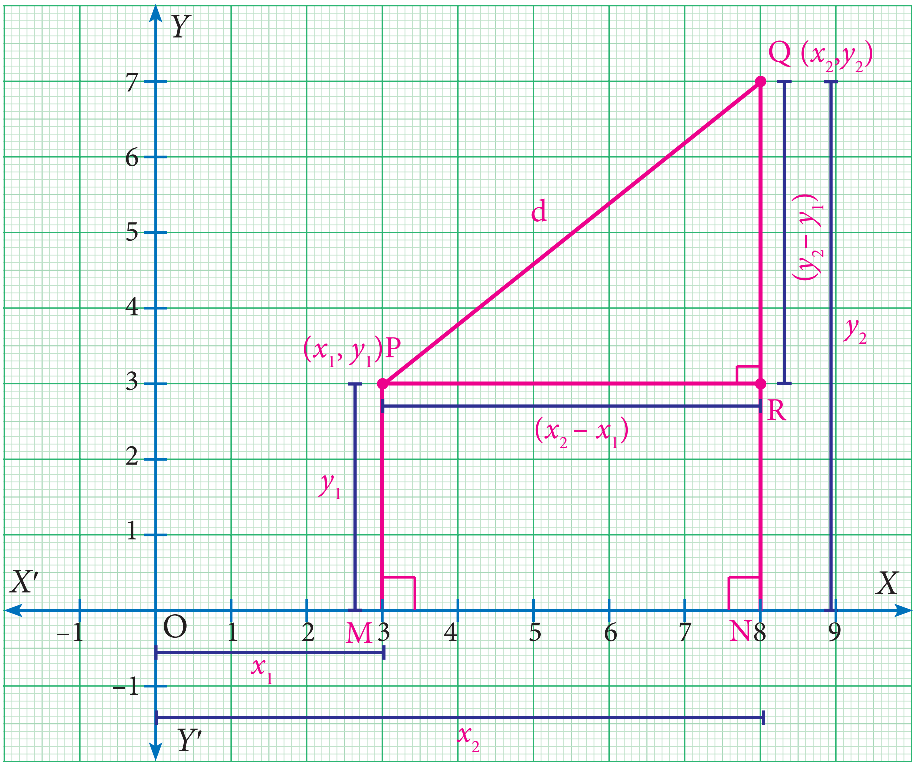


Fig. 5.18

Step 1

By the definition of coordinates,

$OM = x_1 \qquad MP = y_1$

$ON = x_2 \qquad NQ = y_2$

Now, $(PR \perp NQ)$

$PR = MN$ (Opposite sides of the rectangle $MNRP$)

Thinking Corner

A man goes 3 km towards north and then 4 km towards east. How far is he away from the initial position?

$= ON - OM$ (Measuring the distance from O)

$= x_2 - x_1$ ……..(1)

And $RQ = NQ - NR$

$= NQ - MP$ (Opposite sides of the rectangle $MNRP$)

$= y_2 - y_1$ ………..(2)

E768EB

Step 2

Triangle PQR is right angled at R. ($PR \perp NQ$)

$PQ^2 = PR^2 + RQ^2$ (By Pythagoras theorem)

$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (Taking positive square root)

Note

Distance between two points

- Given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the distance between these points is given by the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- The distance between PQ = The distance between QP
  i.e. $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
- The distance of a point $P(x_1, y_1)$ from the origin $O(0,0)$ is $OP = \sqrt{x_1^2 + y_1^2}$

5.3.4 Properties of Distances

We have already seen that distance (A,B) = distance (B,A) for any points A, B on the plane. What other properties have you noticed? In case you have missed them, here are some:

distance $(A,B) = 0$ exactly when A and B denote the identical point: $A = B$.

distance $(A,B) > 0$ for any two distinct points A and B.

Now consider three points A, B and C. If we are given their co-ordinates and we find that their x-co-ordinates are the same then we know that they are collinear, and lie on a line parallel to the y-axis. Similarly, if their y-co-ordinates are the same then we know that they are collinear, and lie on a line parallel to the x-axis. But these are not the only conditions. Points (0,0), (1,1) and (2,2) are collinear as well. Can you think of what relationship should exist between these coordinates for the points to be collinear?

The distance formula comes to our help here. We know that when A, B and C are the vertices of a triangle, we get,

distance(A,B) + distance(B,C) >distance(A,C) (after renaming the vertices suitably).

When do three points on the plane not form a triangle? When they are collinear, of course. In fact, we can show that when, distance(A,B) + distance(B,C) = distance(A,C), the points A, B and C must be collinear.

Similarly, when A, B and C are the vertices of a right angled triangle, $\angle ABC = 90^\circ$ we know that:

distance$(AB)^2$+ distance$(BC)^2$ = distance$(AC)^2$

with appropriate naming of vertices. We can also show that the converse holds: whenever the equality here holds for A, B and C, they must be the vertices of a right angled triangle.

The following examples illustrate how these properties of distances are useful for answering questions about specific geometric shapes.

Example 5.4

Find the distance between the points (–4, 3), (2,–3).

Solution

The distance between the points (-4, 3), (2,-3) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(2+4)^2 + (-3-3)^2}$$

$$= \sqrt{(6^2 + (-6)^2)} = \sqrt{(36+36)}$$

$$= \sqrt{(36 \times 2)}$$

$$= 6\sqrt{2}$$

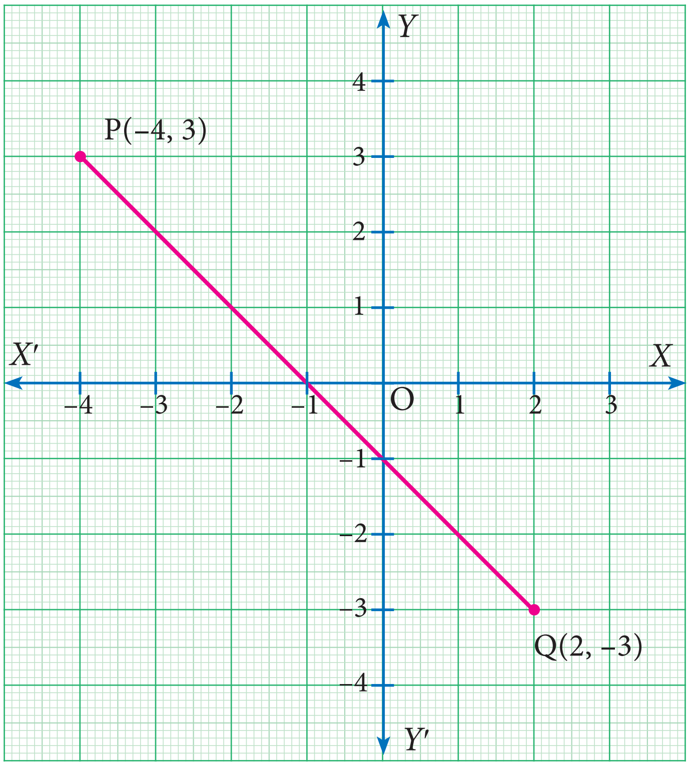


Fig. 5.19

Example 5.5

Show that the following points $A(3,1)$, $B(6,4)$ and $C(8,6)$ lies on a straight line.

Solution

Using the distance formula, we have

$$AB = \sqrt{(6-3)^2 + (4-1)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

$$BC = \sqrt{(8-6)^2 + (6-4)^2} = \sqrt{(4+4)} = \sqrt{8} = 2\sqrt{2}$$

$$AC = \sqrt{(8-3)^2 + (6-1)^2} = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2}$$

$$AB + BC = 3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2} = AC$$

Therefore the points lie on a straight line.

Collinear points

To show the collinearity of three points, we prove that the sum of the distance between two pairs of points is equal to the third pair of points.

In otherwords, points A, B, C are collinear if $AB + BC = AC$

Example 5.6 Show that the points $A(7,10)$, $B(-2,5)$, $C(3,-4)$ are the vertices of a right angled triangle.

Solution

Here $A = (7, 10)$, $B = (-2, 5)$, $C = (3,-4)$

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(-2-7)^2 + (5-10)^2}$$

$$= \sqrt{(-9)^2 + (-5)^2}$$

$$= \sqrt{(81+25)}$$

$$= \sqrt{106}$$

$$AB^2 = 106 \qquad \ldots (1)$$

$$BC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(3-(-2))^2 + (-4-5)^2} = \sqrt{(5)^2 + (-9)^2}$$

$$= \sqrt{25+81} = \sqrt{106}$$

$$BC^2 = 106 \qquad \ldots (2)$$

$$AC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(3-7)^2 + (-4-10)^2} = \sqrt{(-4)^2 + (-14)^2}$$

$$= \sqrt{16+196} = \sqrt{212}$$

$$AC^2 = 212 \qquad \ldots (3)$$

From (1), (2) & (3) we get,

$$AB^2 + BC^2 = 106 + 106 = 212 = AC^2$$

Since $AB^2 + BC^2 = AC^2$

$\therefore$ ΔABC is a right angled triangle, right angled at B.

Right angled triangle

We know that the sum of the squares of two sides is equal to the square of the third side,which is the hypotenuse of a right angled triangle.

Example 5.7

Show that the points $A(-4,-3)$, $B(3,1)$, $C(3,6)$, $D(-4,2)$ taken in that order form the vertices of a parallelogram.

Solution

Let $A(-4, -3)$, $B(3, 1)$, $C(3, 6)$, $D(-4, 2)$ be the four vertices of any quadrilateral $ABCD$. Using the distance formula,

Parallelogram

We know that opposite sides are equal

$$\text{Let} \quad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$AB = \sqrt{(3+4)^2 + (1+3)^2} = \sqrt{49+16} = \sqrt{65}$$

$$BC = \sqrt{(3-3)^2 + (6-1)^2}$$

$$= \sqrt{0+25} = \sqrt{25} = 5$$

$$CD = \sqrt{(-4-3)^2 + (2-6)^2}$$

$$= \sqrt{(-7)^2 + (-4)^2} = \sqrt{49+16} = \sqrt{65}$$

$$AD = \sqrt{(-4+4)^2 + (2+3)^2} = \sqrt{(0)^2 + (5)^2} = \sqrt{25} = 5$$

$AB = CD = \sqrt{65}$ and $BC = AD = 5$

Here, the opposite sides are equal. Hence $ABCD$ is a parallelogram.

Example 5.8

Calculate the distance between the points A (7, 3) and B which lies on the x-axis whose abscissa is 11.

Solution

Since B is on the x-axis, the y-coordinate of B is 0.

So, the coordinates of the point B is (11, 0)

By the distance formula the distance between the points A (7, 3), B (11, 0) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$AB = \sqrt{(11-7)^2 + (0-3)^2} = \sqrt{(4)^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25}$$

$$= 5$$

Example 5.9 Find the value of 'a' such that $PQ = QR$ where P, Q, and R are the points whose coordinates are (6, –1), (1, 3) and (a, 8) respectively.

Solution

Given P (6, –1), Q (1, 3) and R (a, 8)

$$PQ = \sqrt{(1-6)^2+(3+1)^2} = \sqrt{(-5)^2+(4)^2} = \sqrt{41}$$

$$QR = \sqrt{(a-1)^2+(8-3)^2} = \sqrt{(a-1)^2+(5)^2}$$

$$\text{Given } PQ = QR$$

$$\text{Therefore } \sqrt{41} = \sqrt{(a-1)^2+(5)^2}$$

$$41 = (a-1)^2 + 25 \quad \text{[Squaring both sides]}$$

$$(a-1)^2 + 25 = 41$$

$$(a-1)^2 = 41 - 25$$

$$(a-1)^2 = 16$$

$$(a-1) = \pm 4 \quad \text{[taking square root on both sides]}$$

$$a = 1 \pm 4$$

$$a = 1 + 4 \text{ or } a = 1 - 4$$

$$a = 5 \text{ or } a = -3$$

Example 5.10 Let $A(2, 2)$, $B(8, -4)$ be two given points in a plane. If a point P lies on the X- axis (in positive side), and divides AB in the ratio 1: 2, then find the coordinates of P.

Solution

Given points are $A(2, 2)$ and $B(8, -4)$ and let $P = (x, 0)$ [P lies on x axis]

By the distance formula

$$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$AP = \sqrt{(x-2)^2+(0-2)^2} = \sqrt{x^2-4x+4+4} = \sqrt{x^2-4x+8}$$

$$BP = \sqrt{(x-8)^2+(0+4)^2} = \sqrt{x^2-16x+64+16} = \sqrt{x^2-16x+80}$$

Given $AP : PB = 1 : 2$

i.e. $\frac{AP}{BP} = \frac{1}{2} \quad (\because BP = PB)$

$$2AP = BP$$

squaring on both sides,

$$4AP^2 = BP^2$$

$$\begin{aligned} 4(x^2 - 4x + 8) &= (x^2 - 16x + 80) \\ 4x^2 - 16x + 32 &= x^2 - 16x + 80 \\ 3x^2 - 48 &= 0 \\ 3x^2 &= 48 \\ x^2 &= 16 \\ x &= \pm 4 \end{aligned}$$

E7F4FY

As the point P lies on x-axis (positive side), its x- coordinate cannot be –4.

Hence the coordinates of P is(4, 0)

Example 5.11 Show that (4, 3) is the centre of the circle passing through the points (9, 3), (7,–1), (–1,3). Also find its radius.

Solution

Let $P(4, 3)$, $A(9, 3)$, $B(7, -1)$ and $C(-1, 3)$

If P is the centre of the circle which passes through the points A, B, and C, then P is equidistant from A, B and C (i.e.) $PA = PB = PC$

By distance formula,

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ AP = PA &= \sqrt{(4-9)^2 + (3-3)^2} = \sqrt{(-5)^2 + 0} = \sqrt{25} = 5 \\ BP = PB &= \sqrt{(4-7)^2 + (3+1)^2} = \sqrt{(3)^2 + (4)^2} = \sqrt{9+16} = \sqrt{25} = 5 \\ CP = PC &= \sqrt{(4+1)^2 + (3-3)^2} = \sqrt{(5)^2 + 0} = \sqrt{25} = 5 \end{aligned}$$

$$PA = PB = PC$$

Therefore P is the centre of the circle, passing through A, B and C

Radius = PA = 5.

Exercise 5.2

1. Find the distance between the following pairs of points.
   (i) (1, 2) and (4, 3)
   (ii) (3,4) and (– 7, 2)
   (ii) (a, b) and (c, b)
   (iv) (3,– 9) and (–2, 3)

2. Determine whether the given set of points in each case are collinear or not.
   (i) (7,–2),(5,1),(3,4)
   (ii) (a,–2), (a,3), (a,0)

3. Show that the following points taken in order form an isosceles triangle.
(i) $A\ (5,4), B(2,0), C\ (-2,3)$ (ii) $A\ (6,-4), B\ (-2, -4), C\ (2,10)$

4. Show that the following points taken in order form an equilateral triangle in each case.
(i) $A(2, 2), B(-2, -2), C\left(-2\sqrt{3},2\sqrt{3}\right)$ (ii) $A\left(\sqrt{3},2\right), B\ (0,1), C(0,3)$

5. Show that the following points taken in order form the vertices of a parallelogram.
(i) $A(-3, 1), B(-6, -7), C\ (3, -9), D(6, -1)$ (ii) $A\ (-7, -3), B(5,10), C(15,8), D(3, -5)$

6. Verify that the following points taken in order form the vertices of a rhombus.
(i) $A(3,-2), B\ (7,6), C\ (-1,2), D\ (-5, -6)$ (ii) $A\ (1,1), B(2,1), C\ (2,2), D(1,2)$

7. $A\ (-1, 1), B\ (1, 3)$ and $C\ (3, a)$ are points and if $AB = BC$, then find 'a'.

8. The abscissa of a point A is equal to its ordinate, and its distance from the point $B(1, 3)$ is 10 units, What are the coordinates of A?

9. The point (x, y) is equidistant from the points (3,4)and(–5,6). Find a relation between x and y.

10. Let $A(2, 3)$ and $B(2,-4)$ be two points. If P lies on the x-axis, such that $AP = \frac{3}{7}AB$, find the coordinates of P.

11. Show that the point (11,2) is the centre of the circle passing through the points (1,2), (3,–4) and (5,–6)

12. The radius of a circle with centre at origin is 30 units. Write the coordinates of the points where the circle intersects the axes. Find the distance between any two such points.

5.4 The Mid-point of a Line Segment

Fig. 5.20

Imagine a person riding his two-wheeler on a straight road towards East from his college to village A and then to village B. At some point in between A and B, he suddenly realises that there is not enough petrol for the journey. On the way there is no petrol bunk in between these two places. Should he travel back to A or just try his luck moving towards B? Which would be the shorter distance? There is a dilemma. He has to know whether he crossed the half way mid-point or not.

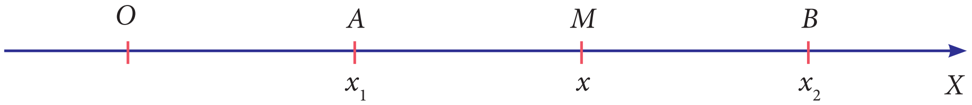


Fig. 5.21

The above Fig. 5.21 illustrates the situation. Imagine college as origin O from which the distances of village A and village B are respectively x_1 and x_2 $(x_1 < x_2)$. Let M be the mid-point of AB then x can be obtained as follows.

$$AM = MB \text{ and so, } \quad x - x_1 = x_2 - x$$

$$\text{and this is simplified to} \quad x = \frac{x_1 + x_2}{2}$$

Now it is easy to discuss the general case. If $A(x_{1,}y_1)$, $B(x_{2,}y_2)$ are any two points and $M(x, y)$ is the mid-point of the line segment AB, then M' is the mid-point of AC (in the Fig. 5.22). In a right triangle the perpendicular bisectors of the sides intersect at the mid-point of the hypotenuse. (Also, this property is due to similarity among the two coloured triangles shown; In such triangles, the corresponding sides will be proportional).

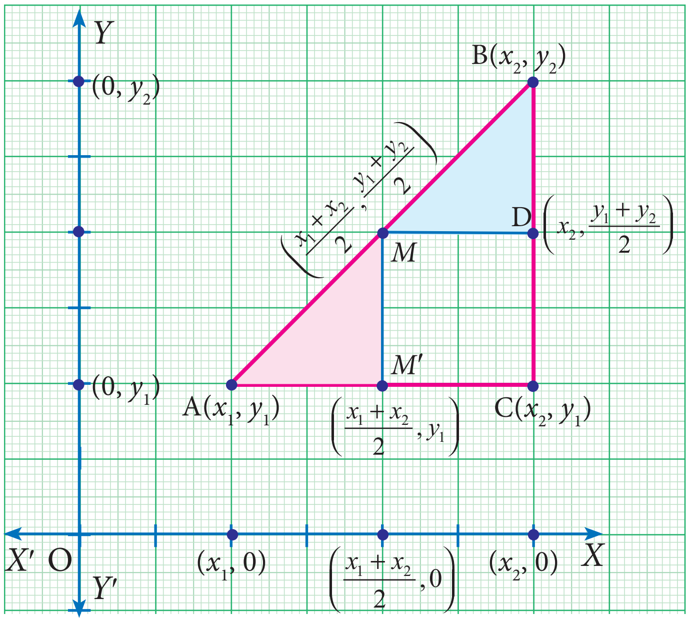


Fig. 5.22

Another way of solving

(Using similarity property)

Let us take the point M as $M(x,y)$

Now, $\Delta AMM'$ and ΔMBD are similar. Therefore,

$$\frac{AM'}{MD} = \frac{MM'}{BD} = \frac{AM}{MB}$$

$$\frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y} = \frac{1}{1} \quad (AM = MB)$$

$$\text{Consider, } \frac{x - x_1}{x_2 - x} = 1$$

$$2x = x_2 + x_1 \Rightarrow x = \frac{x_2 + x_1}{2}$$

$$\text{Similarly, } y = \frac{y_2 + y_1}{2}$$

The x-coordinate of M = the average of the x-coordinates of A and $C = \frac{x_1 + x_2}{2}$ and

similarly, the y-coordinate of M = the average of the y-coordinates of B and $C = \frac{y_1 + y_2}{2}$

The mid-point M of the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Thinking Corner

If D is the mid-point of AC and C is the mid-point of AB, then find the length of AB if $AD = 4cm$.

For example, The mid-point of the line segment joining the points $(-8,-10)$ and $(4,-2)$ is given by $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$ where $x_1=-8$, $x_2=4$, $y_1=-10$ and $y_2=-2$.

The required mid-point is $\left(\frac{-8+4}{2},\frac{-10-2}{2}\right)$ or $(-2,-6)$.

Let us now see the application of mid-point formula in our real life situation, consider the longitude and latitude of the following cities.

Name of the city	Longitude	Latitude
Chennai (Besant Nagar)	80.27° E	13.00° N
Mangaluru (Kuthethoor)	74.85° E	13.00° N
Bengaluru (Rajaji Nagar)	77.56° E	13.00° N

Fig. 5.23

Let us take the longitude and latitude of Chennai (80.27° E, 13.00° N) and Mangaluru (74.85° E , 13.00° N) as pairs. Since Bengaluru is located in the middle of Chennai and Mangaluru, we have to find the average of the coordinates, that is $\left(\frac{80.27+74.85}{2},\frac{13.00+13.00}{2}\right)$. This gives (77.56°E, 13.00°N) which is the longitude and latitude of Bengaluru. In all the above examples, the point exactly in the middle is the mid-point and that point divides the other two points in the same ratio.

Example 5.12

The point $(3,-4)$ is the centre of a circle. If AB is a diameter of the circle and B is $(5,-6)$, find the coordinates of A.

Solution Let the coordinates of A be (x_1, y_1) and the given point is $B(5,-6)$. Since the centre is the mid-point of the diameter AB, we have

$$\frac{x_1+x_2}{2}=3 \qquad x_1+5=6 \qquad x_1=6-5 \qquad x_1=1$$

$$\frac{y_1+y_2}{2}=-4 \qquad y_1-6=-8 \qquad y_1=-8+6 \qquad y_1=-2$$

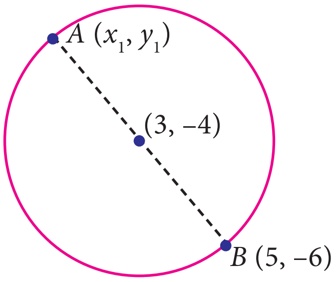


Fig. 5.24

Therefore, the coordinates of A is $(1,-2)$.

Progress Check

(i) Let X be the mid-point of the line segment joining $A(3,0)$ and $B(-5,4)$ and Y be the mid-point of the line segment joining $P(-11,-8)$ and $Q(8,-2)$. Find the mid-point of the line segment XY.

(ii) If $(3,x)$ is the mid-point of the line segment joining the points $A(8,-5)$ and $B(-2,11)$, then find the value of 'x'.

Example 5.13

If $(x,3)$, $(6,y)$, $(8,2)$ and $(9,4)$ are the vertices of a parallelogram taken in order, then find the value of x and y.

Solution Let $A(x,3)$, $B(6,y)$, $C(8,2)$ and $D(9,4)$ be the vertices of the parallelogram $ABCD$. By definition, diagonals AC and BD bisect each other.

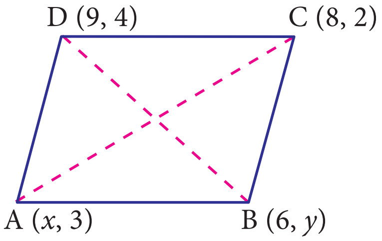


Fig. 5.25

Mid-point of AC = Mid-point of BD

$$\left(\frac{x+8}{2},\frac{3+2}{2}\right)=\left(\frac{6+9}{2},\frac{y+4}{2}\right)$$

equating the coordinates on both sides, we get

$$\frac{x+8}{2}=\frac{15}{2} \quad \text{and} \quad \frac{5}{2}=\frac{y+4}{2}$$

$$x+8=15 \qquad 5=y+4$$

$$x=7 \qquad y=1$$

Hence, $x = 7$ and $y = 1$.

Thinking Corner

$A(6,1)$, $B(8,2)$ and $C(9,4)$ are three vertices of a parallelogram $ABCD$ taken in order. Find the fourth vertex D. If (x_1,y_1), (x_2,y_2), (x_3,y_3) and (x_4,y_4) are the four vertices of the parallelogram, then using the given points, find the value of $(x_1+x_3-x_2, y_1+y_3-y_2)$ and state the reason for your result.

Example 5.14

Find the points which divide the line segment joining $A(-11,4)$ and $B(9,8)$ into four equal parts.

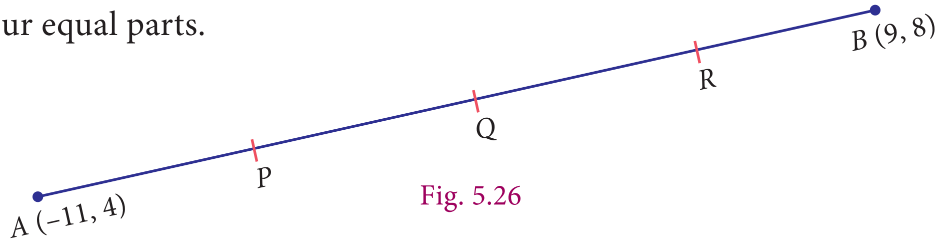


Fig. 5.26

Solution

Let P, Q, R be the points on the line segment joining $A(-11,4)$ and $B(9,8)$ such that $AP = PQ = QR = RB$.

Here Q is the mid-point of AB, P is the mid-point of AQ and R is the mid-point of QB.

$$Q \text{ is the mid-point of } AB = \left(\frac{-11+9}{2}, \frac{4+8}{2}\right) = \left(\frac{-2}{2}, \frac{12}{2}\right) = (-1,6)$$

$$P \text{ is the mid-point of } AQ = \left(\frac{-11-1}{2}, \frac{4+6}{2}\right) = \left(\frac{-12}{2}, \frac{10}{2}\right) = (-6,5)$$

$$R \text{ is the mid-point of } QB = \left(\frac{-1+9}{2}, \frac{6+8}{2}\right) = \left(\frac{8}{2}, \frac{14}{2}\right) = (4,7)$$

Hence the points which divides AB into four equal parts are $P(-6, 5)$, $Q(-1, 6)$ and $R(4, 7)$.

Example 5.15

The mid-points of the sides of a triangle are $(5,1)$, $(3,-5)$ and $(-5,-1)$. Find the coordinates of the vertices of the triangle.

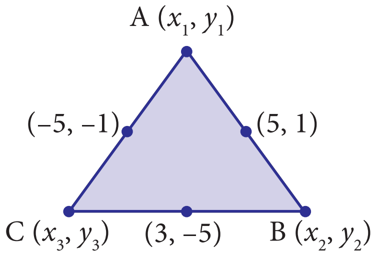


Fig. 5.27

Solution Let the vertices of the ΔABC be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ and the given mid-points of the sides AB, BC and CA are $(5,1)$, $(3,-5)$ and $(-5,-1)$ respectively. Therefore,

$$\frac{x_1+x_2}{2} = 5 \Rightarrow x_1 + x_2 = 10 \quad \text{...(1)}$$

$$\frac{x_2+x_3}{2} = 3 \Rightarrow x_2 + x_3 = 6 \quad \text{...(2)}$$

$$\frac{x_3+x_1}{2} = -5 \Rightarrow x_3 + x_1 = -10 \quad \text{...(3)}$$

Adding (1), (2) and (3)

$$2x_1 + 2x_2 + 2x_3 = 6$$

$$x_1 + x_2 + x_3 = 3 \quad \text{...(4)}$$

$$(4) - (2) \Rightarrow x_1 = 3 - 6 = -3$$

$$(4) - (3) \Rightarrow x_2 = 3 + 10 = 13$$

$$(4) - (1) \Rightarrow x_3 = 3 - 10 = -7$$

$$\frac{y_1+y_2}{2} = 1 \Rightarrow y_1 + y_2 = 2 \quad \text{...(5)}$$

$$\frac{y_2+y_3}{2} = -5 \Rightarrow y_2 + y_3 = -10 \quad \text{...(6)}$$

$$\frac{y_3+y_1}{2} = -1 \Rightarrow y_3 + y_1 = -2 \quad \text{...(7)}$$

Adding (5), (6) and (7),

$$2y_1 + 2y_2 + 2y_3 = -10$$

$$y_1 + y_2 + y_3 = -5 \quad \text{...(8)}$$

$$(8) - (6) \Rightarrow y_1 = -5 + 10 = 5$$

$$(8) - (7) \Rightarrow y_2 = -5 + 2 = -3$$

$$(8) - (5) \Rightarrow y_3 = -5 - 2 = -7$$

Therefore the vertices of the triangles are $A(-3,5)$, $B(13,-3)$ and $C(-7,-7)$.

Thinking Corner

If (a_1,b_1), (a_2,b_2) and (a_3,b_3) are the mid-points of the sides of a triangle, using the mid-points given in example 5.15 find the value of $(a_1+a_3-a_2, b_1+b_3-b_2)$, $(a_1+a_2-a_3, b_1+b_2-b_3)$ and $(a_2+a_3-a_1, b_2+b_3-b_1)$. Compare the results. What do you observe? Give reason for your result?

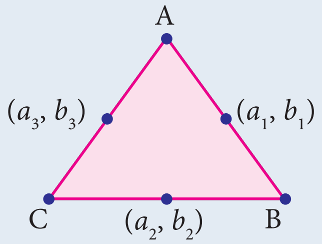


Fig. 5.28

Exercise 5.3

1. Find the mid-points of the line segment joining the points

   (i) $(-2,3)$ and $(-6,-5)$ (ii) $(8,-2)$ and $(-8,0)$

   (iii) (a,b) and $(a+2b,2a-b)$ (iv) $\left(\frac{1}{2},-\frac{3}{7}\right)$ and $\left(\frac{3}{2},\frac{-11}{7}\right)$

2. The centre of a circle is $(-4,2)$. If one end of the diameter of the circle is $(-3,7)$, then find the other end.

3. If the mid-point (x,y) of the line joining $(3,4)$ and $(p,7)$ lies on $2x+2y+1=0$, then what will be the value of p?

4. The mid-point of the sides of a triangle are $(2,4)$, $(-2,3)$ and $(5,2)$. Find the coordinates of the vertices of the triangle.

5. $O(0,0)$ is the centre of a circle whose one chord is AB, where the points A and B are $(8,6)$ and $(10,0)$ respectively. OD is the perpendicular from the centre to the chord AB. Find the coordinates of the mid-point of OD.

6. The points $A(-5,4)$, $B(-1,-2)$ and $C(5,2)$ are the vertices of an isosceles right-angled triangle where the right angle is at B. Find the coordinates of D so that $ABCD$ is a square.

7. The points $A(-3,6)$, $B(0,7)$ and $C(1,9)$ are the mid-points of the sides DE, EF and FD of a triangle DEF. Show that the quadrilateral $ABCD$ is a parallelogram.

8. $A(-3,2)$, $B(3,2)$ and $C(-3,-2)$ are the vertices of the right triangle, right angled at A. Show that the mid-point of the hypotenuse is equidistant from the vertices.

5.5 Points of Trisection of a Line Segment

The mid-point of a line segment is the point of bisection, which means dividing into two parts of equal length. Suppose we want to divide a line segment into three parts of equal length, we have to locate points suitably to effect a trisection of the segment.

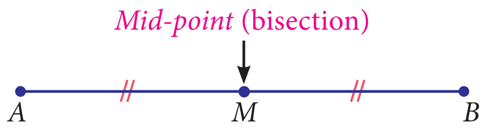


Fig. 5.29

$AM = MB$

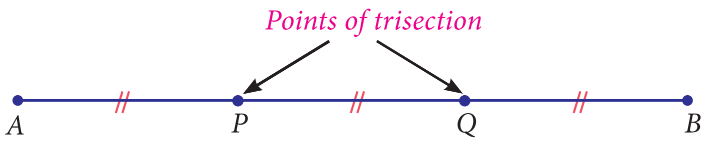


Fig. 5.30

$AP = PQ = QB$

For a given line segment, there are two points of trisection. The method of obtaining this is similar to that of what we did in the case of locating the point of bisection (i.e., the mid-point). Observe the given Fig. 5.31. Here P and Q are the points of trisection of the line segment AB where A is (x_1, y_1) and B is (x_2, y_2). Clearly we know that, P is the mid-point of AQ and Q is the mid-point of PB. Now consider the ΔACQ and ΔPDB (Also, can be verified using similarity property of triangles which will be dealt in detail in higher classes).

$$A'P' = P'Q' = Q'B'$$

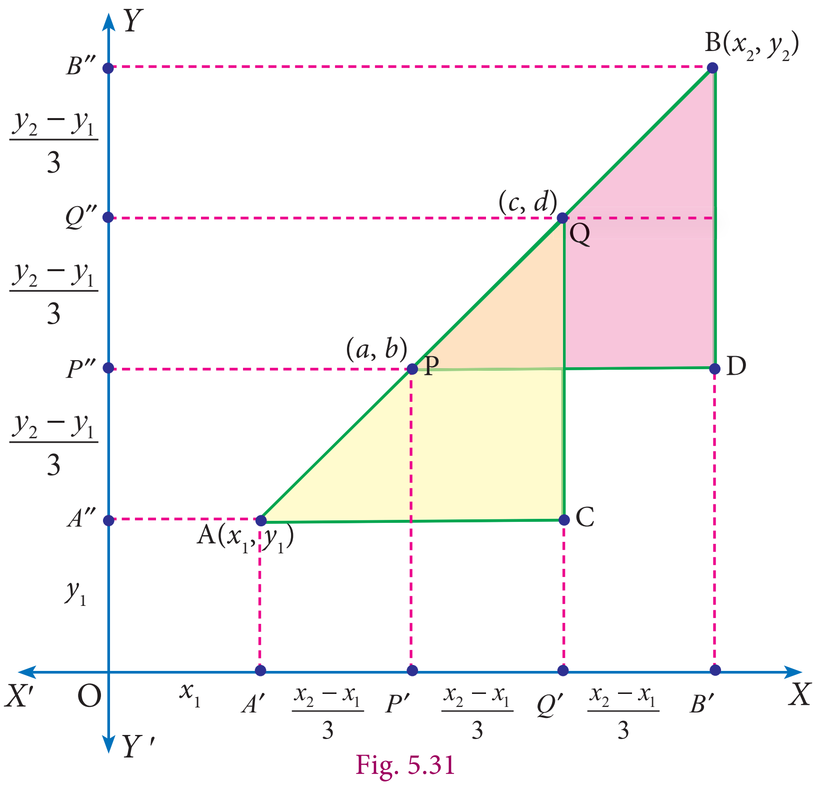


Fig. 5.31

Note that when we divide the segment into 3 equal parts, we are also dividing the horizontal and vertical legs into three equal parts.

If P is (a, b), then

$$a = OP' = OA' + A'P' = x_1 + \frac{x_2 - x_1}{3} = \frac{x_2 + 2x_1}{3};$$

$$b = PP' = OA'' + A''P'' = y_1 + \frac{y_2 - y_1}{3} = \frac{y_2 + 2y_1}{3}$$

Thus we get the point P as $\left(\frac{x_2 + 2x_1}{3}, \frac{y_2 + 2y_1}{3}\right)$

If Q is (c, d), then

$$c = OQ' = OB' - Q'B' = x_2 - \left(\frac{x_2 - x_1}{3}\right) = \frac{2x_2 + x_1}{3}\,;$$

$$d = OQ'' = OB'' - Q''B'' = y_2 - \left(\frac{y_2 - y_1}{3}\right) = \frac{2y_2 + y_1}{3}$$

Thus the required point Q is $\left(\frac{2x_2 + x_1}{3}, \frac{2y_2 + y_1}{3}\right)$

Example 5.16

Find the points of trisection of the line segment joining $(-2,-1)$ and $(4,8)$.

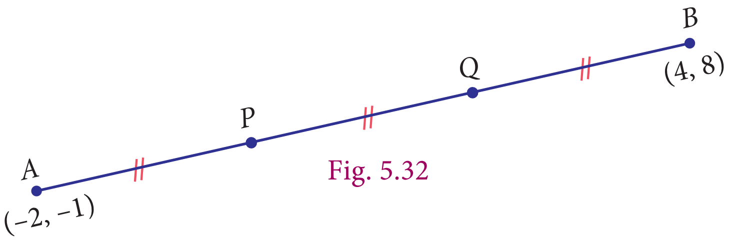


Fig. 5.32

Solution Let $A(-2,-1)$ and $B(4,8)$ are the given points.

Let $P(a,b)$ and $Q(c,d)$ be the points of trisection of AB, so that $AP = PQ = QB$.

By the formula proved above,

P is the point

$$\left(\frac{x_2 + 2x_1}{3}, \frac{y_2 + 2y_1}{3}\right) = \left(\frac{4 + 2(-2)}{3}, \frac{8 + 2(-1)}{3}\right)$$

$$= \left(\frac{4-4}{3}, \frac{8-2}{3}\right) = (0, 2)$$

Q is the point

$$\left(\frac{2x_2 + x_1}{3}, \frac{2y_2 + y_1}{3}\right) = \left(\frac{2(4) - 2}{3}, \frac{2(8) - 1}{3}\right)$$

$$= \left(\frac{8-2}{3}, \frac{16-1}{3}\right) = (2, 5)$$

Progress Check

(i) Find the coordinates of the points of trisection of the line segment joining $(4,-1)$ and $(-2,-3)$.

(ii) Find the coordinates of points of trisection of the line segment joining the point $(6,-9)$ and the origin.

5.6 Section Formula

We studied bisection and trisection of a given line segment. These are only particular cases of the general problem of dividing a line segment joining two points (x_1, y_1) and (x_2, y_2) in the ratio $m : n$.

Given a segment AB and a positive real number r.

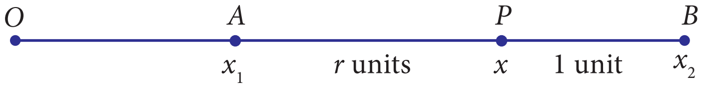


Fig.5.33

We wish to find the coordinate of point P which divides AB in the ratio r :1. This means $\frac{AP}{PB} = \frac{r}{1}$ or $AP = r(PB)$.

This means that $x - x_1 = r(x_2 - x)$

Solving this, $$x = \frac{rx_2 + x_1}{r+1} \qquad \text{….. (1)}$$

We can use this result for points on a line to the general case as follows.

