MATHEMATICS

CLASS - IX

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Preface

When we started working on this textbook, there were some questions that needed to be addressed like what can be done to make the book more interesting, readable and useful? Why should a concept or unit be included in the book? What sort of skills do we want to develop in the children? Can the book help to increase the participation of children in the process of learning mathematics.

While pondering over these questions, a lot of ideas came which were kept in mind while selecting the units. Then the chapters were decided, which were regularly reviewed. Every time some issues were encountered and the chapter was rewritten. In the present edition, simple colloquial language is used. The technical terms been contextualised for the understanding the meaning. Secondly, no rule or principle has directly been stated. Effort has been made to keep the thought process going. While reading the chapters the children are encouraged to discuss logically, listen to each other and then go ahead. We have tried incorporate the historical background so that the children might be acquainted with the growth and development of mathematics in almost all the units.

Most of the chapters begin with some interesting examples related to real life. The concepts have been developed gradually and follow an interactive mode. Simple questions based on a concepts have been solved to explain those concepts, then new situations have been created, which connect them to more difficult concepts, so that children are able to understand the concepts and apply them when needed. In this whole process a lot of mathematical skills like, understanding the abstract ideas, expressing them through mathematical symbols, explaining logically, giving proof, reaching to a conclusion, understanding and using appropriate language while defining, etc., have been developed.

Besides these there are several important things that help making the textbook useful. We have tried to incorporate those in this book. Please read and recognize them. If you feel there is a need to improve something, please do inform us. Your suggestions will help to make the book more useful.

We have received ample guidance and help from Vidya Bhawan Society, Udaipur, Saraswati Educational Institute, Chhattisgarh and Azim Premji Foundation in preparing this book. The council is grateful to them.

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Note for the teacher...

The secondary classes being the final period of general education, the expectation would be a to integrate the mathematical understanding and capability that we want to have in all citizens. The National Curriculum Framework enlarges the scope of this to include aspects of mathematization as an ability not restricted to solving mathematical problem given in the textbook. It goes beyond the obvious utility in daily life and is expected to enrich the scope of thought and visualization available to the student.

The Secondary school mathematics, therefore, on the one hand, needs to focus on consolidation of the conceptual edifice that has been initiated in classes 6 to 8 but also take it forward to help the child explore wider connection and deeper understanding.

The logical formulation and the arguments included in each step along with the precision of presentation are of value to engage with the world in more forceful manner. The broad description of purpose of mathematics for secondary classes, therefore, includes elaborating and consolidating the conceptual edifice, the ability to make logical arguments, the precise and concise formulation of ideas, ability to perceive rules and generalization and mechanisms to prove them. They must have a sense how to understand the notion of proof and the need for organized arguments. It also seeks to expand the ability to visualize, organize and analyze space and spatial transformations. Going beyond numbers to understanding number system in the abstract, forming general rules about numbers, understanding variables and equalities and learning to understand what solutions mean. These are just a few examples of the areas that are covered in the syllabus.

If we talk about the portion of mathematics mentioned in the N.C.F. 2005, we find it impresses on the need to develop an understanding of mathematics and ability to use it in all situations. It also means developing abilities in the learner that would influence his/her life in wider spheres. Mathematics has to move from expecting children to do unnecessarily complex calculations and move toward expecting her/him to reason logically because as a mathematics student she/he needs to understand how mathematics works rather than become an adept calculator or efficient book keeper. Not only she/he must engage with concepts but also enjoy the problems she/he is solving and the tasks that she/he is undertaking. Her/his ability to understand problems and find a way to solve them needs to be built upon so that she/he develops a confidence of being able to attempt any new problem she/he comes across. This does not mean that exercises given in the book or in the classrooms are those that go along the beaten track and are replete with different degree of mechanical complexity rather than they help the child explore the concepts and develop framework for it.

To develop the capacity to solve problems at this stage is important. While it has already been said that the learner should enjoy solving a problem, it also needs to be emphasized that the objective of solving a problem is not to find one correct answer by one correct method. It is also important for teachers to help the children find different approaches to solving problems and learn to realize that many problems have a variety of solutions. Situations need be created in the classrooms that ask children to construct their own problems and bring forth their own definitions so as to be able to understand the question and be able to choose the appropriate concepts and algorithms.

One of the many important things that emerge from the N.C.F. is about how a mathematics classroom should be. Along with that one point that needs to be mentioned is the importance of allowing children to share, exchange and develop ideas in a group jointly. The classes at the secondary stage are particularly important for this because children at the stage like to interact in groups and if these groups can be formed to encourage discussion on interesting issues arising from mathematics, then they will be able to form connections of what is learnt in the school with the real life experiences.

Mathematics in the secondary classes, therefore, has to keep in mind that learners are being prepared for formulating logical ideas and therefore be given tasks that help them understand the notion of proof, help them build the ideas necessary to prove something, understand and absorb basic concepts of mathematics and develop such a level of confidence in their capability that they learn the mathematical ideas and also use them.

Key ideas that you as a teacher should have in mind below entering a mathematics classroom:

- Mathematics is not a subject which should be transacted in the classroom where a teacher is the active entity and students are passive. Students should be given an environment where they can talk, share their thoughts and construct their knowledge.
- Giving the opportunity to students to talk is also a crucial from point of view of learning the language. Students should not only be able to communicate their mathematical ideas using their words and language but should also be able to move forward in the use of formal language of mathematics using symbols, graphs etc.
- The mathematics that students deal with in secondary level might be abstract but they should understand the concepts and form connections between them. They should also be able to make sense of the abstractness to connect with the subject.
- For teachers, textbook should be used as a tool to refer to and it shouldn't become a tool which guides the whole learning process. The teacher should decide themselves of how best they can use this textbook.