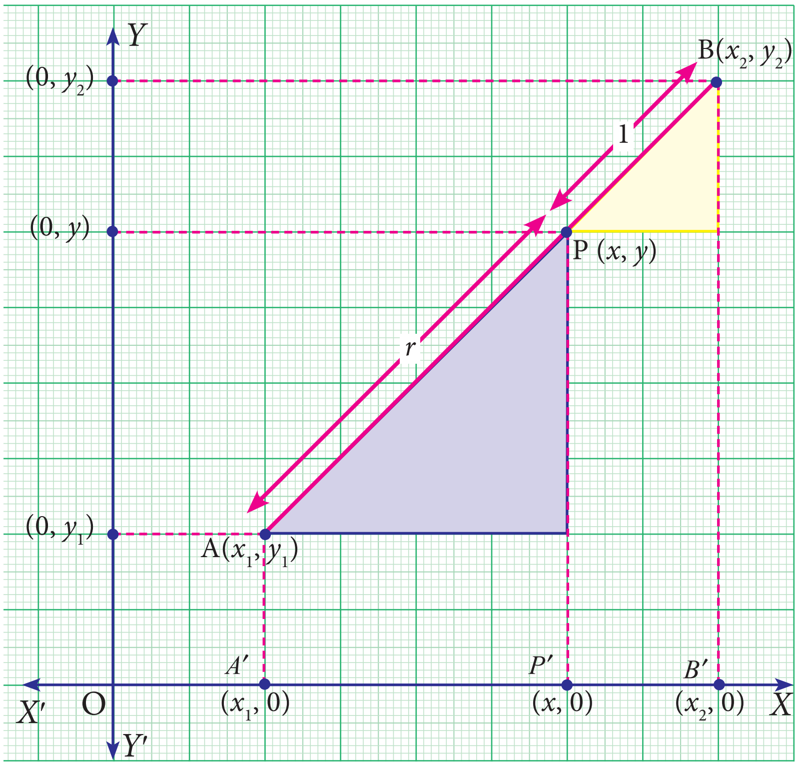


Fig. 5.34

Taking $AP:PB = r:1$, we get $A'P':P'B' = r:1$.

Therefore $\quad A'P' = r(P'B')$

Thus, $\quad (x - x_1) = r(x_2 - x)$

which gives $$x = \frac{rx_2 + x_1}{r+1} \qquad \text{… [see (1)]}$$

Precisely in the same way we can have $y = \dfrac{ry_2 + y_1}{r+1}$

If P is between A and B, and $\dfrac{AP}{PB} = r$, then we have the formula,

$$P \text{ is } \left(\frac{rx_2 + x_1}{r+1}, \frac{ry_2 + y_1}{r+1}\right).$$

If r is taken as $\dfrac{m}{n}$, then the section formula is $\left(\dfrac{mx_2 + nx_1}{m+n}, \dfrac{my_2 + ny_1}{m+n}\right)$, which is the standard form.

Thinking Corner

(i) What happens when $m = n = 1$? Can you identify it with a result already proved?

(ii) $AP : PB = 1 : 2$ and $AQ : QB = 2:1$. What is $AP : AB$? What is $AQ : AB$?

Note

- The line joining the points (x_1, y_1) and (x_2, y_2) is divided by x-axis in the ratio $\frac{-y_1}{y_2}$ and by y-axis in the ratio $\frac{-x_1}{x_2}$.
- If three points are collinear, then one of the points divide the line segment joining the other two points in the ratio $r : 1$.
- Remember that the section formula can be used only when the given three points are collinear.
- This formula is helpful to find the centroid, incenter and excenters of a triangle. It has applications in physics too; it helps to find the center of mass of systems, equilibrium points and many more.

Example 5.17 Find the coordinates of the point which divides the line segment joining the points (3,5) and (8,–10) internally in the ratio 3:2.

Solution

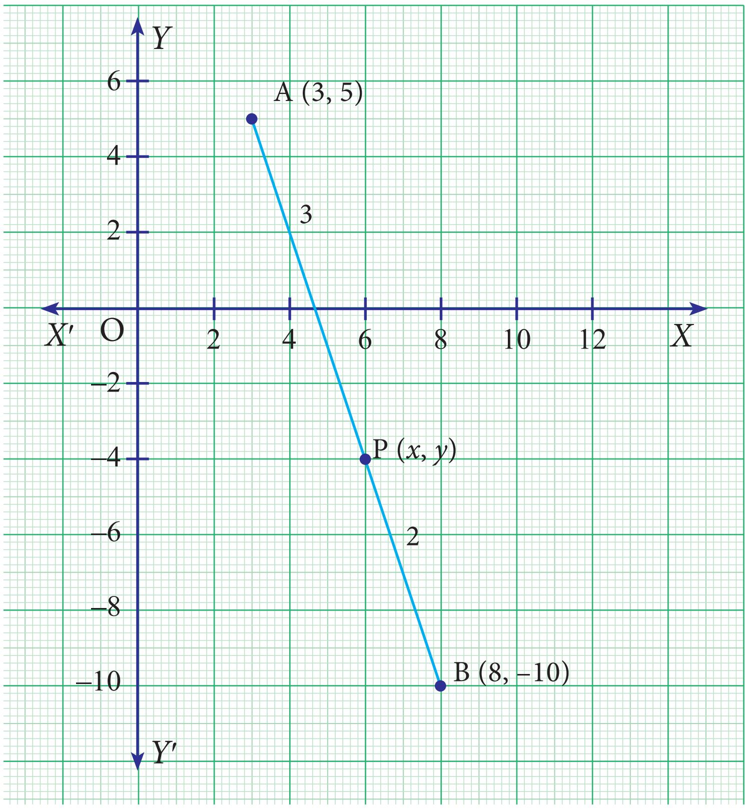


Fig. 5.35

Let $A(3,5)$, $B(8,-10)$ be the given points and let the point $P(x,y)$ divides the line segment AB internally in the ratio 3:2.

By section formula, $P(x, y) = P\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$

Here $x_1 = 3,\ y_1 = 5,\ x_2 = 8,\ y_2 = -10$ and $m = 3,\ n = 2$

Therefore $P(x,y) = P\left(\frac{3(8)+2(3)}{3+2}, \frac{3(-10)+2(5)}{3+2}\right) = P\left(\frac{24+6}{5}, \frac{-30+10}{5}\right) = P\left(6,-4\right)$

Example 5.18

In what ratio does the point $P(-2, 4)$ divide the line segment joining the points $A(-3, 6)$ and $B(1, -2)$ internally?

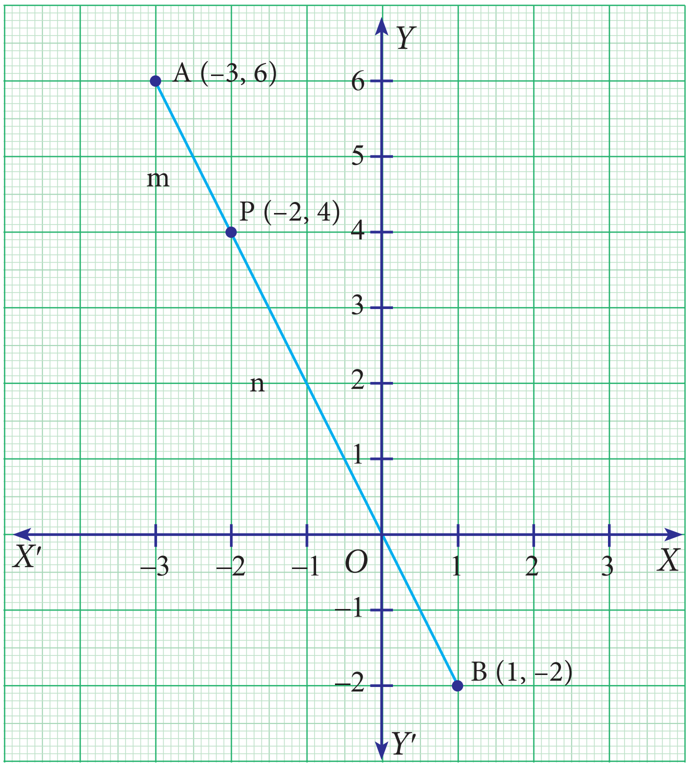


Fig. 5.36

Solution

Given points are $A(-3, 6)$ and $B(1, -2)$. $P(-2, 4)$ divide AB internally in the ratio $m : n$.

By section formula,

$$P(x,y) = P\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$

$$= P\left(-2,4\right) \quad .....(1)$$

Here $x_1 = -3,\ y_1 = 6,\ x_2 = 1,\ y_2 = -2$

$$(1) \Rightarrow \left(\frac{m(1)+n(-3)}{m+n}, \frac{m(-2)+n(6)}{m+n}\right) = P(-2,4)$$

Equating x-coordinates, we get

$$\frac{m-3n}{m+n} = -2 \text{ or } m-3n = -2m-2n$$

$$3m = n$$

$$\frac{m}{n} = \frac{1}{3}$$

$$m:n = 1:3$$

Hence P divides AB internally in the ratio 1:3.

Note

We may arrive at the same result by also equating the y-coordinates.

Try it.

Example 5.19

What are the coordinates of B if point $P(-2,3)$ divides the line segment joining $A(-3,5)$ and B internally in the ratio 1:6?

Solution

Let $A(-3,5)$ and $B(x_2, y_2)$ be the given two points.

Given $P(-2,3)$ divides AB internally in the ratio 1:6.

By section formula, $P\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n}\right) = P(-2,3)$

$$P\left(\frac{1(x_2)+6(-3)}{1+6},\frac{1(y_2)+6(5)}{1+6}\right) = P(-2,3)$$

Equating the coordinates

$$\frac{x_2-18}{7}=-2$$
$$x_2-18=-14$$
$$x_2=4$$

$$\frac{y_2+30}{7}=3$$
$$y_2+30=21$$
$$y_2=-9$$

Therefore, the coordinate of B is $(4,-9)$

Exercise 5.4

1. Find the coordinates of the point which divides the line segment joining the points $A(4,-3)$ and $B(9,7)$ in the ratio 3:2.
2. In what ratio does the point $P(2,-5)$ divide the line segment joining $A(-3,5)$ and $B(4,-9)$.
3. Find the coordinates of a point P on the line segment joining $A(1,2)$ and $B(6,7)$ in such a way that $AP=\frac{2}{5}AB$.
4. Find the coordinates of the points of trisection of the line segment joining the points $A(-5,6)$ and $B(4,-3)$.
5. The line segment joining $A(6,3)$ and $B(-1,-4)$ is doubled in length by adding half of AB to each end. Find the coordinates of the new end points.
6. Using section formula, show that the points $A(7,-5)$, $B(9,-3)$ and $C(13,1)$ are collinear.
7. A line segment AB is increased along its length by 25% by producing it to C on the side of B. If A and B have the coordinates $(-2,-3)$ and $(2,1)$ respectively, then find the coordinates of C.

5.7 The Coordinates of the Centroid

Consider a ΔABC whose vertices are $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$.

Let AD, BE and CF be the medians of the ΔABC.

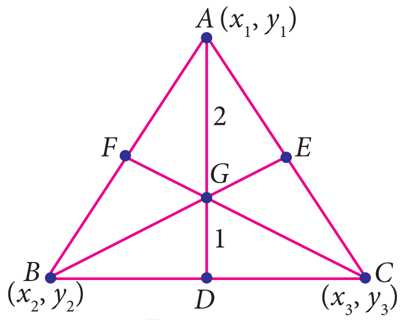


Fig. 5.37

The mid-point of BC is $D\left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2}\right)$

The centroid G divides the median AD internally in the ratio 2:1 and therefore by section formula, the centroid

$$G(x,y) \text{ is } \left(\frac{\frac{2(x_2+x_3)}{2}+1(x_1)}{2+1}, \frac{\frac{2(y_2+y_3)}{2}+1(y_1)}{2+1} \right) = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3} \right)$$

The centroid G of the triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is

$$G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3} \right)$$

Activity - 2

1. Draw ΔABC with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ on the graph sheet.
2. Draw medians and locate the centroid of ΔABC

Observation

(i) The coordinates of the vertices of ΔABC where
$A(x_1, y_1) =$ ______________,
$B(x_2, y_2) =$ ______________
and $C(x_3, y_3) =$ __________

(ii) The coordinates of the centroid G= ___________

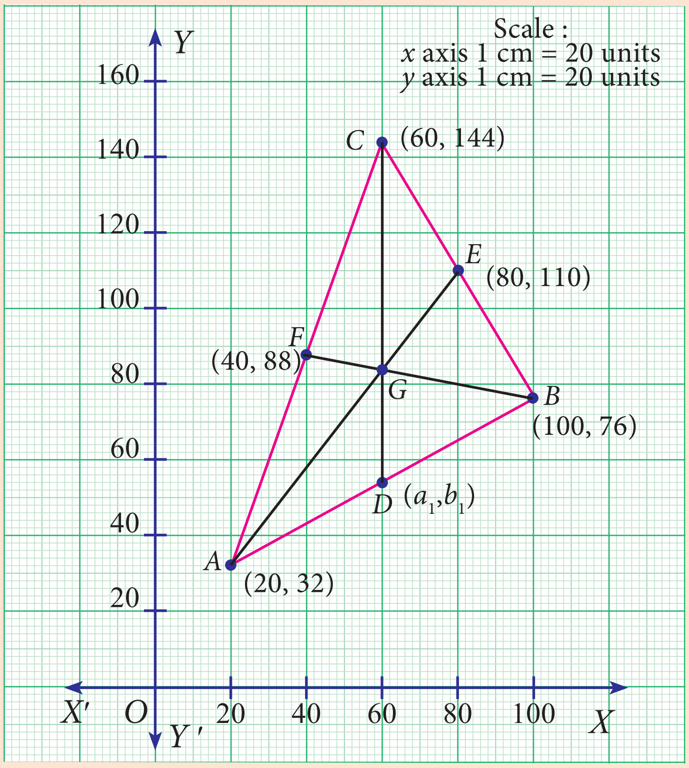


Fig. 5.38

(iii) Use the formula to locate the centroid, whose coordinates are = __________.

(iv) Mid-point of AB is ____________.

(v) Find the point which divides the line segment joining (x_3, y_3) and the mid-point of AB internally in the ratio 2:1 is ____________.

Note

- The medians of a triangle are concurrent and the point of concurrence, the centroid G, is one-third of the distance from the opposite side to the vertex along the median.
- The centroid of the triangle obtained by joining the mid-points of the sides of a triangle is the same as the centroid of the original triangle.
- If (a_1, b_1), (a_2, b_2) and (a_3, b_3) are the mid-points of the sides of a triangle ABC then its centroid G is given by

$$G\left(\frac{a_1+a_2+a_3}{3}, \frac{b_1+b_2+b_3}{3}\right)$$

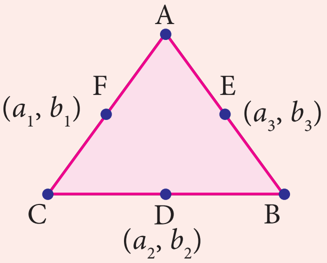


Fig. 5.39

Example 5.20 Find the centroid of the triangle whose veritices are $A(6,-1)$, $B(8,3)$ and $C(10,-5)$.

Solution

The centroid $G(x, y)$ of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$G(x, y) = G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

We have $(x_1, y_1) = (6,-1)$; $(x_2, y_2) = (8,3)$;

$(x_3, y_3) = (10,-5)$

The centroid of the triangle

$$G(x, y) = G\left(\frac{6+8+10}{3}, \frac{-1+3-5}{3}\right)$$

$$= G\left(\frac{24}{3}, \frac{-3}{3}\right) = G(8,-1)$$

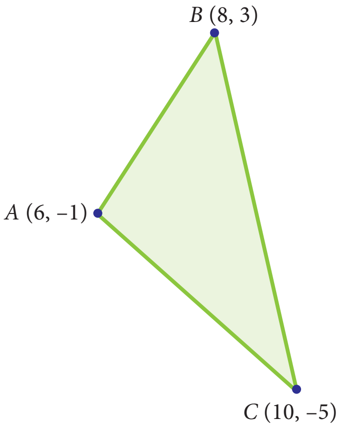


Fig. 5.40

Note

- The Euler line of a triangle is the line that passes through the orthocenter (H), centroid (G) and the circumcenter (S). G divides the line segment $\overline{HS}$ in the ratio 2:1 from the orthocenter. That is centroid divides orthocenter and circumcenter internally in the ratio 2:1 from the Orthocentre.
- In an equilateral triangle, orthocentre, incentre, centroid and circumcentre are all the same.

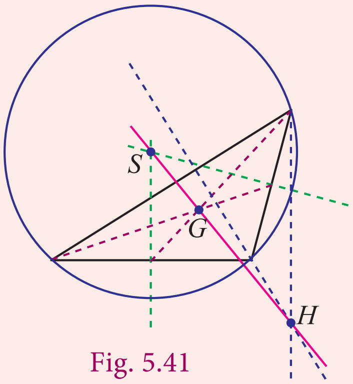


Fig. 5.41

Example 5.21 If the centroid of a triangle is at $(-2,1)$ and two of its vertices are $(1,-6)$ and $(-5,2)$, then find the third vertex of the triangle.

Solution Let the vertices of a triangle be

$A(1,-6),\ B(-5,2)$ and $C(x_3, y_3)$

Given the centroid of a triangle as $(-2,1)$ we get,

$$\frac{x_1+x_2+x_3}{3}=-2 \qquad \frac{y_1+y_2+y_3}{3}=1$$

$$\frac{1-5+x_3}{3}=-2 \qquad \frac{-6+2+y_3}{3}=1$$

$$-4+x_3=-6 \qquad -4+y_3=3$$

$$x_3=-2 \qquad y_3=7$$

Therefore, third vertex is (–2,7).

Thinking Corner

(i) Master gave a trianglular plate with vertices A(5, 8), B(2, 4), C(8, 3) and a stick to a student. He wants to balance the plate on the stick. Can you help the boy to locate that point which can balance the plate.

(ii) Which is the centre of gravity for this triangle? why?

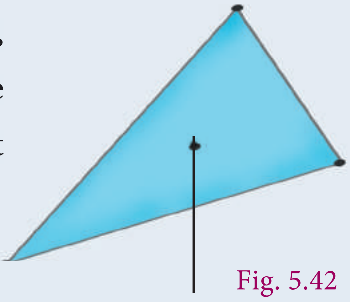


Fig. 5.42

Exercise 5.5

1. Find the centroid of the triangle whose vertices are

   (i) (2,–4), (–3,–7) and (7,2) (ii) (–5,–5), (1,–4) and (–4,–2)

2. If the centroid of a triangle is at (4,–2) and two of its vertices are (3,–2) and (5,2) then find the third vertex of the triangle.

3. Find the length of median through A of a triangle whose vertices are $A(-1,3)$, $B(1,-1)$ and $C(5,1)$.

4. The vertices of a triangle are (1,2), $(h,-3)$ and $(-4,k)$. If the centroid of the triangle is at the point (5,–1) then find the value of $\sqrt{(h+k)^2+(h+3k)^2}$.

5. Orthocentre and centroid of a triangle are $A(-3,5)$ and $B(3,3)$ respectively. If C is the circumcentre and AC is the diameter of this circle, then find the radius of the circle.

6. ABC is a triangle whose vertices are $A(3,4)$, $B(-2,-1)$ and $C(5,3)$. If G is the centroid and $BDCG$ is a parallelogram then find the coordinates of the vertex D.

7. If $\left(\frac{3}{2},5\right)$, $\left(7,\frac{-9}{2}\right)$ and $\left(\frac{13}{2},\frac{-13}{2}\right)$ are mid-points of the sides of a triangle, then find the centroid of the triangle.

Exercise 5.6

FZGA4G

Multiple Choice Questions

1. If the y-coordinate of a point is zero, then the point always lies ______

(1)in the I quadrant (2) in the II quadrant (3)on x-axis (4) on y-axis

2. The points (–5, 2) and (2, –5) lie in the ________

(1) same quadrant (2) II and III quadrant respectively
(3) II and IV quadrant respectively (4) IV and II quadrant respectively

3. On plotting the points $O(0,0)$, $A(3, -4)$, $B(3, 4)$ and $C(0, 4)$ and joining OA, AB, BC and CO, which of the following figure is obtained?

(1) Square (2) Rectangle (3) Trapezium (4) Rhombus

4. If $P(-1,1)$, $Q(3,-4)$, $R(1,-1)$, $S(-2,-3)$ and $T(-4,4)$ are plotted on a graph paper, then the points in the fourth quadrant are __________

(1) P and T (2) Q and R (3) only S (4) P and Q

5. The point whose ordinate is 4 and which lies on the y-axis is _______________

(1)(4, 0) (2) (0, 4) (3) (1, 4) (4) (4, 2)

6. The distance between the two points (2, 3) and (1, 4) is ______

(1) 2 (2) $\sqrt{56}$ (3) $\sqrt{10}$ (4) $\sqrt{2}$

7. If the points A (2,0), B (-6,0), C (3, a–3) lie on the x-axis then the value of a is _____

(1) 0 (2) 2 (3) 3 (4) –6

8. If (x+2, 4) = (5, y–2), then the coordinates (x,y) are _____

(1) (7, 12) (2) (6, 3) (3) (3, 6) (4) (2, 1)

9. If Q_1, Q_2, Q_3, Q_4 are the quadrants in a Cartesian plane then $Q_2 \cap Q_3$ is ____________

(1) $Q_1 \cup Q_2$ (2) $Q_2 \cup Q_3$ (3) Null set (4) Negative x-axis.

10. The distance between the point (5, –1) and the origin is _________

(1) $\sqrt{24}$ (2) $\sqrt{37}$ (3) $\sqrt{26}$ (4) $\sqrt{17}$

11. The coordinates of the point C dividing the line segment joining the points $P(2,4)$ and $Q(5,7)$ internally in the ratio 2:1 is

(1) $\left(\frac{7}{2},\frac{11}{2}\right)$ (2) (3,5) (3) (4,4) (4) (4,6)

12. If $P\left(\frac{a}{3},\frac{b}{2}\right)$ is the mid-point of the line segment joining $A(-4,3)$ and $B(-2,4)$ then (a,b) is

(1) (−9,7) (2) $\left(-3,\frac{7}{2}\right)$ (3) (9, −7) (4) $\left(3,-\frac{7}{2}\right)$

13. In what ratio does the point $Q(1,6)$ divide the line segment joining the points $P(2,7)$ and $R(-2,3)$

(1) 1:2 (2) 2:1 (3) 1:3 (4) 3:1

14. If the coordinates of one end of a diameter of a circle is (3,4) and the coordinates of its centre is (−3,2), then the coordinate of the other end of the diameter is

(1) (0,−3) (2) (0,9) (3) (3,0) (4) (−9,0)

15. The ratio in which the x-axis divides the line segment joining the points $A(a_1,b_1)$ and $B(a_2,b_2)$ is

(1) $b_1:b_2$ (2) $-b_1:b_2$ (3) $a_1:a_2$ (4) $-a_1:a_2$

16. The ratio in which the x-axis divides the line segment joining the points (6,4) and (1, −7) is

(1) 2:3 (2) 3:4 (3) 4:7 (4) 4:3

17. If the coordinates of the mid-points of the sides AB, BC and CA of a triangle are (3,4), (1,1) and (2,−3) respectively, then the vertices A and B of the triangle are

(1) (3,2), (2,4) (2) (4,0), (2,8) (3) (3,4), (2,0) (4) (4,3), (2,4)

18. The mid-point of the line joining $(-a,2b)$ and $(-3a,-4b)$ is

(1) $(2a,3b)$ (2) $(-2a,-b)$ (3) $(2a,b)$ (4) $(-2a,-3b)$

19. In what ratio does the y-axis divides the line joining the points (−5,1) and (2,3) internally

(1) 1:3 (2) 2:5 (3) 3:1 (4) 5:2

20. If (1,−2), (3,6), $(x,10)$ and (3,2) are the vertices of the parallelogram taken in order, then the value of x is

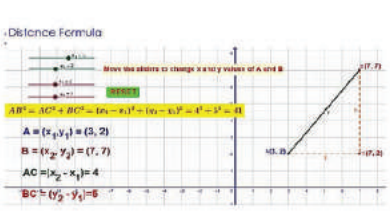
(1) 6 (2) 5 (3) 4 (4) 3

Points to remember

- If x_1, x_2 are the x-coordinates of two points on the x-axis then the distance between them is $x_2 - x_1$, if $x_2 > x_1$.
- If y_1, y_2 are the y-coordinates of two points on the y-axis then the distance between them is $|\, y_1 - y_2 \,|$.
- Distance between the two points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- Distance between (x_1, y_1) and the origin $(0, 0)$ is $\sqrt{x_1^2 + y_1^2}$
- The mid-point M of the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
- The point P which divides the line segment joining the two points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m{:}n$ is $P\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$
- The centroid G of the triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is $G\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$
- The centroid of the triangle obtained by joining the mid-points of the sides of a triangle is the same as the centroid of the original triangle.

ICT Corner-1

Expected Result is shown in this picture



Distance Formula

Move the sliders to change x and y values of A and B

RESET

$A = (x_1, y_1) = (3, 2)$

$B = (x_2, y_2) = (7, 7)$

$AC = |x_2 - x_1| = 4$

$BC = |y_2 - y_1| = 5$

Step – 1

Open the Browser and copy and paste the Link given below (or) by typing the URL given (or) Scan the QR Code.

Step - 2

GeoGebra work book called "IX Analytical Geometry" will open. There are several worksheets given. Select the one you want. For example, open" Distance Formula"

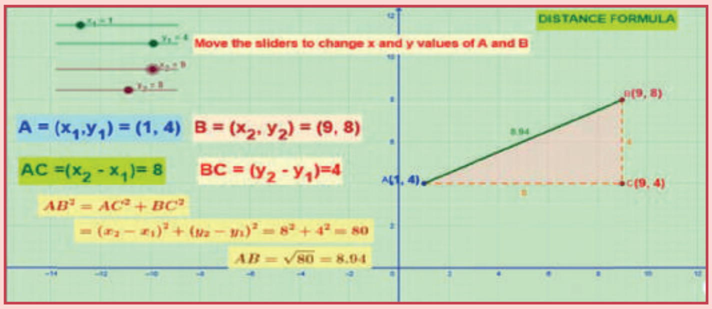
Step-3

Move the sliders x_1, x_2, y_1, y_2 to change the co-ordinates of A and B. Now you calculate the distance AB using theDistance formula in a piece of paper and check your answer

B566_9_MAT_EM_T3

ICT Corner-2

Expected Result is shown in this picture



Move the sliders to change x and y values of A and B

$A = (x_1, y_1) = (1, 4)$ $B = (x_2, y_2) = (9, 8)$

$AC = (x_2 - x_1) = 8$ $BC = (y_2 - y_1) = 4$

$AB^2 = AC^2 + BC^2$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2 = 8^2 + 4^2 = 80$

$AB = \sqrt{80} = 8.94$

Step – 1

Open the Browser by typing the URL Link given below (or) Scan the QR Code. GeoGebra work sheet named "Co-ordinate Geometry" will open. There are two worksheets under the title Distance Formula and Section Formula.

Step - 2

Move the sliders of the respective values to change the points and ratio. Work out the solution and check and click on the respective check box and check the answer.

Scan the QR Code.

B566_9_MAT_EM_T3

Activity

Plot the points $A(1, 0)$, $B(-7, 2)$, $C(-3, 7)$ on a graph sheet and join them to form a triangle.

Plot the point $G(-3, 3)$.

Join AG and extend it to intersect BC at D.

Join BG and extend it to intersect AC at E.

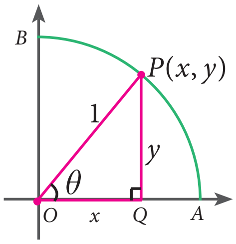
What do you infer when you measure the distance between BD and DC and the distance between CE and EA?

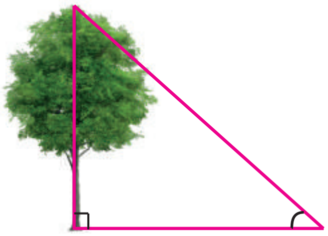
Using distance formula find the lengths of CG and GF, where F in on AB.

Write your inference about $AG: GD$, $BG: GE$ and $CG: GF$.

Note: **G** is the **centroid** of the triangle and AD, BE and CF are the three medians of the triangle.

6





TRIGONOMETRY

> *There is perhaps nothing which so occupies the middle position of mathematics as Trigonometry.- J. F. Herbart*

Leonhard Euler
(AD (CE) 1707 - 1783)

Euler, like Newton, was the greatest mathematician of his generation. He studied all areas of mathematics and continued to work hard after he had gone blind. Euler made discoveries in many areas of mathematics, especially Calculus and Trigonometry. He was the first to prove several theorems in Geometry.

FZJWSD

Learning Outcomes

- To understand the relationship among various trigonometric ratios.
- To recognize the values of trigonometric ratios and their reciprocals.
- To use the concept of complementary angles.
- To understand the usage of trigonometric tables.

6.1 Introduction

Trigonometry (which comes from Greek words trigonon means triangle and metron means measure) is the branch of mathematics that studies the relationships involving lengths of sides and measures of angles of triangles. It is a useful tool for engineers, scientists, and surveyors and is applied even in seismology and navigation.

Observe the three given right angled triangles; in particular scrutinize their measures. The corresponding angles shown in the three triangles are of the same size. Draw your attention to the lengths of “opposite” sides (meaning the side opposite to the given angle) and the “adjacent” sides (which is the side adjacent to the given angle) of the triangle.

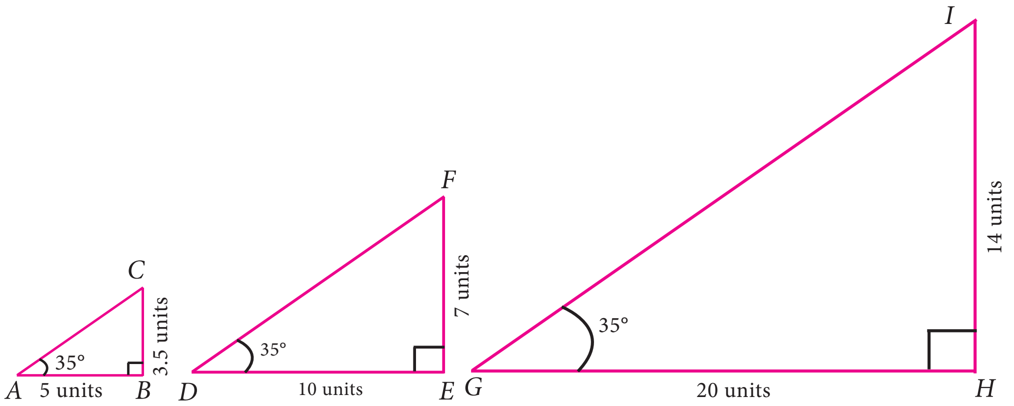


Fig. 6.1

What can you say about the ratio $\left(\dfrac{\textit{opposite side}}{\textit{adjacent side}}\right)$ in each case? Every right angled triangle given here has the same ratio 0.7 ; based on this finding, now what could be the length of the side marked '*x*' in the Fig 6.2? Is it 15?

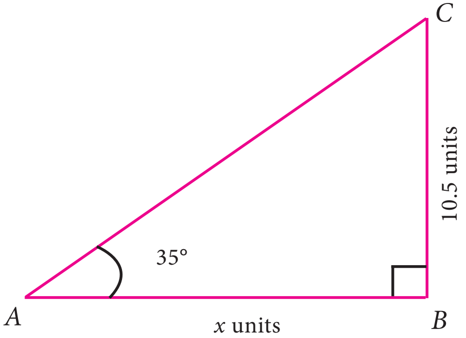
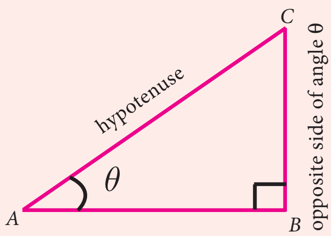
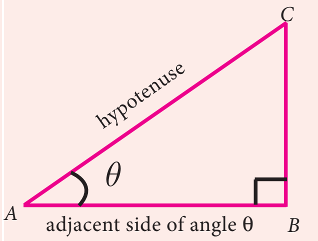
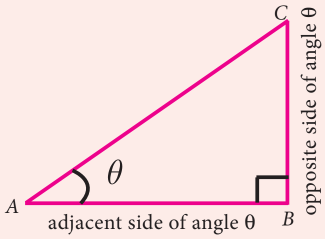


Fig. 6.2

Such remarkable ratios stunned early mathematicians and paved the way for the subject of trigonometry.

There are three basic ratios in trigonometry, each of which is one side of a right-angled triangle divided by another.

The three ratios are:

Name of the angle	**sine**	**cosine**	**tangent**
Short form	sin	cos	tan
Related measurements			
Relationship	$\sin\theta = \dfrac{\textit{opposite side}}{\textit{hypotenuse}}$	$\cos\theta = \dfrac{\textit{adjacent side}}{\textit{hypotenuse}}$	$\tan\theta = \dfrac{\textit{opposite side}}{\textit{adjacent side}}$

Example 6.1 For the measures in the figure, compute sine, cosine and tangent ratios of the angle θ.

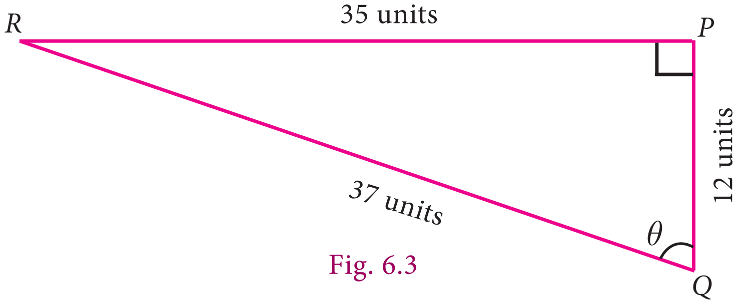


Fig. 6.3

Solution

In the given right angled triangle, note that for the given angle θ, PR is the 'opposite' side and PQ is the 'adjacent' side.

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{PR}{QR} = \frac{35}{37}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{PQ}{QR} = \frac{12}{37}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{PR}{PQ} = \frac{35}{12}$$

FZQ6G4

It is enough to leave the ratios as fractions. In case, if you want to simplify each ratio neatly in a terminating decimal form, you may opt for it, but that is not obligatory.

Note

- Since trigonometric ratios are defined in terms of ratios of sides, they are unitless numbers.
- Ratios like $\sin\theta$, $\cos\theta$, $\tan\theta$ are not to be treated like $(\sin)\times(\theta)$, $(\cos)\times(\theta)$,$(\tan)\times(\theta)$.

Thinking Corner

The given triangles ABC, DEF and GHI have measures 3-4-5, 6-8-10 and 12-16-20. Are they all right triangles? How do you know?

The angles at the vertices B, E and H are of equal size (each angle is equal to θ).

With these available details, fill up the following table and comment on the ratios that you get.

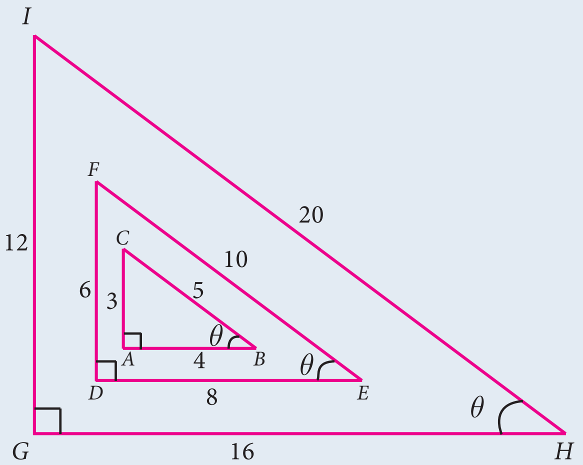


Fig. 6.4

In ΔABC	In ΔDEF	In ΔGHI
$\sin\,\theta = \frac{3}{5}$	$\sin\,\theta = \frac{6}{10} = ?$	$\sin\,\theta = \frac{12}{20} = ?$
$\cos\,\theta = ?$	$\cos\,\theta = ?$	$\cos\,\theta = ?$
$\tan\theta = \frac{3}{4}$	$\tan\theta = ?$	$\tan\theta = ?$

Reciprocal ratios

We defined three basic trigonometric ratios namely, sine, cosine and tangent. The reciprocals of these ratios are also often useful during calculations. We define them as follows:

Basic Trigonometric Ratios	Its reciprocal	Short form
$\sin\,\theta = \frac{\textit{opposite side}}{\textit{hypotenuse}}$	$\text{cosecant}\,\theta = \frac{\textit{hypotenuse}}{\textit{opposite side}}$	$\text{cosec}\,\theta = \frac{\textit{hypotenuse}}{\textit{opposite side}}$
$\cos\,\theta = \frac{\textit{adjacent side}}{\textit{hypotenuse}}$	$\text{secant}\,\theta = \frac{\textit{hypotenuse}}{\textit{adjacent side}}$	$\sec\,\theta = \frac{\textit{hypotenuse}}{\textit{adjacent side}}$
$\tan\,\theta = \frac{\textit{opposite side}}{\textit{adjacent side}}$	$\text{cotangent}\,\theta = \frac{\textit{adjacent side}}{\textit{opposite side}}$	$\cot\,\theta = \frac{\textit{adjacent side}}{\textit{opposite side}}$

From the above ratios we can observe easily the following relations:

$$\text{cosec}\,\theta = \frac{1}{\sin\theta} \qquad \sec\,\theta = \frac{1}{\cos\theta} \qquad \cot\,\theta = \frac{1}{\tan\theta}$$

$$\sin\theta = \frac{1}{\text{cosec}\,\theta} \qquad \cos\,\theta = \frac{1}{\sec\theta} \qquad \tan\,\theta = \frac{1}{\cot\theta}$$

$(\sin\theta)\times(\text{cosec}\,\theta) = 1$. We usually write this as $\sin\theta\ \ \text{cosec}\,\theta = 1$.

$(\cos\theta)\times(\sec\theta) = 1$. We usually write this as $\cos\theta\ \ \sec\theta = 1$.

$(\tan\theta)\times(\cot\theta) = 1$. We usually write this as $\tan\theta\ \ \cot\theta = 1$.

Example 6.2

Find the six trigonometric ratios of the angle θ using the given diagram.

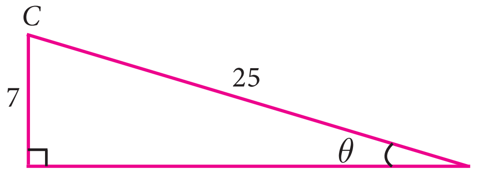


Fig. 6.5

Solution

By Pythagoras theorem,

$$AB = \sqrt{BC^2 - AC^2}$$

$$= \sqrt{(25)^2 - 7^2}$$

$$= \sqrt{625 - 49} = \sqrt{576} = 24$$

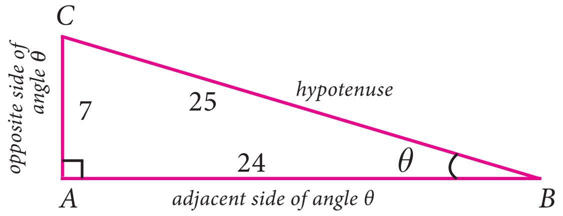


Fig. 6.6

The six trignometric ratios are

$$\sin\theta = \frac{opposite\,side}{hypotenuse} = \frac{7}{25} \qquad \cos\theta = \frac{adjacent\ side}{hypotenuse} = \frac{24}{25}$$

$$\tan\theta = \frac{opposite\,side}{adjacent\ side} = \frac{7}{24} \qquad \text{cosec}\theta = \frac{hypotenuse}{opposite\,side} = \frac{25}{7}$$

$$\sec\theta = \frac{hypotenuse}{adjacent\,side} = \frac{25}{24} \qquad \cot\theta = \frac{adjacent\ side}{opposite\,side} = \frac{24}{7}$$

Example 6.3 If $\tan A = \frac{2}{3}$, then find all the other trigonometric ratios.

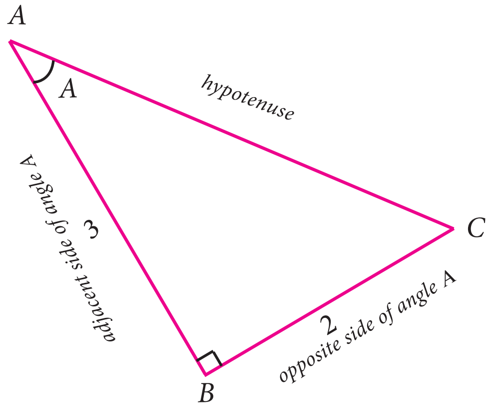


Fig. 6.7

Solution

$$\tan A = \frac{opposite\,side}{adjacent\ side} = \frac{2}{3}$$

By Pythagoras theorem,

$$AC = \sqrt{AB^2 + BC^2}$$

$$= \sqrt{3^2 + 2^2} = \sqrt{9+4} = \sqrt{13}$$

$$AC = \sqrt{13}$$

$$\sin A = \frac{opposite\,side}{hypotenuse} = \frac{2}{\sqrt{13}} \qquad \cos A = \frac{adjacent\ side}{hypotenuse} = \frac{3}{\sqrt{13}}$$

$$\text{cosec}A = \frac{hypotenuse}{opposite\,side} = \frac{\sqrt{13}}{2} \qquad \sec A = \frac{hypotenuse}{adjacent\ side} = \frac{\sqrt{13}}{3}$$

$$\cot A = \frac{adjacent\ side}{opposite\,side} = \frac{3}{2}$$

Example 6.4 If $\sec\theta = \frac{13}{5}$, then show that $\frac{2\sin\theta - 3\cos\theta}{4\sin\theta - 9\cos\theta} = 3$

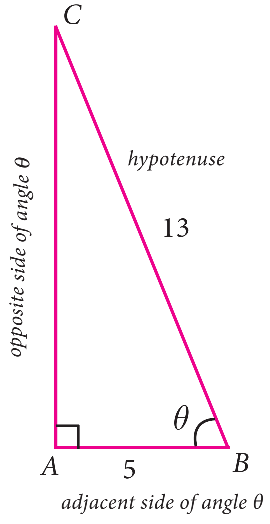


Fig. 6.8

Solution:

Let $BC = 13$ and $AB = 5$

$$\sec\theta = \frac{hypotenuse}{adjacent\ side} = \frac{BC}{AB} = \frac{13}{5}$$

By the Pythagoras theorem,

$$AC = \sqrt{BC^2 - AB^2}$$

$$= \sqrt{13^2 - 5^2}$$

$$= \sqrt{169 - 25} \quad = \quad \sqrt{144} = 12$$

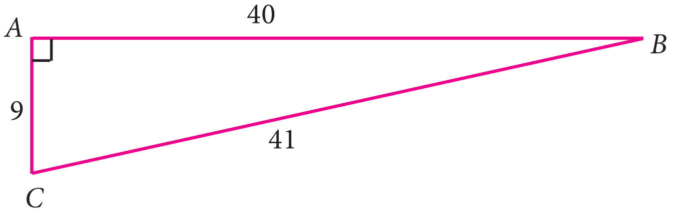
Therefore, $\sin\theta = \dfrac{AC}{BC} = \dfrac{12}{13}$; $\cos\theta = \dfrac{AB}{BC} = \dfrac{5}{13}$

$$LHS = \frac{2\sin\theta - 3\cos\theta}{4\sin\theta - 9\cos\theta} = \frac{2\times\frac{12}{13} - 3\times\frac{5}{13}}{4\times\frac{12}{13} - 9\times\frac{5}{13}} = \frac{\frac{24-15}{13}}{\frac{48-45}{13}} = \frac{9}{3} = 3 \;= RHS$$

Note: We can also take the angle 'θ' at the vertex 'C' and proceed in the same way.

Exercise 6.1

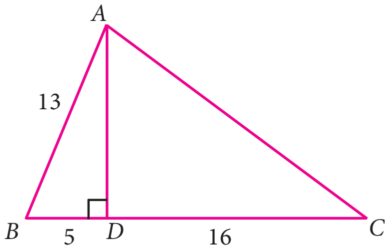
1. From the given figure, find all the trigonometric ratios of angle B.



2. From the given figure, find the values of

   (i) $\sin B$ (ii) $\sec B$ (iii) $\cot B$

   (iv) $\cos C$ (v) $\tan C$ (vi) $\operatorname{cosec} C$



3. If $2\cos\theta = \sqrt{3}$, then find all the trigonometric ratios of angle θ.

4. If $\cos A = \dfrac{3}{5}$, then find the value of $\dfrac{\sin A - \cos A}{2\tan A}$.

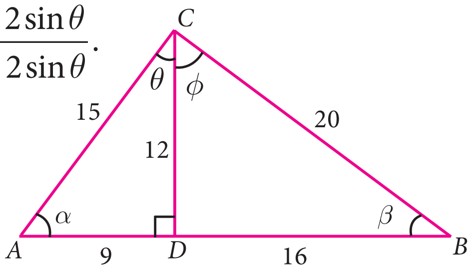
5. If $\cos A = \dfrac{2x}{1+x^2}$, then find the values of $\sin A$ and $\tan A$ in terms of x.

6. If $\sin\theta = \dfrac{a}{\sqrt{a^2+b^2}}$, then show that $b\sin\theta = a\cos\theta$.

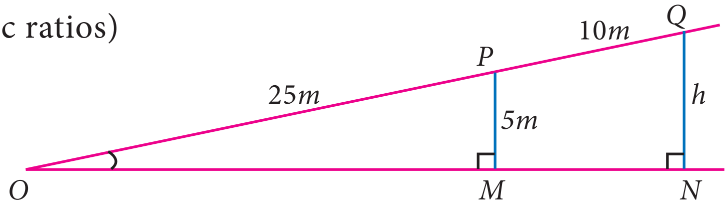
7. If $3\cot A = 2$, then find the value of $\dfrac{4\sin A - 3\cos A}{2\sin A + 3\cos A}$.

8. If $\cos\theta : \sin\theta = 1:2$, then find the value of $\dfrac{8\cos\theta - 2\sin\theta}{4\cos\theta + 2\sin\theta}$.

9. From the given figure, prove that $\theta + \phi = 90°$. Also prove that there are two other right angled triangles. Find $\sin\alpha$, $\cos\beta$ and $\tan\phi$.



10. A boy standing at a point O finds his kite flying at a point P with distance $OP=25m$. It is at a height of $5m$ from the ground. When the thread is extended by $10m$ from P, it reaches a point Q. What will be the height QN of the kite from the ground? (use trigonometric ratios)



6.2 Trigonometric Ratios of Some Special Angles

The values of trigonometric ratios of certain angles can be obtained geometrically. Two special triangles come to help here.

6.2.1 Trigonometric ratios of 45°

Consider a triangle ABC with angles 45°, 45° and 90° as shown in the figure 6.9.

It is the shape of half a square, cut along the square's diagonal. Note that it is also an isosceles triangle (both legs have the same length, a units).

Use Pythagoras theorem to check if the diagonal is of length $a\sqrt{2}$.

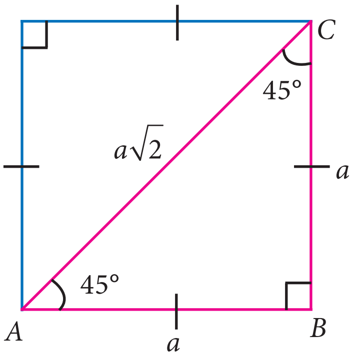


Fig. 6.9

Now, from the right-angled triangle ABC,

$$\sin 45^o = \frac{opposite\,side}{hypotenuse} = \frac{BC}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\cos 45^o = \frac{adjacent\,side}{hypotenuse} = \frac{AB}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\tan 45^o = \frac{opposite\,side}{adjacent\,side} = \frac{BC}{AB} = \frac{a}{a} = 1$$

The reciprocals of these ratio can be easily foundout to be

$\text{cosec}\,45° = \sqrt{2}$;

$\sec 45° = \sqrt{2}$ and

$\cot 45° = 1$

6.2.2 Trigonometric Ratios of 30° and 60°

Consider an equilateral triangle PQR of side length 2 units.

Draw a bisector of $\angle P$. Let it meet QR at M.

$PQ = QR = RP = 2$ units.

$QM = MR = 1$ unit (Why?)

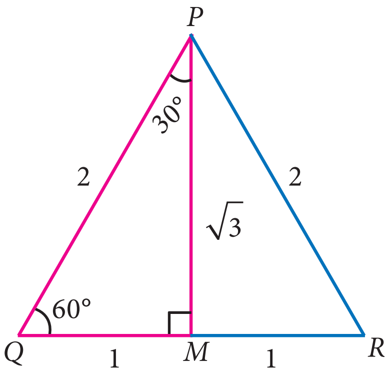


Fig. 6.10

Knowing PQ and QM, we can find PM, using Pythagoras theorem, we find that $PM = \sqrt{3}$ units.

Now, from the right-angled triangle PQM,

$$\sin 30^o = \frac{opposite\,side}{hypotenuse} = \frac{QM}{PQ} = \frac{1}{2}$$

$$\cos 30^O = \frac{adjacent\ side}{hypotenuse} = \frac{PM}{PQ} = \frac{\sqrt{3}}{2}$$

$$\tan 30^O = \frac{opposite\ side}{adjacent\ side} = \frac{QM}{PM} = \frac{1}{\sqrt{3}}$$

The reciprocals of these ratio can be easily foundout to be

$$\operatorname{cosec} 30° = 2, \sec 30° = \frac{2}{\sqrt{3}}$$

and $\cot 30° = \sqrt{3}$

We will use the same triangle but the other angle of measure 60° now.

$$\sin 60^o = \frac{opposite\,side}{hypotenuse} = \frac{PM}{PQ} = \frac{\sqrt{3}}{2}$$

$$\cos 60^o = \frac{adjacent\ side}{hypotenuse} = \frac{QM}{PQ} = \frac{1}{2}$$

$$\tan 60^o = \frac{opposite\ side}{adjacent\ side} = \frac{PM}{QM} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

The reciprocals of these ratio can be easily foundout to be

$$\text{cosec}60° = \frac{2}{\sqrt{3}};\ \sec 60° = 2$$

and $\cot 60° = \frac{1}{\sqrt{3}}$

6.2.3 Trigonometric ratios of 0° and 90°

To find the trigonometric ratios of 0° and 90°, we take the help of what is known as a unit circle.

A unit circle is a circle of unit radius (that is of radius 1 unit), centred at the origin.

Why make a circle where the radius is 1unit?

This means that every reference triangle that we create here has a hypotenuse of 1unit, which makes it so much easier to compare angles and ratios.

We will be interested only in the positive values since we consider 'lengths' and it is hence enough to concentrate on the first quadrant.

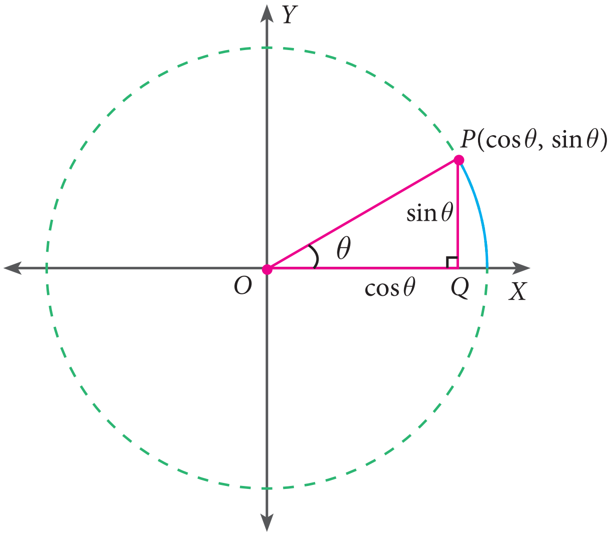


Fig. 6.11

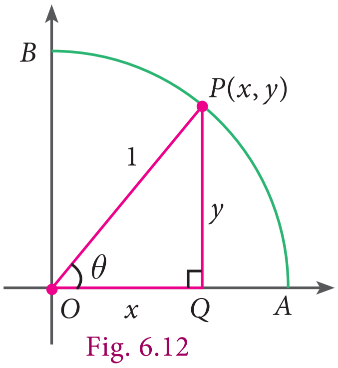


Fig. 6.12

We can see that if $P(x,y)$ be any point on the unit circle in the first quadrant and $\angle POQ = \theta$

$$\sin\theta = \frac{PQ}{OP} = \frac{y}{1} = y \;;\quad \cos\theta = \frac{OQ}{OP} = \frac{x}{1} = x \;;\quad \tan\theta = \frac{PQ}{OQ} = \frac{y}{x}$$

When $\theta = 0°$, OP coincides with OA, where A is $(1,0)$ giving $x = 1,\ y = 0$.

We get thereby,

$$\sin 0° = 0 \;;\quad \text{cosec}\, 0° = \text{not defined (why?)}$$

$$\cos 0° = 1 \;;\quad \sec 0° = 1$$

$$\tan 0° = \frac{0}{1} = 0 \;;\quad \cot 0° = \text{not defined (why?)}$$

When $\theta = 90°$, OP coincides with OB, where B is $(0,1)$ giving $x = 0,\ y = 1$.

Hence,

$$\sin 90° = 1 \;;\quad \text{cosec}\, 90° = 1$$

$$\cos 90° = 0 \;;\quad \sec 90° = \text{not defined}$$

$$\tan 90° = \frac{1}{0} = \text{not defined} \;;\quad \cot 90° = 0$$

Let us summarise all the results in the table given below:

θ Trigonometric ratio	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined
$\text{cosec}\,\theta$	not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined
$\cot\theta$	not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Example 6.5 Evaluate: (i) $\sin 30^\circ + \cos 30^\circ$ (ii) $\tan 60^\circ \cot 60^\circ$

(iii) $\dfrac{\tan 45^\circ}{\tan 30^\circ + \tan 60^\circ}$

(iv) $\sin^2 45^\circ + \cos^2 45^\circ$

Note

(i) $(\sin\theta)^2$ is written as $\sin^2\theta = (\sin\theta)\times(\sin\theta)$

(ii) $(\sin\theta)^2$ is not written as $\sin\theta^2$, because it may mean as $\sin(\theta\times\theta)$.

Solution

(i) $\sin 30^\circ + \cos 30^\circ = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2} = \dfrac{1+\sqrt{3}}{2}$

(ii) $\tan 60^\circ \cot 60^\circ = \sqrt{3} \times \dfrac{1}{\sqrt{3}} = 1$

(iii) $\dfrac{\tan 45^\circ}{\tan 30^\circ + \tan 60^\circ} = \dfrac{1}{\dfrac{1}{\sqrt{3}} + \dfrac{\sqrt{3}}{1}} = \dfrac{1}{\dfrac{1+\left(\sqrt{3}\right)^2}{\sqrt{3}}} = \dfrac{1}{\dfrac{1+3}{\sqrt{3}}} = \dfrac{\sqrt{3}}{4}$

(iv) $\sin^2 45^\circ + \cos^2 45^\circ = \left(\dfrac{1}{\sqrt{2}}\right)^2 + \left(\dfrac{1}{\sqrt{2}}\right)^2 = \dfrac{1^2}{\left(\sqrt{2}\right)^2} + \dfrac{1^2}{\left(\sqrt{2}\right)^2} = \dfrac{1}{2} + \dfrac{1}{2} = 1$

Thinking Corner

The set of three numbers are called as Pythagorean triplets as they form the sides of a right angled triangle. For example,

(i) 3, 4, 5 (ii) 5, 12, 13 (iii) 7, 24, 25

Multiply each number in any of the above Pythagorean triplet by a non-zero constant. Verify whether each of the resultant set so obtained is also a Pythagorean triplet or not.

Example 6.6 Find the values of the following:

(i) $(\cos 0^\circ + \sin 45^\circ + \sin 30^\circ)(\sin 90^\circ - \cos 45^\circ + \cos 60^\circ)$

(ii) $\tan^2 60^\circ - 2\tan^2 45^\circ - \cot^2 30^\circ + 2\sin^2 30^\circ + \dfrac{3}{4}\operatorname{cosec}^2 45^\circ$

Solution

(i) $(\cos 0^\circ + \sin 45^\circ + \sin 30^\circ)(\sin 90^\circ - \cos 45^\circ + \cos 60^\circ)$

$$= \left[1 + \frac{1}{\sqrt{2}} + \frac{1}{2}\right]\left[1 - \frac{1}{\sqrt{2}} + \frac{1}{2}\right]$$

$$=\left[\frac{2\sqrt{2}+2+\sqrt{2}}{2\sqrt{2}}\right]\left[\frac{2\sqrt{2}-2+\sqrt{2}}{2\sqrt{2}}\right]=\left[\frac{3\sqrt{2}+2}{2\sqrt{2}}\right]\left[\frac{3\sqrt{2}-2}{2\sqrt{2}}\right]$$

$$=\frac{18-4}{4\left(\sqrt{2}\right)^2}=\frac{14}{4\times 2}=\frac{7}{4}$$

(ii) $\tan^2 60°-2\tan^2 45°-\cot^2 30°+2\sin^2 30°+\frac{3}{4}\text{cosec}^2 45°$

$$=\left(\sqrt{3}\right)^2-2(1)^2-\left(\sqrt{3}\right)^2+2\left(\frac{1}{2}\right)^2+\frac{3}{4}\left(\sqrt{2}\right)^2$$

$$=3-2-3+\frac{1}{2}+\frac{3}{2}$$

$$=-2+\frac{4}{2}=-2+2=0$$

Note

(i) In a right angled triangle, if the angles are in the ratio $45°:45°:90°$, then the sides are in the ratio $1:1:\sqrt{2}$.

(ii) Similarly, if the angles are in the ratio $30°:60°:90°$, then the sides are in the ratio $1:\sqrt{3}:2$.

(The two set squares in your geometry box is one of the best example for the above two types of triangles).

Exercise 6.2

1. Verify the following equalities:

   (i) $\sin^2 60°+\cos^2 60°=1$

   (ii) $1+\tan^2 30°=\sec^2 30°$

   (iii) $\cos 90°=1-2\sin^2 45°=2\cos^2 45°-1$

   (iv) $\sin 30°\cos 60°+\cos 30°\sin 60°=\sin 90°$

2. Find the value of the following:

   (i) $\frac{\tan 45°}{\text{cosec}30°}+\frac{\sec 60°}{\cot 45°}-\frac{5\sin 90°}{2\cos 0°}$

   (ii) $(\sin 90°+\cos 60°+\cos 45°)\times(\sin 30°+\cos 0°-\cos 45°)$

   (iii) $\sin^2 30°-2\cos^3 60°+3\tan^4 45°$

3. Verify $\cos 3A=4\cos^3 A-3\cos A$, when $A=30°$

4. Find the value of $8\sin 2x\ \cos 4x\ \sin 6x$, when $x=15°$

6.3 Trigonometric Ratios for Complementary Angles

Recall that two acute angles are said to be complementary if the sum of their measures is equal to 90°.

What can we say about the acute angles of a right-angled triangle?

In a right angled triangle the sum of the two acute angles is equal to 90°. So, the two acute angles of a right angled triangle are always complementary to each other.

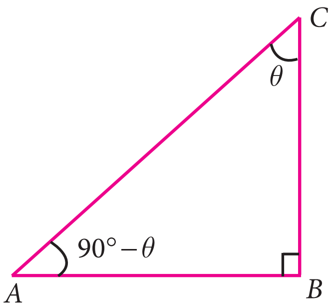


Fig. 6.13

In the above figure 6.13, the triangle is right-angled at B. Therefore, if $\angle C$ is θ, then $\angle A = 90^\circ - \theta$.

We find that

$$\left.\begin{array}{ll} \sin\theta = \dfrac{AB}{AC} & \operatorname{cosec}\theta = \dfrac{AC}{AB} \\ \cos\theta = \dfrac{BC}{AC} & \sec\theta = \dfrac{AC}{BC} \\ \tan\theta = \dfrac{AB}{BC} & \cot\theta = \dfrac{BC}{AB} \end{array}\right\}\ldots.(1)$$

Similarly for the angle (90° – θ), We have

$$\left.\begin{array}{ll} \sin(90^\circ-\theta) = \dfrac{BC}{AC} & \operatorname{cosec}(90^\circ-\theta) = \dfrac{AC}{BC} \\ \cos(90^\circ-\theta) = \dfrac{AB}{AC} & \sec(90^\circ-\theta) = \dfrac{AC}{AB} \\ \tan(90^\circ-\theta) = \dfrac{BC}{AB} & \cot(90^\circ-\theta) = \dfrac{AB}{BC} \end{array}\right\}\ldots..(2)$$

Comparing (1) and (2), we get

$$\begin{array}{lcl|lcl} \sin\theta & = & \cos(90^\circ-\theta) & \operatorname{cosec}\theta & = & \sec(90^\circ-\theta) \\ \cos\theta & = & \sin(90^\circ-\theta) & \sec\theta & = & \operatorname{cosec}(90^\circ-\theta) \\ \tan\theta & = & \cot(90^\circ-\theta) & \cot\theta & = & \tan(90^\circ-\theta) \end{array}$$

Example 6.7

Express (i) $\sin 74^\circ$ in terms of cosine (ii) $\tan 12^\circ$ in terms of cotangent (iii) $\operatorname{cosec} 39^\circ$ in terms of secant

Solution

(i) $\sin 74^\circ = \sin(90^\circ - 16^\circ)$ (since, $90^\circ - 16^\circ = 74^\circ$)

RHS is of the form $\sin(90^\circ - \theta) = \cos\theta$

Therefore $\sin 74^\circ = \cos 16^\circ$

(ii) $\tan 12^\circ = \tan(90^\circ - 78^\circ)$ (since, $12^\circ = 90^\circ - 78^\circ$)

RHS is of the form $\tan(90^\circ - \theta) = \cot\theta$

Therefore $\tan 12^\circ = \cot 78^\circ$

(iii) $\operatorname{cosec} 39^\circ = \operatorname{cosec}(90^\circ - 51^\circ)$ (since $39^\circ = 90^\circ - 51^\circ$)

RHS is of the form $\operatorname{cosec}(90^\circ - \theta) = \sec\theta$

Therefore $\operatorname{cosec} 39^\circ = \sec 51^\circ$

Example 6.8 Evaluate: (i) $\dfrac{\sin 49^\circ}{\cos 41^\circ}$ (ii) $\dfrac{\sec 63^\circ}{\operatorname{cosec} 27^\circ}$

Solution

(i) $\dfrac{\sin 49^\circ}{\cos 41^\circ}$

$\sin 49^\circ = \sin(90^\circ - 41^\circ) = \cos 41^\circ$, since $49^\circ + 41^\circ = 90^\circ$ (complementary),

Hence on substituting $\sin 49^\circ = \cos 41^\circ$ we get, $\dfrac{\cos 41^\circ}{\cos 41^\circ} = 1$

(ii) $\dfrac{\sec 63^\circ}{\operatorname{cosec} 27^\circ}$

$\sec 63^\circ = \sec(90^\circ - 27^\circ) = \operatorname{cosec} 27^\circ$, here 63° and 27° are complementary angles.

we have $\dfrac{\sec 63^\circ}{\operatorname{cosec} 27^\circ} = \dfrac{\operatorname{cosec} 27^\circ}{\operatorname{cosec} 27^\circ} = 1$

Example 6.9 Find the values of (i) $\tan 7^\circ \tan 23^\circ \tan 60^\circ \tan 67^\circ \tan 83^\circ$

(ii) $\dfrac{\cos 35^\circ}{\sin 55^\circ} + \dfrac{\sin 12^\circ}{\cos 78^\circ} - \dfrac{\cos 18^\circ}{\sin 72^\circ}$

Solution

(i) $\tan 7^\circ \tan 23^\circ \tan 60^\circ \tan 67^\circ \tan 83^\circ$

$= \tan 7^\circ \tan 83^\circ \tan 23^\circ \tan 67^\circ \tan 60^\circ$ (Grouping complementary angles)

$= \tan 7^\circ \tan(90^\circ - 7^\circ) \tan 23^\circ \tan(90^\circ - 23^\circ) \tan 60^\circ$

$= (\tan 7^\circ . \cot 7^\circ)(\tan 23^\circ . \cot 23^\circ) \tan 60^\circ$

$= (1) \times (1) \times \tan 60^\circ$

$= \tan 60^\circ \quad = \sqrt{3}$

(ii) $\dfrac{\cos 35^\circ}{\sin 55^\circ} + \dfrac{\sin 12^\circ}{\cos 78^\circ} - \dfrac{\cos 18^\circ}{\sin 72^\circ}$

$$= \frac{\cos(90^\circ - 55^\circ)}{\sin 55^\circ} + \frac{\sin(90^\circ - 78^\circ)}{\cos 78^\circ} - \frac{\cos(90^\circ - 72^\circ)}{\sin 72^\circ} \quad \left[\begin{array}{l} \text{since} \\ \cos 35^\circ = \cos(90^\circ - 55^\circ) \\ \sin 12^\circ = \sin(90^\circ - 78^\circ) \\ \cos 18^\circ = \cos(90^\circ - 72^\circ) \end{array}\right]$$

$$= \frac{\sin 55^\circ}{\sin 55^\circ} + \frac{\cos 78^\circ}{\cos 78^\circ} - \frac{\sin 72^\circ}{\sin 72^\circ}$$

$$= 1 + 1 - 1 = 1$$

Example 6.10 (i) If $\operatorname{cosec} A = \sec 34^\circ$, then find A (ii) If $\tan B = \cot 47^\circ$, then find B.

Solution

(i) We know that $\operatorname{cosec} A = \sec(90^\circ - A)$

$$\sec(90^\circ - A) = \sec(34^\circ)$$

$$90^\circ - A = 34^\circ$$

We get $A = 90^\circ - 34^\circ$

$$A = 56^\circ$$

(ii) We know that $\tan B = \cot(90^\circ - B)$

$$\cot(90^\circ - B) = \cot 47^\circ$$

$$90^\circ - B = 47^\circ$$

We get $B = 90^\circ - 47^\circ$

$$B = 43^\circ$$

Exercise 6.3

Find the value of the following:

(i) $\left(\frac{\cos 47^\circ}{\sin 43^\circ}\right)^2 + \left(\frac{\sin 72^\circ}{\cos 18^\circ}\right)^2 - 2\cos^2 45^\circ$ (ii) $\frac{\cos 70^\circ}{\sin 20^\circ} + \frac{\cos 59^\circ}{\sin 31^\circ} + \frac{\cos\theta}{\sin(90^\circ - \theta)} - 8\cos^2 60^\circ$

(iii) $\tan 15^\circ \tan 30^\circ \tan 45^\circ \tan 60^\circ \tan 75^\circ$

(iv) $\frac{\cot\theta}{\tan(90^\circ - \theta)} + \frac{\cos(90^\circ - \theta)\tan\theta\sec(90^\circ - \theta)}{\sin(90^\circ - \theta)\cot(90^\circ - \theta)\operatorname{cosec}(90^\circ - \theta)}$

Thinking Corner

(i) What are the minimum and maximum values of $\sin\theta$?

(ii) What are the minimum and maximum values of $\cos\theta$?

6.4 Method of using Trigonometric Table

We have learnt to calculate the trigonometric ratios for angles 0°, 30°, 45°, 60° and 90°. But during certain situations we need to calculate the trigonometric ratios of all the other acute angles. Hence we need to know the method of using trigonometric tables.

One degree (1°) is divided into 60 minutes ($60'$) and one minute ($1'$) is divided into 60 seconds ($60''$). Thus, $1^\circ = 60'$ and $1' = 60''$.

The trigonometric tables give the values, correct to four places of decimals for the angles from 0° to 90° spaced at intervals of $60'$. A trigonometric table consists of three parts.

A column on the extreme left which contains degrees from 0° to 90°, followed by ten columns headed by $0'$, $6'$, $12'$, $18'$, $24'$, $30'$, $36'$, $42'$, $48'$ and $54'$.

Five columns under the head mean difference has values from 1,2,3,4 and 5.

For angles containing other measures of minutes (that is other than $0'$, $6'$, $12'$, $18'$, $24'$, $30'$, $36'$, $42'$, $48'$ and $54'$), the appropriate adjustment is obtained from the mean difference columns.

The mean difference is to be added in the case of sine and tangent while it is to be subtracted in the case of cosine.

Now let us understand the calculation of values of trigonometric angle from the following examples.

Example 6.11 Find the value of $\sin 64°34'$.

Solution

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Difference				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
64°						0.9026								5	

Write $64°34' = 64°30' + 4'$

From the table we have, $\sin 64°30' = 0.9026$

Mean difference for $4' =$ 5(Mean difference to be added for sine)

$\sin 64°34' = 0.9031$

Example 6.12 Find the value of $\cos 19°59'$

Solution

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Difference				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
19°										0.9403					5

Write $19°59' = 19°54' + 5'$

From the table we have,

$\cos 19°54' = 0.9403$

Mean difference for $5' =$ 5 (Mean difference to be subtracted for cosine)

$\cos 19°59' = 0.9398$

Example 6.13

Find the value of $\tan 70°13'$

Solution

	$0'$	$6'$	$12'$	$18'$	$24'$	$30'$	$36'$	$42'$	$48'$	$54'$	Mean Difference				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
70°			2.7776								26				

Write $70°13' = 70°12' + 1'$

From the table we have, $\tan 70°12' = 2.7776$

Mean difference for $\underline{\quad 1' = \quad 26}$ (Mean difference to be added for tan)

$\underline{\tan 70°13' = 2.7802}$

Example 6.14

Find the value of

(i) $\sin 38°36' + \tan 12°12'$ (ii) $\tan 60°25' - \cos 49°20'$

Solution

(i) $\sin 38°36' + \tan 12°12'$

$\sin 38°36' = 0.6239$

$\tan 12°12' = 0.2162$

$\sin 38°36' + \tan 12°12' = 0.8401$

(ii) $\tan 60°25' - \cos 49°20'$

$\tan 60°25' = 1.7603 + 0.0012 = 1.7615$

$\cos 49°20' = 0.6521 - 0.0004 = 0.6517$

$\tan 60°25' - \cos 49°20' = 1.1098$

Example 6.15

Find the value of θ if

(i) $\sin\theta = 0.9858$ (ii) $\cos\theta = 0.7656$

Solution

(i) $\sin\theta = 0.9858 = 0.9857 + 0.0001$

From the sine table $0.9857 = 80°18'$

Mean difference $1 = \quad 2'$

$0.9858 = 80°20'$

$\sin\theta = 0.9858 = \sin 80°20'$

$\theta = 80°20'$

(ii) $\cos\theta = 0.7656 = 0.7660 - 0.0004$

From the cosine table

$0.7660 = 40°0'$

Mean difference $4 = \quad 2'$

$0.7656 = 40°2'$

$\cos\theta = 0.7656 = \cos 40°2'$

$\theta = 40°2'$

Example 6.16 Find the area of the right angled triangle with hypotenuse 5*cm* and one of the acute angle is $48°30'$

Solution

From the figure,

$$\begin{aligned} \sin\theta &= \frac{AB}{AC} \\ \sin 48°30' &= \frac{AB}{5} \\ 0.7490 &= \frac{AB}{5} \\ 5\times 0.7490 &= AB \\ AB &= 3.7450\ cm \end{aligned}$$

$$\begin{aligned} \cos\theta &= \frac{BC}{AC} \\ \cos 48°30' &= \frac{BC}{5} \\ 0.6626 &= \frac{BC}{5} \\ 0.6626\times 5 &= BC \\ BC &= 3.313\ cm \end{aligned}$$

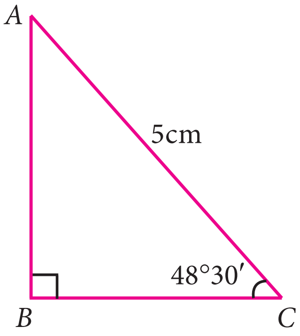
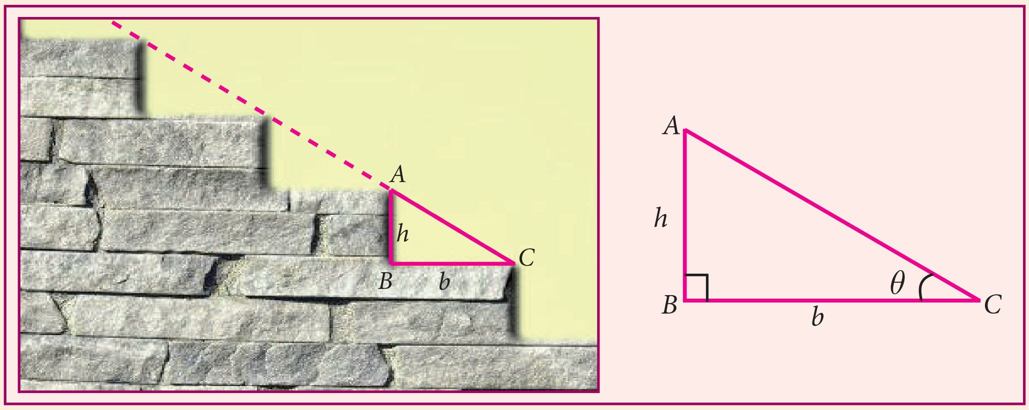


Fig. 6.14

$$\begin{aligned} \text{Area of right triangle} &= \frac{1}{2}bh \\ &= \frac{1}{2}\times BC\times AB \\ &= \frac{1}{2}\times 3.3130\times 3.7450 \\ &= 1.6565\times 3.7450 \quad = 6.2035925\ cm^2 \end{aligned}$$

Activity

Observe the steps in your home. Measure the breadth and the height of one step. Enter it in the following picture and measure the angle (of elevation) of that step.



(i) Compare the angles (of elevation) of different steps of same height and same breadth and discuss your observation.

(ii) Sometimes few steps may not be of same height. Compare the angles (of elevation) of different steps of those different heights and same breadth and dicuss your observation.

Exercise 6.4

1. Find the value of the following:

   (i) $\sin 49°$ (ii) $\cos 74°39'$ (iii) $\tan 54°26'$ (iv) $\sin 21°21'$ (v) $\cos 33°53'$ (vi) $\tan 70°17'$

2. Find the value of θ if

   (i) $\sin\theta = 0.9975$ (ii) $\cos\theta = 0.6763$ (iii) $\tan\theta = 0.0720$

   (iv) $\cos\theta = 0.0410$ (v) $\tan\theta = 7.5958$

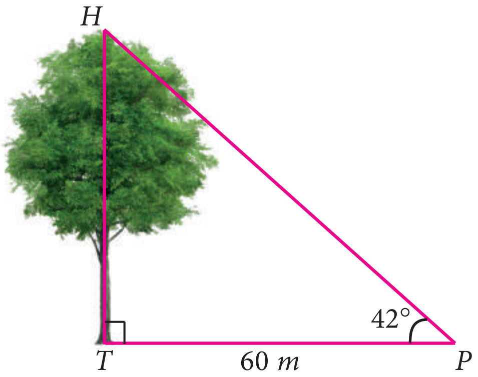
3. Find the value of the following:

   (i) $\sin 65°39' + \cos 24°57' + \tan 10°10'$ (ii) $\tan 70°58' + \cos 15°26' - \sin 84°59'$

4. Find the area of a right triangle whose hypotenuse is 10*cm* and one of the acute angle is $24°24'$

5. Find the angle made by a ladder of length 5*m* with the ground, if one of its end is 4*m* away from the wall and the other end is on the wall.

6. In the given figure, *HT* shows the height of a tree standing vertically. From a point *P*, the angle of elevation of the top of the tree (that is $\angle P$) measures $42°$ and the distance to the tree is 60 metres. Find the height of the tree.



Exercise 6.5

Multiple choice questions

1. If $\sin 30° = x$ and $\cos 60° = y$, then $x^2 + y^2$ is

   (1) $\frac{1}{2}$ (2) 0 (3) $\sin 90°$ (4) $\cos 90°$

2. If $\tan\theta = \cot 37°$, then the value of θ is

   (1) 37° (2) 53° (3) 90° (4) 1°

3. The value of $\tan 72° \tan 18°$ is

   (1) 0 (2) 1 (3) 18° (4) 72°

4. The value of $\frac{2\tan 30°}{1-\tan^2 30°}$ is equal to

   (1) $\cos 60°$ (2) $\sin 60°$ (3) $\tan 60°$ (4) $\sin 30°$

FZTSU1

5. If $2\sin 2\theta = \sqrt{3}$, then the value of θ is

(1) 90° (2) 30° (3) 45° (4) 60°

6. The value of $3\sin 70^\circ \sec 20^\circ + 2\sin 49^\circ \sec 51^\circ$ is

(1) 2 (2) 3 (3) 5 (4) 6

7. The value of $\dfrac{1-\tan^2 45^\circ}{1+\tan^2 45^\circ}$ is

(1) 2 (2) 1 (3) 0 (4) $\dfrac{1}{2}$

8. The value of $\text{cosec}(70^\circ+\theta)-\sec(20^\circ-\theta)+\tan(65^\circ+\theta)-\cot(25^\circ-\theta)$ is

(1) 0 (2) 1 (3) 2 (4) 3

9. The value of $\tan 1^\circ \tan 2^\circ \tan 3^\circ \ldots \tan 89^\circ$ is

(1) 0 (2) 1 (3) 2 (4) $\dfrac{\sqrt{3}}{2}$

10. Given that $\sin\alpha = \dfrac{1}{2}$ and $\cos\beta = \dfrac{1}{2}$, then the value of $\alpha+\beta$ is

(1) 0° (2) 90° (3) 30° (4) 60°

Points to Remember

- Trigonometric ratios are

$\sin\theta = \dfrac{\textit{opposite side}}{\textit{hypotenuse}}$	$\text{cosec}\ \theta = \dfrac{\textit{hypotenuse}}{\textit{opposite side}}$
$\cos\theta = \dfrac{\textit{adjacent side}}{\textit{hypotenuse}}$	$\sec\theta = \dfrac{\textit{hypotenuse}}{\textit{adjacent side}}$
$\tan\theta = \dfrac{\textit{opposite side}}{\textit{adjacent side}}$	$\cot\theta = \dfrac{\textit{adjacent side}}{\textit{opposite side}}$

- Reciprocal trigonometric ratios

$\sin\theta = \dfrac{1}{\cos ec\,\theta}$ $\cos\theta = \dfrac{1}{\sec\theta}$ $\tan\theta = \dfrac{1}{\cot\theta}$

$\text{cosec}\ \theta = \dfrac{1}{\sin\theta}$ $\sec\theta = \dfrac{1}{\cos\theta}$ $\cot\theta = \dfrac{1}{\tan\theta}$

- Complementary angles

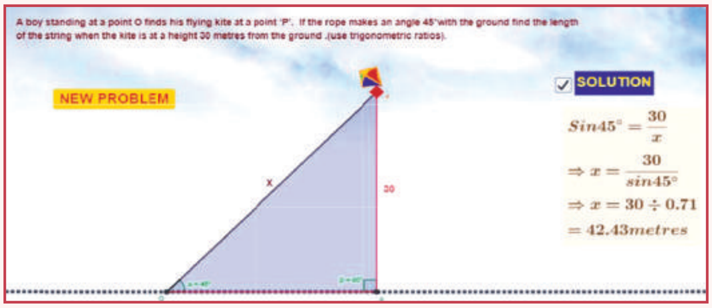
$\sin\theta = \cos(90^\circ-\theta)$ $\text{cosec}\theta = \sec(90^\circ-\theta)$

$\cos\theta = \sin(90^\circ-\theta)$ $\sec\theta = \text{cosec}(90^\circ-\theta)$

$\tan\theta = \cot(90^\circ-\theta)$ $\cot\theta = \tan(90^\circ-\theta)$

ICT Corner

Expected Result is shown in this picture



A boy standing at a point O finds his flying kite at a point 'P'. If the rope makes an angle 45°with the ground find the length of the string when the kite is at a height 30 metres from the ground .(use trigonometric ratios).

$$Sin45^\circ = \frac{30}{x}$$
$$\Rightarrow x = \frac{30}{sin45^\circ}$$
$$\Rightarrow x = 30 \div 0.71$$
$$= 42.43 metres$$

Step – 1

Open the Browser by typing the URL Link given below (or) Scan the QR Code. GeoGebra work sheet named "Trigonometry" will open. There are three worksheets under the title Trigonometric ratios and Complementary angles and kite problem.