Authors

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It is difficult to tell exactly where and in what form mathematics came into existence. Even before the existence of oldest written documents, we come across a few pictures or symbols which indicate towards basic knowledge of math. For example, Palaeontologists have discovered Ochre rocks in the caves of South Africa where some geometrical patterns can be seen which have been made by etching. It is believed that Ishango bone near Nile River in Congo is the oldest representation of a series of prime numbers. This might be 20, 000 years old.

Around 5000 B.C., some geometrical spatial designs were depicted by the people of Egypt. In ancient India, the oldest known knowledge of Mathematics goes back to 3000-2600 B.C. Indus Valley civilization of North India developed a system of same weights and measures. We also get the proof of a surprisingly advanced brick technique in which ratio and proportion was used. Roads which cut each other at right angles, cuboidal, conical and cylindrical, circular and triangular shapes indicate that math was highly developed at that time.

Mohan-jo-daro

The oldest example of Chinese Mathematics goes back to Chang Dynasty (1600-1046 B.C) where etched on the shells of a tortoise were found. We also get the proof of written math from Sumerian civilization that developed the oldest civilization of Mesopotamia. They had developed a very complicated method of measurement science around 3000-2500 B.C.

Ancient civilizations of Egypt, Greek Babylon and Arabia have contributed a great deal in the field of Math. After Christ, math has been developed continuously in different parts of the world. This knowledge has become richer by sharing with each other.

The story of development of math can be gathered from different sources. In class 9th, we have included a part of the development of Indian mathematics. Hope it will motivate you to see and understand how mathematics was developed in the world.

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History of Mathematics

Mathematics is the backbone of Science and Technology. Hence, in Vedang Jyotish, Rishi Lagadh has written:

यथा शिखा मयूराणाम् नागानाम् मणयो यथा ।

तद्वद् वेदांगशास्त्राणाम् गणितम् मूर्धनिस्थितम् ।।

Meaning, mathematics adorns the head of all Vedang Shastras(Sciences) like the plume on top of a peacock's head or the bead on top of the cobra's head.

When we see the history of mathematics, the contribution of India has been very distinctive and famous. Work on the various branches of mathematics have been done since the ancient times in India. We will discuss the same briefly.

Arithmetic: Arithmetic is the main branch of mathematics. Its usage can be seen a lot in the day to day affairs. The basis of Arithmetic is the Number system in which zero has a special position.

Discovery of zero: The concept of zero is very much a part of the Vedas. In the Richa of Yajurveda, chapter 40, Verse 17 'Aum kham brahma' has used the word 'kham'. The word 'kham' indicates the sky and also Shunya (zero). So in books like the Jyotish, the word 'kham' has been used to indicate zero.

Bhaskaracharya, in his book Lilavati has used 'kham' as a zero in Shunya Parikarmaashtak (शून्य परिकर्माष्टक).

योगेखंक्षेपसमं, वर्गादौखं, खभाजितो राशिः ।

खहरः स्यात्, खगुणः खं, खगुणश्चिन्त्यश्च शेष विधौ ।।

On adding any number to zero, the sum is the number itself. The square,cube etc. of zero is zero. If any number is divided by zero, the denominator of that numeric is zero. On multiplying a number by zero, the product is zero.

Note: Shlokas have just been given in context of the Vedic hymn and so would not be desirable to ask to be quoted in the examination.

The credit of conceptualizing zero, has been given to the great Sanskrit Grammarian, Panini (500 B.C.) and to Pingal (200 B.C.) We also find a reference to the discovery of zero being made by Vedic Rishi Grutsamad.

The first proof of the symbol assigned to zero has been found in the Bakshaali Pandulipi (300- 400 AD). The existence of zero and its symbol in the number system of ancient India have been the most important contribution.

Prof. G.B.Halstead has said:-

"None of the concepts in Mathematics have proved to be as important as zero for the development of the brain and brawn."

Number System: Since ancient times, various countries have been using different methods of representing numbers. Devnagri, Roman and Hindu-Arabic systems are some which we have studied before. We will now see the historical background of these.

Prof.Guinsberg says: Around 770 C.E. a Hindu Scholar by the name of Kank was invited to the famous court of Baghdad by Abbasayyed Khaleefa Alamansur. In this way the Hindu numeric system came to Arabia. Kank taught Astrological Science and Mathematics to the Arab Scholars. With Kank's help, they also translated Brahmagupta's "Brahmasphut Siddhanta" into Arabic.

The discovery of French Scholar, M.F.Nau, proves that in the mid 7th century, Hindu numerics were well known in Syria and they were considered praiseworthy.

B.B.Dutt says: "The Indian number system went slowly from Arabia, Egypt and Northern Arabia towards Europe and by eleventh century it had completely reached Europe. Europeans referred to them as Arabic numbers because they had got them from Arabia, but the Arabs were unanimous in acknowledging them as Hindu Numbers (Al- Arkaan Al-Hindu). These ten numbers were referred to by the Arabs as "Hindsa"."

Place Value: To express any number using ten digits, including zero and to give each digit a face value and place value has made this number system scientific. The place value system is the specialty of the modern number system(the Hindu-Arabic system).

The great French Mathematician, Pierre Laplace has written: It was India that gave us an excellent system of expressing every number using the ten digits(where each digit has a face and a place value). The base of the decimal system is ten. That is the reason this system is called the Metric or the Decimal number system.

History of Hindu Numerals and big numbers: The Hindu numerals developed as follows:

- Kharoshthi System (fourth century b.c.)
- Brahmi system (third century b.c.)
- Gwalior system(ninth century)
- Devnagri system(eleventh century)
- Modern system

From fourth century B.C. to second century A.D., one can find the use of the Kharoshthi system. In the Brahmi system, besides ten, multipliers of ten till hundred and multipliers of hundred upto nine hundred have been found.

In Yajurveda Samhita, Ramayan and religious books thereafter have given numbers from 1 to 10^{53} different names:

- Niyutam 10^{11}
- Utsang 10^{21}
- Hetuheelam 10^{31}
- Nitravaadyam 10^{41}
- Tallakshana 10^{53}

Introduction to Coded Numbers (कूटांक): When an alphabet is used to represent a number, it is called a "coded" number. Ancient Mathematicians had used this concept to express numbers. The use of which can be seen in the Astrology and other vedic books.

- Alphanumeric (वर्णांक) system
- Shabdank (शब्दांक) system
- Vyanjanank (व्यंजनांक) system

Algebra: There are quite a few similarities in the formation and principles of Algebra and Arithmetic. The major difference in these two is Arithmetic deals with expressed (known) quantities and Algebra deals with unexpressed (unknown) quantities. By unexpressed quantities we mean quantities that are not known in the beginning. This is known as an algebraic quantity, and so the branch is Unexpressed Mathematics or Algebra.

The use of Algebra can be seen in era of Shulvsutras when there came up several problems while constructing of Yajna altars(vedis), requiring the use of finding solutions to linear and infinite equations. The contribution of Aryabhatt is creditable in both the fields of Arithmetic as well as Algebra. Algebra developed as an independent branch right during the time of Brahmagupta. It is also known as Coded Mathematics or Implicit Mathematics. Mathematician called Pruthudak swami (860 CE) named it Beej Ganit.

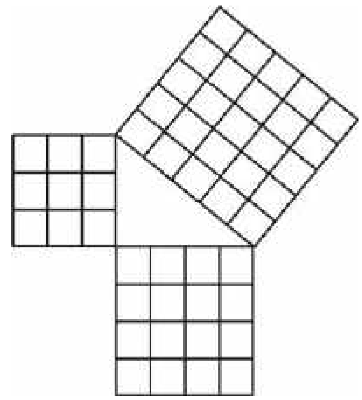
Geometry: When we see the history of Indian Mathematics, we realize that the base of Geometry had been already laid during the Vedic Period. We get to see the mathematics in a Vedang named "Kalp" in the form of Shulvasutras. The rope used in measuring the altars was known as shulva. Sutra means to express the information in the precise form. The shulva sutras were named after their creators - Baudhayan, Aapstambha, Kaatyaayan, Maanav, Maitraayan etc. The shulvasutras contain the information of how to make vedis (altars) of different shapes: • Garun Vedi • Koorma Vedi • Shri Yantra

The **examples** of the shulvasutras geometry are as follows:-

The formation of triangles, squares, rectangles and other complicated geometrical shapes, forming such geometrical shapes whose area are equal to the sum or difference of the areas of some given shapes.

The contributions made by Aryabhatt (476-550 CE), Brahmagupta (600 CE), Bhaskar first(629 CE), Mahavir (850 CE) in the field of Geometry have been commendable.

Baudhayan Theorem



दीर्घ=चतुरस्त्रस्य=अक्षण्या=रज्जु:=पार्श्वमानी=तिर्यक्=मानी=च ।
यत=पृथग्भूते=कुरूत:=तत्=उभयं=करोति=(इति=क्षेत्र=ज्ञानम्) ।।

।।=48=(1)=बौधायन=शुल्व=सूत्र ।।

Meaning: The area of the square drawn on the diagonal of a rectangle is equal to the sum of the areas of the squares drawn on the two sides of the rectangle. We know that the diagonal of the rectangle divides it into two right angled triangles and in such a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the remaining two sides.

This relation between the sides of the right angled triangle has been known as the Pythagoras Theorem. However, in the book by Dr. Brijmohan - History of Mathematics (page 243), there is a reference that now a lot of historians agree that this Pythagoras Theorem was known to the shulva sutra writers some hundreds of years prior to the birth of Pythagoras. The life of the Greek philosopher Pythagoras is believed to have been from 572 B.C. to 501 B.C. Whereas Indian Mathematician Baudhayan has described this theorem in great detail several years before that. Hence this theorem is in fact the Baudhayan Theorem. Or also known as Shulva Theorem.

Indian history of Pi (®) :

(1) **Aryabhatt (476 - 550 AD)** was the first Mathematician to give a reasonably close estimate of the value of Pi (π) which is the ratio of the circumference and diameter.

चतुरधिकम्=शतमष्टगुणं=द्वाषष्टिस्तथा=सहस्त्राणाम् ।
अयुतद्वय=विष्कम्भस्य=आसन्नो=वृत्त=परिपाह:=।।

Add 4 to 100, multiply the sum by 8 and add it to 62000. This sum will be approximately the circumference of a circle whose diameter is 20000 that is a circle with 20000 diameter will have a circumference of 62832.

$$\text{Pi (®)} = \frac{\text{Circumference}}{\text{Diameter}} = \frac{62832}{20000}$$

Thus, according to him, Pi = 3.1416, which is correct upto four decimals even today.