Step - 2

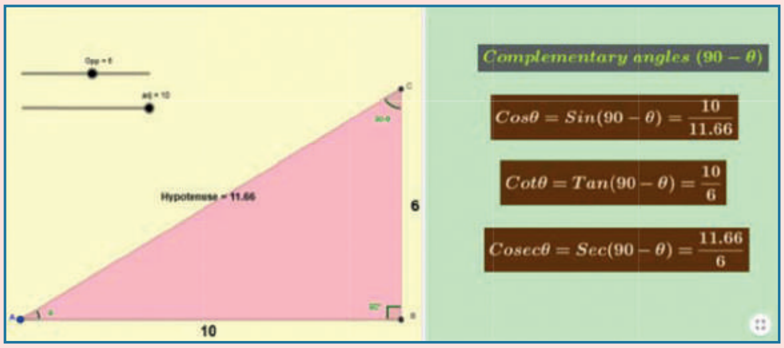
Move the sliders of the respective values to change the points and ratio. Work out the solution and check.

For the kite problem click on "NEW PROBLEM" to change the question and work it out. Click the check box for solution to check your answer.

Step 1

$$Sin\theta = \frac{Opp}{Hyp} = \frac{3}{5}$$
$$Cos\theta = \frac{Adj}{Hyp} = \frac{4}{5}$$
$$Tan\theta = \frac{Opp}{Adj} = \frac{3}{4}$$
$$Cosec\theta = \frac{1}{Sin\theta} = \frac{Hyp}{Opp} = \frac{5}{3}$$
$$Sec\theta = \frac{1}{Cos\theta} = \frac{Hyp}{Adj} = \frac{5}{4}$$
$$Cot\theta = \frac{1}{tan\theta} = \frac{Cos\theta}{Sin\theta} = \frac{Adj}{Opp} = \frac{4}{3}$$

Step 2



Complementary angles $(90 - \theta)$

$$Cos\theta = Sin(90 - \theta) = \frac{10}{11.66}$$
$$Cot\theta = Tan(90 - \theta) = \frac{10}{6}$$
$$Cosec\theta = Sec(90 - \theta) = \frac{11.66}{6}$$

Scan the QR Code.

B566_9_MAT_EM_T3

NATURAL SINES

Degree	0´	6´	12´	18´	24´	30´	36´	42´	48´	54´	Mean Difference				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
0	0.0000	0.0017	0.0035	0.0052	0.0070	0.0087	0.0105	0.0122	0.0140	0.0157	3	6	9	12	15
1	0.0175	0.0192	0.0209	0.0227	0.0244	0.0262	0.0279	0.0297	0.0314	0.0332	3	6	9	12	15
2	0.0349	0.0366	0.0384	0.0401	0.0419	0.0436	0.0454	0.0471	0.0488	0.0506	3	6	9	12	15
3	0.0523	0.0541	0.0558	0.0576	0.0593	0.0610	0.0628	0.0645	0.0663	0.0680	3	6	9	12	15
4	0.0698	0.0715	0.0732	0.0750	0.0767	0.0785	0.0802	0.0819	0.0837	0.0854	3	6	9	12	15
5	0.0872	0.0889	0.0906	0.0924	0.0941	0.0958	0.0976	0.0993	0.1011	0.1028	3	6	9	12	14
6	0.1045	0.1063	0.1080	0.1097	0.1115	0.1132	0.1149	0.1167	0.1184	0.1201	3	6	9	12	14
7	0.1219	0.1236	0.1253	0.1271	0.1288	0.1305	0.1323	0.1340	0.1357	0.1374	3	6	9	12	14
8	0.1392	0.1409	0.1426	0.1444	0.1461	0.1478	0.1495	0.1513	0.1530	0.1547	3	6	9	12	14
9	0.1564	0.1582	0.1599	0.1616	0.1633	0.1650	0.1668	0.1685	0.1702	0.1719	3	6	9	12	14
10	0.1736	0.1754	0.1771	0.1788	0.1805	0.1822	0.1840	0.1857	0.1874	0.1891	3	6	9	12	14
11	0.1908	0.1925	0.1942	0.1959	0.1977	0.1994	0.2011	0.2028	0.2045	0.2062	3	6	9	11	14
12	0.2079	0.2096	0.2113	0.2130	0.2147	0.2164	0.2181	0.2198	0.2215	0.2233	3	6	9	11	14
13	0.2250	0.2267	0.2284	0.2300	0.2317	0.2334	0.2351	0.2368	0.2385	0.2402	3	6	8	11	14
14	0.2419	0.2436	0.2453	0.2470	0.2487	0.2504	0.2521	0.2538	0.2554	0.2571	3	6	8	11	14
15	0.2588	0.2605	0.2622	0.2639	0.2656	0.2672	0.2689	0.2706	0.2723	0.2740	3	6	8	11	14
16	0.2756	0.2773	0.2790	0.2807	0.2823	0.2840	0.2857	0.2874	0.2890	0.2907	3	6	8	11	14
17	0.2924	0.2940	0.2957	0.2974	0.2990	0.3007	0.3024	0.3040	0.3057	0.3074	3	6	8	11	14
18	0.3090	0.3107	0.3123	0.3140	0.3156	0.3173	0.3190	0.3206	0.3223	0.3239	3	6	8	11	14
19	0.3256	0.3272	0.3289	0.3305	0.3322	0.3338	0.3355	0.3371	0.3387	0.3404	3	5	8	11	14
20	0.3420	0.3437	0.3453	0.3469	0.3486	0.3502	0.3518	0.3535	0.3551	0.3567	3	5	8	11	14
21	0.3584	0.3600	0.3616	0.3633	0.3649	0.3665	0.3681	0.3697	0.3714	0.3730	3	5	8	11	14
22	0.3746	0.3762	0.3778	0.3795	0.3811	0.3827	0.3843	0.3859	0.3875	0.3891	3	5	8	11	14
23	0.3907	0.3923	0.3939	0.3955	0.3971	0.3987	0.4003	0.4019	0.4035	0.4051	3	5	8	11	14
24	0.4067	0.4083	0.4099	0.4115	0.4131	0.4147	0.4163	0.4179	0.4195	0.4210	3	5	8	11	13
25	0.4226	0.4242	0.4258	0.4274	0.4289	0.4305	0.4321	0.4337	0.4352	0.4368	3	5	8	11	13
26	0.4384	0.4399	0.4415	0.4431	0.4446	0.4462	0.4478	0.4493	0.4509	0.4524	3	5	8	10	13
27	0.4540	0.4555	0.4571	0.4586	0.4602	0.4617	0.4633	0.4648	0.4664	0.4679	3	5	8	10	13
28	0.4695	0.4710	0.4726	0.4741	0.4756	0.4772	0.4787	0.4802	0.4818	0.4833	3	5	8	10	13
29	0.4848	0.4863	0.4879	0.4894	0.4909	0.4924	0.4939	0.4955	0.4970	0.4985	3	5	8	10	13
30	0.5000	0.5015	0.5030	0.5045	0.5060	0.5075	0.5090	0.5105	0.5120	0.5135	3	5	8	10	13
31	0.5150	0.5165	0.5180	0.5195	0.5210	0.5225	0.5240	0.5255	0.5270	0.5284	2	5	7	10	12
32	0.5299	0.5314	0.5329	0.5344	0.5358	0.5373	0.5388	0.5402	0.5417	0.5432	2	5	7	10	12
33	0.5446	0.5461	0.5476	0.5490	0.5505	0.5519	0.5534	0.5548	0.5563	0.5577	2	5	7	10	12
34	0.5592	0.5606	0.5621	0.5635	0.5650	0.5664	0.5678	0.5693	0.5707	0.5721	2	5	7	10	12
35	0.5736	0.5750	0.5764	0.5779	0.5793	0.5807	0.5821	0.5835	0.5850	0.5864	2	5	7	10	12
36	0.5878	0.5892	0.5906	0.5920	0.5934	0.5948	0.5962	0.5976	0.5990	0.6004	2	5	7	9	12
37	0.6018	0.6032	0.6046	0.6060	0.6074	0.6088	0.6101	0.6115	0.6129	0.6143	2	5	7	9	12
38	0.6157	0.6170	0.6184	0.6198	0.6211	0.6225	0.6239	0.6252	0.6266	0.6280	2	5	7	9	11
39	0.6293	0.6307	0.6320	0.6334	0.6347	0.6361	0.6374	0.6388	0.6401	0.6414	2	4	7	9	11
40	0.6428	0.6441	0.6455	0.6468	0.6481	0.6494	0.6508	0.6521	0.6534	0.6547	2	4	7	9	11
41	0.6561	0.6574	0.6587	0.6600	0.6613	0.6626	0.6639	0.6652	0.6665	0.6678	2	4	7	9	11
42	0.6691	0.6704	0.6717	0.6730	0.6743	0.6756	0.6769	0.6782	0.6794	0.6807	2	4	6	9	11
43	0.6820	0.6833	0.6845	0.6858	0.6871	0.6884	0.6896	0.6909	0.6921	0.6934	2	4	6	8	11
44	0.6947	0.6959	0.6972	0.6984	0.6997	0.7009	0.7022	0.7034	0.7046	0.7059	2	4	6	8	10

NATURAL SINES

Degree	0´	6´	12´	18´	24´	30´	36´	42´	48´	54´	Mean Difference				
	0.0˚	0.1˚	0.2˚	0.3˚	0.4˚	0.5˚	0.6˚	0.7˚	0.8˚	0.9˚	1	2	3	4	5
45	0.7071	0.7083	0.7096	0.7108	0.7120	0.7133	0.7145	0.7157	0.7169	0.7181	2	4	6	8	10
46	0.7193	0.7206	0.7218	0.7230	0.7242	0.7254	0.7266	0.7278	0.7290	0.7302	2	4	6	8	10
47	0.7314	0.7325	0.7337	0.7349	0.7361	0.7373	0.7385	0.7396	0.7408	0.7420	2	4	6	8	10
48	0.7431	0.7443	0.7455	0.7466	0.7478	0.7490	0.7501	0.7513	0.7524	0.7536	2	4	6	8	10
49	0.7547	0.7559	0.7570	0.7581	0.7593	0.7604	0.7615	0.7627	0.7638	0.7649	2	4	6	8	9
50	0.7660	0.7672	0.7683	0.7694	0.7705	0.7716	0.7727	0.7738	0.7749	0.7760	2	4	6	7	9
51	0.7771	0.7782	0.7793	0.7804	0.7815	0.7826	0.7837	0.7848	0.7859	0.7869	2	4	5	7	9
52	0.7880	0.7891	0.7902	0.7912	0.7923	0.7934	0.7944	0.7955	0.7965	0.7976	2	4	5	7	9
53	0.7986	0.7997	0.8007	0.8018	0.8028	0.8039	0.8049	0.8059	0.8070	0.8080	2	3	5	7	9
54	0.8090	0.8100	0.8111	0.8121	0.8131	0.8141	0.8151	0.8161	0.8171	0.8181	2	3	5	7	8
55	0.8192	0.8202	0.8211	0.8221	0.8231	0.8241	0.8251	0.8261	0.8271	0.8281	2	3	5	7	8
56	0.8290	0.8300	0.8310	0.8320	0.8329	0.8339	0.8348	0.8358	0.8368	0.8377	2	3	5	6	8
57	0.8387	0.8396	0.8406	0.8415	0.8425	0.8434	0.8443	0.8453	0.8462	0.8471	2	3	5	6	8
58	0.8480	0.8490	0.8499	0.8508	0.8517	0.8526	0.8536	0.8545	0.8554	0.8563	2	3	5	6	8
59	0.8572	0.8581	0.8590	0.8599	0.8607	0.8616	0.8625	0.8634	0.8643	0.8652	1	3	4	6	7
60	0.8660	0.8669	0.8678	0.8686	0.8695	0.8704	0.8712	0.8721	0.8729	0.8738	1	3	4	6	7
61	0.8746	0.8755	0.8763	0.8771	0.8780	0.8788	0.8796	0.8805	0.8813	0.8821	1	3	4	6	7
62	0.8829	0.8838	0.8846	0.8854	0.8862	0.8870	0.8878	0.8886	0.8894	0.8902	1	3	4	5	7
63	0.8910	0.8918	0.8926	0.8934	0.8942	0.8949	0.8957	0.8965	0.8973	0.8980	1	3	4	5	6
64	0.8988	0.8996	0.9003	0.9011	0.9018	0.9026	0.9033	0.9041	0.9048	0.9056	1	3	4	5	6
65	0.9063	0.9070	0.9078	0.9085	0.9092	0.9100	0.9107	0.9114	0.9121	0.9128	1	2	4	5	6
66	0.9135	0.9143	0.9150	0.9157	0.9164	0.9171	0.9178	0.9184	0.9191	0.9198	1	2	3	5	6
67	0.9205	0.9212	0.9219	0.9225	0.9232	0.9239	0.9245	0.9252	0.9259	0.9265	1	2	3	4	6
68	0.9272	0.9278	0.9285	0.9291	0.9298	0.9304	0.9311	0.9317	0.9323	0.9330	1	2	3	4	5
69	0.9336	0.9342	0.9348	0.9354	0.9361	0.9367	0.9373	0.9379	0.9385	0.9391	1	2	3	4	5
70	0.9397	0.9403	0.9409	0.9415	0.9421	0.9426	0.9432	0.9438	0.9444	0.9449	1	2	3	4	5
71	0.9455	0.9461	0.9466	0.9472	0.9478	0.9483	0.9489	0.9494	0.9500	0.9505	1	2	3	4	5
72	0.9511	0.9516	0.9521	0.9527	0.9532	0.9537	0.9542	0.9548	0.9553	0.9558	1	2	3	3	4
73	0.9563	0.9568	0.9573	0.9578	0.9583	0.9588	0.9593	0.9598	0.9603	0.9608	1	2	2	3	4
74	0.9613	0.9617	0.9622	0.9627	0.9632	0.9636	0.9641	0.9646	0.9650	0.9655	1	2	2	3	4
75	0.9659	0.9664	0.9668	0.9673	0.9677	0.9681	0.9686	0.9690	0.9694	0.9699	1	1	2	3	4
76	0.9703	0.9707	0.9711	0.9715	0.9720	0.9724	0.9728	0.9732	0.9736	0.9740	1	1	2	3	3
77	0.9744	0.9748	0.9751	0.9755	0.9759	0.9763	0.9767	0.9770	0.9774	0.9778	1	1	2	3	3
78	0.9781	0.9785	0.9789	0.9792	0.9796	0.9799	0.9803	0.9806	0.9810	0.9813	1	1	2	2	3
79	0.9816	0.9820	0.9823	0.9826	0.9829	0.9833	0.9836	0.9839	0.9842	0.9845	1	1	2	2	3
80	0.9848	0.9851	0.9854	0.9857	0.9860	0.9863	0.9866	0.9869	0.9871	0.9874	0	1	1	2	2
81	0.9877	0.9880	0.9882	0.9885	0.9888	0.9890	0.9893	0.9895	0.9898	0.9900	0	1	1	2	2
82	0.9903	0.9905	0.9907	0.9910	0.9912	0.9914	0.9917	0.9919	0.9921	0.9923	0	1	1	2	2
83	0.9925	0.9928	0.9930	0.9932	0.9934	0.9936	0.9938	0.9940	0.9942	0.9943	0	1	1	1	2
84	0.9945	0.9947	0.9949	0.9951	0.9952	0.9954	0.9956	0.9957	0.9959	0.9960	0	1	1	1	2
85	0.9962	0.9963	0.9965	0.9966	0.9968	0.9969	0.9971	0.9972	0.9973	0.9974	0	0	1	1	1
86	0.9976	0.9977	0.9978	0.9979	0.9980	0.9981	0.9982	0.9983	0.9984	0.9985	0	0	1	1	1
87	0.9986	0.9987	0.9988	0.9989	0.9990	0.9990	0.9991	0.9992	0.9993	0.9993	0	0	0	1	1
88	0.9994	0.9995	0.9995	0.9996	0.9996	0.9997	0.9997	0.9997	0.9998	0.9998	0	0	0	0	0
89	0.9998	0.9999	0.9999	0.9999	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	0	0	0	0	0

NATURAL COSINES

(Numbers in mean difference columns to be subtracted, not added)

Degree	0´	6´	12´	18´	24´	30´	36´	42´	48´	54´	Mean Difference				
	0.0˚	0.1˚	0.2˚	0.3˚	0.4˚	0.5˚	0.6˚	0.7˚	0.8˚	0.9˚	1	2	3	4	5
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9999	0	0	0	0	0
1	0.9998	0.9998	0.9998	0.9997	0.9997	0.9997	0.9996	0.9996	0.9995	0.9995	0	0	0	0	0
2	0.9994	0.9993	0.9993	0.9992	0.9991	0.9990	0.9990	0.9989	0.9988	0.9987	0	0	0	1	1
3	0.9986	0.9985	0.9984	0.9983	0.9982	0.9981	0.9980	0.9979	0.9978	0.9977	0	0	1	1	1
4	0.9976	0.9974	0.9973	0.9972	0.9971	0.9969	0.9968	0.9966	0.9965	0.9963	0	0	1	1	1
5	0.9962	0.9960	0.9959	0.9957	0.9956	0.9954	0.9952	0.9951	0.9949	0.9947	0	1	1	1	2
6	0.9945	0.9943	0.9942	0.9940	0.9938	0.9936	0.9934	0.9932	0.9930	0.9928	0	1	1	1	2
7	0.9925	0.9923	0.9921	0.9919	0.9917	0.9914	0.9912	0.9910	0.9907	0.9905	0	1	1	2	2
8	0.9903	0.9900	0.9898	0.9895	0.9893	0.9890	0.9888	0.9885	0.9882	0.9880	0	1	1	2	2
9	0.9877	0.9874	0.9871	0.9869	0.9866	0.9863	0.9860	0.9857	0.9854	0.9851	0	1	1	2	2
10	0.9848	0.9845	0.9842	0.9839	0.9836	0.9833	0.9829	0.9826	0.9823	0.9820	1	1	2	2	3
11	0.9816	0.9813	0.9810	0.9806	0.9803	0.9799	0.9796	0.9792	0.9789	0.9785	1	1	2	2	3
12	0.9781	0.9778	0.9774	0.9770	0.9767	0.9763	0.9759	0.9755	0.9751	0.9748	1	1	2	3	3
13	0.9744	0.9740	0.9736	0.9732	0.9728	0.9724	0.9720	0.9715	0.9711	0.9707	1	1	2	3	3
14	0.9703	0.9699	0.9694	0.9690	0.9686	0.9681	0.9677	0.9673	0.9668	0.9664	1	1	2	3	4
15	0.9659	0.9655	0.9650	0.9646	0.9641	0.9636	0.9632	0.9627	0.9622	0.9617	1	2	2	3	4
16	0.9613	0.9608	0.9603	0.9598	0.9593	0.9588	0.9583	0.9578	0.9573	0.9568	1	2	2	3	4
17	0.9563	0.9558	0.9553	0.9548	0.9542	0.9537	0.9532	0.9527	0.9521	0.9516	1	2	3	3	4
18	0.9511	0.9505	0.9500	0.9494	0.9489	0.9483	0.9478	0.9472	0.9466	0.9461	1	2	3	4	5
19	0.9455	0.9449	0.9444	0.9438	0.9432	0.9426	0.9421	0.9415	0.9409	0.9403	1	2	3	4	5
20	0.9397	0.9391	0.9385	0.9379	0.9373	0.9367	0.9361	0.9354	0.9348	0.9342	1	2	3	4	5
21	0.9336	0.9330	0.9323	0.9317	0.9311	0.9304	0.9298	0.9291	0.9285	0.9278	1	2	3	4	5
22	0.9272	0.9265	0.9259	0.9252	0.9245	0.9239	0.9232	0.9225	0.9219	0.9212	1	2	3	4	6
23	0.9205	0.9198	0.9191	0.9184	0.9178	0.9171	0.9164	0.9157	0.9150	0.9143	1	2	3	5	6
24	0.9135	0.9128	0.9121	0.9114	0.9107	0.9100	0.9092	0.9085	0.9078	0.9070	1	2	4	5	6
25	0.9063	0.9056	0.9048	0.9041	0.9033	0.9026	0.9018	0.9011	0.9003	0.8996	1	3	4	5	6
26	0.8988	0.8980	0.8973	0.8965	0.8957	0.8949	0.8942	0.8934	0.8926	0.8918	1	3	4	5	6
27	0.8910	0.8902	0.8894	0.8886	0.8878	0.8870	0.8862	0.8854	0.8846	0.8838	1	3	4	5	7
28	0.8829	0.8821	0.8813	0.8805	0.8796	0.8788	0.8780	0.8771	0.8763	0.8755	1	3	4	6	7
29	0.8746	0.8738	0.8729	0.8721	0.8712	0.8704	0.8695	0.8686	0.8678	0.8669	1	3	4	6	7
30	0.8660	0.8652	0.8643	0.8634	0.8625	0.8616	0.8607	0.8599	0.8590	0.8581	1	3	4	6	7
31	0.8572	0.8563	0.8554	0.8545	0.8536	0.8526	0.8517	0.8508	0.8499	0.8490	2	3	5	6	8
32	0.8480	0.8471	0.8462	0.8453	0.8443	0.8434	0.8425	0.8415	0.8406	0.8396	2	3	5	6	8
33	0.8387	0.8377	0.8368	0.8358	0.8348	0.8339	0.8329	0.8320	0.8310	0.8300	2	3	5	6	8
34	0.8290	0.8281	0.8271	0.8261	0.8251	0.8241	0.8231	0.8221	0.8211	0.8202	2	3	5	7	8
35	0.8192	0.8181	0.8171	0.8161	0.8151	0.8141	0.8131	0.8121	0.8111	0.8100	2	3	5	7	8
36	0.8090	0.8080	0.8070	0.8059	0.8049	0.8039	0.8028	0.8018	0.8007	0.7997	2	3	5	7	9
37	0.7986	0.7976	0.7965	0.7955	0.7944	0.7934	0.7923	0.7912	0.7902	0.7891	2	4	5	7	9
38	0.7880	0.7869	0.7859	0.7848	0.7837	0.7826	0.7815	0.7804	0.7793	0.7782	2	4	5	7	9
39	0.7771	0.7760	0.7749	0.7738	0.7727	0.7716	0.7705	0.7694	0.7683	0.7672	2	4	6	7	9
40	0.7660	0.7649	0.7638	0.7627	0.7615	0.7604	0.7593	0.7581	0.7570	0.7559	2	4	6	8	9
41	0.7547	0.7536	0.7524	0.7513	0.7501	0.7490	0.7478	0.7466	0.7455	0.7443	2	4	6	8	10
42	0.7431	0.7420	0.7408	0.7396	0.7385	0.7373	0.7361	0.7349	0.7337	0.7325	2	4	6	8	10
43	0.7314	0.7302	0.7290	0.7278	0.7266	0.7254	0.7242	0.7230	0.7218	0.7206	2	4	6	8	10
44	0.7193	0.7181	0.7169	0.7157	0.7145	0.7133	0.7120	0.7108	0.7096	0.7083	2	4	6	8	10

NATURAL COSINES

(Numbers in mean difference columns to be subtracted, not added)

Degree	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Difference				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
45	0.7071	0.7059	0.7046	0.7034	0.7022	0.7009	0.6997	0.6984	0.6972	0.6959	2	4	6	8	10
46	0.6947	0.6934	0.6921	0.6909	0.6896	0.6884	0.6871	0.6858	0.6845	0.6833	2	4	6	8	11
47	0.6820	0.6807	0.6794	0.6782	0.6769	0.6756	0.6743	0.6730	0.6717	0.6704	2	4	6	9	11
48	0.6691	0.6678	0.6665	0.6652	0.6639	0.6626	0.6613	0.6600	0.6587	0.6574	2	4	7	9	11
49	0.6561	0.6547	0.6534	0.6521	0.6508	0.6494	0.6481	0.6468	0.6455	0.6441	2	4	7	9	11
50	0.6428	0.6414	0.6401	0.6388	0.6374	0.6361	0.6347	0.6334	0.6320	0.6307	2	4	7	9	11
51	0.6293	0.6280	0.6266	0.6252	0.6239	0.6225	0.6211	0.6198	0.6184	0.6170	2	5	7	9	11
52	0.6157	0.6143	0.6129	0.6115	0.6101	0.6088	0.6074	0.6060	0.6046	0.6032	2	5	7	9	12
53	0.6018	0.6004	0.5990	0.5976	0.5962	0.5948	0.5934	0.5920	0.5906	0.5892	2	5	7	9	12
54	0.5878	0.5864	0.5850	0.5835	0.5821	0.5807	0.5793	0.5779	0.5764	0.5750	2	5	7	9	12
55	0.5736	0.5721	0.5707	0.5693	0.5678	0.5664	0.5650	0.5635	0.5621	0.5606	2	5	7	10	12
56	0.5592	0.5577	0.5563	0.5548	0.5534	0.5519	0.5505	0.5490	0.5476	0.5461	2	5	7	10	12
57	0.5446	0.5432	0.5417	0.5402	0.5388	0.5373	0.5358	0.5344	0.5329	0.5314	2	5	7	10	12
58	0.5299	0.5284	0.5270	0.5255	0.5240	0.5225	0.5210	0.5195	0.5180	0.5165	2	5	7	10	12
59	0.5150	0.5135	0.5120	0.5105	0.5090	0.5075	0.5060	0.5045	0.5030	0.5015	3	5	8	10	13
60	0.5000	0.4985	0.4970	0.4955	0.4939	0.4924	0.4909	0.4894	0.4879	0.4863	3	5	8	10	13
61	0.4848	0.4833	0.4818	0.4802	0.4787	0.4772	0.4756	0.4741	0.4726	0.4710	3	5	8	10	13
62	0.4695	0.4679	0.4664	0.4648	0.4633	0.4617	0.4602	0.4586	0.4571	0.4555	3	5	8	10	13
63	0.4540	0.4524	0.4509	0.4493	0.4478	0.4462	0.4446	0.4431	0.4415	0.4399	3	5	8	10	13
64	0.4384	0.4368	0.4352	0.4337	0.4321	0.4305	0.4289	0.4274	0.4258	0.4242	3	5	8	11	13
65	0.4226	0.4210	0.4195	0.4179	0.4163	0.4147	0.4131	0.4115	0.4099	0.4083	3	5	8	11	13
66	0.4067	0.4051	0.4035	0.4019	0.4003	0.3987	0.3971	0.3955	0.3939	0.3923	3	5	8	11	14
67	0.3907	0.3891	0.3875	0.3859	0.3843	0.3827	0.3811	0.3795	0.3778	0.3762	3	5	8	11	14
68	0.3746	0.3730	0.3714	0.3697	0.3681	0.3665	0.3649	0.3633	0.3616	0.3600	3	5	8	11	14
69	0.3584	0.3567	0.3551	0.3535	0.3518	0.3502	0.3486	0.3469	0.3453	0.3437	3	5	8	11	14
70	0.3420	0.3404	0.3387	0.3371	0.3355	0.3338	0.3322	0.3305	0.3289	0.3272	3	5	8	11	14
71	0.3256	0.3239	0.3223	0.3206	0.3190	0.3173	0.3156	0.3140	0.3123	0.3107	3	6	8	11	14
72	0.3090	0.3074	0.3057	0.3040	0.3024	0.3007	0.2990	0.2974	0.2957	0.2940	3	6	8	11	14
73	0.2924	0.2907	0.2890	0.2874	0.2857	0.2840	0.2823	0.2807	0.2790	0.2773	3	6	8	11	14
74	0.2756	0.2740	0.2723	0.2706	0.2689	0.2672	0.2656	0.2639	0.2622	0.2605	3	6	8	11	14
75	0.2588	0.2571	0.2554	0.2538	0.2521	0.2504	0.2487	0.2470	0.2453	0.2436	3	6	8	11	14
76	0.2419	0.2402	0.2385	0.2368	0.2351	0.2334	0.2317	0.2300	0.2284	0.2267	3	6	8	11	14
77	0.2250	0.2233	0.2215	0.2198	0.2181	0.2164	0.2147	0.2130	0.2113	0.2096	3	6	9	11	14
78	0.2079	0.2062	0.2045	0.2028	0.2011	0.1994	0.1977	0.1959	0.1942	0.1925	3	6	9	11	14
79	0.1908	0.1891	0.1874	0.1857	0.1840	0.1822	0.1805	0.1788	0.1771	0.1754	3	6	9	11	14
80	0.1736	0.1719	0.1702	0.1685	0.1668	0.1650	0.1633	0.1616	0.1599	0.1582	3	6	9	12	14
81	0.1564	0.1547	0.1530	0.1513	0.1495	0.1478	0.1461	0.1444	0.1426	0.1409	3	6	9	12	14
82	0.1392	0.1374	0.1357	0.1340	0.1323	0.1305	0.1288	0.1271	0.1253	0.1236	3	6	9	12	14
83	0.1219	0.1201	0.1184	0.1167	0.1149	0.1132	0.1115	0.1097	0.1080	0.1063	3	6	9	12	14
84	0.1045	0.1028	0.1011	0.0993	0.0976	0.0958	0.0941	0.0924	0.0906	0.0889	3	6	9	12	14
85	0.0872	0.0854	0.0837	0.0819	0.0802	0.0785	0.0767	0.0750	0.0732	0.0715	3	6	9	12	15
86	0.0698	0.0680	0.0663	0.0645	0.0628	0.0610	0.0593	0.0576	0.0558	0.0541	3	6	9	12	15
87	0.0523	0.0506	0.0488	0.0471	0.0454	0.0436	0.0419	0.0401	0.0384	0.0366	3	6	9	12	15
88	0.0349	0.0332	0.0314	0.0297	0.0279	0.0262	0.0244	0.0227	0.0209	0.0192	3	6	9	12	15
89	0.0175	0.0157	0.0140	0.0122	0.0105	0.0087	0.0070	0.0052	0.0035	0.0017	3	6	9	12	15

NATURAL TANGENTS

Degree	0´	6´	12´	18´	24´	30´	36´	42´	48´	54´	Mean Difference				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
0	0.0000	0.0017	0.0035	0.0052	0.0070	0.0087	0.0105	0.0122	0.0140	0.0157	3	6	9	12	15
1	0.0175	0.0192	0.0209	0.0227	0.0244	0.0262	0.0279	0.0297	0.0314	0.0332	3	6	9	12	15
2	0.0349	0.0367	0.0384	0.0402	0.0419	0.0437	0.0454	0.0472	0.0489	0.0507	3	6	9	12	15
3	0.0524	0.0542	0.0559	0.0577	0.0594	0.0612	0.0629	0.0647	0.0664	0.0682	3	6	9	12	15
4	0.0699	0.0717	0.0734	0.0752	0.0769	0.0787	0.0805	0.0822	0.0840	0.0857	3	6	9	12	15
5	0.0875	0.0892	0.0910	0.0928	0.0945	0.0963	0.0981	0.0998	0.1016	0.1033	3	6	9	12	15
6	0.1051	0.1069	0.1086	0.1104	0.1122	0.1139	0.1157	0.1175	0.1192	0.1210	3	6	9	12	15
7	0.1228	0.1246	0.1263	0.1281	0.1299	0.1317	0.1334	0.1352	0.1370	0.1388	3	6	9	12	15
8	0.1405	0.1423	0.1441	0.1459	0.1477	0.1495	0.1512	0.1530	0.1548	0.1566	3	6	9	12	15
9	0.1584	0.1602	0.1620	0.1638	0.1655	0.1673	0.1691	0.1709	0.1727	0.1745	3	6	9	12	15
10	0.1763	0.1781	0.1799	0.1817	0.1835	0.1853	0.1871	0.1890	0.1908	0.1926	3	6	9	12	15
11	0.1944	0.1962	0.1980	0.1998	0.2016	0.2035	0.2053	0.2071	0.2089	0.2107	3	6	9	12	15
12	0.2126	0.2144	0.2162	0.2180	0.2199	0.2217	0.2235	0.2254	0.2272	0.2290	3	6	9	12	15
13	0.2309	0.2327	0.2345	0.2364	0.2382	0.2401	0.2419	0.2438	0.2456	0.2475	3	6	9	12	15
14	0.2493	0.2512	0.2530	0.2549	0.2568	0.2586	0.2605	0.2623	0.2642	0.2661	3	6	9	12	16
15	0.2679	0.2698	0.2717	0.2736	0.2754	0.2773	0.2792	0.2811	0.2830	0.2849	3	6	9	13	16
16	0.2867	0.2886	0.2905	0.2924	0.2943	0.2962	0.2981	0.3000	0.3019	0.3038	3	6	9	13	16
17	0.3057	0.3076	0.3096	0.3115	0.3134	0.3153	0.3172	0.3191	0.3211	0.3230	3	6	10	13	16
18	0.3249	0.3269	0.3288	0.3307	0.3327	0.3346	0.3365	0.3385	0.3404	0.3424	3	6	10	13	16
19	0.3443	0.3463	0.3482	0.3502	0.3522	0.3541	0.3561	0.3581	0.3600	0.3620	3	7	10	13	16
20	0.3640	0.3659	0.3679	0.3699	0.3719	0.3739	0.3759	0.3779	0.3799	0.3819	3	7	10	13	17
21	0.3839	0.3859	0.3879	0.3899	0.3919	0.3939	0.3959	0.3979	0.4000	0.4020	3	7	10	13	17
22	0.4040	0.4061	0.4081	0.4101	0.4122	0.4142	0.4163	0.4183	0.4204	0.4224	3	7	10	14	17
23	0.4245	0.4265	0.4286	0.4307	0.4327	0.4348	0.4369	0.4390	0.4411	0.4431	3	7	10	14	17
24	0.4452	0.4473	0.4494	0.4515	0.4536	0.4557	0.4578	0.4599	0.4621	0.4642	4	7	11	14	18
25	0.4663	0.4684	0.4706	0.4727	0.4748	0.4770	0.4791	0.4813	0.4834	0.4856	4	7	11	14	18
26	0.4877	0.4899	0.4921	0.4942	0.4964	0.4986	0.5008	0.5029	0.5051	0.5073	4	7	11	15	18
27	0.5095	0.5117	0.5139	0.5161	0.5184	0.5206	0.5228	0.5250	0.5272	0.5295	4	7	11	15	18
28	0.5317	0.5340	0.5362	0.5384	0.5407	0.5430	0.5452	0.5475	0.5498	0.5520	4	8	11	15	19
29	0.5543	0.5566	0.5589	0.5612	0.5635	0.5658	0.5681	0.5704	0.5727	0.5750	4	8	12	15	19
30	0.5774	0.5797	0.5820	0.5844	0.5867	0.5890	0.5914	0.5938	0.5961	0.5985	4	8	12	16	20
31	0.6009	0.6032	0.6056	0.6080	0.6104	0.6128	0.6152	0.6176	0.6200	0.6224	4	8	12	16	20
32	0.6249	0.6273	0.6297	0.6322	0.6346	0.6371	0.6395	0.6420	0.6445	0.6469	4	8	12	16	20
33	0.6494	0.6519	0.6544	0.6569	0.6594	0.6619	0.6644	0.6669	0.6694	0.6720	4	8	13	17	21
34	0.6745	0.6771	0.6796	0.6822	0.6847	0.6873	0.6899	0.6924	0.6950	0.6976	4	9	13	17	21
35	0.7002	0.7028	0.7054	0.7080	0.7107	0.7133	0.7159	0.7186	0.7212	0.7239	4	9	13	18	22
36	0.7265	0.7292	0.7319	0.7346	0.7373	0.7400	0.7427	0.7454	0.7481	0.7508	5	9	14	18	23
37	0.7536	0.7563	0.7590	0.7618	0.7646	0.7673	0.7701	0.7729	0.7757	0.7785	5	9	14	18	23
38	0.7813	0.7841	0.7869	0.7898	0.7926	0.7954	0.7983	0.8012	0.8040	0.8069	5	9	14	19	24
39	0.8098	0.8127	0.8156	0.8185	0.8214	0.8243	0.8273	0.8302	0.8332	0.8361	5	10	15	20	24
40	0.8391	0.8421	0.8451	0.8481	0.8511	0.8541	0.8571	0.8601	0.8632	0.8662	5	10	15	20	25
41	0.8693	0.8724	0.8754	0.8785	0.8816	0.8847	0.8878	0.8910	0.8941	0.8972	5	10	16	21	26
42	0.9004	0.9036	0.9067	0.9099	0.9131	0.9163	0.9195	0.9228	0.9260	0.9293	5	11	16	21	27
43	0.9325	0.9358	0.9391	0.9424	0.9457	0.9490	0.9523	0.9556	0.9590	0.9623	6	11	17	22	28
44	0.9657	0.9691	0.9725	0.9759	0.9793	0.9827	0.9861	0.9896	0.9930	0.9965	6	11	17	23	29

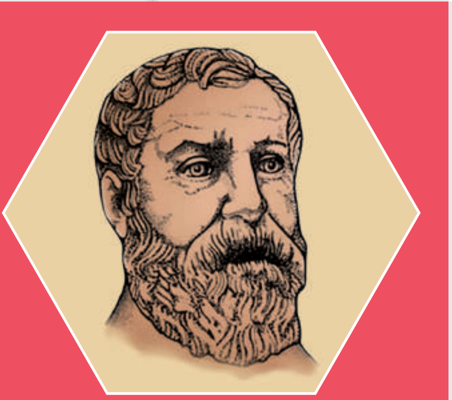
NATURAL TANGENTS

Degree	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Difference				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
45	1.0000	1.0035	1.0070	1.0105	1.0141	1.0176	1.0212	1.0247	1.0283	1.0319	6	12	18	24	30
46	1.0355	1.0392	1.0428	1.0464	1.0501	1.0538	1.0575	1.0612	1.0649	1.0686	6	12	18	25	31
47	1.0724	1.0761	1.0799	1.0837	1.0875	1.0913	1.0951	1.0990	1.1028	1.1067	6	13	19	25	32
48	1.1106	1.1145	1.1184	1.1224	1.1263	1.1303	1.1343	1.1383	1.1423	1.1463	7	13	20	27	33
49	1.1504	1.1544	1.1585	1.1626	1.1667	1.1708	1.1750	1.1792	1.1833	1.1875	7	14	21	28	34
50	1.1918	1.1960	1.2002	1.2045	1.2088	1.2131	1.2174	1.2218	1.2261	1.2305	7	14	22	29	36
51	1.2349	1.2393	1.2437	1.2482	1.2527	1.2572	1.2617	1.2662	1.2708	1.2753	8	15	23	30	38
52	1.2799	1.2846	1.2892	1.2938	1.2985	1.3032	1.3079	1.3127	1.3175	1.3222	8	16	24	31	39
53	1.3270	1.3319	1.3367	1.3416	1.3465	1.3514	1.3564	1.3613	1.3663	1.3713	8	16	25	33	41
54	1.3764	1.3814	1.3865	1.3916	1.3968	1.4019	1.4071	1.4124	1.4176	1.4229	9	17	26	34	43
55	1.4281	1.4335	1.4388	1.4442	1.4496	1.4550	1.4605	1.4659	1.4715	1.4770	9	18	27	36	45
56	1.4826	1.4882	1.4938	1.4994	1.5051	1.5108	1.5166	1.5224	1.5282	1.5340	10	19	29	38	48
57	1.5399	1.5458	1.5517	1.5577	1.5637	1.5697	1.5757	1.5818	1.5880	1.5941	10	20	30	40	50
58	1.6003	1.6066	1.6128	1.6191	1.6255	1.6319	1.6383	1.6447	1.6512	1.6577	11	21	32	43	53
59	1.6643	1.6709	1.6775	1.6842	1.6909	1.6977	1.7045	1.7113	1.7182	1.7251	11	23	34	45	56
60	1.7321	1.7391	1.7461	1.7532	1.7603	1.7675	1.7747	1.7820	1.7893	1.7966	12	24	36	48	60
61	1.8040	1.8115	1.8190	1.8265	1.8341	1.8418	1.8495	1.8572	1.8650	1.8728	13	26	38	51	64
62	1.8807	1.8887	1.8967	1.9047	1.9128	1.9210	1.9292	1.9375	1.9458	1.9542	14	27	41	55	68
63	1.9626	1.9711	1.9797	1.9883	1.9970	2.0057	2.0145	2.0233	2.0323	2.0413	15	29	44	58	73
64	2.0503	2.0594	2.0686	2.0778	2.0872	2.0965	2.1060	2.1155	2.1251	2.1348	16	31	47	63	78
65	2.1445	2.1543	2.1642	2.1742	2.1842	2.1943	2.2045	2.2148	2.2251	2.2355	17	34	51	68	85
66	2.2460	2.2566	2.2673	2.2781	2.2889	2.2998	2.3109	2.3220	2.3332	2.3445	18	37	55	73	92
67	2.3559	2.3673	2.3789	2.3906	2.4023	2.4142	2.4262	2.4383	2.4504	2.4627	20	40	60	79	99
68	2.4751	2.4876	2.5002	2.5129	2.5257	2.5386	2.5517	2.5649	2.5782	2.5916	22	43	65	87	108
69	2.6051	2.6187	2.6325	2.6464	2.6605	2.6746	2.6889	2.7034	2.7179	2.7326	24	47	71	95	119
70	2.7475	2.7625	2.7776	2.7929	2.8083	2.8239	2.8397	2.8556	2.8716	2.8878	26	52	78	104	131
71	2.9042	2.9208	2.9375	2.9544	2.9714	2.9887	3.0061	3.0237	3.0415	3.0595	29	58	87	116	145
72	3.0777	3.0961	3.1146	3.1334	3.1524	3.1716	3.1910	3.2106	3.2305	3.2506	32	64	96	129	161
73	3.2709	3.2914	3.3122	3.3332	3.3544	3.3759	3.3977	3.4197	3.4420	3.4646	36	72	108	144	180
74	3.4874	3.5105	3.5339	3.5576	3.5816	3.6059	3.6305	3.6554	3.6806	3.7062	41	81	122	163	204
75	3.7321	3.7583	3.7848	3.8118	3.8391	3.8667	3.8947	3.9232	3.9520	3.9812	46	93	139	186	232
76	4.0108	4.0408	4.0713	4.1022	4.1335	4.1653	4.1976	4.2303	4.2635	4.2972	53	107	160	213	267
77	4.3315	4.3662	4.4015	4.4373	4.4737	4.5107	4.5483	4.5864	4.6252	4.6646					
78	4.7046	4.7453	4.7867	4.8288	4.8716	4.9152	4.9594	5.0045	5.0504	5.0970					
79	5.1446	5.1929	5.2422	5.2924	5.3435	5.3955	5.4486	5.5026	5.5578	5.6140					
80	5.6713	5.7297	5.7894	5.8502	5.9124	5.9758	6.0405	6.1066	6.1742	6.2432					
81	6.3138	6.3859	6.4596	6.5350	6.6122	6.6912	6.7720	6.8548	6.9395	7.0264					
82	7.1154	7.2066	7.3002	7.3962	7.4947	7.5958	7.6996	7.8062	7.9158	8.0285					
83	8.1443	8.2636	8.3863	8.5126	8.6427	8.7769	8.9152	9.0579	9.2052	9.3572					
84	9.5144	9.6768	9.8448	10.0187	10.1988	10.3854	10.5789	10.7797	10.9882	11.2048					
85	11.4301	11.6645	11.9087	12.1632	12.4288	12.7062	12.9962	13.2996	13.6174	13.9507					
86	14.3007	14.6685	15.0557	15.4638	15.8945	16.3499	16.8319	17.3432	17.8863	18.4645					
87	19.0811	19.7403	20.4465	21.2049	22.0217	22.9038	23.8593	24.8978	26.0307	27.2715					
88	28.6363	30.1446	31.8205	33.6935	35.8006	38.1885	40.9174	44.0661	47.7395	52.0807					
89	57.2900	63.6567	71.6151	81.8470	95.4895	114.5887	143.2371	190.9842	286.4777	572.9572					

7

MENSURATION

The most beautiful plane figure is the circle and the most beautiful solid figure is the sphere. - *Pythagoras.*



Heron
A.D (C.E) 10-75

Heron of Alexandria was a Greek mathematician. He wrote books on mathematics, mechanics and physics. His famous book 'Metrica' consists of three volumes. This book shows the way to calculate area and volume of plane and solid figures. Heron has derived the formula for the area of triangle when three sides are given.

Learning Outcomes

- To use Heron's formula for calculating area of triangles and quadrilaterals.
- To find Total Surface Area (TSA), Lateral Surface Area (LSA) and Volume of cuboids and cubes.

FZZ27R

7.1 Introduction

Mensuration is the branch of mathematics which deals with the study of areas and volumes of different kinds of geometrical shapes. In the broadest sense, it is all about the process of measurement.

Mensuration is used in the field of architecture, medicine, construction, etc. It is necessary for everyone to learn formulae used to find the perimeter and area of two dimensional figures as well as the surface area and volume of three dimensional solids in day to day life. In this chapter we deal with finding the area of triangles (using Heron's formula), surface area and volume of cuboids and cubes.

For a closed plane figure (a quadrilateral or a triangle), what do we call the distance around its boundary? What is the measure of the region covered inside the boundary?

In general, the area of a triangle is calculated by the formula

$$Area = \frac{1}{2} \times Base \times Height \quad \text{sq. units}$$

That is, $A = \frac{1}{2} \times b \times h$ sq. units

where, b is base and h is height of the triangle.

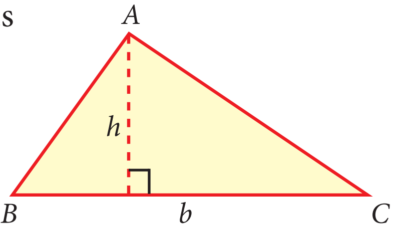


Fig. 7.1

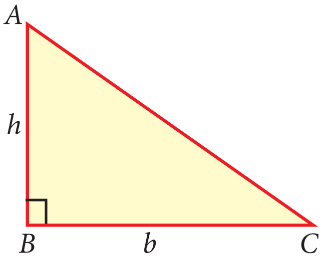


Fig. 7.2

From the above, we know how to find the area of a triangle when its ‘base’ and ‘height’ (that is altitude) are given.

7.2 Heron’s Formula

How will you find the area of a triangle, if the height is not known but the lengths of the three sides are known?

For this, Heron has given a formula to find the area of a triangle.

If a, b and c are the sides of a triangle, then the area of a triangle $= \sqrt{s(s-a)(s-b)(s-c)}$ sq.units. where $s = \frac{a+b+c}{2}$, ‘s’ is the semi-perimeter (that is half of the perimeter) of the triangle.

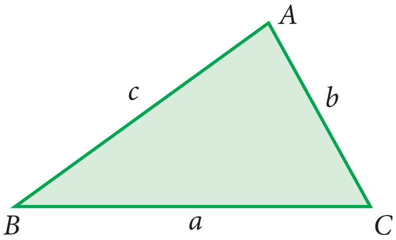


Fig. 7.3

Note

If we assume that the sides are of equal length that is $a = b = c$, then Heron’s formula will be $\frac{\sqrt{3}}{4}a^2$ sq.units, which is the area of an equilateral triangle.

Example 7.1

The lengths of sides of a triangular field are 28 m, 15 m and 41 m. Calculate the area of the field. Find the cost of levelling the field at the rate of ₹ 20 per m^2.

Solution

Let $a = 28\,m$, $b = 15\,m$ and $c = 41\,m$

Then, $s = \frac{a+b+c}{2} = \frac{28+15+41}{2} = \frac{84}{2} = 42\ m$

Area of triangular field $= \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{42(42-28)(42-15)(42-41)}$$

$$= \sqrt{42 \times 14 \times 27 \times 1}$$

$$= \sqrt{2 \times 3 \times 7 \times 7 \times 2 \times 3 \times 3 \times 3 \times 1}$$

$$= 2 \times 3 \times 7 \times 3$$

$$= 126 \ m^2$$

Given the cost of levelling is ₹ 20 per m^2.

The total cost of levelling the field $= 20 \times 126 \ =$ ₹ 2520.

Example 7.2

Three different triangular plots are available for sale in a locality. Each plot has a perimeter of 120 m. The side lengths are also given:

Shape of plot	Perimeter	Length of sides
Right angled triangle	120 m	30 m, 40 m, 50 m
Acute angled triangle	120 m	35 m, 40 m, 45 m
Equilateral triangle	120 m	40 m, 40 m, 40 m

Help the buyer to decide which among these will be more spacious.

Solution

For clarity, let us draw a rough figure indicating the measurements:

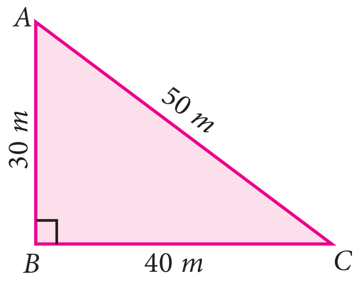


Fig. 7.4

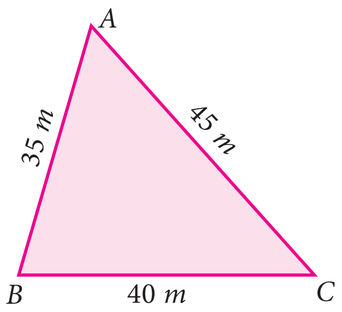


Fig. 7.5

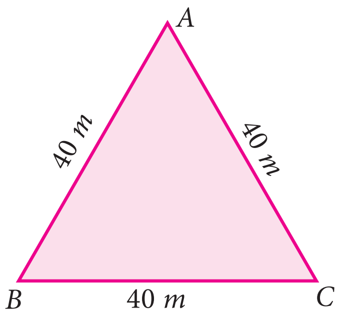


Fig. 7.6

(i) The semi-perimeter of Fig.7.4, $s = \dfrac{30+40+50}{2} = 60\, m$

Fig.7.5, $s = \dfrac{35+40+45}{2} = 60 \ m$

Fig.7.6, $s = \dfrac{40+40+40}{2} = 60\, m$

Note that all the semi-perimeters are equal.

(ii) Area of triangle using Heron's formula:

In Fig.7.4, Area of triangle $= \sqrt{60(60-30)(60-40)(60-50)}$

$$= \sqrt{60 \times 30 \times 20 \times 10}$$

$$= \sqrt{30\times 2\times 30\times 2\times 10\times 10}$$

$$= 600\ m^2$$

In Fig.7.5, Area of triangle $= \sqrt{60(60-35)(60-40)(60-45)}$

$$= \sqrt{60\times 25\times 20\times 15}$$

$$= \sqrt{20\times 3\times 5\times 5\times 20\times 3\times 5}$$

$$= 300\sqrt{5} \qquad (\text{ Since } \sqrt{5} = 2.236\ )$$

$$= 670.8\ \ m^2$$

In Fig.7.6, Area of triangle $= \sqrt{60(60-40)(60-40)(60-40)}$

$$= \sqrt{60\times 20\times 20\times 20}$$

$$= \sqrt{3\times 20\times 20\times 20\times 20}$$

$$= 400\sqrt{3} \qquad (\text{ Since } \sqrt{3} = 1.732\ )$$

$$= 692.8\ m^2$$

We find that though the perimeters are same, the areas of the three triangular plots are different. The area of the triangle in Fig 7.6 is the greatest among these; the buyer can be suggested to choose this since it is more spacious.

Note

If the perimeter of different types of triangles have the same value, among all the types of triangles, the equilateral triangle possess the greatest area. We will learn more about maximum areas in higher classes.

7.3 Application of Heron's Formula in Finding Areas of Quadrilaterals

A plane figure bounded by four line segments is called a quadrilateral.

Let $ABCD$ be a quadrilateral. To find the area of a quadrilateral, we divide the quadrilateral into two triangular parts and use Heron's formula to calculate the area of the triangular parts.

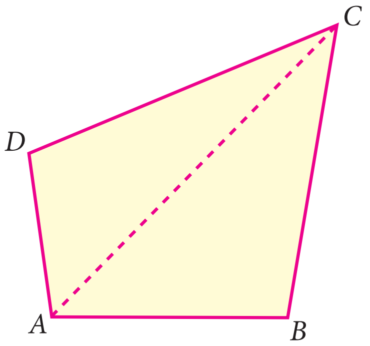


Fig. 7.7

In Fig 7.7,

Area of quadrilateral $ABCD$ = Area of triangle ABC + Area of triangle ACD

Example 7.3

A farmer has a field in the shape of a rhombus. The perimeter of the field is 400m and one of its diagonal is 120m. He wants to divide the field into two equal parts to grow two different types of vegetables. Find the area of the field.

Solution

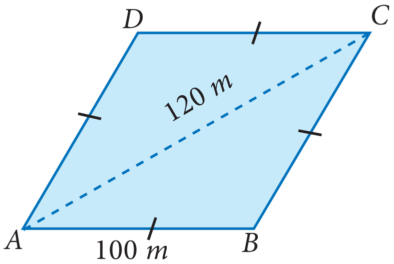


Fig. 7.8

Let $ABCD$ be the rhombus.

Its perimeter = 4 × side = 400 m

Therefore, each side of the rhombus = 100 m

Given the length of the diagonal AC = 120 m

In ΔABC, let $a = 100\ m$, $b = 100\ m$, $c = 120\ m$

$$s = \frac{a+b+c}{2} = \frac{100+100+120}{2} = 160\ m$$

$$\text{Area of } \Delta ABC = \sqrt{160(160-100)(160-100)(160-120)}$$

$$= \sqrt{160\times60\times60\times40}$$

$$= \sqrt{40\times2\times2\times60\times60\times40}$$

$$= 40\times2\times60 = 4800\,m^2$$

Therefore, Area of the field $ABCD = 2\times$ Area of ΔABC $= 2\times4800 = 9600\ m^2$

Exercise 7.1

1. Using Heron's formula, find the area of a triangle whose sides are

   (i) 10 cm, 24 cm, 26 cm (ii) 1.8m, 8 m, 8.2 m

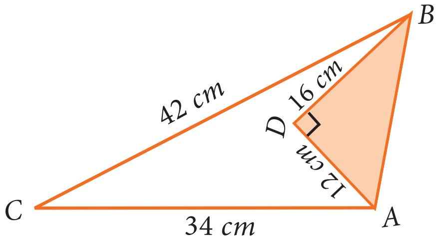
2. The sides of the triangular ground are 22 m, 120 m and 122 m. Find the area and cost of levelling the ground at the rate of ₹ 20 per m^2.

3. The perimeter of a triangular plot is 600 m. If the sides are in the ratio 5:12:13, then find the area of the plot.

4. Find the area of an equilateral triangle whose perimeter is 180 cm.

5. An advertisement board is in the form of an isosceles triangle with perimeter 36m and each of the equal sides are 13 m. Find the cost of painting it at ₹ 17.50 per square metre.

6. Find the area of the unshaded region.



7. Find the area of a quadrilateral $ABCD$ whose sides are $AB = 13\,cm$, $BC = 12\,cm$, $CD = 9\,cm$, $AD = 14\,cm$ and diagonal $BD = 15\,cm$.

8. A park is in the shape of a quadrilateral. The sides of the park are 15 m, 20 m, 26 m and 17 m and the angle between the first two sides is a right angle. Find the area of the park.

9. A land is in the shape of rhombus. The perimeter of the land is 160 m and one of the diagonal is 48 m. Find the area of the land.

10. The adjacent sides of a parallelogram measures 34 m, 20 m and the measure of one of the diagonal is 42 m. Find the area of parallelogram.

7.4 Surface Area of Cuboid and Cube

We have learnt in the earlier classes about 3-Dimension structures. The 3D shapes are those which do not lie completely in a plane. Any 3D shape has dimensions namely length, breadth and height.

G18X9E

7.4.1 Cuboid and its Surface Area

Cuboid: A cuboid is a closed solid figure bounded by six rectangular plane regions. For example, match box, Brick, Book.

A cuboid has 6 faces, 12 edges and 8 vertices. Ultimately, a cuboid has the shape of a rectangular box.

Total Surface Area (TSA) of a cuboid is the sum of the areas of all the faces that enclose the cuboid. If we leave out the areas of the top and bottom of the cuboid we get what is known as its Lateral Surface Area (LSA).

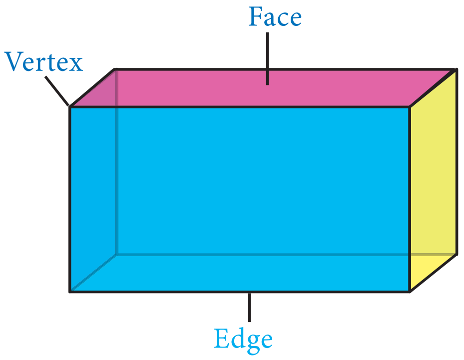


Fig. 7.9

In the Fig 7.10, l, b and h represents length, breadth and height respectively.

(i) Total Surface Area (TSA) of a cuboid

Top and bottom	$2 \times lb$
Front and back	$2 \times bh$
Left and Right sides	$2 \times lh$

$= 2\,(lb + bh + lh)$ sq. units.

(ii) Lateral Surface Area (LSA) of a cuboid

Front and back	$2 \times bh$
Left and Right sides	$2 \times lh$

$= 2\,(l+b)h$ sq. units.

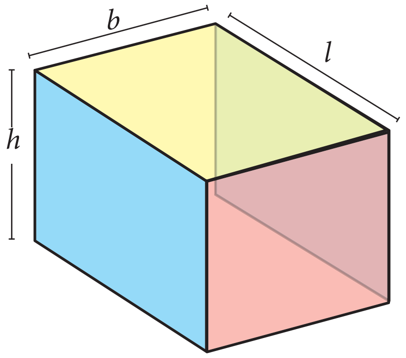


Fig. 7.10

We are using the concept of Lateral Surface Area (LSA) and Total Surface Area (TSA) in real life situations. For instance a room can be cuboidal in shape that has different length, breadth and height. If we require to find areas of only the walls of a room, avoiding floor and ceiling then we can use LSA. However if we want to find the surface area of the whole room then we have to calculate the TSA.

If the length, breadth and height of a cuboid are l, b and h respectively. Then

(i) Total Surface Area $= 2\,(lb + bh + lh)$ sq.units.

(ii) Lateral Surface Area $= 2\,(l+b)h$ sq.units.

Note

- The top and bottom area in a cuboid is independent of height. The total area of top and bottom is $2lb$. Hence LSA is obtained by removing $2lb$ from $2(lb+bh+lh)$.
- The units of length, breadth and height should be same while calculating surface area of the cuboid.

Example 7.4 Find the TSA and LSA of a cuboid whose length, breadth and height are 7.5 m, 3 m and 5 m respectively.

Solution

Given the dimensions of the cuboid;

that is length $(l) = 7.5\ m$, breadth $(b) = 3\ m$ and height $(h) = 5\ m$.

$$\begin{aligned}\text{TSA} &= 2(lb + bh + lh)\\ &= 2\left[(7.5\times 3)+(3\times 5)+(7.5\times 5)\right]\\ &= 2(22.5+15+37.5)\\ &= 2\times 75\\ &= 150\ m^2\end{aligned}$$

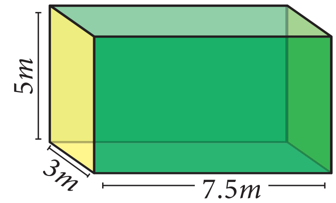


Fig. 7.11

$$\begin{aligned}\text{LSA} &= 2(l+b)\times h\\ &= 2(7.5+3)\times 5\\ &= 2\times 10.5\times 5\\ &= 105\ m^2\end{aligned}$$

Example 7.5 The length, breadth and height of a hall are 25 *m*, 15 *m* and 5 *m* respectively. Find the cost of renovating its floor and four walls at the rate of ₹80 per m^2.

Solution

Here, length (l) = 25 *m*, breadth (b) =15 *m*, height (h) = 5 *m*.

Area of four walls = LSA of cuboid

$$= 2(l+b)\times h$$

$$= 2(25+15)\times 5$$

$$= 80\times 5 = 400\ \ m^2$$

Area of the floor $= l\times b$

$$= 25\times 15$$

$$= 375\ \ m^2$$

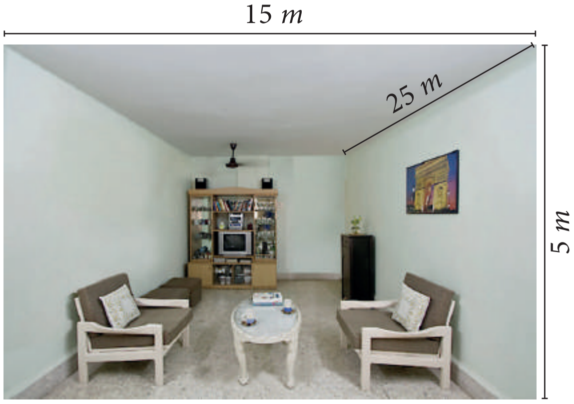


Fig. 7.12

Total renovating area of the hall

= (Area of four walls + Area of the floor)

$$= (400+375)\ \ m^2\quad = 775\ \ m^2$$

Therefore, cost of renovating at the rate of ₹80 per $m^2 = 80\times 775$

= ₹ 62,000

7.4.2 Cube and its Surface Area

Cube: A cuboid whose length, breadth and height are all equal is called as a cube. That is a cube is a solid having six square faces. Here are some real-life examples.

Dice Ice cubes Sugar cubes

Fig. 7.13

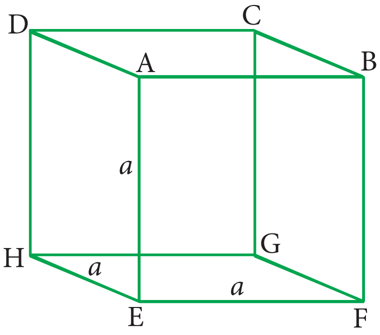


Fig. 7.14

A cube being a cuboid has 6 faces, 12 edges and 8 vertices.

Consider a cube whose sides are '*a*' units as shown in the Fig 7.14. Now,

(i) Total Surface Area of the cube

= sum of area of the faces $(ABCD+EFGH+AEHD+BFGC+ABFE+CDHG)$

$=(a^2+a^2+a^2+a^2+a^2+a^2)$

$=6a^2$ sq. units

(ii) Lateral Surface Area of the cube

= sum of area of the faces $(AEHD+BFGC+ABFE+CDHG)$

$=(a^2+a^2+a^2+a^2)$

$=4a^2$ sq. units

If the side of a cube is a units, then,

(i) The Total Surface Area = $6a^2$ sq.units

(ii) The Lateral Surface Area = $4a^2$ sq.units

Thinking Corner

Can you get these formulae from the corresponding formula of Cuboid?

Example 7.6 Find the Total Surface Area and Lateral Surface Area of the cube, whose side is 5 cm.

Solution

The side of the cube (a) $= 5\ cm$

Total Surface Area $=6a^2=6(5^2)=150\ sq.\ cm$

Lateral Surface Area $=4a^2=4(5^2)=100\ sq.\ cm$

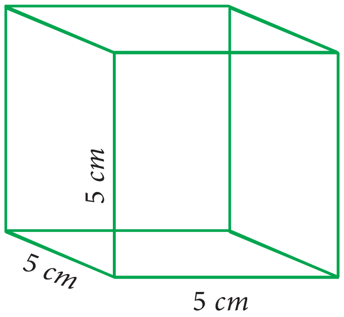


Fig. 7.15

Example 7.7 A cube has the Total Surface Area of 486 cm^2. Find its lateral surface area.

Solution

Here, Total Surface Area of the cube = 486 cm^2

$6a^2=486 \Rightarrow a^2=\frac{486}{6}$ and so, $a^2=81$. This gives a = 9.

The side of the cube $=9\ cm$

Lateral Surface Area $=4a^2\ =4\times 9^2\ =4\times 81\ =324\ cm^2$

Example 7.8 Two identical cubes of side 7 *cm* are joined end to end. Find the Total and Lateral surface area of the new resulting cuboid.

Solution

Side of a cube = 7 *cm*

Now length of the resulting cuboid (l) = 7+7 =14 *cm*

Breadth (b) = 7 *cm*, Height (h) = 7 *cm*

So, Total Surface Area $= 2(lb + bh + lh)$

$$= 2\left[(14\times7)+(7\times7)+(14\times7)\right]$$

$$= 2(98+49+98)$$

$$= 2\times245$$

$$= 490\ cm^2$$

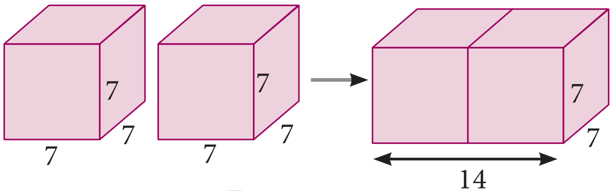


Fig. 7.16

Lateral Surface Area $= 2(l+b)\times h$

$$= 2(14+7)\times7 = 2\times21\times7$$

$$= 294\ cm^2$$

Exercise 7.2

1. Find the Total Surface Area and the Lateral Surface Area of a cuboid whose dimensions are: length = 20 *cm*, breadth = 15 *cm* and height = 8 *cm*
2. The dimensions of a cuboidal box are 6 *m* × 400 *cm* × 1.5 *m*. Find the cost of painting its entire outer surface at the rate of ₹22 per m^2.
3. The dimensions of a hall is 10 *m* × 9 *m* × 8 *m*. Find the cost of white washing the walls and ceiling at the rate of ₹8.50 per m^2.
4. Find the TSA and LSA of the cube whose side is (i) 8 *m* (ii) 21 *cm* (iii) 7.5 *cm*
5. If the total surface area of a cube is 2400 cm^2 then, find its lateral surface area.
6. A cubical container of side 6.5 *m* is to be painted on the entire outer surface. Find the area to be painted and the total cost of painting it at the rate of ₹24 per m^2.
7. Three identical cubes of side 4 *cm* are joined end to end. Find the total surface area and lateral surface area of the new resulting cuboid.

7.5 Volume of Cuboid and Cube

G1CJXB

All of us have tasted 50 *ml* and 100 *ml* of ice cream. Take one such 100 *ml* ice cream cup. This cup can contain 100 *ml* of water, which means that the capacity or volume of that cup is 100 *ml*. Take a 100 *ml* cup and find out how many such cups of water can fill a jug. If 10 such 100 *ml* cups can fill a jug then the capacity or volume of the jug is 1 litre $\left(10\times100\,ml=1000\,ml=1l\right)$. Further check how many such jug of water can fill a bucket. That is the capacity or volume of the bucket. Likewise we can calculate the volume or capacity of any such things.

Volume is the measure of the amount of space occupied by a three dimensional solid. Cubic centimetres (cm^3) , cubic metres (m^3) are some cubic units to measure volume.

Note

Unit Cube :

A cube with side 1 unit.

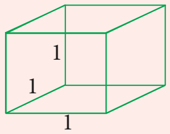


Fig. 7.17

Volume of the solid is the product of 'base area' and 'height'. This can easily be understood from a practical situation. You might have seen the bundles of A4 size paper. Each paper is rectangular in shape and has an area (=*lb*). When you pile them up, it becomes a bundle in the form of a cuboid; *h* times *lb* make the cuboid.

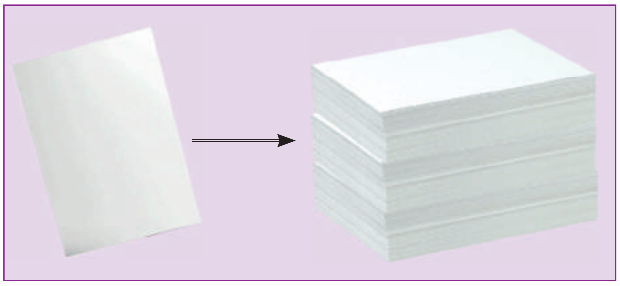


Fig. 7.18

7.5.1 Volume of a Cuboid

Let the length, breadth and height of a cuboid be *l*, *b* and *h* respectively.

Then, volume of the cuboid

$$V = \text{(cuboid's base area)} \times \text{height}$$

$$= (l\times b)\times h \;=\; lbh \text{ cubic units}$$

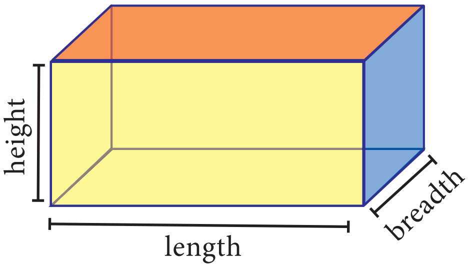


Fig. 7.19

Note

The units of length, breadth and height should be same while calculating the volume of a cuboid.

Example 7.9 The length, breadth and height of a cuboid is 120 *mm*, 10 *cm* and 8 *cm* respectively. Find the volume of 10 such cuboids.

Solution

Since both breadth and height are given in *cm*, it is necessary to convert the length also in *cm*.

So we get, $l = 120\ mm = \frac{120}{10} = 12\ cm$ and take $b = 10\ cm$, $h = 8\ cm$ as such.

$$\text{Volume of a cuboid} = l \times b \times h$$

$$= 12 \times 10 \times 8$$

$$= 960\ cm^3$$

$$\text{Volume of 10 such cuboids} = 10 \times 960$$

$$= 9600\ cm^3$$

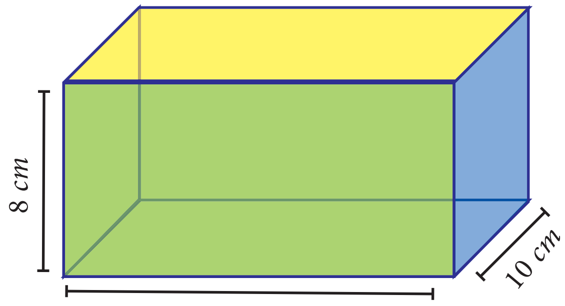


Fig. 7.20

Example 7.10 The length, breadth and height of a cuboid are in the ratio 7:5:2. Its volume is 35840 cm^3. Find its dimensions.

Solution

Let the dimensions of the cuboid be

$$l = 7x,\ b = 5x \text{ and } h = 2x.$$

Given that volume of cuboid = 35840 cm^3

$$l \times b \times h = 35840$$

$$(7x)(5x)(2x) = 35840$$

$$70x^3 = 35840$$

$$x^3 = \frac{35840}{70}$$

$$x^3 = 512$$

$$x = \sqrt[3]{8 \times 8 \times 8}$$

$$x = 8\ cm$$

$$\text{Length of cuboid} = 7x = 7 \times 8 = 56\ cm$$

$$\text{Breadth of cuboid} = 5x = 5 \times 8 = 40\ cm$$

$$\text{Height of cuboid} = 2x = 2 \times 8 = 16\ cm$$

THINKING CORNER

Each cuboid given below has the same volume 120 cm^3. Can you find the missing dimensions?

(i)

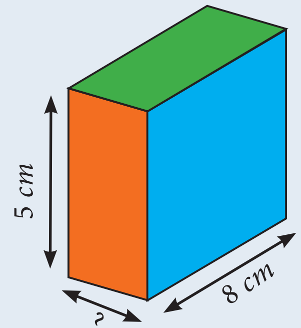


Fig. 7.21

(ii)

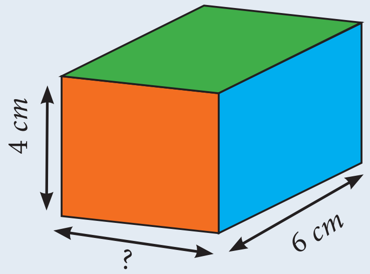


Fig. 7.22

Example 7.11 The dimensions of a fish tank are 3.8 *m* × 2.5 *m* × 1.6 *m*. How many litres of water it can hold?

Fig. 7.23

Solution

Length of the fish tank $l = 3.8\ m$

Breadth of the fish tank $b = 2.5\ m$, Height of the fish tank $h = 1.6\ m$

$$\text{Volume of the fish tank} = l \times b \times h$$
$$= 3.8 \times 2.5 \times 1.6$$
$$= 15.2\ m^3$$
$$= 15.2 \times 1000\ litres$$
$$= 15200\ litres$$

Note

A few important conversions

$1\ cm^3 = 1\ ml$, $1000\ cm^3 = 1\ litre$,
$1 m^3 = 1000\ litres$

Example 7.12 The dimensions of a sweet box are 22 *cm* × 18 *cm* × 10 *cm*. How many such boxes can be packed in a carton of dimensions 1 *m* × 88 *cm* × 63 *cm*?

Solution

Here, the dimensions of a sweet box are Length (l) = 22cm, breadth (b) = 18cm, height (h) = 10 *cm*.

Volume of a sweet box = $l \times b \times h$

$$= 22 \times 18 \times 10\ cm^3$$

The dimensions of a carton are

Length (l) = 1*m*= 100 *cm*, breadth (b) = 88 *c*

height (h) = 63 *cm*.

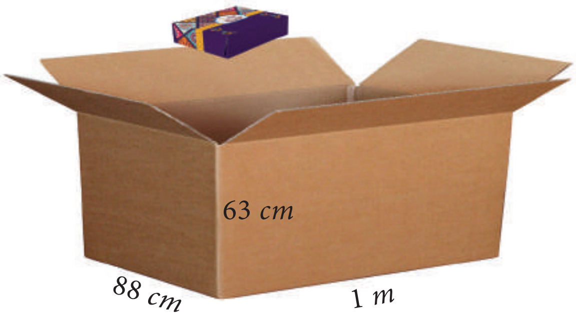


Fig. 7.24

Volume of the carton = $l \times b \times h$

$$= 100 \times 88 \times 63\ cm^3$$

$$\text{The number of sweet boxes packed} = \frac{\text{volume of the carton}}{\text{volume of a sweet box}}$$
$$= \frac{100 \times 88 \times 63}{22 \times 18 \times 10}$$
$$= 140\ \text{boxes}$$

7.5.2 Volume of a Cube

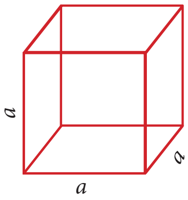


Fig. 7.25

It is easy to get the volume of a cube whose side is a units. Simply put $l = b = h = a$ in the formula for the volume of a cuboid. We get volume of cube to be a^3 cubic units.

If the side of a cube is 'a' units then the Volume of the cube (V) = a^3 cubic units.

Note

For any two cubes, the following results are true.

- Ratio of surface areas = (Ratio of sides)^2
- Ratio of volumes = (Ratio of sides)^3
- (Ratio of surface areas)^3 = (Ratio of volumes)^2

Example 7.13

Find the volume of cube whose side is 10 *cm*.

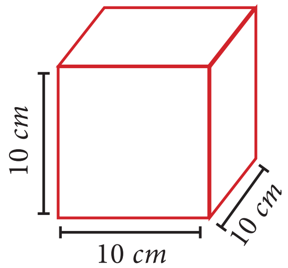


Fig. 7.26

Solution

Given that side (a) = 10 *cm*

$$\text{volume of the cube} = a^3$$

$$= 10 \times 10 \times 10$$

$$= 1000\ cm^3$$

Example 7.14

A cubical tank can hold 64,000 litres of water. Find the length of its side in metres.

Solution

Let 'a' be the side of cubical tank.

Here, volume of the tank $= 64{,}000$ litres

$$\text{i.e., } a^3 = 64{,}000 = \frac{64000}{1000} \quad [\text{since}, 1000 \text{ litres} = 1m^3]$$

$$a^3 = 64\ m^3$$

$$a = \sqrt[3]{64} \qquad a = 4\ m$$

Therefore, length of the side of the tank is 4 metres.

Example 7.15

The side of a metallic cube is 12 *cm*. It is melted and formed into a cuboid whose length and breadth are 18 *cm* and 16 *cm* respectively. Find the height of the cuboid.

Solution

Cube	**Cuboid**
Side $(a) = 12\,cm$	length $(l) = 18cm$ breadth $(b) = 16cm$ height $(h) = ?$

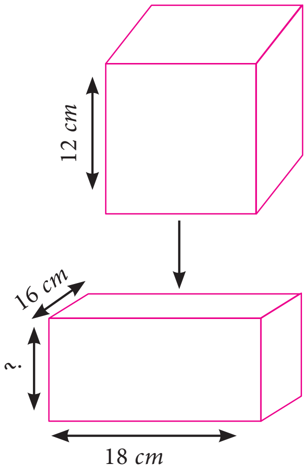


Fig. 7.27

Here, Volume of the Cuboid $=$ Volume of the Cube

$$l \times b \times h = a^3$$

$$18 \times 16 \times h = 12 \times 12 \times 12$$

$$h = \frac{12 \times 12 \times 12}{18 \times 16}$$

$$h = 6\,cm$$

Therefore, the height of the cuboid is 6 *cm*.

Activity

Take some square sheets of paper / chart paper of given dimension 18 *cm* × 18 *cm*. Remove the squares of same sizes from each corner of the given square paper and fold up the flaps to make a open cuboidal box. Then tabulate the dimensions of each of the cuboidal boxes made. Also find the volume each time and complete the table. The side measures of corner squares that are to be removed is given in the table below.

Side of the corner square	Dimensions of boxes			Volume
	l	b	h	V
2 *cm*				
3 *cm*				
4 *cm*				
5 *cm*				

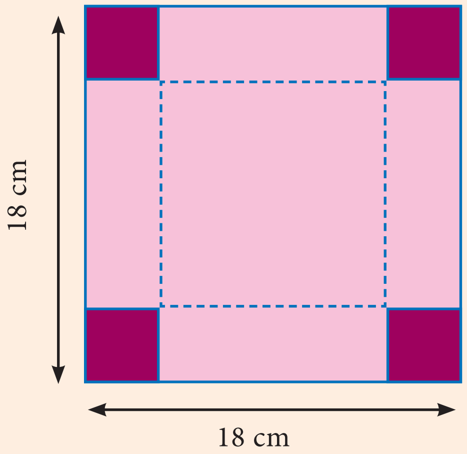


Fig. 7.28

Observe the above table and answer the following:

(i) What is the greatest possible volume?