(2) **Bhaskaracharya (1114-1193 AD)** has given the value of Pi in his Granth Leelawati.

व्यासे=भनन्दाग्नि=हते=विभक्ते=खबाण=सूर्यै:=परिधि:=स=सूक्ष्म: ।
द्वाविंशतिघ्ने=विहृतेऽथशैलै:=स्थूलोऽथवास्याद्=व्यवहार्=योग्य: ।।

If you multiply the diameter by 3927 and divide this by 1250, we get the precise circumference. Or if we multiply the diameter by 22 and divide that by 7 we get a useable approximate value of the circumference.

(3) **Swaamibharti Krishnateertha(1884-1960 A.D.)** has given the value of pi/10 in the well known Anushtup verses, using the alphabets as codes:

गोपी=भाग्यमधुव्रात–श्रृंगिशोदधिसंधिग ।

खलजीवित=खाताव=गलहाला=रसंधर=।।

According to swamiji, there can be three useful interpretations of these verses. In the first interpretation, it is a praise of the Lord Krishna. In the second interpretation, it can be considered the praise of Lord Shiva and the third interpretation is that it is the value of pi/10 correct upto 32 decimal places.

$\pi/10 = 0.31415926535897932384626433832792$

(4) **Shrinivas Ramanujan (1887-1920 AD)** The first research paper published in Europe by Ramanujan was titled "Modular equation and approximation to π". He found several formulae to get the approximate value of π.

Trigonometry: Trigonometry is that branch of Mathematics in which the relation between the sides and angles of a triangle are studied. This is a very old and important branch of Mathematics. The use of this knowledge is made in calculating the positions of the planets in Indian astrology and astronomy.

The ancient Indian Mathematicians like Aryabhatt, Varahmihira and Brahmagupta made significant contributions to this field.

You can see the description of the trigonometric concepts, formulae and statements in "Soorya Siddhanta"(400 AD), in the Panch Siddhanta by Varahmihira and in Brahmasphoot Siddhanta(630 AD) by Brahmagupta.

There is a clear reference in the book by Dr. Brijmohan, "History of Mathematics" (Page 314), that there is no doubt that three of the trigonometric functions have been defined first by the Hindus.

Aaryabhatt was the first to have used the word 'jya' (around 510 AD). He was also the first to give the tables related to jya and utkram jya (ujjya).

So the word 'jya' went from India to Arab countries, where it became 'Jeeba'. After a while this became 'Jaib'. In Arabic 'jaib' means 'breast' or 'bust'. When the translations were made into Latin in 1150 AD, 'jaib' was replaced by 'Sinus', which has the same meaning in Latin is also breast.

Brahmagupta used the word 'kramjya' to imply jya. It was so named to differentiate it from 'utkram jya'. In Arabic this was then converted to 'karaj'. Alkhwarijamee also used 'karaj'. Indian jya and kotijyaa became sine and cosine in European languages.

The use of trigonometry is seen in Astrology, Astronomy, Engineering and Navigation as also in the study of heights and distances.

Excercise - 1.1

Q.1. Match the following:

Bharti Krishnateertha	Brahmasphoot Siddhanta
Varahamihira	Siddhanta Shiromani
Brahmagupta	Aaryabhatiya
Bhaskaracharya	Panch Siddhanta
Aaryabhatt	Vedic Mathematics

Q.2. Fill in the blanks:

1. The word used for the sky and zero was ____________.
2. Tallakshana was used to denote the number _____________.
3. The alphanumeric system was used by the Mathematician ___________ in his book ________________________.
4. ________________ was also known as Implicit Mathematics.
5. Vedic Mathematics have the __________Sutras.

Q.3 Write the importance of the Modern Number system.

Q.4 Write a few points on the discovery of zero.

Q.5 Give a brief description of the Alphanumeric system.

Q.6 What is the Baudhaayan Theorem?

Q.7 Write the contribution of Aaryabhatt in the estimation of π.

Q.8 Write the name of the creator of Vedic Mathematics. Also give a brief description of its contents.

Activity:

(1) Form a Mathematics Society in your school.

(2) Make a collection of Mathematical books in your school.

(3) Develop a Mathematical Laboratory in your school.

Simple techniques for multiplication: We are going to study some simple multiplying techniques, before which we shall obtain a brief introduction of the book and its researcher mathematician.

The unparalleled mathematician, Swami Bharti Krishna Teertha Shankaraachrya Govardhanmath Jagganaathpuri (1884-1960 AD) created a book named "Vedic Mathematics" and thus made an innovative contribution.

In this book, he has described 16 exceptional sutras and 13 upsutras with their properties and experiments. This book has 40 chapters. It has been presented with a very unique viewpoint.

Digit sum (बीजांक): When you obtain a single digit after adding the different digits in a number, that is known as its Digit sum.

EXAMPLE-1. Find the Digit sum of 10, 11, 321and 78

SOLUTION : Digit sum of 10 → 1 + 0 → 1

Digit sum of 11 → 1 + 1 → 2

Digit sum of 321→ 3 + 2 + 1 → 6

Digit sum of 78 → 7 + 8 → 15, but 15 is not a Digit sum, so we have to add these digits further 1 + 5 → 6, hence the Digit sum of 78 is 6

EXAMPLE-2. Find the Digit sum of 8756904

SOLUTION :

Method 1: Add all the digits of the given number.

8 + 7 + 5 + 6 + 9 + 0 + 4 = 39, again adding these we get 3 + 9 = 12, further adding, we get, 1+2 = 3, which is the Digit sum.

Method 2: Consecutively keep adding the digits until you get a single digit number.

8 + 7 → 15 → 1 + 5 → 6 + 5 → 11 → 1 + 1 → 2 + 6 → 8 + 9 → 17 → 1 + 7 → 8 + 0 → 8 + 4 → 12 → 1 + 2 → 3 is the Digit sum.

Method 3: Inspect the digits of the number 8756904, Add the digits besides 0,9 and other pair which adds to give 9 i.e 4 + 5, so 8 + 7 + 6 → 21 →2 + 1 →3 is the Digit sum.

Points to be noted when finding a Digit sum:

(1) As soon as you add the digits, if you get a two digit number, add the digits further to obtain the Digit sum.

(2) Adding 0 and 9 or leaving them out, will not affect the Digit sum.

(3) The Digit sum of a number is actually the remainder left when you divide the number by 9. Thus finding a Digit sum is the same as dividing the number by 9 and finding the remainder.

(4) A number whose Digit sum is 9, is completely divisible by 9. In that case the Digit sum of the number is 9 not 0.

(5) We can also test the divisibility of a number by 3. So a number which has a Digit sum of 3, 6 or 9 will be divisible by 3.

(6) You can also test the solution you have obtained using Digit sum, hence one needs to have an adequate knowledge of finding Digit sums orally.

Excercise - 1.2

Q.1 What do you mean by Digit sum?

Q.2 Find the Digit sum of the following numbers:

15, 38, 88, 99, 412, 867, 4852, 9875, 24601, 48956701

Q.3 What should be added to the following numbers to make them divisible by 9?

241, 861, 4441, 83504

Q.4 Write the usefulness of Digit sums.

Use of Digit sums in checking the solutions: In Vedic Mathematics there are several ways of checking the correctness of a solution. We will see here how Digit sum helps in checking the correctness of a solution.

Adding:

Check for additions: The digit sum of the numbers to be added should be = to the sum of the digits in the solution.

EXAMPLE-3. Find the sum of 3469, 2220 and 1239 and check if the solution is right using Digit sum.

SOLUTION : Check

		Check
	3469	4
	2220	6
+	1239	6
	6928	7

(1) The Digit sums of the numbers are respectively 4, 6 and 6.

(2) The Digit sum of the Digit sums of the numbers is

$4 + 6 + 6 = 16 \rightarrow \quad 1 + 6 \rightarrow \quad 7$

(3) The Digit sum of the sum of the three numbers is

$6 + 9 + 2 + 8 = 25 \rightarrow \quad 2 + 5 \rightarrow 7$

(4) Since the Digit sum of both (2) and (3) is 7, hence the solution is correct.

Subtraction:

Check for subtraction : In this the Digit sum of subtracting (below) number + the Digit sum of the answer = Digit sum of the upper number.

EXAMPLE-4. 7816 - 3054. Solve this and check the answer.

SOLUTION : Check

		Check
	7816	4
–	3054	3
	4762	1

(1) The digit sum of the quantity being reduced(lower) is 3

And the Digit sum of the answer 4762 is 1.

(2) The sum of the two digit sums is 3 + 1 = 4

(3) The digit sum of the upper quantity 7816 is = 4.

(4) Since both (2) and (3) have 4 as their digit sum, the answer is correct.

Multiplication:

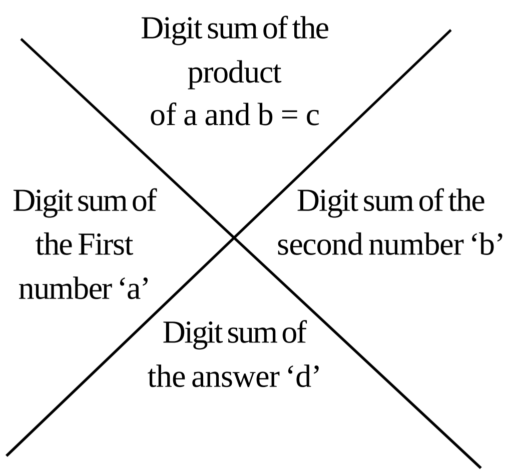
Check for multiplication: Digit sum of first number × Digit sum of second number = Digit sum of the product of the two numbers.

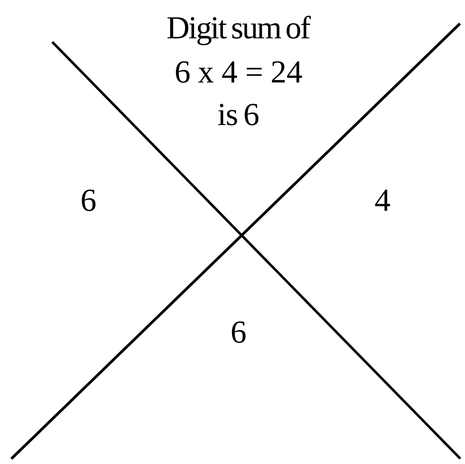
EXAMPLE-5. 456 × 814. Find the solution and check the answer.

SOLUTION :

456 × 814	(1) 6 × 4 = 24, the digit sum of which is 6
1824	(2) The product of numbers is 371184, the Digit sum of which is 6
4560	
364800	(3) Since (1) and (2) are the same digit sums that is 6, the answer is correct.
371184	

Another way of putting this is





If 'c' and 'd' are different then the answer is definitely wrong.

Division:

Check for division : Digit sum of the dividend = (Digit sum of divisor x digit sum of quotient) + Digit sum of remainder

EXAMPLE-6. 7481 ÷ 31. Find the solution and check your answer.