(ii) What is the side of the square that when removed produces the greatest volume?

Exercise 7.3

1. Find the volume of a cuboid whose dimensions are
   (i) length = 12 *cm*, breadth = 8 *cm*, height = 6 *cm*
   (ii) length = 60 *m*, breadth = 25 *m*, height = 1.5 *m*

2. The dimensions of a match box are 6 *cm* × 3.5 *cm* × 2.5 *cm*. Find the volume of a packet containing 12 such match boxes.

3. The length, breadth and height of a chocolate box are in the ratio 5:4:3. If its volume is $7500\ cm^3$, then find its dimensions.

4. The length, breadth and depth of a pond are 20.5 *m*, 16 *m* and 8 *m* respectively. Find the capacity of the pond in litres.

5. The dimensions of a brick are 24 *cm* × 12 *cm* × 8 *cm*. How many such bricks will be required to build a wall of 20 *m* length, 48 *cm* breadth and 6 *m* height?

6. The volume of a container is $1440\ m^3$. The length and breadth of the container are 15 *m* and 8 *m* respectively. Find its height.

7. Find the volume of a cube each of whose side is (i) 5 *cm* (ii) 3.5 *m* (iii) 21 *cm*

8. A cubical milk tank can hold 125000 litres of milk. Find the length of its side in metres.

9. A metallic cube with side 15 *cm* is melted and formed into a cuboid. If the length and height of the cuboid is 25 *cm* and 9 *cm* respectively then find the breadth of the cuboid.

Exercise 7.4

Multiple choice questions

1. The semi-perimeter of a triangle having sides 15 *cm*, 20 *cm* and 25 *cm* is
   (1) 60 *cm* (2) 45 *cm* (3) 30 *cm* (4) 15 *cm*

2. If the sides of a triangle are 3 *cm*, 4 *cm* and 5 *cm*, then the area is
   (1) $3\ cm^2$ (2) $6\ cm^2$ (3) $9\ cm^2$ (4) $12\ cm^2$

G1HTB2

3. The perimeter of an equilateral triangle is 30 *cm*. The area is
   (1) $10\sqrt{3}\ cm^2$ (2) $12\sqrt{3}\ cm^2$ (3) $15\sqrt{3}\ cm^2$ (4) $25\sqrt{3}\ cm^2$

4. The lateral surface area of a cube of side 12 *cm* is
   (1) $144\ cm^2$ (2) $196\ cm^2$ (3) $576\ cm^2$ (4) $664\ cm^2$

5. If the lateral surface area of a cube is 600 cm^2, then the total surface area is

(1) 150 cm^2 (2) 400 cm^2 (3) 900 cm^2 (4) 1350 cm^2

6. The total surface area of a cuboid with dimension 10 $cm \times 6\ cm \times 5\ cm$ is

(1) 280 cm^2 (2) 300 cm^2 (3) 360 cm^2 (4) 600 cm^2

7. If the ratio of the sides of two cubes are 2:3, then ratio of their surface areas will be

(1) 4:6 (2) 4:9 (3) 6:9 (4) 16:36

8. The volume of a cuboid is 660 cm^3 and the area of the base is 33 cm^2. Its height is

(1) 10 cm (2) 12 cm (3) 20 cm (4) 22 cm

9. The capacity of a water tank of dimensions 10 $m \times 5\ m \times 1.5\ m$ is

(1) 75 litres (2) 750 litres (3) 7500 litres (4) 75000 litres

10. The number of bricks each measuring 50 $cm \times 30\ cm \times 20\ cm$ that will be required to build a wall whose dimensions are 5 $m \times 3\ m \times 2\ m$ is

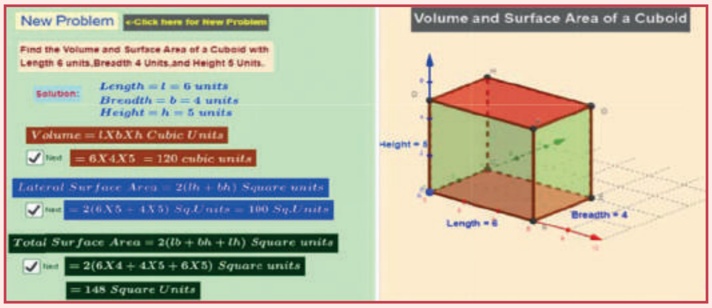
(1) 1000 (2) 2000 (3) 3000 (4) 5000

Points to Remember

- If a, b and c are the sides of a triangle, then the area of a triangle $= \sqrt{s(s-a)(s-b)(s-c)}$ sq.units, where $s = \dfrac{a+b+c}{2}$.
- If the length, breadth and height of the cuboid are l, b and h respectively, then
  - (i) Total Surface Area(TSA) $= 2(lb+bh+lh)$ sq.units
  - (ii) Lateral Surface Area(LSA) $= 2(l+b)h$ sq.units
- If the side of a cube is 'a' units, then
  - (i) Total Surface Area(TSA) $= 6a^2$ sq.units
  - (ii) Lateral Surface Area(LSA) $= 4a^2$ sq.units
- If the length, breadth and height of the cuboid are l, b and h respectively, then the Volume of the cuboid $(V) = lbh$ cu.units
- If the side of a cube is 'a' units then, the Volume of the cube $(V) = a^3$ cu.units.

ICT Corner

Expected Result is shown in this picture



New Problem ⇐Click here for New Problem

Find the Volume and Surface Area of a Cuboid with Length 6 units,Breadth 4 Units,and Height 5 Units.

Solution: $Length = l = 6$ units
$Breadth = b = 4$ units
$Height = h = 5$ units

$Volume = lXbXh$ Cubic Units

☑ Next $= 6X4X5 = 120$ cubic units

Lateral Surface Area $= 2(lh + bh)$ Square units

☑ Next $= 2(6X5 + 4X5)$ Sq.Units $= 100$ Sq.Units

Total Surface Area $= 2(lb + bh + lh)$ Square units

☑ Next $= 2(6X4 + 4X5 + 6X5)$ Square units
$= 148$ Square Units

Volume and Surface Area of a Cuboid

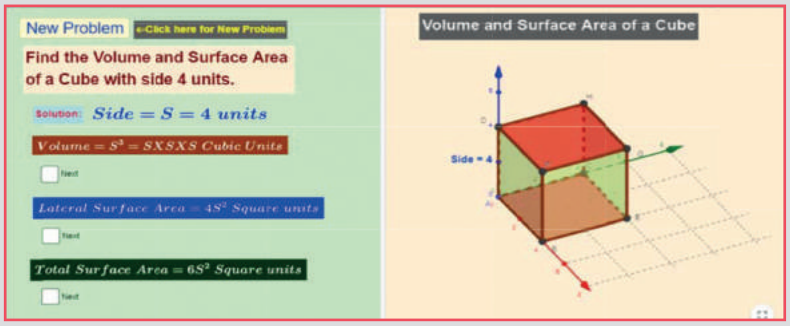
Step – 1

Open the Browser by typing the URL Link given below (or) Scan the QR Code. GeoGebra work sheet named "Mensuration" will open. There are two worksheets under the title CUBE and CUBOID.

Step - 2

Click on "New Problem". Volume, Lateral surface and Total surface area are asked. Work out the solution, and click on the respective check box and check the answer.

Step 1



New Problem ⇐Click here for New Problem

Find the Volume and Surface Area of a Cube with side 4 units.

Solution: $Side = S = 4$ units

$Volume = S^3 = SXSXS$ Cubic Units

☐ Next

Lateral Surface Area $= 4S^2$ Square units

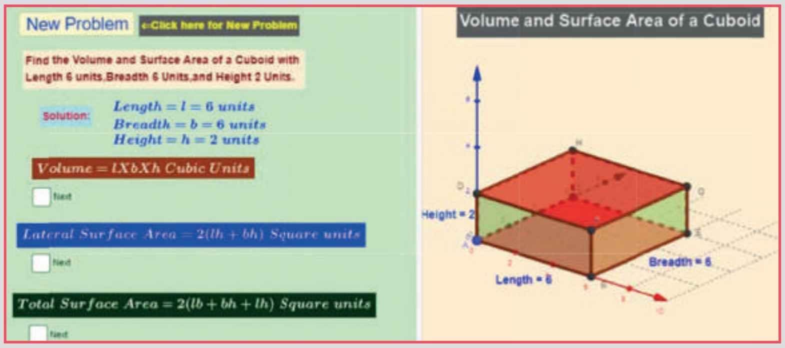
☐ Next

Total Surface Area $= 6S^2$ Square units

☐ Next

Volume and Surface Area of a Cube

Step 2



New Problem ⇐Click here for New Problem

Find the Volume and Surface Area of a Cuboid with Length 6 units,Breadth 6 Units,and Height 2 Units.

Solution: $Length = l = 6$ units
$Breadth = b = 6$ units
$Height = h = 2$ units

$Volume = lXbXh$ Cubic Units

☐ Next

Lateral Surface Area $= 2(lh + bh)$ Square units

☐ Next

Total Surface Area $= 2(lb + bh + lh)$ Square units

☐ Next

Volume and Surface Area of a Cuboid

Scan the QR Code.

B566_9_MAT_EM_T3

8

$$l+\frac{\left(\frac{N}{2}-m\right)}{f}\times c \qquad \bar{X}=\frac{\sum fx}{\sum f}$$

STATISTICS

"Lack of statistics is to hide inconvenient facts."

- Albert Bertilsson

Sir Ronald Aylmer Fisher
(AD (CE) 1890 - 1962)

Sir Ronald Aylmer Fisher was a British Statistician and Biologist. Also he was known as the Father of Modern Statistics and Experimental Design. Fisher did experimental agricultural research, which saved millions from starvation. He was awarded the Linnean Society of London's prestigious Darwin-Wallace Medal in 1958.

BEC9CY

Learning Outcomes

- To recall different types of averages known already.
- To recall the methods of computing the Mean, Median and Mode for ungrouped data.
- To compute the Mean, Median and Mode for the grouped data.

8.1 Introduction

Statistics is the science of collecting, organising, analysing and interpreting data in order to make decisions. In everyday life, we come across a wide range of quantitative and qualitative information. These have profound impact on our lives.

Data means the facts, mostly numerical, that are gathered; statistics implies collection of data. We analyse the data to make decisions. The methods of statistics are tools to help us in this.

Cricket News

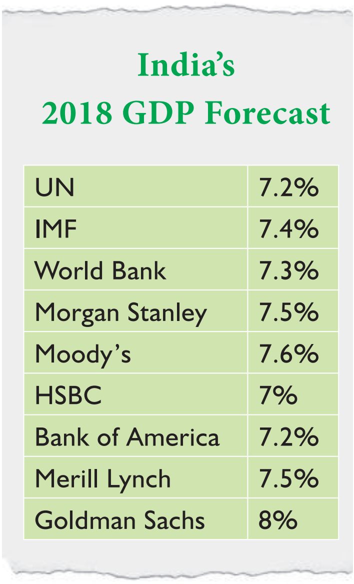
Team U19	Matches	Won	Lost	NR/Tied
India	71	52	18	0/1
Australia	67	50	15	0/2
Pakistan	69	50	19	0/0
Bangladesh	64	45	17	1/1
West Indies	71	44	27	0/0
South Africa	61	43	17	0/1
England	69	40	28	0/1
Sri Lanka	68	36	31	0/1
New Zealand	66	30	35	0/1
Zimbabwe	62	28	34	0/0
Ireland	49	16	32	1/0
Afghanistan	24	11	13	0/0
Namibia	47	9	37	1/0
Kenya	17	5	12	0/0
Canada	29	4	23	1/1
PNG	41	3	38	0/0

Customer Satisfaction Survey

Hotel Tamilnadu

Tell us how you were satisfied with our service

Very Satisfied
Satisfied
Neutral
Unsatisfied
Annoyed



India's 2018 GDP Forecast

UN	7.2%
IMF	7.4%
World Bank	7.3%
Morgan Stanley	7.5%
Moody's	7.6%
HSBC	7%
Bank of America	7.2%
Merill Lynch	7.5%
Goldman Sachs	8%

8.2 Collection of Data

Primary data are first-hand original data that we collect ourselves. Primary data collection can be done in a variety of ways such as by conducting personal interviews (by phone, mail or face-to-face), by conducting experiments, etc.

Secondary data are the data taken from figures collected by someone else. For example, government-published statistics, available research reports etc.

Progress Check

Identify the primary data

(i) Customer surveys
(ii) Medical researches
(iii) Economic predictions
(iv) School results
(v) Political polls
(vi) Marketing details
(vii) Sales forecasts
(viii) Price index details

8.2.1 Getting the Facts Sorted Out

When data are initially collected and before it is edited and not processed for use, they are known as Raw data. It will not be of much use because it would be too much for the human eye to analyse.

For example, study the marks obtained by 50 students in mathematics in an examination, given below:

Activity - 1

Prepare an album of pictures, tables, numeric details etc that exhibit data. Discuss how they are related to daily life situations.

61	60	44	49	31	60	79	62	39	51	67	65	43	54	51	42
52	43	46	40	60	63	72	46	34	55	76	55	30	67	44	57
62	50	65	58	25	35	54	59	43	46	58	58	56	59	59	45
42	44														

In this data, if you want to locate the five highest marks, is it going to be easy? You have to search for them; in case you want the third rank among them, it is further complicated. If you need how many scored less than, say 56, the task will be quite time consuming.

Hence **arrangement of an array of marks** will make the job simpler.

With some difficulty you may note in the list that 79 is the highest mark and 25 is the least. Using these you can subdivide the data into convenient classes and place each mark into the appropriate class. Observe how one can do it.

Class Interval	Marks
25-30	30, 25
31-35	31, 34, 35
36-40	39, 40
41-45	44, 43, 42, 43, 44, 43, 45, 42, 44
46-50	49, 46, 46, 50, 46
51-55	51, 54, 51, 52, 55, 55, 54
56-60	60, 60, 60, 57, 58, 59, 58, 58, 56, 59, 59
61-65	61, 62, 65, 63, 62, 65
66-70	67, 67
71-75	72
76-80	79, 76

From this table can you answer the questions raised above? To answer the question, "how many scored below 56", you do not need the actual marks. You just want "how many" were there. To answer such cases, which often occur in a study, we can modify the table slightly and just note down how many items are there in each class. We then may have a slightly simpler and more useful arrangement, as given in the table.

Class Interval	Number of items
25-30	2
31-35	3
36-40	2
41-45	9
46-50	5
51-55	7
56-60	11
61-65	6
66-70	2
71-75	1
76-80	2

This table gives us the number of items in each class; each such number tells you how many times the required item occurs in the class and is called the frequency in that class.

The table itself is called a frequency table.

We use what are known as tally marks to compute the frequencies. (Under the column 'number of items', we do not write the actual marks but just tally marks). For example, against the class 31-35, instead of writing the actual marks 31, 34, 35 we simply put |||. You may wonder if for the class 56-60 in the example one has to write |||||||||||, making it difficult to count. To avoid confusion, every fifth tally mark is put across the four preceding it, like this 𝍸 . For example, 11 can be written as 𝍸 𝍸 | . The frequency table for the above illustration will be seen as follows:

Class Interval	Tally Marks	Frequency
25-30	||	2
31-35	|||	3
36-40	||	2
41-45	𝍸 ||||	9
46-50	𝍸	5
51-55	𝍸 ||	7
56-60	𝍸 𝍸 |	11
61-65	𝍸 |	6
66-70	||	2
71-75	|	1
76-80	||	2
	TOTAL	**50**

Note

Consider any class, say 56- 60; then 56 is called the lower limit and 60 is called the upper limit of the class.

Progress Check

Form a frequency table for the following data:

23	44	12	11	45	55	79	20
52	37	77	97	82	56	28	71
62	58	69	24	12	99	55	78
21	39	80	65	54	44	59	65
17	28	65	35	55	68	84	97
80	46	30	49	50	61	59	33
11	57						

8.3 Measures of Central Tendency

It often becomes necessary in everyday life to express a quantity that is typical for a given data. Suppose a researcher says that on an average, people watch TV serials for 3 hours per day, it does not mean that everybody does so; some may watch more and some less. The average is an acceptable indicator of the data regarding programmes watched on TV.

Averages summarise a large amount of data into a single value and indicate that there is some variability around this single value within the original data.

A mathematician's view of an average is slightly different from that of the commoner. There are three different definitions of average known as the Mean, Median and Mode. Each of them is found using different methods and when they are applied to the same set of original data they often result in different average values. It is important to figure out what each of these measures of average tells you about the original data and consider which one is the most appropriate to calculate.

8.4 Arithmetic Mean

8.4.1 Arithmetic Mean-Raw Data

The Arithmetic Mean of a data is the most commonly used of all averages and is found by adding together all the values and dividing by the number of items.

For example, a cricketer, played eight (T20) matches and scored the following scores 25, 32, 36, 38, 45, 41, 35, 36.

Then, the mean of his scores (that is the arithmetic average of the scores) is obtained by

$$\overline{X} = \frac{\sum x}{n} = \frac{25 + 32 + 36 + 38 + 45 + 41 + 35 + 36}{8} = \frac{288}{8} = 36 .$$

In general, if we have n number of observations $x_1, x_2, x_3, \ldots, x_n$ then their arithmetic mean denoted by $\overline{X}$ (read as X bar) is given by

$$\overline{X} = \frac{\text{sum of all the observations}}{\text{number of observations}} \qquad \overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

We express this as a formula: $\overline{X} = \frac{\sum x}{n}$

Assumed Mean method: Sometimes we can make calculations easy by working from an entry that we guess to be the right answer. This guessed number is called the assumed mean.

In the example above on cricket scores, let us assume that 38 is the assumed mean. We now list the differences between the assumed mean and each score entered:

Note

It does not matter which number is chosen as the assumed mean; we need a number that would make our calculations simpler. Perhaps a choice of number that is closer to most of the entries would help; it need not even be in the list given.

25–38 = –13, 32–38 = –6, 36–38 = –2, 38–38 = 0,

45–38 = 7, 41–38 = 3, 35–38 = –3, 36–38 = –2

The average of these differences is $\frac{-13-6-2+0+7+3-3-2}{8} = \frac{-16}{8} = -2$

We add this 'mean difference' to the assumed mean to get the correct mean.

Thus the correct mean = Assumed Mean +Mean difference = 38 – 2 = 36 .

This method will be very helpful when large numbers are involved.

8.4.2 Arithmetic Mean-Ungrouped Frequency Distribution

Consider the following list of heights (in cm) of 12 students who are going to take part in an event in the school sports.

140, 142, 150, 150, 140, 148, 140, 147, 145, 140, 147, 145.

How will you find the Mean height?

There are several options.

(i) You can add all the items and divide by the number of items.

$$\frac{140+142+150+150+140+148+140+147+145+140+147+145}{12}=\frac{1734}{12}=144.5$$

(ii) You can use Assumed mean method. Assume, 141 as the assumed mean.

Then the mean will be given by

$$=141+\frac{(-1)+(1)+(9)+(9)+(-1)+(7)+(-1)+(6)+(4)+(-1)+(6)+(4)}{12}$$

$$=141+\frac{-4+46}{12}=141+\frac{42}{12}=141+3.5=144.5$$

(iii) A third method is to deal with an ungrouped frequency distribution. You find that 140 has occurred 4 times, (implying 4 is the frequency of 140), 142 has occurred only once (indicating that 1 is the frequency of 142) and so on. This enables us to get the following frequency distribution.

Height(cm)	140	142	150	148	145	147
No. of students	4	1	2	1	2	2

You find that there are four 140s; their total will be 140 × 4 = 560

There is only one 142; so the total in this case is 142 × 1 = 142

There are two 150s; their total will be 150 × 2 = 300 etc.

These details can be neatly tabulated as follows:

Height (x)	Frequency (f)	fx
140	4	560
142	1	142
150	2	300
148	1	148
145	2	290
147	2	294
	12	1734

$$\text{Mean} = \frac{\text{Sum of all } fx}{\text{No.of items}}$$

$$=\frac{1734}{12}=144.5 \text{ cm}$$

Looking at the procedure in general terms, you can obtain a formula for ready use. If $x_1, x_2, x_3, \ldots x_n$ are n observations whose corresponding frequencies are $f_1, f_2, f_3, \ldots f_n$ then the mean is given by

Note

Study each step and understand the meaning of each symbol.

$$\bar{X}=\frac{f_1x_1+f_2x_2+\ldots+f_nx_n}{f_1+f_2+\ldots f_n}=\frac{\sum_{i=1}^{n} f_ix_i}{\sum_{i=1}^{n} f_i}=\frac{\sum fx}{\sum f}$$

Can you adopt the above method combining with the assumed mean method? Here is an attempt in that direction:

(iv) Let the assumed mean be 145. Then we can prepare the following table:

Height(x)	**d = deviation from the assumed mean**	**Frequency (f)**	fd
140	140 – 145 = – 5	4	–20
142	142– 145 = – 3	1	– 3
150	150 – 145 = + 5	2	+10
148	148 – 145 = + 3	1	+ 3
145 (Assumed)	145 – 145 = 0	2	0
147	147 – 145 = + 2	2	+ 4
Total		$\sum f = 12$	$\sum fd = -23 + 17 = -6$

Arithmetic mean = Assumed mean + Average of the sum of deviations

$$= A + \frac{\sum fd}{\sum f} = 145 + \left(\frac{-6}{12}\right) = 145.0 - 0.5 = 144.5$$

When large numbers are involved, this method could be useful.

8.4.3 Arithmetic Mean-Grouped Frequency Distribution

When data are grouped in class intervals and presented in the form of a frequency table, we get a frequency distribution like this one:

Age (in years)	10-20	20-30	30 – 40	40 – 50	50 - 60
Number of customers	80	120	50	22	8

The above table shows the number of customers in the various age groups. For example, there are 120 customers in the age group 20 – 30, but does not say anything about the age of any individual. (When we form a grouped frequency table the identity of the individual observations is lost). Hence we need a value that represents the particular class interval. Such a value is called mid value (mid-point or class mark) The mid-point or class mark can be found using the formula given below.

Mid Value $= \frac{UCL + LCL}{2}$, *UCL – Upper Class Limit, LCL – Lower Class Limit*

In grouped frequency distribution, arithmetic mean may be computed by applying any one of the following methods.

(i) Direct Method (ii) Assumed Mean Method (iii) Step Deviation Method

Direct Method

When direct method is used, the formula for finding the arithmetic mean is

$$\bar{X} = \frac{\sum fx}{\sum f}$$

Where x is the mid-point of the class interval and f is the corresponding frequency

Steps

(i) Obtain the mid-point of each class and denote it by x

(ii) Multiply those mid-points by the respective frequency of each class and obtain the sum of fx

(iii) Divide Σfx by Σf to obtain mean

Example 8.1 The following data gives the number of residents in an area based on their age. Find the average age of the residents.

Age	0-10	10-20	20-30	30-40	40-50	50-60
Number of Residents	2	6	9	7	4	2

Solution

Age	Number of Residents(f)	Midvalue(x)	fx
0-10	2	5	10
10-20	6	15	90
20-30	9	25	225
30-40	7	35	245
40-50	4	45	180
50-60	2	55	110
	$\sum f = 30$		$\sum fx = 860$

$$\text{Mean} = \bar{X} = \frac{\sum fx}{\sum f} = \frac{860}{30} = 28.67$$

Hence the average age $= 28.67$.

Assumed Mean Method

We have seen how to find the arithmetic mean of a grouped data quickly using the direct method formula. However, if the observations are large, finding the products of the observations and their corresponding frequencies, and then adding them is not only difficult and time consuming but also has chances of errors. In such cases, we can use the Assumed Mean Method to find the arithmetic mean of grouped data.

Steps

1. Assume any value of the observations as the Mean (A). Preferably, choose the middle value.
2. Calculate the deviation $d = x - A$ for each class
3. Multiply each of the corresponding frequency 'f' with 'd' and obtain Σfd
4. Apply the formula $\overline{X} = A + \frac{\sum fd}{\sum f}$

Example 8.2 Find the mean for the following frequency table:

Class Interval	100-120	120-140	140-160	160-180	180-200	200-220	220-240
Frequency	10	8	4	4	3	1	2

Solution

Let Assumed mean $A = 170$

Class Interval	Frequency f	Mid value x	$d = x - A$ $d = x - 170$	fd
100-120	10	110	–60	–600
120-140	8	130	–40	–320
140-160	4	150	–20	–80
160-180	4	170	0	0
180-200	3	190	20	60
200-220	1	210	40	40
220-240	2	230	60	120
	$\sum f = 32$			$\sum fd = -780$

Mean $\overline{X} = A + \frac{\sum fd}{\sum f}$

$$= 170 + \left(\frac{-780}{32}\right)$$

Therefore , $\overline{X} = 170 - 24.375$

$$= 145.625$$

Step Deviation Method

In order to simplify the calculation, we divide the deviation by the width of class intervals (i.e. calculate $\frac{x - A}{c}$) and then multiply by c in the formula for getting the mean of the data. The formula to calculate the Arithmetic Mean is

$$\bar{X} = A + \left[\frac{\sum fd}{\sum f} \times c\right], \text{ where } d = \frac{x - A}{c}$$

Example 8.3

Find the mean of the following distribution using Step Deviation Method.

Class Interval	0-8	8-16	16-24	24-32	32-40	40-48
Frequency (f)	10	20	14	16	18	22

Solution

Let Assumed mean $A = 28$, class width c =8

Class Interval	Mid Value x	Frequency f	$d = \frac{x-A}{c}$	fd
0-8	4	10	-3	-30
8-16	12	20	-2	-40
16-24	20	14	-1	-14
24-32	28	16	0	0
32-40	36	18	1	18
40-48	44	22	2	44
		$\sum f = 100$		$\sum fd = -22$

Mean

$$\bar{X} = A + \frac{\sum fd}{\sum f} \times c$$

$$= 28 + \left(\frac{-22}{100}\right) \times 8$$

$$= 28 - 1.76 = 26.24$$

Note

- When x_i and f_i are small, then Direct Method is the appropriate choice.
- When x_i and f_i are numerically large numbers, then Assumed Mean Method or Step Deviation Method can be used.
- When class sizes are unequal and d numerically large,we can still use Step Deviation Method.

8.4.4 A special property of the Arithmetic Mean

1. *The sum of the deviations of the entries from the arithmetic mean is always zero.*

If $x_1, x_2, x_3, \ldots, x_n$ are n observations taken from the arithmetic mean $\bar{X}$

then $\left(x_1 - \bar{X}\right) + \left(x_2 - \bar{X}\right) + \left(x_3 - \bar{X}\right) + \ldots + \left(x_n - \bar{X}\right) = 0$ .Hence $\sum_{i=1}^{n}(x_i - \bar{X}) = 0$

2. *If each observation is increased or decreased by k (constant) then the arithmetic mean is also increased or decreased by k respectively.*

3. *If each observation is multiplied or divided by k, $k \neq 0$, then the arithmetic mean is also multiplied or divided by the same quantity k respectively.*

Example 8.4 Find the sum of the deviations from the arithmetic mean for the following observations:

21, 30, 22, 16, 24, 28, 18, 17

Solution

$$\overline{X} = \frac{\sum_{i=1}^{8} x_i}{n} = \frac{21+30+22+16+24+28+18+17}{8} = \frac{176}{8} = 22$$

Deviation of an entry x_i from the arithmetic mean $\overline{X}$ is $x_i - \overline{X}$, $i = 1, 2, \ldots 8$.

Sum of the deviations

$$= (21\text{-}22)+(30\text{-}22)+(22\text{-}22)+(16\text{-}22)+(24\text{-}22)+(28\text{-}22)+(18\text{-}22)+(17\text{-}22)$$

$= 16 - 16 = 0.$ or equivalently, $\sum_{i=1}^{8}(x_i - \overline{X}) = 0$

Hence, we conclude that sum of the deviations from the Arithmetic Mean is zero.

Example 8.5 The arithmetic mean of 6 values is 45 and if each value is increased by 4, then find the arithmetic mean of new set of values.

Solution

Let $x_1, x_2, x_3, x_4, x_5, x_6$ be the given set of values then $\frac{\sum_{i=1}^{6} x_i}{6} = 45.$

If each value is increased by 4, then the mean of new set of values is

$$\text{New A.M. } \overline{X} = \frac{\sum_{i=1}^{6}(x_i + 4)}{6}$$

$$= \frac{(x_1+4)+(x_2+4)+(x_3+4)+(x_4+4)+(x_5+4)+(x_6+4)}{6}$$

$$= \frac{\sum_{i=1}^{6} x_i + 24}{6} = \frac{\sum_{i=1}^{6} x_i}{6} + 4$$

$$\overline{X} = 45 + 4 = 49.$$

Progress Check

Mean of 10 observations is 48 and 7 is subtracted to each observation, then mean of new observation is ___________

Example 8.6 If the arithmetic mean of 7 values is 30 and if each value is divided by 3, then find the arithmetic mean of new set of values

Solution

Let X represent the set of seven values $x_1, x_2, x_3, x_4, x_5, x_6, x_7$.

Then $\overline{X} = \frac{\sum_{i=1}^{7} x_i}{7} = 30$ or $\sum_{i=1}^{7} x_i = 210$

If each value is divided by 3, then the mean of new set of values is

Progress Check

1. The Mean of 12 numbers is 20. If each number is multiplied by 6, then the new mean is __
2. The Mean of 30 numbers is 16. If each number is divided by 4, then the new mean is___

$$\frac{\sum_{i=1}^{7}\frac{x_i}{3}}{7} = \frac{\left(\frac{x_1}{3}+\frac{x_2}{3}+\frac{x_3}{3}+\frac{x_4}{3}+\frac{x_5}{3}+\frac{x_6}{3}+\frac{x_7}{3}\right)}{7}$$

$$= \frac{\sum_{i=1}^{7} x_i}{21} = \frac{210}{21} = 10$$

Aliter

If Y is the set of values obtained by dividing each value of X by 3.

Then, $\bar{Y} = \frac{\bar{X}}{3} = \frac{30}{3} = 10.$

Example 8.7 The average mark of 25 students was found to be 78.4. Later on, it was found that score of 96 was misread as 69. Find the correct mean of the marks.

Solution

Given that the total number of students $n = 25, \bar{X} = 78.4$

So, Incorrect $\sum x = \bar{X} \times n \quad = 78.4 \times 25 = 1960$

Correct $\Sigma x = incorrect \sum x - wrong\ entry + correct\ entry$

$= 1960 - 69 + 96 = 1987$

Correct $\bar{X} = \frac{correct \sum x}{n} = \frac{1987}{25} = 79.48$

Progress Check

There are four numbers. If we leave out any one number, the average of the remaining three numbers will be 45, 60, 65 or 70. What is the average of all four numbers?

Exercise 8.1

1. In a week, temperature of a certain place is measured during winter are as follows 26°C, 24°C, 28°C, 31°C, 30°C, 26°C, 24°C. Find the mean temperature of the week.

2. The mean weight of 4 members of a family is 60kg.Three of them have the weight 56kg, 68kg and 72kg respectively. Find the weight of the fourth member.

3. In a class test in mathematics, 10 students scored 75 marks, 12 students scored 60 marks, 8 students scored 40 marks and 3 students scored 30 marks. Find the mean of their score.

4. In a research laboratory scientists treated 6 mice with lung cancer using natural medicine. Ten days later, they measured the volume of the tumor in each mouse and given the results in the table.

Mouse marking	1	2	3	4	5	6
Tumor Volume(mm^3)	145	148	142	141	139	140

Find the mean.

5. If the mean of the following data is 20.2, then find the value of p

Marks	10	15	20	25	30
No. of students	6	8	p	10	6

6. In the class, weight of students is measured for the class records. Calculate mean weight of the class students using Direct method.

Weight in kg	15-25	25-35	35-45	45-55	55-65	65-75
No. of students	4	11	19	14	0	2

7. Calculate the mean of the following distribution using Assumed Mean Method:

Class Interval	0-10	10-20	20-30	30-40	40-50
Frequency	5	7	15	28	8

8. Find the Arithmetic Mean of the following data using Step Deviation Method:

Age	15-19	20-24	25-29	30-34	35-39	40-44
No. of persons	4	20	38	24	10	9

8.5 Median

The arithmetic mean is typical of the data because it 'balances' the numbers; it is the number in the 'middle', pulled up by large values and pulled down by smaller values.

Suppose four people of an office have incomes of ₹5000, ₹6000, ₹7000 and ₹8000. Their mean income can be calculated as $\frac{5000+6000+7000+8000}{4}$ which gives ₹6500. If a fifth person with an income of ₹ 29000 is added to this group, then the arithmetic mean of all the five would be $\frac{5000+6000+7000+8000+29000}{5} = \frac{55000}{5}$ = ₹11000. Can one say that the average income of ₹11000 truly represents the income status of the individuals in the office? Is it not, misleading? The problem here is that an extreme score affects the Mean and can move the mean away from what would generally be considered the central area.

In such situations, we need a different type of average to provide reasonable answers.

Median is the value which occupies the middle position when all the observations are arranged in an ascending or descending order. It is a positional average.

For example, the height of nine students in a class are 122 cm, 124 cm, 125 cm, 135 cm, 138 cm, 140cm, 141cm, 147 cm, and 161 cm.

(i) Usual calculation gives Arithmetic Mean to be 137 cm.

(ii) If the heights are neatly arranged in, say, ascending order, as follows 122 cm, 124 cm, 125 cm, 135 cm, 138 cm, 140cm, 141cm, 147 cm, 161 cm, one can observe the value 138 cm is such that equal number of items lie on either side of it. Such a value is called the Median of given readings.

(iii) Suppose a data set has 11 items arranged in order. Then the median is the 6th item because it will be the middlemost one. If it has 101 items, then 51st item will be the Median.

If we have an odd number of items, one can find the middle one easily. In general, if a data set has n items and n is odd, then the median will be the $\left(\frac{n+1}{2}\right)^{th}$ item.

(iv) If there are 6 observations in the data, how will you find the Median? It will be the average of the middle two terms. (Shall we denote it as 3.5^{th} term?) If there are 100 terms in the data, the Median will be 50.5^{th} term!

In general, if a data set has n items and n is even, then the Median will be the average of $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{\text{th}}$ items.

Example 8.8 The following are scores obtained by 11 players in a cricket match 7, 21, 45, 12, 56, 35, 25, 0, 58, 66, 29. Find the median score.

Solution

Let us arrange the values in ascending order.

0,7,12,21,25,29,35,45,56,58,66

The number of values = 11 which is odd

$$\text{Median} = \left(\frac{11+1}{2}\right)^{\text{th}} \text{ value}$$

$$= \left(\frac{12}{2}\right)^{\text{th}} \text{ value} = 6^{\text{th}} \text{ value} = 29$$

Example 8.9 For the following ungrouped data 10, 17, 16, 21, 13, 18, 12, 10, 19, 22. Find the median.

Solution

Arrange the values in ascending order.

10, 10, 12, 13, 16, 17, 18, 19, 21, 22.

The number of values = 10

$$\text{Median} = \text{Average of } \left(\frac{10}{2}\right)^{\text{th}} \text{ and } \left(\frac{10}{2}+1\right)^{th} \text{ values}$$

$$= \text{Average of } 5^{\text{th}} \text{ and } 6^{\text{th}} \text{ values}$$

$$= \frac{16+17}{2} = \frac{33}{2} = 16.5$$

Example 8.10 The following table represents the marks obtained by a group of 12 students in a class test in Mathematics and Science.

Marks (Mathematics)	52	55	32	30	60	44	28	25	50	75	33	62
Marks (Science)	54	42	48	49	27	25	24	19	28	58	42	69

Indicate in which subject, the level of achievement is higher?

Solution

Let us arrange the marks in the two subjects in ascending order.

Marks (Mathematics)	25	28	30	32	33	44	50	52	55	60	62	75
Marks (Science)	19	24	25	27	28	42	42	48	49	54	58	69

Since the number of students is 12, the marks of the middle-most student would be the mean mark of 6th and 7th students.

$$\text{Therefore , Median mark in Mathematics} = \frac{44+50}{2} = 47$$

$$\text{Median mark in Science} = \frac{42+42}{2} = 42$$

Here the median mark in Mathematics is greater than the median mark in Science. Therefore, the level of achievement of the students is higher in Mathematics than Science.

8.5.1 Median-Ungrouped Frequency Distribution

(i) Arrange the data in ascending (or) decending order of magnitude.

(ii) Construct the cumulative frequency distribution. Let N be the total frequency.

(iii) If N is odd, median $= \left(\frac{N+1}{2}\right)^{th}$ observation.

(iv) If N is even, median $= \left(\frac{\left(\frac{N}{2}\right)^{\text{th}} \text{observation} + \left(\frac{N}{2}+1\right)^{\text{th}} \text{observation}}{2}\right)$

Example 8.11 Calculate the median for the following data:

Height (cm)	160	150	152	161	156	154	155
No. of Students	12	8	4	4	3	3	7

Solution

Let us arrange the marks in ascending order and prepare the following data:

Height (cm)	Number of students (f)	Cumulative frequency (cf)
150	8	8
152	4	12
154	3	15
155	7	22
156	3	25
160	12	37
161	4	41

Here N = 41

Median = size of $\left(\frac{N+1}{2}\right)^{th}$ value = size of $\left(\frac{41+1}{2}\right)^{th}$ value = size of 21^{st} value.

If the 41 students were arranged in order (of height), the 21^{st} student would be the middle most one, since there are 20 students on either side of him/her. We therefore need to find the height against the 21^{st} student. 15 students (see cumulative frequency) have height less than or equal to 154 cm. 22 students have height less than or equal to 155 cm. This means that the 21^{st} student has a height 155 cm.

Therefore, Median = 155 cm

8.5.2 Median - Grouped Frequency Distribution

In a grouped frequency distribution, computation of median involves the following

Steps

(i) Construct the cumulative frequency distribution.

(ii) Find $\left(\frac{N}{2}\right)^{th}$ term.

(iii) The class that contains the cumulative frequency $\frac{N}{2}$ is called the median class.

(iv) Find the median by using the formula:

$$\text{Median} = l + \frac{\left(\frac{N}{2} - m\right)}{f} \times c$$

Where l = *Lower limit of the median class,* f = *Frequency of the median class*

c = *Width of the median class,* N = *The total frequency* $\left(\sum f\right)$

m = *cumulative frequency of the class preceeding the median class*

Example 8.12 The following table gives the weekly expenditure of 200 families. Find the median of the weekly expenditure.

Weekly expenditure (₹)	0-1000	1000-2000	2000-3000	3000-4000	4000-5000
Number of families	28	46	54	42	30

Solution

Weekly Expenditure	Number of families (f)	Cumulative frequency (cf)
0-1000	28	28
1000-2000	46	74
2000-3000	54	128
3000-4000	42	170
4000-5000	30	200
	N=200	

$$\text{Median class} = \left(\frac{N}{2}\right)^{\text{th}} \text{value} = \left(\frac{200}{2}\right)^{\text{th}} \text{value}$$

$$= 100^{\text{th}} \text{ value}$$

$$\text{Median class} = 2000 - 3000$$

$$\frac{N}{2} = 100 \quad l = 2000$$

$$m = 74, \quad c = 1000, \quad f = 54$$

$$\text{Median} = l + \frac{\left(\frac{N}{2} - m\right)}{f} \times c$$

Progress Check

1. The median of the first four whole numbers ______.
2. If 4 is also included to the collection of first four whole numbers then median value is _________.
3. The difference between two median is ______.

$$= 2000 + \left(\frac{100 - 74}{54}\right) \times 1000$$

$$= 2000 + \left(\frac{26}{54}\right) \times 1000 = 2000 + 481.5$$

$$= 2481.5$$

Example 8.13 The Median of the following data is 24. Find the value of x.

Class Interval (CI)	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50
Frequency (f)	6	24	x	16	9

Solution

Class Interval (CI)	Frequency (f)	Cumulative frequency (cf)
0-10	6	6
10-20	24	30
20-30	x	$30 + x$
30-40	16	$46 + x$
40-50	9	$55 + x$
	$N = 55 + x$	

Since the median is 24 and median class is 20 – 30

$l = 20 \quad N = 55 + x, \quad m = 30, \quad c = 10, \quad f = x$

$$\text{Median} = l + \frac{\left(\frac{N}{2} - m\right)}{f} \times c$$

$$24 = 20 + \frac{\left(\frac{55 + x}{2} - 30\right)}{x} \times 10$$

$$4 = \frac{5x - 25}{x} \quad \text{(after simplification)}$$

$$4x = 5x - 25$$

$$5\text{x} - 4x = 25$$

$$x = 25$$

Note

The median is a good measure of the average value when the data include extremely high or low values, because these have little influence on the outcome.

Exercise 8.2

1. Find the median of the given values : 47, 53, 62, 71, 83, 21, 43, 47, 41.
2. Find the Median of the given data: 36, 44, 86, 31, 37, 44, 86, 35, 60, 51

3. The median of observation 11, 12, 14, 18, x+2, x+4, 30, 32, 35, 41 arranged in ascending order is 24. Find the values of x.
4. A researcher studying the behavior of mice has recorded the time (in seconds) taken by each mouse to locate its food by considering 13 different mice as 31, 33, 63, 33, 28, 29, 33, 27, 27, 34, 35, 28, 32. Find the median time that mice spent in searching its food.
5. The following are the marks scored by the students in the Summative Assessment exam

Class	0-10	10-20	20-30	30-40	40-50	50-60
No. of Students	2	7	15	10	11	5

Calculate the median.

6. The mean of five positive integers is twice their median. If four of the integers are 3, 4, 6, 9 and median is 6, then find the fifth integer.

8.6 Mode

(i) The votes obtained by three candidates in an election are as follows:

Name of the Candidate	Votes Polled
Mr. X	4, 12, 006
Mr. Y	9, 87, 991
Mr. Z	7, 11, 973
Total	21, 11, 970

Who will be declared as the winner? Mr. Y will be the winner, because the number of votes secured by him is the highest among all the three candidates. Of course, the votes of Mr. Y do not represent the majority population (because there are more votes against him). However, he is declared winner because the mode of selection here depends on the highest among the candidates.

BEV1G9

(ii) An Organisation wants to donate sports shoes of same size to maximum number of students of class IX in a School. The distribution of students with different shoe sizes is given below.

Shoe Size	5	6	7	8	9	10
No. of Students	10	12	27	31	19	1

If it places order, shoes of only one size with the manufacturer, which size of the shoes will the organization prefer?

In the above two cases, we observe that mean or median does not fit into the situation. We need another type of average, namely the Mode.

The mode is the number that occurs most frequently in the data.

When you search for some good video about Averages on You Tube, you look to watch the one with maximum views. Here you use the idea of a mode.

8.6.1 Mode - Raw Data

For an individual data mode is the value of the variable which occurs most frequently.

Example 8.14 In a rice mill, seven labours are receiving the daily wages of ₹500, ₹600, ₹600, ₹800, ₹800, ₹800 and ₹1000, find the modal wage.

Solution

In the given data ₹800 occurs thrice.Hence the mode is ₹ 800.

Example 8.15 Find the mode for the set of values 17, 18, 20, 20, 21, 21, 22, 22.

Solution

In this example, three values 20, 21, 22 occur two times each. There are three modes for the given data!

Note

- A distribution having only one mode is called unimodal.
- A distribution having two modes is called bimodal.
- A distribution having Three modes is called trimodal.
- A distribution having more than three modes is called multimodal.

8.6.2 Mode for Ungrouped Frequency Distribution

In a ungrouped frequency distribution, the value of the item having maximum frequency is taken as the mode.

Example 8.16 A set of numbers consists of five 4's, four 5's, nine 6's,and six 9's. What is the mode.

Solution

Size of item	4	5	6	9
Frequency	5	4	9	6

6 has the maximum frequency 9. Therefore 6 is the mode.

8.6.3 Mode – Grouped Frequency Distribution:

In case of a grouped frequency distribution, the exact values of the variables are not known and as such it is very difficult to locate mode accurately. In such cases, if the class intervals are of equal width, an appropriate value of the mode may be determined by

$$\text{Mode} = l + \left(\frac{f - f_1}{2f - f_1 - f_2}\right) \times c$$

The class interval with maximum frequency is called the modal class.

Where l - lower limit of the modal class; f - frequency of the modal class

f_1 - frequency of the class just preceding the modal class

f_2 - frequency of the class succeeding the modal class

c - width of the class interval

Example 8.17 Find the mode for the following data.

Marks	1-5	6-10	11-15	16-20	21-25
No. of students	7	10	16	32	24

Solution

Marks	f
0.5-5.5	7
5.5-10.5	10
10.5-15.5	16
15.5-20.5	32
20.5-25.5	24

Note

To convert discontinuous class interval into continuous class interval, 0.5 is to be substracted at the lower limit and 0.5 is to be added at the upper limit for each class interval.

Modal class is 16 -20 since it has the maximum frequency.

$l = 15.5$, $f = 32$, $f_1 = 16$, $f_2=24$, $c = 20.5–15.5 = 5$

$$\text{Mode} = l + \left(\frac{f - f_1}{2f - f_1 - f_2}\right) \times c$$

$$= 15.5 + \left(\frac{32 - 16}{64 - 16 - 24}\right) \times 5$$

$$= 15.5 + \left(\frac{16}{24}\right) \times 5 = 15.5 + 3.33 = 18.83.$$

8.6.4 An Empirical Relationship between Mean, Medan and Mode

We have seen that there is an approximate relation that holds among the three averages discussed earlier, when the frequencies are nearly symmetrically distributed.

$$\text{Mode} \approx 3\ \text{Median} - 2\ \text{Mean}$$

Example 8.18 In a distribution, the mean and mode are 66 and 60 respectively. Calculate the median.

Solution

Given, Mean = 66 and Mode = 60.

Using,

$$\text{Mode} \approx 3\text{Median} - 2\text{Mean}$$

$$60 \approx 3\text{Median} - 2(66)$$

$$3\text{ Median} \approx 60 + 132$$

$$\text{Therefore, Median} \approx \frac{192}{3} \approx 64$$

Exercise 8.3

1. The monthly salary of 10 employees in a factory are given below :
₹5000, ₹7000, ₹5000, ₹7000, ₹8000, ₹7000, ₹7000, ₹8000, ₹7000, ₹5000
Find the mean, median and mode.

2. Find the mode of the given data : 3.1, 3.2, 3.3, 2.1, 1.3, 3.3, 3.1

3. For the data 11, 15, 17, x+1, 19, x–2, 3 if the mean is 14 , find the value of x. Also find the mode of the data.

4. The demand of track suit of different sizes as obtained by a survey is given below:

Size	38	39	40	41	42	43	44	45
No. of Persons	36	15	37	13	26	8	6	2

Which size is in greater demanded?

5. Find the mode of the following data:

Marks	0-10	10-20	20-30	30-40	40-50
Number of students	22	38	46	34	20

6. Find the mode of the following distribution:

Weight(in kgs)	25-34	35-44	45-54	55-64	65-74	75-84
Number of students	4	8	10	14	8	6

Exercise 8.4

Multiple choice questions

1. Let m be the mid point and b be the upper limit of a class in a continuous frequency distribution. The lower limit of the class is

   (1) $2m - b$ (2) $2m+b$ (3) $m-b$ (4) $m-2b$.

2. The mean of a set of seven numbers is 81. If one of the numbers is discarded , the mean of the remaining numbers is 78 . The value of discarded number is

   (1) 101 (2) 100 (3) 99 (4) 98.

3. A particular observation which occurs maximum number of times in a given data is called its

   (1) Frequency (2) range (3) mode (4) Median.

4. For which set of numbers do the mean, median and mode all have the same values?

   (1) 2,2,2,4 (2) 1,3,3,3,5 (3) 1,1,2,5,6 (4) 1,1,2,1,5.

5. The algebraic sum of the deviations of a set of n values from their mean is

   (1) 0 (2) n-1 (3) n (4) n+1.

6. The mean of a,b,c,d and e is 28 . If the mean of a, c and e is 24 , then mean of b and d is_

   (1) 24 (2) 36 (3) 26 (4) 34

7. If the mean of five observations x, x+2 , x+4 , x+6 , x+8 , is 11 , then the mean of first three observations is

   (1) 9 (2) 11 (3) 13 (4) 15.

8. The mean of 5, 9, x, 17, and 21 is 13, then find the value of x

   (1) 9 (2) 13 (3) 17 (4) 21

9. The mean of the square of first 11 natural numbers is

   (1) 26 (2) 46 (3) 48 (4) 52.

10. The mean of a set of numbers is $\bar{X}$. If each number is multiplied by z, the mean is

    (1) $\bar{X}+z$ (2) $\bar{X} - z$ (3) $z\ \bar{X}$ (4) $\bar{X}$

Project

1. Prepare a frequency table of the top speeds of 20 different land animals. Find mean, median and mode. Justify your answer.
2. From the record of students particulars of the class,
   (i) Find the mean age of the class (using class interval)
   (ii) Calculate the mean height of the class(using class intervals)

Points to Remember

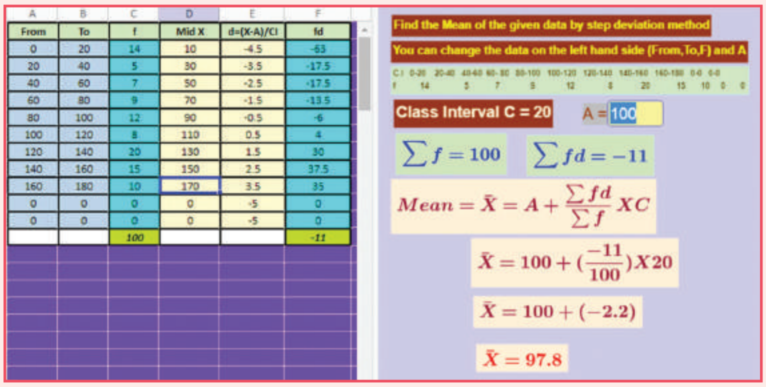
- The information collected for a definite purpose is called data.
- The data collected by the investigator are known as primary data. When the information is gathered from an external source,the data are called secondary data.
- Initial data obtained through unorganized form are called Raw data.
- Mid Value $= \frac{UCL + LCL}{2}$ (where UCL–Upper Class Limit, LCL–Lower Class Limit).
- Size of the class interval = $UCL - LCL$.
- The mean for grouped data:

Direct Method	Assumed Mean Method	Step-Deviation Method
$\bar{X} = \frac{\sum fx}{\sum f}$	$\bar{X} = A + \frac{\sum fd}{\sum f}$	$\bar{X} = A + \left[\frac{\sum fd}{\sum f} \times c\right]$

- The cumulative frequency of a class is the frequency obtained by adding the frequency of all up to the classes preceeding the given class.
- Formula to find the median for grouped data: Median $= l + \frac{\left(\frac{N}{2} - m\right)}{f} \times c$.
- Formula to find the mode for grouped data: Mode $= l + \left(\frac{f - f_1}{2f - f_1 - f_2}\right) \times c$.

ICT Corner

Expected Result is shown in this picture



From	To	f	Mid X	d=(X-A)/CI	fd
0	20	14	10	-4.5	-63
20	40	5	30	-3.5	-17.5
40	60	7	50	-2.5	-17.5
60	80	9	70	-1.5	-13.5
80	100	12	90	-0.5	-6
100	120	8	110	0.5	4
120	140	20	130	1.5	30
140	160	15	150	2.5	37.5
160	180	10	170	3.5	35
0	0	0	0	-5	0
0	0	0	0	-5	0
		100			-11

Find the Mean of the given data by step deviation method

You can change the data on the left hand side (From,To,F) and A

Class Interval C = 20

A = 100

$\sum f = 100$ $\quad \sum fd = -11$

$$Mean = \bar{X} = A + \frac{\sum fd}{\sum f} XC$$

$$\bar{X} = 100 + (\frac{-11}{100})X20$$

$$\bar{X} = 100 + (-2.2)$$

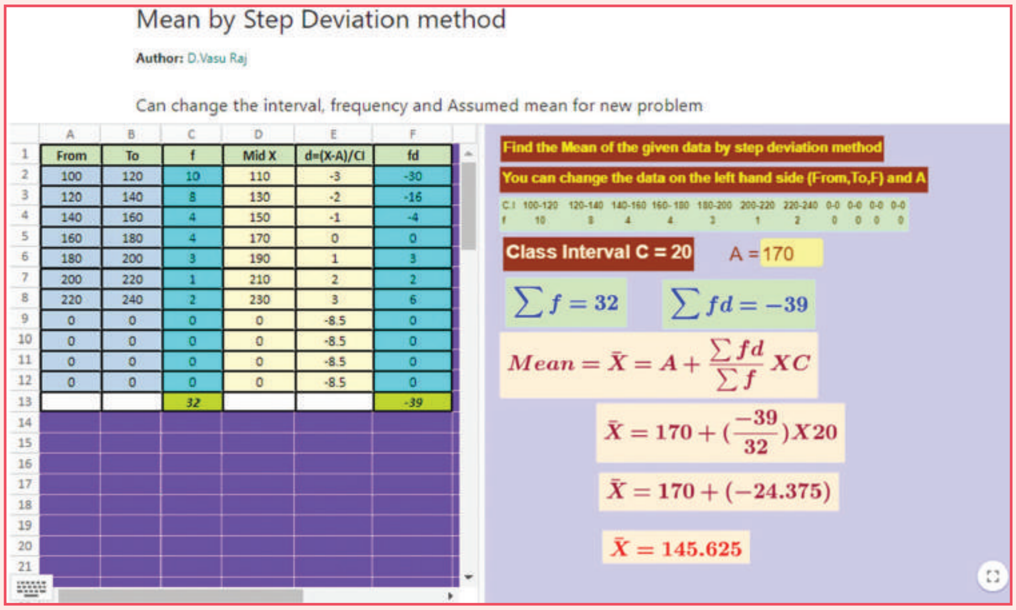
$$\bar{X} = 97.8$$

Step

Open the browser, type the URL Link given below (or) Scan the QR Code. GeoGebra work sheet named "Mean by step deviation method" will open.

In the work sheet Example 5.5 is given. Observe the steps. You can change the question by typing new data "From", "To" and "Frequency f" in the spread sheet on Left hand side. After that change, the Assumed mean on the right-hand side and check the calculation.

Step 1



Mean by Step Deviation method

Author: D.Vasu Raj

Can change the interval, frequency and Assumed mean for new problem

	From	To	f	Mid X	d=(X-A)/CI	fd
2	100	120	10	110	-3	-30
3	120	140	8	130	-2	-16
4	140	160	4	150	-1	-4
5	160	180	4	170	0	0
6	180	200	3	190	1	3
7	200	220	1	210	2	2
8	220	240	2	230	3	6
9	0	0	0	0	-8.5	0
10	0	0	0	0	-8.5	0
11	0	0	0	0	-8.5	0
12	0	0	0	0	-8.5	0
13			32			-39

Find the Mean of the given data by step deviation method

You can change the data on the left hand side (From,To,F) and A

C.I	100-120	120-140	140-160	160-180	180-200	200-220	220-240	0-0	0-0	0-0	0-0
f	10	8	4	4	3	1	2	0	0	0	0

Class Interval C = 20

A = 170

$\sum f = 32$ $\quad \sum fd = -39$

$$Mean = \bar{X} = A + \frac{\sum fd}{\sum f} XC$$

$$\bar{X} = 170 + (\frac{-39}{32})X20$$

$$\bar{X} = 170 + (-24.375)$$

$$\bar{X} = 145.625$$

Scan the QR Code

B566_9_MAT_EM_T3

9

PROBABILITY

Probability theory is nothing more than common sense reduced to calculation. - *Pierre Simon Laplace.*

Richard Von Mises
(AD (CE) 1883-1953)

The statistical or empirical, attitude towards probability has been developed mainly by R.F.Fisher and R.Von Mises. The notion of sample space comes from R.Von Mises. This notion made it possible to build up a strictly mathematical theory of probability based on measure theory. Such an approach emerged gradually in the last century under the influence of many authors.

Learning Outcomes

- To understand the basic concepts of probability.
- To understand the classical and empirical approach of probability.
- To familiarise the types of events in probability.

G1RPCP

9.1 Introduction

To understand the notion of probability, we look into some real life situations that involve some traits of uncertainty.

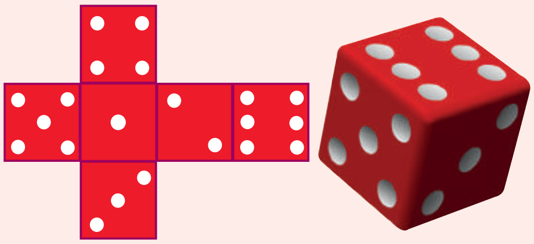
A life-saving drug is administered to a patient admitted in a hospital. The patient's relatives may like to know the probability with which the drug will work; they will be happy if the doctor tells that out of 100 patients treated with the drug, it worked well with more than 80 patients. This percentage of success is illustrative of the concept of probability; it is based on the frequency of occurrence. It helps one to arrive at a conclusion under uncertain conditions. Probability is thus a way of quantifying or measuring uncertainty.

You should be familiar with the usual complete pack of 52 playing cards. It has 4 suits(Hearts ♥, Clubs ♣, Diamonds ♦, Spades♠), each with 13 cards. Choose one of the suits or cards, say spades. Keep these 13 cards facing downwards on the table. Shuffle them well and pick up any one card. What is the chance that it will be a King? Will the chances vary if you do not want a King but an Ace? You will be quick to see that in either case, the chances are 1 in 13 (Why?). It will be the same whatever single card you choose to pick up. The word 'Probability' means precisely the same thing as 'chances' and has the same value, but instead of saying 1 in 13 we write it as a fraction $\frac{1}{13}$. (It would be easy to manipulate with fractions when we combine probabilities). It is 'the ratio of the favourable cases to the total number of possible cases'.

Have you seen a 'dice' ? (Some people use the word 'die' for a single 'dice'; we use 'dice' here, both for the singular and plural cases). A standard dice is a cube, with each side having a different number of spots on it, ranging from one to six, rolled and used in gambling and other games involving chance.

Note

In a fair die the sum of the numbers turning on the opposite sides will always be equal to 7.



If you throw a dice, what is the probability of getting a five? a two? a seven?

In all the answers you got for the questions raised above, did you notice anything special about the concept of probability? Could there be a maximum value for probability? or the least value? If you are sure of a certain occurrence what could be its probability? For a better clarity, we will try to formalize the notions in the following paragraphs.

9.2 Basic Ideas

When we carry out experiments in science repeatedly under identical conditions, we get almost the same result. Such experiments are known as *deterministic*. For example, the experiments to verify Archimedes principle or to verify Ohm's law are *deterministic*. The outcomes of the experiments can be predicted well in advance.

But, there are experiments in which the outcomes may be different even when performed under identical conditions. For example, when a fair dice is rolled, a fair coin is flipped or while selecting the balls from an urn, we cannot predict the exact outcome of these experiments; these are *random experiments*. Each performance of a random experiment is called a trial and the result of each trial is called an outcome. (*Note: Many statisticians use the words 'experiment' and 'trial' synonymously.*)

Now let us see some of the important terms related to probability.

Trial : Rolling a dice and flipping a coin are trials. A *trial* is an action which results in one or several outcomes.

Outcome : While flipping a coin we get Head or Tail . Head and Tail are called outcomes. The result of the trial is called an *outcome*.

Sample point : While flipping a coin, each outcome H or T are the sample points. Each outcome of a random experiment is called a *sample point*.

Sample space : In a single flip of a coin, the collection of sample points is given by $S = \{H,T\}$.

If two coins are tossed the collection of sample points $S=\{(HH),(HT),(TH),(TT)\}$.

The set of all possible outcomes (or Sample points) of a random experiment is called the *Sample space*. It is denoted by S. The number of elements in it are denoted by $n(S)$.

Event : If a dice is rolled, it shows 4 which is called an outcome (since, it is a result of a single trial). In the same experiment the event of getting an even number is {2,4,6}. So any subset of a sample space is called an *event*. Hence an event can be one or more than one outcome.

For example

(i) Random experiment : Flipping a coin

Possible outcomes : Head(H) or Tail(T)

Sample space : $S = \{H,T\}$

Subset of S : $A=\{H\}$ or $A=\{T\}$

Thus, in this example A is an event.

(ii) When we roll a single dice, the collection of all sample points is $S = \{1,2,3,4,5,6\}$. (iii) When we select a day in a week the collection of sample points is S = {Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}.

Activity - 1

Perform the experiment of tossing two coins at a time. List out the following in the above experiment.	Perform the experiment of throwing two dice at a time. List out the following in this experiment also.
Random experiment : Possible outcomes : Sample space : Any three subsets of S : (or any 3 events)	Random experiment : Possible outcomes : Sample space : Any three subsets of S : (or any 3 events)

Activity - 2

Each student is asked to flip a coin 10 times and tabulate the number of heads and tails obtained in the following table.

Number of tosses	**Number of times head comes up**	**Number of times tail comes up**
⋮	⋮	⋮

(i) Fraction 1 : $$\frac{Number\ of\ times\ head\ comes\ up}{Total\ number\ of\ times\ the\ coin\ is\ tossed}$$

(ii) Fraction 2 : $$\frac{Number\ of\ times\ tail\ comes\ up}{Total\ number\ of\ times\ the\ coin\ is\ tossed}$$

Repeat it by tossing the coin 20, 30, 40, 50 times and find the fractions.

Activity - 3

Divide the class students into groups of pairs. In each pair, the first one tosses a coin 50 times, and the second one records the outcomes of tosses. Then prepare a table given below.

Group	**Number of times head comes up**	**Number of times tail comes up**	$\frac{\textbf{Number of times head comes up}}{\textbf{Total number of times the coin is tossed}}$	$\frac{\textbf{Number of times tail comes up}}{\textbf{Total number of times the coin is tossed}}$
1				
2				
3				
⋮	⋮	⋮	⋮	⋮

9.3 Classical Approach

The chance of an event happening when expressed quantitatively is probability.

For example, An urn contains 4 Red balls and 6 Blue balls. You choose a ball at random from the urn. What is the probability of choosing a Red ball?

The phrase ‘at random’ assures you that each one of the 10 balls has the same chance (that is, probability) of getting chosen. You may be blindfolded and the balls may be mixed up for a “fair” experiment. This makes the outcomes “equally likely”.

The probability that the Red Ball is chosen is $\frac{4}{10}$ (You may also give it as $\frac{2}{5}$ or 0.4).

What would be the probability for choosing a Blue ball? It is $\frac{6}{10}$ (or $\frac{3}{5}$ or 0.6).

Note that the sum of the two probabilities is 1. This means that no other outcome is possible.

The approach we adopted in the above example is classical. It is calculating a *priori probability*. (The Latin phrase a *priori* means ‘*without investigation or sensory experience*’). Note that the above treatment is possible only when the outcomes are equally likely.

Classical probability is so named, because it was the first type of probability studied formally by mathematicians during the 17th and 18th centuries.

Thinking Corner

If the probability of success of an experiment is 0.4, what is the probability of failure?

Let S be the set of all equally likely outcomes of a random experiment. (S is called the sample space for the experiment.)

Let E be some particular outcome or combination of outcomes of an experiment. (E is called an event.)

The probability of an event E is denoted as $P(E)$.

$$P(E) = \frac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}} = \frac{n(E)}{n(S)}$$

G21KEC

The empirical approach (relative frequency theory) of probability holds that if an experiment is repeated for an extremely large number of times and a particular outcome occurs at a percentage of the time, then that particular percentage is close to the probability of that outcome.

9.4 Empirical Approach

For example, A manufacturer produces 10,000 electric switches every month and1,000 of them are found to be defective. What is the probability of the manufacturer producing a defective switch every month?

Note

The number of trials has to be large to decide this probability. The larger the number of trials, the better will be the estimate of probability.

The required probability, according to relative frequency concept, is nearly 1000 out of 10000, which is 0.1

Let us formalize the definition: "If, the total number of trials, say n, we find r of the outcomes in an event E, then the probability of event E, denoted by $P(E)$, is given by

$$P(E) = \frac{r}{n}.$$

Is there a guarantee that this value will settle down to a constant value when the number of trials gets larger and larger? One cannot say; the concept being experimental, it is quite possible to get distinct relative frequency each time the experiment is repeated.

However, there is a security range: the value of probability can at the least take the value 0 and at the most take the value 1. We can state this mathematically as

$$0 \le P(E) \le 1.$$

Thinking Corner

For a question on probability the student's answer was $\frac{3}{2}$.
The teacher told that
the answer was wrong. Why?

Let us look at this in a little detail.

First, we know that r cannot be larger than n.

This means $\frac{r}{n} < 1$. That is $P(E) < 1$. … (1)

Next, if $r = 0$, it means either the event cannot happen or has not occurred in a large number of trials. (Can you get a 7, when you roll a dice?).

Thus , in this case $\frac{r}{n} = \frac{0}{n} = 0$. … (2)

Lastly, if $r = n$, the event must occur (in every trial or in a large number of trials).

In such a situation, $\frac{r}{n} = \frac{n}{n} = 1$. … (3)

(getting any number from 1 to 6 when you roll a dice)

From (1), (2) and (3) we find $0 \le P(E) \le 1$.

Progress Check

A random experiment was conducted. Which of these cannot be considered as a probability of an outcome?

(i) 1/5 (ii) – 1/7 (iii) 0.40 (iv) – 0.52 (v) 0

(vi) 1.3 (vii) 1 (viii) 72% (ix) 107%

Example 9.1 When a dice is rolled, find the probability to get the number which is greater than 4?

Solution

The outcomes $S = \{1,2,3,4,5,6\}$

Let E be the event of getting a number greater than 4

$$E = \{5,6\}$$

$$P(E) = \frac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}}$$

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{6} = 0.333...$$

G2AGFZ

Example 9.2 In an office, where 42 staff members work, 7 staff members use cars, 20 staff members use two-wheelers and the remaining 15 staff members use cycles. Find the relative frequencies.

Solution

Total number of staff members = 42.

The relative frequencies:

$$\text{Car users} = \frac{7}{42} = \frac{1}{6}$$

$$\text{Two-wheeler users} = \frac{20}{42} = \frac{10}{21}$$

$$\text{Cycle users} = \frac{15}{42} = \frac{5}{14}$$

In this example note that the total probability does not exceed 1 that is,

$$\frac{1}{6} + \frac{10}{21} + \frac{5}{14} = \frac{7}{42} + \frac{20}{42} + \frac{15}{42} = 1$$

Example 9.3 Team I and Team II play 10 cricket matches each of 20 overs. Their total scores in each match are tabulated in the table as follows:

Match numbers	1	2	3	4	5	6	7	8	9	10
Team I	200	122	111	88	156	184	99	199	121	156
Team II	143	123	156	92	164	72	100	201	98	157

What is the relative frequency of Team I winning?

Solution

In this experiment, each trial is a match where Team I faces Team II.

We are concerned about the winning status of Team I.

There are 10 trials in total; out of which Team I wins in the 1st, 6th and 9th matches.

The relative frequency of Team I winning the matches $= \frac{3}{10}$ or 0.3.

(**Note :** The relative frequency depends on the sequence of outcomes that we observe during the course of the experiment).

Exercise 9.1

1. You are walking along a street. If you just choose a stranger crossing you, what is the probability that his next birthday will fall on a sunday?

2. What is the probability of drawing a King or a Queen or a Jack from a deck of cards?

3. What is the probability of throwing an even number with a single standard dice of six faces?

4. There are 24 balls in a pot. If 3 of them are Red, 5 of them are Blue and the remaining are Green then, what is the probability of picking out (i) a Blue ball, (ii) a Red ball and (iii) a Green ball?

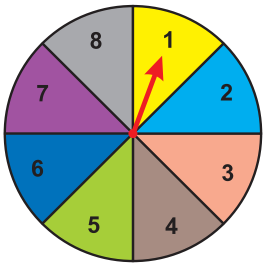
5. When two coins are tossed, what is the probability that two heads are obtained?

6. Two dice are rolled, find the probability that the sum is

   i) equal to 1 ii) equal to 4 iii) less than 13

7. A manufacturer tested 7000 LED lights at random and found that 25 of them were defective. If a LED light is selected at random, what is the probability that the selected LED light is a defective one.

8. In a football match, a goalkeeper of a team can stop the goal, 32 times out of 40 attempts tried by a team. Find the probability that the opponent team can convert the attempt into a goal.



9. What is the probability that the spinner will not land on a multiple of 3?

10. Frame two problems in calculating probability, based on the spinner shown here.

9.5 Types of Events

We have seen some important cases of events already.

When the likelihood of happening of two events are same they are known as equally likely events.

- If we toss a coin, getting a head or a tail are equally likely events.
- If a dice is rolled, then getting an odd number and getting an even number are equally likely events, whereas getting an even number and getting 1 are not equally likely events.

When probability is 1, the event is sure to happen. Such an event is called a sure or certain event. The other extreme case is when the probability is 0, which is known as an impossible event.

If $P(E) = 1$ then E is called Certain event or Sure event.

If $P(E) = 0$ then E is known is an Impossible event.

Consider a "coin flip". When you flip a coin, you cannot get both heads and tails simultaneously. (Of course, the coin must be fair; it should not have heads or tails on both sides!). If two events cannot occur simultaneously (at the same time), in a single trial they are said to be mutually exclusive events. Are rain and sunshine mutually exclusive? What about choosing Kings and Hearts from a pack of 52 cards?

A dice is thrown. Let E be the event of getting an "even face". That is getting 2, 4 or 6. Then the event of getting an "odd face" is complementary to E and is denoted by E' or E^c. In the above sense E and E' are complementary events.

Note

- The events E and E' are mutually exclusive. (how?)
- The probability of E + the probability of $E'=1$. Also E and E' are mutually exclusive and exhaustive.
- Since $P(E)+P(E')=1$, if you know any one of them, you can find the other.

Progress Check

Which among the following are mutually exclusive?

Sl.No.	Trial	Event 1	Event 2
1	Roll a dice	getting a 5	getting an odd number
2	Roll a dice	getting a 5	getting an even number
3	Draw a card from a standard pack	getting a Spade Card	getting a black
4	Draw a card from a standard pack	getting a Picture Card	getting a 5
5	Draw a card from a standard pack	getting a Heart Card	getting a 7

Example 9.4 The probability that it will rain tomorrow is $\frac{91}{100}$. What is the probability that it will not rain tomorrow?

Solution

Let E be the event that it will rain tomorrow. Then E' is the event that it will not rain tomorrow.

Since $P(E)$ = 0.91, we have $P(E') = 1-0.91$ (how?)

$= 0.09$

Therefore, the probability that it will not rain tomorrow

$= 0.09$

Example 9.5

In a recent year, of the 1184 centum scorers in various subjects in tenth standard public exams, 233 were in mathematics. 125 in social science and 106 in science. If one of the student is selected at random, find the probability of that selected student,

(i) is a centum scorer in Mathematics (ii) is not a centum scorer in Science

Solution

Total number of centum scorers =1184

Therefore $n = 1184$

(i) Let E_1 be the event of getting a centum scorer in Mathematics.

Therefore $n(E_1) = 233$, That is, $r_1 = 233$

$$P(E_1) = \frac{r_1}{n} = \frac{233}{1184}$$

(ii) Let E_2 be the event of getting a centum scorer in Science.

Therefore $n(E_2) = 106$, That is, $r_2 = 106$

$$P(E_2) = \frac{r_2}{n} = \frac{106}{1184}$$

$$P(E_2') = 1 - P(E_2)$$

$$= 1 - \frac{106}{1184}$$

$$= \frac{1078}{1184}$$

Exercise 9.2

1. A company manufactures 10000 Laptops in 6 months. Out of which 25 of them are found to be defective. When you choose one Laptop from the manufactured, what is the probability that selected Laptop is a good one.

2. In a survey of 400 youngsters aged 16-20 years, it was found that 191 have their voter ID card. If a youngster is selected at random, find the probability that the youngster does not have their voter ID card.

3. The probability of guessing the correct answer to a certain question is $\frac{x}{3}$. If the probability of not guessing the correct answer is $\frac{x}{5}$, then find the value of x.

4. If a probability of a player winning a particular tennis match is 0.72. What is the probability of the player loosing the match?

5. 1500 families were surveyed and following data was recorded about their maids at homes

Type of maids	Only part time	Only full time	Both
Number of families	860	370	250

A family is selected at random. Find the probability that the family selected has

(i)Both types of maids (ii)Part time maids (iii)No maids

Exercise 9.3

Multiple choice questions

1. A number between 0 and 1 that is used to measure uncertainty is called

(1) Random variable (2) Trial (3) Simple event (4) Probability

2. Probability lies between

(1) -1 and $+1$ (2) 0 and 1 (3) 0 and n (4) 0 and ∞

3. The probability based on the concept of relative frequency theory is called

(1) Empirical probability (2) Classical probability

(3) Both (1) and (2) (4) Neither (1) nor (2)

4. The probability of an event cannot be

(1) Equal to zero (2) Greater than zero (3) Equal to one (4) Less than zero

5. The probability of all possible outcomes of a random experiment is always equal to

(1) One (2) Zero (3) Infinity (4) Less than one

6. If A is any event in S and its complement is A' then, $P(A')$ is equal to

(1) 1 (2) 0 (3) $1-A$ (4) $1-\text{P}(A)$

7. Which of the following cannot be taken as probability of an event?

(1) 0 (2) 0.5 (3) 1 (4) -1

8. A particular result of an experiment is called

(1) Trial (2) Simple event (3) Compound event (4) Outcome

9. A collection of one or more outcomes of an experiment is called

(1) Event (2) Outcome (3) Sample point (4) None of the above

10. The six faces of the dice are called equally likely if the dice is

(1) Small (2) Fair (3) Six-faced (4) Round

Points to Remember

- If we are able to predict the exact outcome of an experiment then it is called deterministic experiment.
- If we cannot predict the exact outcome of an experiment then it is called random experiment.
- Sample space S for a random experiment is the set of all possible outcomes of a random experiment.
- An event is a particular outcome or combination of outcomes of an experiment.
- Empirical probability states that probability of an outcome is close to the percentage of occurrence of the outcome.
- If the likelyhood of happening of two events are same then they are known as equally likely events.
- If two events cannot occur simultaneously in single trial then they are said to be mutually exclusive events.
- Two events E and E' are said to be complementary events if $P(E)+P(E')=1$.
- An event which is sure to happen is called certain or sure event. The probability of a sure event is always one.
- An event which never happen is called impossible event. The probability of an impossible event is always zero.

ICT Corner

Expected Result is shown
in this picture

There were 6 Red balls, 3 Blue balls and 6 Yellow balls in an urn.
Find the probability of (i) Red Balls (ii) Blue Balls and (iii) Yellow balls.

No. of Red Balls = 6
No. of Blue balls = 3
No.of Yellow Balls = 6
Total No. of Balls =6+3+ 6 = 15

$Probability = \frac{favourable}{Total}$

☐ Probability of Red Balls =
☑ Probability of Blue Balls = $\frac{No.\ of\ Blue\ Balls}{Total No.of Balls} = \frac{3}{15}$
☐ Probability of Yellow Balls =

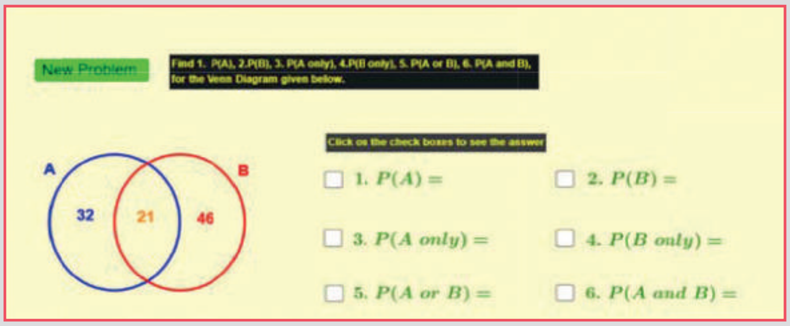
Step – 1

Open the Browser by typing the URL Link given below (or) Scan the QR Code. GeoGebra work sheet named “Probability” will open. There are two worksheets under the title Venn diagram and Basic probability.

Step - 2

Click on “New Problem”. Work out the solution, and click on the respective check box and check the answer.

Step 1

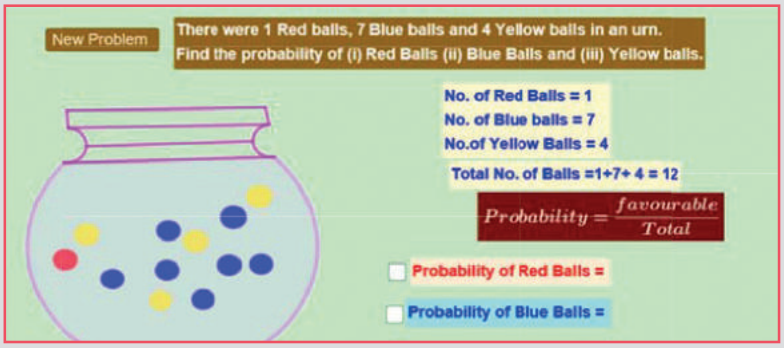


Find 1. P(A), 2.P(B), 3. P(A only), 4.P(B only), 5. P(A or B), 6. P(A and B), for the Venn Diagram given below.

Click on the check boxes to see the answer

☐ 1. $P(A) =$
☐ 2. $P(B) =$
☐ 3. $P(A\ only) =$
☐ 4. $P(B\ only) =$
☐ 5. $P(A\ or\ B) =$
☐ 6. $P(A\ and\ B) =$

Step 2



There were 1 Red balls, 7 Blue balls and 4 Yellow balls in an urn.
Find the probability of (i) Red Balls (ii) Blue Balls and (iii) Yellow balls.

No. of Red Balls = 1
No. of Blue balls = 7
No.of Yellow Balls = 4
Total No. of Balls =1+7+ 4 = 12

$Probability = \frac{favourable}{Total}$

☐ Probability of Red Balls =
☐ Probability of Blue Balls =

can the QR Code.