SOLUTION :

Divisor dividend quotient

$$
\begin{array}{l}
31)\ 7481\ (241 \\
\underline{-62} \\
\quad 128 \\
\underline{-124} \\
\quad 0041 \\
\quad \underline{-31} \\
\qquad 10
\end{array}
$$

Digit sum of dividend → 7 + 4 + 8 + 1 → 20 → 2

Digit sum of divisor x digit sum of quotient + digit sum of remainder

(7 × 4) + 1 → 28 + 1 → 29 →11 →2

Since above two are equal, the answer is correct.

Vedic Methods to multiply

(1) **Urdhva Tiryak Vidhi,** using the sutra Urdhvatiryagbhyam.

The meaning of the sutra is Urdhva = Vertical (↑)

And tiryak = diagonal ↖↗ = ⤧

EXAMPLE-7.

$$
\begin{array}{r}
41 \\
\times\ 38 \\
\hline
\end{array}
$$

SOLUTION :

$$
\begin{array}{r}
41 \\
\times\ 38 \\
\hline
15|5|8 \\
3\ \ \ \ \ \
\end{array}
$$

$\boxed{1558 \quad \text{(answer)}}$

Formula: Urdhwatiryagbhyaam

(1) First column (multiplication of units place with units place)

$$
\begin{array}{r}
1 \\
\times 8 \\
\hline
8
\end{array}
$$

↑ Urdhwaguna

(Vertical multiplication)

and write the product in units position in the solution.

(2) First and second column (units and tens place)

$$
\begin{array}{r}
41 \\
\times 38 \\
\hline
\end{array}
$$

Multiply diagonally and add

(4×8) + (1×3)

32+3=35 (5 of the 35 is written in tens place and 3 is carried to hundreds place)

(3) Second column (tens place)

$$\begin{array}{r} 4 \\ \times 3 \\ \hline 12 \end{array}$$ ↑ Vertical multiplication

+ 3 (carried over) = 15 (write this as it is to the left)

EXAMPLE-8.

$$\begin{array}{r} 56 \\ \times 82 \\ \hline \end{array}$$

SOLUTION :

$$\begin{array}{r} 56 \\ \times 82 \\ \hline \end{array}$$

Formula:Urdhwatiryagbhyaam (observe and understand)

45	9	2
5	1	

Vertical multiplication

5 5 6 6

8 8 2 2

Vertical multiplication

Add after doing Diagonal multiplication

EXAMPLE-9.

$$\begin{array}{r} 231 \\ \times 425 \\ \hline \end{array}$$

Solve using formula Urdhwatiryagbhyaam.

SOLUTION :

$$\begin{array}{r} 231 \\ \times 425 \\ \hline \end{array}$$

9	8	1	7	5
1	2	1		

2 2 3   2 3 1   3 1 1

4 4 2   4 2 5   2 5 5

(1) frist column (units place with units place product)

$$\begin{array}{r} 1 \\ \times 5 \\ \hline 5 \end{array}$$ ↑ Vertical multiplication

(2) First and second column (units and tens place product)

$$\begin{array}{c} 31 \\ \times 25 \\ \hline (3 \times 5) + (1 \times 2) \end{array}$$ Diagonal multiplication

15 + 2 = 17 (write the 7 in tens place and carry over 1)

(3) First, second and third column (units, tens and hundreds place)

$$\begin{array}{c} 231 \\ \times 425 \\ \hline (2\times5)+(1\times4)+(3\times2) \end{array}$$

As shown by arrows, product of first by third digit of these two numbers and add it to the product of second by second digits of both.

10 + 4 + 6 = 20

20 + 1 (carried over) = 21 (keep 1 in hundreds place and carry over 2)

(4) Second and third column (drop the units place)

$$\begin{array}{r} 23 \\ \times\ 42 \\ \hline (2\times2) + (3\times4) \end{array}$$

4 + 12 = 16

16 + 2 (carried over) = 18 (write 8 in the thousands position and keep 1 as a carry over)

(5) Third column (drop the units and tens place)

$$\begin{array}{r} 2 \\ \times\ 4 \\ \hline 8 \end{array}$$ Vertical multiplication

8 + 1 (carried over) = 9 (write 9 in the ten thousands place)

Exercise - 1.3

Find the following products using formula Urdhwatirygbhyaam:

(1) $\begin{array}{r} 23 \\ \times 32 \\ \hline \end{array}$ (2) $\begin{array}{r} 44 \\ \times\ 52 \\ \hline \end{array}$ (3) $\begin{array}{r} 92 \\ \times\ 37 \\ \hline \end{array}$ (4) $\begin{array}{r} 55 \\ \times\ 55 \\ \hline \end{array}$

(5) $\begin{array}{r} 123 \\ \times\ 321 \\ \hline \end{array}$ (6) $\begin{array}{r} 414 \\ \times\ 232 \\ \hline \end{array}$ (7) $\begin{array}{r} 504 \\ \times\ 618 \\ \hline \end{array}$ (8) $\begin{array}{r} 812 \\ \times\ 453 \\ \hline \end{array}$

(2) Ekanyunena Poorvena Vidhi (method) (meaning one less than before) : This formula is used when one of the numbers is made up of 9s. There are three conditions that occur between the multiplier and multiplicand:

1. The number of digits is same.
2. The number of digits is more in the multiplier than the multiplicand that is there are more 9s.
3. The number of digits is less in the multiplier than the multiplicand that is there are less 9s.

Condition 1:

EXAMPLE-10. Solve 63 x 99 using Ekanyunenapoorven method.

SOLUTION : 63×99

62 | 37

(1) There are two digits in both.

(2) The left side of the answer is 1 less than the multiplicand. i.e. 1 less than 63 is 62.

(3) The right side of the answer is 99 - 62 = 37

EXAMPLE-11. 3452×9999

SOLUTION : 3452×9999

3451 | 6548

(1) Left side of the answer 3451 (1 less than 3452)

(2) Right side of the answer is 9999 - 3451 = 6548

Condition 2.

EXAMPLE-12. Find the product 43×999 using formula Ekanyunena poorvena.

SOLUTION : 043×999

042 | 957

$\boxed{42957 \text{ Answer}}$

(1) Add a zero to the left of 43, to make the digits equal

(2) The left side of the answer is 1 less than 043 ie 042

(3) The right side of the answer is 999-042 = 957

EXAMPLE-13. Solve 347×99999 using the formula Ekanyunenpoorvena.

SOLUTION : 00347×99999

00346 | 99653

$\boxed{34699653 \text{ Answer}}$

(1) Add two zeros to the left of 347 and make the digits equal

(2) The left side of the answer is 1 less tha 00347 is 00346

(3) The right side of the answer is

$$\begin{array}{r} 99999 \\ -\ 00346 \\ \hline 99653 \end{array}$$

Condition 3:

EXAMPLE-14. Solve 438 x 99 using Ekanyunenapoorvena.

SOLUTION : 438×99

$$\begin{array}{r} 43799 \\ -\ 437 \\ \hline 43362 \end{array}$$

(1) Reduce 438 by 1 and keep 99 as it is following this. Then subtract 437 from this number to get the solution. Thus subtract 437 from 43799 to get 43362

Exercise - 1.4

Solve using formula Eknyunenapoorvena (also check your answers):

(1) 57×99 (2) 4378×9999 (3) 87×999

(4) 345×99999 (5) 48×9 (6) 9457×999

(3) Ekaadhikena poorvena Vidhi (method) : Here the use of the sutra Ekaadhikenapoovena and Antyayordeshakepi is made.

This method is used when the sum of the units place digits of the two numbers is 10 and the remaining digits are the same.

EXAMPLE-15. Find the product 12 x 18.

SOLUTION : 12×18

2 | 16

Formula Ekaadhikenapoorvena and antyayordeshakepi

(1) Left side of the answer (one more than the tens place x tens digit) $= 2 \times 1 = 2$

(2) Right side of the answer = product of units digits $= 2 \times 8 = 16$

$\therefore\ 12 \times 18 = 216$

EXAMPLE-16. 21×29

SOLUTION : 21×29

6 | 09

Formula Ekaadhikenapoorvena and Antyardeshkepi.

(1) Left side of the answer (one more than tens digit x tens digit) $= 3 \times 2 = 6$

(2) Right side of the answer = product of the units digits $= 1 \times 9 = 9$

$\therefore\ 21 \times 29 = 609$

Note: The sum of the units digits is 10. So the product of these units digits should have two digits, since there is only one, we add a 0 before 9.

EXAMPLE-17. Solve 102×108

SOLUTION : 102×108

110 | 16

Formula Ekaadhikenapoorvena and Antyardeshkepi.

(1) Left side of the answer (1 more than 10 x ten) $= 11 \times 10 = 110$

(2) Right side of the answer = product of units

digits $= 2 \times 8 = 16$

$\therefore 102 \times 108 = 11016$

EXAMPLE-18. Solve 194 × 196.

SOLUTION :

$$\begin{array}{c|c} \multicolumn{2}{c}{194 \times 196} \\ \hline 380 & 24 \end{array}$$

Formula Ekaadhikenapoorvena and Antyardeshkepi.

(1) Left side of the answer = (1 more than19×19)

= 20 × 19 = 380

(2) Right side of the answer = product of units digits. = 4 × 6 = 24

∴ 194 × 380 = 38024

Exercise-1.5

Use the formula Ekaadhikenapoorvena and Antyardeshkepi to find the answers and check using Digit sums:

(1) 13 × 17	(2) 22 × 28	(3) 34 × 36	(4) 91 × 99
(5) 35 × 35	(6) 42 × 48	(7) 72 × 78	(8) 93 × 97
(9) 104 × 106	(10) 105 × 105	(11) 203 × 207	(12) 405 × 405
(13) 502 × 508	(14) 603 × 607	(15) 704 × 706	(16) 905 × 905
(17) 193 × 197	(18) 292 × 298	(19) 392 × 398	(20) 495 × 495

Nikhilam Vidhi : This method is used to find products, when the numbers are close in value to the base or the sub base.

Base : 10,100,1000............etc. are called bases

Sub base: 20,30,......200,300, etc. are called sub bases.

Deviation from the base:

(1) Firstly find the base which is a power of 10 and is closest to the number.

(2) If the given number is greater than this base, subtract the base from the number and write the deviation with a positive sign.

(3) If the given number is less than the base, then subtract it from the base and write the deviation with a negative sign..

Number	Base	Deviation
12	10	+2
9	10	-1
104	100	+4
98	100	-2
1002	1000	+002
992	1000	-008

.... And so on.