B566_9_MAT_EM_T3

ANSWERS

1 Set Language

Exercise 1.1

1. (i) set (ii) not a set (iii) Set (iv) not a set

2. (i) {I, N, D, A} (ii) {P, A, R, L, E, O, G, M} (iii) {M, I, S, P}

(iv) {C, Z, E, H, O, S, L, V, A, K, I}

3. (a) (i) True (ii) True (iii) False (iv) True (v) False (vi) False

(b) (i) A (ii) C (iii) $\notin$ (iv) $\in$

4. (i) $A = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$ (ii) $B = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}\right\}$

(iii) $C = \{64, 125\}$ (iv) $D = \{-4, -3, -2, -1, 0, 1, 2\}$

5. (i) $B = \{x : x$ is an Indian player who scored double centuries in One Day International$\}$

(ii) $C = \left\{x : x = \frac{n}{n+1}, n \in \mathbb{N}\right\}$ (iii) $D = \{x : x$ is a tamil month in a year$\}$

(iv) $E = \{x : x$ is an odd whole number less than 9$\}$

6. (i) P = The set of English months starting with letter '*J*'

(ii) Q = The set of Prime numbers between 5 and 31

(iii) R = The set of natural numbers less than 5

(iv) S = The set of English consonants

Exercise 1.2

1. (i) $n(M) = 6$ (ii) $n(P) = 5$ (iii) $n(Q) = 3$ (iv) $n(R) = 10$ (v) $n(S) = 5$

2. (i) finite (ii) infinite (iii) infinite (iv) finite

3. (i) Equivalent sets (ii) Unequal sets (iii) Equal sets (iv) Equivalent sets

4. (i) null set (ii) null set (iii) singleton set (iv) null set

5. (i) overlapping (ii) disjoint (iii) overlapping

6. (i) {square, rhombus} (ii) {circle} (iii) {triangle} (iv) { }

7. $\{\ \}, \{a\}, \{a, b\}, \{a, \{a, b\}\}$

8. (i) $\{\{\ \}, \{a\}, \{b\}, \{a, b\}\}$

(ii) $\{\{\ \}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\ \}, \{1, 2, 3\}\}$

(iii) $\{\{\ \}, \{p\}, \{q\}\ \{r\}, \{s\}, \{p, q\}, \{p, r\}, \{p, s\}, \{q, r\}, \{q, s\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}, \{p, q, r, s\}\}$ (iv) $P(E)= \{\{\ \}\}$

9. (i) 8, 7 (ii) 1024, 1023

10. (i) 16 (ii) 1 (iii) 8

Exercise 1.3

1. (i) {2, 4, 7, 8, 10} (ii) {3, 4, 6, 7, 9, 11} (iii) {2, 3, 4, 6, 7, 8, 9, 10, 11}

(iv) {4, 7} (v) {2, 8, 10} (vi) {3, 6, 9, 11}

(vii) {1, 3, 6, 9, 11, 12} (viii) {1, 2, 8, 10, 12}

(ix) {1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12}

2. (i) {2, 5, 6, 10, 14, 16}, {2, 14}, {6, 10}, {5, 16}

(ii) $\{a, b, c, e, i, o, u\}, \{a, e, u\}, \{b, c\}, \{i, o\}$

(iii) {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, {1, 2, 3, 4, 5}, {6, 7, 8, 9, 10,}, {0}

(iv) $\{m, a, t, h, e, i, c, s, g, o, r, y\}, \{e, m, t,\}, \{a, h, i, c, s\}, \{g, o, r, y\}$

3. (i) $\{a, c, e, g\}$ (ii) $\{b, c, f, g\}$ (iii) $\{a, b, c, e, f, g\}$ (iv) $\{c, g\}$ (v) $\{c, g\}$

(vi) $\{a, b, c, e, f, g\}$ (vii) $\{b, d, f, h\}$ (viii) $\{a, d, e, h\}$

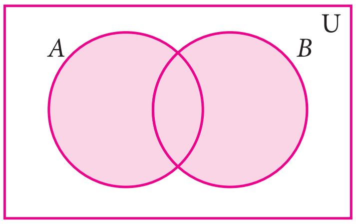
4. (i) {0, 2, 4, 6} (ii) {1, 4, 6} (iii) {0, 1, 2, 4, 6} (iv) {4, 6} (v) {4, 6}

(vi) {0, 1, 2, 4, 6} (vii) {1, 3, 5, 7} (viii) {0, 2, 3, 5, 7}

5. (i) {1, 2, 7} (ii) {*m*, *o*, *p*, *q*, *j*} (iii) {6, 9, 10}

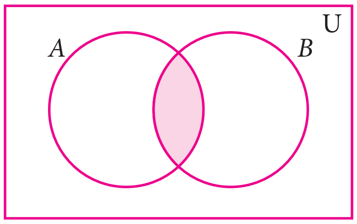
6. (i) $Y-X$ (ii) $(X\cup Y)'$ (iii) $(X-Y)\cup(Y-X)$

7. (i)



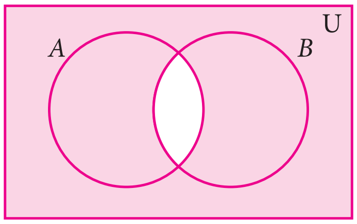
$A\cup B$

(ii)



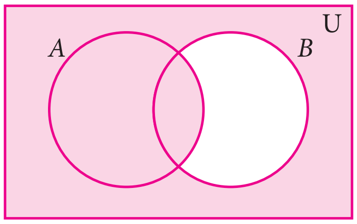
$A\cap B$

(iii)



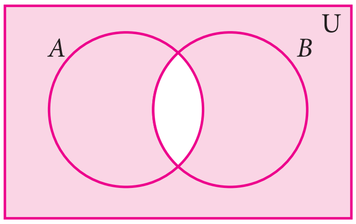
$(A\cap B)'$

(iv)



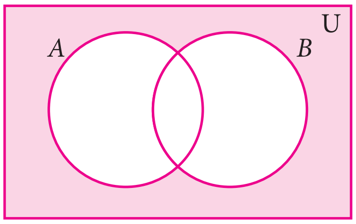
$(B-A)'$

(v)



$A'\cup B'$

(vi)



$A'\cap B'$

(vii) $(A\cap B)' = A'\cup B'$

Exercise 1.4

1. (i) {1,2,3,4,5,7,9,11} (ii) {2,5} (iii) {3,5 }

Exercise 1.5

1. (i) $\{3,4,6\}$ (ii) $\{-1,5,7\}$ (iii) $\{-3,0,1,2,3,4,5,6,7,8\}$

(iv) $\{-3,0,1,2\}$ (v) $\{1,2,4,6\}$ (vi) $\{4,6\}$ (vii) $\{-1,3,4,6\}$

2. (i) $\{a,b,c,d,e,f\}$ (ii) $\{a,b,d\}$ (iii) { a,b,c,d,e,f } (iv) {a, b, d }

Exercise 1.6

1. (i) 15, 65 (ii) 250, 600 4. (1) 17 (ii) 22 (iii) 47

5. (i) 10 (ii) 10 (iii) 25 6. 1000 7. 8 8. Not correct

9. (i) 185 (ii) 141 (iii) 326 10. 70

11. $x=20$, $y=40$, $z=30$ 12. (i) 5 (ii) 7 (iii)8

13. 5

Exercise 1.7

1. (2) 2. (1) 3. (3) 4. (2) 5. (4) 6. (1) 7. (2) 8. (4) 9. (3) 10. (4)

11. (2) 12. (1) 13. (1) 14. (3) 15. (4) 16. (1) 17. (4) 18. (3) 19. (3) 20. (1)

2 Real Numbers

Exercise 2.1

1. D 2. $-\frac{6}{11}, -\frac{5}{11}, -\frac{4}{11}, \dots \frac{1}{11}$

3. (i) $\frac{9}{40}, \frac{19}{80}, \frac{39}{160}, \frac{79}{320}, \frac{159}{640}$;
The given answer is one of the answers. There can be many more answers

(ii) 0.101, 0.102, ... 0.109
The given answer is one of the answers. There can be many more answers

(iii) $-\frac{3}{2}, -\frac{5}{4}, -\frac{9}{8}, -\frac{17}{16}, -\frac{33}{32}$
The given answer is one of the answers. There can be many more answers

Exercise 2.2

1. (i) 0.2857142..., Non terminating and recurring (ii) $-5.\overline{27}$, Non terminating and recurring

(iii) $7.\overline{3}$, Non terminating and recurring (iv) 1.635, Terminating

2. $0.\overline{076293}$, 6 3. $0.03\overline{03}$, $2.\overline{15}$

4. (i) $\frac{24}{99}$ (ii) $\frac{2325}{999}$ (iii) $-\frac{1283}{250}$ (iv) $\frac{143}{45}$ (v) $\frac{5681}{330}$ (vi) $-\frac{190924}{9000}$

5. (i) Terminating (ii) Terminating (iii) Non terminating (iv) Non terminating

Exercise 2.3

2. (i) 0.301202200222..., 0.301303300333... (ii) 0.8616611666111 ..., 0.8717711777111 ...
(iii) 1.515511555..., 1.616611666...

3. 2.2362, 2.2363

Exercise 2.5

1.(i) 5^4 (ii) 5^{-1} (iii) $5^{\frac{1}{2}}$ (iv) $5^{\frac{3}{2}}$

2.(i) 4^2 (ii) $4^{\frac{3}{2}}$ (iii) $4^{\frac{5}{2}}$

3.(i) 7 (ii) 9 (iii) $\frac{1}{27}$ (iv) $\frac{25}{16}$

4.(i) $5^{\frac{1}{2}}$ (ii) $7^{\frac{1}{2}}$ (iii) $7^{\frac{10}{3}}$ (iv) $10^{\frac{-14}{3}}$

5.(i) 2 (ii) 3 (iii) 10 (iv) $\frac{4}{5}$

Exercise 2.6

1.(i) $21\sqrt{3}$ (ii) $3\sqrt[3]{5}$ (iii) $26\sqrt{3}$ (iv) $8\sqrt[3]{5}$

2. (i) $\sqrt{30}$ (ii) $\sqrt{5}$ (iii) 30 (iv) $49a-25b$

(v) $\frac{5}{16}$ 3.(i) 1.852 (ii) 23.978

4. (i) $\sqrt[3]{5} > \sqrt[6]{3} > \sqrt[9]{4}$ (ii) $\sqrt{\sqrt{3}} > \sqrt[2]{\sqrt[3]{5}} > \sqrt[3]{\sqrt[4]{7}}$

5. (i) yes (ii) yes (iii) yes (iv) yes

6. (i) yes (ii) yes (iii) yes (iv) yes

Exercise 2.7

1.(i) $\frac{\sqrt{2}}{10}$ (ii) $\frac{\sqrt{5}}{3}$ (iii) $\frac{5\sqrt{6}}{6}$ (iv) $\frac{\sqrt{30}}{2}$

2. (i) $\frac{4}{3}(5+2\sqrt{6})$ (ii) $13-4\sqrt{6}$ (iii) $\frac{9+4\sqrt{30}}{21}$ (iv) $-2\sqrt{5}$

3. $a=\frac{-4}{3}, b=\frac{11}{3}$ 4. $x^2+\frac{1}{x^2}=18$ 5. 5.414

Exercise 2.8

1. (i) 5.6943×10^{11} (ii) 2.00057×10^{3} (iii) 6.0×10^{-7} (iv) 9.000002×10^{-4}

2. (i) 3459000 (ii) 56780 (iii) 0.0000100005 (iv) 0.0000002530009

3. (i) 1.44×10^{28} (ii) 8.0×10^{-60} (iii) 2.5×10^{-36}

4.(i) 7.0×10^{9} (ii) 9.4605284×10^{15} km (iii) 9.1093822×10^{-31} kg

5. (i) 1.505×10^{8} (ii) 1.5522×10^{17} (iii) 1.224×10^{7} (iv) 1.9558×10^{-1}

Exercise 2.9

1. (4) 2. (3) 3. (2) 4. (1) 5. (4) 6. (2) 7. (2) 8. (2) 9. (4) 10. (1)

11. (4) 12. (4) 13. (4) 14. (2) 15. (2) 16. (3) 17. (2) 18. (4) 19. (2) 20. (3)

3 Algebra

Exercise 3.1

1. (i) not a polynomial (ii) polynomial (iii) not a polynomial
(iv) polynomial (v) polynomial (vi) not a polynomial

2.

	Cooefficient of x^2	Cooefficient of x
(i)	$\frac{2}{5}$	-3
(ii)	-2	$-\sqrt{7}$
(iii)	π	-1
(iv)	$\sqrt{3}$	$\sqrt{2}$
(v)	1	$-\frac{7}{2}$

3. (i) 7 (ii) 4 (iii) 5 (iv) 6 (v) 4

4.

	Descending order	Ascending order
(i)	$\sqrt{7}\,x^3 + 6x^2 + x - 9$	$-9 + x + 6x^2 + \sqrt{7}\,x^3$
(ii)	$-\frac{7}{2}x^4 - 5x^3 + \sqrt{2}\,x^2 + x$	$x + \sqrt{2}\,x^2 - 5x^3 - \frac{7}{2}x^4$
(iii)	$7x^3 - \frac{6}{5}x^2 + 4x - 1$	$-1 + 4x - \frac{6}{5}x^2 + 7x^3$
(iv)	$9y^4 + \sqrt{5}\,y^3 + y^2 - \frac{7}{3}y - 11$	$-11 - \frac{7}{3}y + y^2 + \sqrt{5}\,y^3 + 9y^4$

5. (i) $6x^3+6x^2-14x+17$, 3 (ii) $7x^3+7x^2+11x-8$, 3 (iii) $16x^4-6x^3-5x^2+7x-6$, 4

6. (i) $7x^2+8$, 2 (ii) $-y^3+6y^2-14y+2$, 3 (iii) $z^5-6z^4-6z^2-9z+7$, 5

7. $x^3-8x^2+11x+7$ 8. $2x^4-3x^3+5x^2-5x+6$

9. (i) $6x^4+7x^3-56x^2-63x+18$, 4 (ii) $105x^2-33x-18$, 2 (iii) $30x^3-77x^2+54x-7$, 3

10. x^2+y^2+2xy, ₹ 225 11. $9x^2-4$, 3596 sq. units

12. cubic polynomial or polynomial of degree 3

Exercise 3.2

1. (i) 6 (ii) –6 (iii) 3 2. 1 3. (i) 3 (ii) $-\frac{5}{2}$ (iii) $\frac{3}{2}$ (iv) 0 (v) 0 (vi) $-\frac{b}{a}$

4. (i) $\frac{6}{5}$ (ii) –3 (iii) $-\frac{9}{10}$ (iv) $\frac{4}{9}$

6. (i) 2 (ii) 3 (iii) 0 (iv) 1 (v) 1

Exercise 3.3

1. p(x) is not a multiple of g(x)

2. (i) Remainder : 0 (ii) Remainder : $\frac{3}{2}$ (iii) Remainder : 62

3. Remainder : –143 4. Remainder : 2019 5. $K = 8$

6. $a = -3$, Remainder : 27 7. (i) $(x-1)$ is a factor (ii) $(x-1)$ is not a factor

8. $(x-5)$ is a factor of $p(x)$ 9. $m = 10$ 11. $k = 3$ 12. Yes

Exercise 3.4

1. (i) $4x^2 + 9y^2 + 16z^2 + 12xy + 24yz + 16xz$ (ii) $p^2 + 4q^2 + 9r^2 - 4pq + 12qr - 6pr$

(iii) $8p^3 - 24p^2 - 14p + 60$ (iv) $27a^3 + 27a^2 - 18a - 8$

2.(i) 18,107,210 (ii) −32, −6, +90

3. (i) 14 (ii) $\frac{59}{70}$ (iii) 78 (iv) $\frac{78}{70}$

4. (i) $27a^3 - 64b^3 - 108a^2b + 144ab^2$ (ii) $x^3 + \frac{1}{y^3} + \frac{3x^2}{y} + \frac{3x}{y^2}$

5.(i) 941192 (ii) 1003003001

6. 29 7. 280 8. 335 9. 198 10. ± 5, ± 110

11. 36 12.(i) $8a^3 + 27b^3 + 64c^3 - 72abc$ (ii) $x^3 - 8y^3 + 27z^3 + 18xyz$

13.(i) −630 (ii) $\frac{-9}{4}$

14. 72xyz

Exercise 3.5

1.(i) $2a^2(1 + 2b + 4c)$ (ii) $(a - m)(b - c)$

2.(i) $(x+2)^2$ (ii) $3(a - 4b)^2$

(iii) $x(x+2)(x-2)(x^2+4)$ (iv) $\left(m + \frac{1}{m} + 5\right)\left(m + \frac{1}{m} - 5\right)$

(v) $6(1+6x)(1-6x)$ (vi) $\left(a - \frac{1}{a} + 4\right)\left(a - \frac{1}{a} - 4\right)$

3. (i) $(2x + 3y + 5z)^2$

(ii) $(-5x + 2y + 3z)^2$ (or) $(5x - 2y - 3z)^2$

4. (i) $(2x + 5y)(4x^2 - 10xy + 25y^2)$

(ii) $(3x - 2y)(9x^2 + 6xy + 4y^2)$

(iii) $(a+2)(a-2)(a^2 + 4 - 2a)(a^2 + 4 + 2a)$

5. (i) $(x + 2y - 1)(x^2 + 4y^2 + 1 - 2xy + 2y + x)$

(ii) $(l - 2m - 3n)(l^2 + 4m^2 + 9n^2 + 2lm - 6mn + 3ln)$

Exercise 3.6

1.(i) $(x+6)(x+4)$

(ii) $(z+6)(z-2)$

(iii) $(p-8)(p+2)$

(iv) $(t-9)(t-8)$

(v) $(y-20)(y+4)$

(vi) $(a+30)(a-20)$

2. (i) $(2a+5)(a+2)$

(ii) $(x-7y)(5x+6y)$ (iii) $(2x-3)(4x-3)$ (iv) $2(3x+2y)(x+2y)$

(v) $3x^2(3y+2)^2$ (vi) $(a+b+6)(a+b+3)$

3. (i) $(p-q-8)(p-q+2)$

(ii) $(m+6n)(m-4n)$ (iii) $\left(a+\sqrt{5}\right)\left(\sqrt{5}a-3\right)$ (iv) $(a+1)(a-1)(a^2-2)$

(v) $m(4m+5n)(2m-3n)$

(vi) $\left(\frac{1}{x}+\frac{1}{y}\right)^2$

Exercise 3.7

1. (i) Quotient : $4x^2-6x-5$, Remainder : 33 (ii) Quotient : $4y^2-6y+5$, Remainder : –10

(iii) Quotient : $4x^2+2x+1$, Remainder : 0 (iv) Quotient : $8z^2-6z+2$, Remainder : 10

2. Length : $x+4$ 3. Height : $5x-4$ 4. Mean : $x^2-5x+25$

5. (i) $x^2+4x+5,\ 12$ (ii) $(x^2-1),\ -2$

(iii) $3x^2-11x+40,\ -125$ (iv) $2x^3-\frac{x^2}{2}-\frac{3x}{8}+\frac{51}{32},\ \frac{109}{32}$

6. $4x^3-2x^2+3,\ p=-2,\ q=0,$ remainder=–10

7. $a=20,\ b=94$ & remainder=388

Exercise 3.8

1.(i) $(x-2)(x+3)(x-4)$ (ii) $(x+1)(x-2)(2x-1)$

(iii) $(x-1)(2x-1)(2x+3)$ (iv) $(x+2)(x+3)(x-4)$

(v) $(x-1)(x-2)(x+3)$ (vi) $(x-1)(x-10)(x+1)$

Exercise 3.9

1. (i) p^5 (ii) 1 (iii) $3a^2b^2c^3$ (iv) $16x^6$

(v) abc (vi) $7xyz^2$ (vii) 25ab (viii) 1

2. (i) 1 (ii) a^{m+1} (iii) $(2a+1)$ (iv) 1

(v) $(x+1)(x-1)$ (vi) $(a-3x)$

Exercise 3.10

2. (i) (5,2) (ii) Infinite number of solutions (iii) no solution

(iv) (−3, −3) (v) (1,3) (vi) (−3, 3)

3. 75*km*/*hr*, 25*km*/*hr*

Exercise 3.11

1.(i)(2, −1) (ii) (4,2) (iii) (40,100) (iv) $\left(\sqrt{8},\sqrt{3}\right)$

(2) 45 (3) 409

Exercise 3.12

1.(i) (2,1) (ii) (7,2) (iii) (80,30) (iv) $\left(1,\frac{3}{2}\right)$

(v) $\left(\frac{1}{3},-1\right)$ (vi) (2,4) (2) ₹30000, ₹40000 (3) 75, 15

Exercise 3.13

1.(i) (3,4) (ii) (3, −1) (iii) $\left(-\frac{1}{2},\frac{1}{3}\right)$

(2) Number of 2 rupee coins 60; Number of 5 rupee coins 20

(3) Larger pipe 40 hours; Smaller pipe 60 hours

Exercise 3.14

1. 64
2. $\frac{5}{7}$
3. $\angle A = 120°$, $\angle B = 70°$, $\angle C = 60°$, $\angle D = 110°$
4. Price of TV = ₹20000; Price of fridge = ₹10000
5. 40, 48
6. 1 Indian – 18 days; 1 Chinese – 36 days

Exercise 3.15

1. (4) 2. (3) 3. (4) 4. (4) 5. (2) 6. (1) 7. (4) 8. (4) 9. (4) 10. (3)

11. (2) 12. (3) 13. (3) 14. (2) 15. (2) 16. (3) 17. (3) 18. (4) 19. (2) 20. (3)

21. (2) 22. (4) 23. (3) 24. (2) 25. (1) 26. (2) 27. (3) 28. (1) 29. (3) 30. (2)

4 Geometry

Exercise 4.1

1. (i) 70° (ii) 288° (iii) 89° 2. 30°, 60°, 90° 5. 80°, 85 °, 15 °

Exercise 4.2

1. (i) 40°, 80°, 100°, 140° 2. 62°, 114° , 66° 3. 44° 4. 10cm

7. (i) 30° (ii) 105° (iii) 75° (iv) 105° 8. 122° , 29°

9. Ratios are equal 10. d = 7.6

Exercise 4.3

1. 24cm 2. 17cm 3. 8cm, 45°, 45°

4. 18cm 5. 14 cm 6. 6 cm

7. (i) 45° (ii) 10° (iii) 55° (iv) 120° (v) 60°

8. $\angle BDC = 25^\circ, \angle DBA = 65^\circ, \angle COB = 50^\circ$

Exercise 4.4

1. 30° 2.(i) $\angle ACD = 55^\circ$ (ii) $\angle ACB = 50^\circ$ (iii) $\angle DAE = 25^\circ$

3. $\angle A = 64^\circ$; $\angle B = 80^\circ$; $\angle C = 116^\circ$; $\angle D = 100^\circ$

4.(i) $\angle CAD = 40^\circ$ (ii) $\angle BCD = 80^\circ$ 5. Radius=5cm 6. 3.25m

7. $\angle OAC = 30^\circ$ 8. 5.6m 9. $\angle RPO = 60^\circ$

Exercise 4.7

1. (2) 2. (3) 3. (1) 4. (4) 5. (4) 6. (3) 7. (2) 8. (2) 9. (4) 10. (2)

11. (1) 12. (3) 13. (1) 14. (1) 15. (4) 16. (2) 17. (2) 18. (3) 19. (2) 20. (4)

5 Coordinate Geometry

Exercise 5.1

1. $P(-7,6)$ = II Quadrant; $Q(7,-2)$ = IV Quadrant; $R(-6, -7)$ = III Quadrant; $S(3,5)$ = I Quadrant; and $T(3,9)$ = I Quadrant

2. (i) $P = (-4,4)$ (ii) $Q = (3,3)$ (iii) $R=(4,-2)$ (iv) $S = (-5,-3)$

3. (i) Straight line parallel to x -axis (ii) Straight line which lie on y -axis.

4. (i) Square (ii) Trapezium

Exercise 5.2

1. (i) $\sqrt{10}$ units (ii) $2\sqrt{26}$ units (iii) $c-a$ (iv) 13 units

2. (i) Collinear (ii) Collinear 7. 5 or 1

8. Coordinates of A (9, 9) or (–5,–5) 9. $y = 4x+9$ 10. Coordinates of $P(2,0)$

12. $30\sqrt{2}$

Exercise 5.3

1.(i) $(-4,-1)$ (ii) $(0,-1)$ (iii) $(a+b,a)$ (iv) $(1,-1)$

2. $(-5,-3)$ 3. $P=-15$ 4. $(9,3)(-5,5)$ and $(1,1)$

5. $\left(\frac{9}{2},\frac{3}{2}\right)$ 6. $(1,8)$

Exercise 5.4

1. $(7,3)$ 2. 5:2 3. $(3,4)$

4. $(-2,3),(1,0)$ 5. $\left(\frac{19}{2},\frac{13}{2}\right),\left(\frac{-9}{2},\frac{-15}{2}\right)$ 7. $(3,2)$

Exercise 5.5

1.(i) $(2,-3)$ (ii) $\left(\frac{-8}{3},\frac{-11}{3}\right)$ 2. $(4,-6)$ 3. 5 units

4. 20 5. $3\sqrt{\frac{5}{2}}$ units 6. $(1,0)$ 7. $(5,-2)$

Exercise 5.6

1. (3) 2. (3) 3. (3) 4. (2) 5. (2) 6. (4) 7. (3) 8. (3) 9. (3) 10. (3)

11. (4) 12. (1) 13. (3) 14. (4) 15. (2) 16. (3) 17. (2) 18. (2) 19. (4) 20. (2)

6 Trigonometry

Exercise 6.1

1. $\sin B=\frac{9}{41};\cos B=\frac{40}{41};\tan B=\frac{9}{40};\ \text{cosec}B=\frac{41}{9};\sec B=\frac{41}{40};\cot B=\frac{40}{9}$

2. (i) $\sin B=\frac{12}{13}$ (ii) $\sec B=\frac{13}{5}$ (iii) $\cot B=\frac{5}{12}$ (iv) $\cos C=\frac{4}{5}$

(v) $\tan C=\frac{3}{4}$ (vi) $\text{cosec}C=\frac{5}{3}$

3. $\sin\theta=\frac{1}{2};\cos\theta=\frac{\sqrt{3}}{2};\tan\theta=\frac{1}{\sqrt{3}};\ \text{cosec}\theta=\frac{2}{1};\sec\theta=\frac{2}{\sqrt{3}};\cot\theta=\sqrt{3}$

4. $\frac{3}{40}$ 5. $\sin A=\frac{1-x^2}{1+x^2};\tan A=\frac{1-x^2}{2x}$ 7. $\frac{1}{2}$

8. $\frac{1}{2}$ 9. $\sin\alpha=\frac{4}{5};\cos\beta=\frac{4}{5};\tan\phi=\frac{4}{3}$ 10. 7m

Exercise 6.2

2.(i) 0 (ii) $\frac{7}{4}$ (iii) 3 4. 2

Exercise 6.3

1.(i) 1 (ii) 1 (iii) 1 (iv) 2

Exercise 6.4

1.(i) 0.7547 (ii) 0.2648 (iii) 1.3985 (iv) 0.3641

(v) 0.8302 (vi) 2.7907 2.(i) 85°57′ (or) 85°58′ (or) 85°59′

(ii) 47°27′ (iii) 4°7′ (iv) 87°39′ (v) 82°30′

3.(i) 1.9970 (ii) 2.8659 4. 18.81 cm^2 5. 36°52′

6. 54.02 m

Exercise 6.5

1. (1) 2. (2) 3. (2) 4. (3) 5. (2) 6. (3) 7. (3) 8. (1) 9. (2) 10. (2)

7 Mensuration

Exercise 7.1

1.(i) 120 cm^2 (ii) 7.2 m^2 2. 1320 m^2, ₹26400 3. 12000 m^2

4. 1558.8 cm^2 5. ₹ 1050 6. 240 cm^2 7. 138 cm^2

8. 354m^2 9. 1536 m^2 10. 672 m^2

Exercise 7.2

1. $1160cm^2$, $560cm^2$ 2. ₹1716 3. ₹3349

4.(i) $384\ m^2$, $256\ m^2$ (ii) $2646\ cm^2$, $1764\ cm^2$ (iii) $337.5\ cm^2$, $225\ cm^2$

5. $1600\ cm^2$ 6. $253.50m^2$, ₹6084 7. $224cm^2$, $128cm^2$

Exercise 7.3

1.(i) $576\ cm^3$ (ii) $2250\ m^3$ 2. $630\ cm^3$

3. $25\ cm$, $20\ cm$, $15\ cm$ 4. 2624000 *litres* 5. 25000

6. $12\ m$ 7.(i) $125\ cm^3$ (ii) $42.875\ m^3$ (iii) $9261\ cm^3$

8. $5\ m$ 9. $15\ cm$

Exercise 7.4

1. (3) 2. (2) 3. (4) 4. (3) 5. (3) 6. (1) 7. (2) 8. (3) 9. (4) 10. (1)

8 Statistics

Exercise 8.1

1. $27°C$ 2. 44kg 3. 56.96 (or) 57 (approximately)

4. $142.5\ mm^3$ 5. $p = 20$ 6. 40.2 7. 29.29 8. 29.05

Exercise 8.2

1. 47 2. 44 3. 21 4. 32

5. 31 6. 38

Exercise 8.3

1. 6600, 7000, 7000 2. 3.1 and 3.3 (bimodal) 3. 15

4. 40 5. 24 6. 58.5

Exercise 8.4

1. (1) 2. (3) 3. (3) 4. (2) 5. (1) 6. (4) 7. (1) 8. (2) 9. (2) 10. (3)

9 Probability

Exercise 9.1

1. $\frac{1}{7}$
2. $\frac{3}{13}$
3. $\frac{1}{2}$
4.(i) $\frac{5}{24}$
(ii) $\frac{1}{8}$
(iii) $\frac{2}{3}$
5. $\frac{1}{4}$
6.(i) 0
(ii) $\frac{1}{12}$
(iii) 1
7. $\frac{1}{280}$
8. $\frac{1}{5}$
9. $\frac{3}{4}$

Exercise 9.2

1. 0.9975
2. $\frac{209}{400}$
3. $\frac{15}{8}$
4. 0.28
5.(i) $\frac{1}{6}$
(ii) $\frac{43}{75}$
(iii) $\frac{1}{75}$

Exercise 9.3

1. (4) 2. (2) 3. (1) 4. (4) 5. (1) 6. (4) 7. (4) 8. (4) 9. (1) 10. (2)

MATHEMATICAL TERMS

Abscissa	X-அச்சின் தொலைவு (கிடைஅச்சுத் தொலைவு)
Acute triangle	குறுங்கோண முக்கோணம்
Adjacent angles	அடுத்துள்ள கோணங்கள்
Algebraic expression	இயற்கணிதக் கோவை
Alternate angles	ஒன்றுவிட்ட கோணங்கள்
Altitudes of a triangle	முக்கோணத்தின் குத்துக்கோடுகள்
Angle	கோணம்
Angle of elevation	ஏற்றக் கோணம்
Arithmetic mean	கூட்டுச் சராசரி
Associative property	சேர்ப்புப் பண்பு
Assumed mean	ஊகச் சராசரி
Binomial expression	ஈருறுப்புக் கோவை
Binomial surds	ஈருறுப்பு முறுடுகள்
Cardinal number of a set	கணத்தின் ஆதி எண்
Cartesian plane	கார்டீசியன் தளம்
Cartesian coordinate system	கார்டீசியன் அச்சுத் தொகுப்பு
Centre	மையம்
Centriod	நடுக்கோட்டு மையம்
Chord	நாண்
Circum radius	முக்கோணத்தின் சுற்றுவட்ட ஆரம்
Circumcentre of a triangle	முக்கோணத்தின் சுற்றுவட்டமையம்
Circumcircle	சுற்றுவட்டம்
Circumference	பரிதி
Classical probability	தொன்மை நிகழ்தகவு
Ordinate	Y–அச்சின் தொலைவு (செங்குத்து அச்சுத்தொலைவு)
Co-efficient	கெழு
Coinciding lines	ஒன்றின் மீது ஒன்று பொருந்தும் கோடுகள்
Collections	தொகுப்பு

Commutative property	பரிமாற்றுப் பண்பு
Compound surds	கூட்டு முறுடுகள்
Complement of a set	நிரப்புக் கணம்
Complementary angles	நிரப்புக் கோணங்கள்
Complementary events	நிரப்பு நிகழ்ச்சிகள்
Concentric circle	பொதுமைய வட்டங்கள்
Concurrent lines	ஒரு புள்ளி வழிக் கோடுகள்
Congruent circle	சர்வசம வட்டங்கள்
Congruent triangles	சர்வசம முக்கோணங்கள்
Conjugate	இணை
Consistent	ஒருங்கமைவன
Constant	மாறிலி
Coordinate axes	ஆய அச்சுகள்
Corresponding angles	ஒத்த கோணங்கள்
Cube	கனச் சதுரம்
Cubic polynomial	முப்படி பல்லுறுப்புக் கோவை
Cuboid	கனச் செவ்வகம்
Cyclic Quadrilateral	வட்ட நாற்கரம்
Decimal expansion	தசம விரிவாக்கம்
Decimal representation	தசம குறியீடு
Degree of polynomial	பல்லுறுப்புக்கோவையின் படி
Denseness property	அடர்த்திப் பண்பு
Descriptive form	விவரித்தல் முறை
Deterministic Experiment	உறுதியான சோதனை
Diagonal	மூலைவிட்டம்
Diameter	விட்டம்
Dice	பகடைகள்
Difference of two sets	கணங்களின் வித்தியாசம்
Disjoint sets	வெட்டாக் கணங்கள்
Division algorithm	வகுத்தல் படிமுறை
Division Algoritham of polynomial	பல்லுறுப்புக் கோவையின் வகுத்தல் படிமுறை
Edge	விளிம்பு

Empirical probability	சோதனை நிகழ்தகவு
Empty set/Null set	வெற்றுக் கணம்
Equal sets	சமகணங்கள்
Equally likely event	சமவாய்ப்பு நிகழ்ச்சி
Equiangular triangle	சமகோண முக்கோணம்
Equilateral triangle	சமபக்க முக்கோணம்
Equilibrium	சமநிலை
Equivalent sets	சமான கணங்கள்
Event	நிகழ்ச்சி
Excentre	வெளிவட்ட மையம்
Externally	வெளிப்புறமாக
Face of a solid	ஒரு திண்மத்தின் முகப்பு
Factor theorem	காரணித் தேற்றம்
Factorisation	காரணிப்படுத்துதல்
Finite set	முடிவுறு கணம்
Frequency table	நிகழ்வெண் பட்டியல்
Greatest Common Divisor (G.C.D)	மீப்பெரு பொது வகுத்தி
Grouped data	தொகுக்கப்பட்ட தரவுகள்
Hypotenuse	கர்ணம்
Identities	முற்றொருமைகள்
Impossible event	இயலா நிகழ்ச்சி
Incentre	உள்வட்ட மையம்
Incircle	உள்வட்டம்
Inconsistent	ஒருங்கமையாத
Indices	அடுக்குகள்
Infinite set	முடிவிலி கணம்
Inradius	உள்வட்ட ஆரம்
Interior angles	உட்கோணங்கள்
Internally	உட்புறமாக
Intersecting lines	வெட்டும் கோடுகள்
Intersection of two sets	கணங்களின் வெட்டு

Irrational numbers	விகிதமுறா எண்கள்
Isosceles Trapezium	இருசமபக்க சரிவகம்
Isosceles triangles	இருசமபக்க முக்கோணம்
Lateral surface area	பக்கப் பரப்பு
Linear equations	நேரியச் சமன்பாடுகள்
Linear pair of angles	நேரிய கோணச் சோடிகள்
Linear polynomial expression	நேரியப் பல்லுறுப்புக்கோவை
Major sector	பெரிய வட்டக்கோணப்பகுதி
Mean difference	பொது வித்தியாசம்
Measures of central tendency	மையப்போக்கு அளவைகள்
Median	இடைநிலை
Median of a triangle	முக்கோணத்தின் நடுக்கோடு
Mid point	நடுப்புள்ளி
Minor sector	சிறிய வட்டக் கோணப்பகுதி
Mixed surds	கலப்பு முறுடுகள்
Mode	முகடு
Monomial expression	ஓருறுப்புக் கோவை
Mutually exclusive events	ஒன்றையொன்று விலக்கும் நிகழ்ச்சிகள்
Negative integers	குறை முழுக்கள்
Non-terminating decimals	முடிவுறா தசம எண்கள்
Obtuse traingle	விரிகோண முக்கோணம்
Opposite side	எதிர்ப்பக்கம்
Ortho centre of a triangle	முக்கோணத்தின் செங்கோட்டு மையம்
Outcome	விளைவு
Overlapping sets	வெட்டும் கணங்கள்
Parallel lines	இணை கோடுகள்
Period of decimals	தசம எண்களின் காலமுறைமை
Polygon	பல கோணம்
Polynomial equation	பல்லுறுப்புக்கோவைச் சமன்பாடு

Positive integers	மிகை முழுக்கள்
Power set	அடுக்குக்கணம்
Primary data	முதல்நிலைத் தரவுகள்
Probability	நிகழ்தகவு
Proper sub set	தகு உட்கணம்
Pure surds	முழுமையான முறுடுகள்
Quadrant	காற்பகுதி
Quadratic polynomial	இருபடி பல்லுறுப்புக் கோவை
Quadrilateral	நாற்கரம்
Radical	மூலக்குறியீடு
Radicand	மூல அடிமானம்
Random Experiment	சமவாய்ப்பு சோதனை
Rational numbers	விகிதமுறு எண்கள்
Rationalisation	விகிதப்படுத்துதல்
Raw data	செப்பனிடப்படாத தரவுகள்
Real number	மெய்யெண்கள்
Recurring decimals	சுழல் தன்மையுள்ள தசம எண்கள்
Remainder theroem	மீதித் தேற்றம்
Right triangle	செங்கோண முக்கோணம்
Roots of a polynomial	பல்லுறுப்புக் கோவையின் மூலங்கள்
Sample point	கூறுபுள்ளி
Sample space	கூறுவெளி
Scientific notation	அறிவியல் குறியீடு
Secondary data	இரண்டாம்நிலைத் தரவுகள்
Section formula	பிரிவு வாய்ப்பாடு
Sector	வட்டக்கோணப்பகுதி
Segment	வட்டத்துண்டு
Semi-circle	அரைவட்டம்
Set complementation	கண நிரப்பி
Set difference	கண வித்தியாசம்
Set operations	கணச்செயல்கள்
Sets	கணங்கள்
Singular set/Singleton set	ஓருறுப்புக் கணம்
Square root	வர்க்க மூலம்
Step-deviation method	படி விலக்க முறை
Substitution method	பிரதியிடும் முறை
Subset	உட்கணம்
Surds	முறுடுகள்
Sure event	உறுதியான நிகழ்ச்சி
Symmetric difference of sets	கணங்களின் சமச்சீர் வித்தியாசம்
Synthetic division	தொகுமுறை வகுத்தல்
Terminating Decimals	முடிவுறு தசம எண்கள்
Total surface area	மொத்தப் பரப்பு (அல்லது) மொத்தப் புறப்பரப்பு
Transversal	குறுக்குவெட்டி
Trapezium	சரிவகம்
Trial	முயற்சி
Trigonometry	முக்கோணவியல்
Trinomial expression	மூவுறுப்புக் கோவை
Uncertainty	உறுதியற்ற (அ) நிச்சயமற்ற
Ungrouped data	தொகுக்கப்படாத தரவுகள்
Union of sets	கணங்களின் சேர்ப்பு
Universal set	அனைத்துக் கணம்
Venn Diagram	வென்படம்
Vertically opposite angles	குத்தெதிர் கோணங்கள்
Vertex	முனை
Volume	கன அளவு
Well defined	நன்கு வரையறுக்கப்பட்ட
Zero / Factor	பூச்சியம் / காரணி
Zeros of polynomial	பல்லுறுப்புக்கோவையின் பூச்சியங்கள்

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