Gunaa Nikhilam : Base

EXAMPLE-19. Solve 12×14

SOLUTION :

Number	Deviation
12	+2
$\times$14	+4
16	8

168 Answer.

(1) Both numbers have a base 10

(2) The deviation of 12 from 10 is +2

(3) The deviation of 14 from 10 is +4

(4) The right side of the answer is product of the deviations, $2 \times 4 = 8$

(5) The left side of the answer = (First number + deviation of the second number) or (Second number + deviation of the first number)

$= 12 + (+4)$ or $14 + (+2)$

$= 16$

Note: In this method we keep the same number of digits in the right side of the answer as the number of zeros in the base

EXAMPLE-20. Solve 16×15 using Nikhilam Formula.

SOLUTION :

Number	Deviation
16	+6
$\times$ 15	+5
24	0

(1) Both the numbers have the same base 10.

(2) The deviation of 16 from 10 is +6

The deviation of 15 from 10 is +5

(3) The right side of the answer

= Product of the deviations

$= 6 \times 5 = 30$

(4) The left side of the answer

= One number + deviation of other

$= 16 + 5 + 3$ (carried over) $= 24$

Or $= 15 + 6 + 3$ (carried over) $= 24$

Thus answer is 240

EXAMPLE-21. Solve 8×13 using the formula Nikhilam.

SOLUTION :

Number	Deviation
8	-2
$\times$ 13	+3
11	$\bar{6}$

(1) Base is 10

(2) Right side of answer = Product of the deviations

$= -2 \times (+3) = -6$

(3) Left side of the answer = One number + deviation of other

= 8 + (+3) = 11

Or = 13 + (-2) = 11

Hence answer is 110 - 6 = 104

EXAMPLE-22. Use Nikhilam Sutra to find 104×108

SOLUTION :

$$\begin{array}{r|r} \text{Number} & \text{Deviation} \\ 104 & +04 \\ \times 108 & +08 \\ \hline 112 & 32 \end{array}$$

Answer is 11232

(1) The base is 100. This has two zeros hence the right side of the answer will also have two digits.

(2) The right side of the answer = product of deviations

= $04 \times 08 = 32$

(3) The left side of the answer = One number + deviation of other

= 104 + (+8) = 112

or = 108 + (+4) = 112

EXAMPLE-23. Solve 103×101 using Nikhilam Sutra.

SOLUTION :

$$\begin{array}{r|r} \text{Number} & \text{Deviation} \\ 103 & +03 \\ \times 101 & +01 \\ \hline 104 & 03 \end{array}$$

Hence answer is 10403

(1) The right side of the answer = Product of deviations

= $3 \times 1 = 3$

(2) This right side will have two digits as the base has two zeros, hence we take it as 03

(3) The left side of the answer = One number + deviation of other

= 103 + (+1) = 104

Or = 101 + (+3) = 104

EXAMPLE-24. Solve using the Nikhilam sutra : 92×107

SOLUTION :

$$\begin{array}{r|r} \text{Number} & \text{Deviation} \\ 92 & -08 \\ \times 107 & +07 \\ \hline 99 & \overline{56} \end{array}$$

(1) The base is 100 and the deviations are respectively, -08 and +07.

(2) The right side of the answer = Product of deviations

= $-8 \times (+7) = -56$ or $\overline{56}$

(3) The left side of the answer = One number + deviation of other

= 92 + (+7) = 99

Or = 107 + (-8) = 99

Hence answer is 9900 - 56 = 9844

EXAMPLE-25. Solve 1014 × 994 using Nikhilam Sutra.

SOLUTION :

Number	Deviation
1014	+014
×994	–006
1008	$\overline{084}$

(1) Base is 1000. The deviations are respectively, +014, -006 .

(2) The right side of the answer = Product of deviations

= 14 × (-006) = -084 = $\overline{084}$

(3) The left side of the answer = One number + deviation of other

= 1014 + (–6) = 1008

Or = 994 + (+ 014)= 1008

(4) There are 3 zeros in the base so there has to be three digits in the right side of the answer, so we take it as 1008000.

Answer is 1008000 - 084 = 1007916

Exercise 1.6

Find the products using Nikhilam Sutra and verify the answers using Digit sums:

(1) 13 × 13 (2) 104 × 102 (3) 105 × 106 (4) 98 × 94

(5) 122 × 102 (6) 96 × 107 (7) 1012 × 1004 (8) 998 × 974 (9) 1016 × 998

Square

We have seen four methods of finding a product:

(1) Urdhwatiryak Method (2) Ekanyunena poorvena Method (3) Ekaadhikena poorvena Method (4) Nikhilam Method. We could use these easily to find the square of a number.

We will now find the square of a number using Ekaadhikena poorvena method and some other special methods.

(1) Ekaadhikena poorvena and Antyardeshkepi : We can find the square of numbers that have 5 in the units place orally.

EXAMPLE-26. Find 65^2.

SOLUTION : $65^2 = 65 \times 65$ Using the formula Ekaadhikena poorvena

42 | 25

(1) Left side of the answer = tens digit × one more than tens digit = $6 \times 7 = 42$

(2) Right side of the answer = $5 \times 5 = 25$

Hence answer is 4225

Exercise - 1.7

Find the following squares orally, using Ekaadhikena Poorvena method:

$15^2, 25^2, 35^2, 45^2, 55^2, 75^2, 85^2, 95^2, 105^2, 115^2$

(2) Anurupyena Vidhi : This method is used normally to find the square of a two digit number. We know that $(a + b)^2 = a^2 + 2ab + b^2$, which we will write as follows:

$(a \mid b)^2 = a^2 \mid 2ab \mid b^2$ and using this for the two digits of the number to be squared, we start with the digit in the units place and keep one digit in each of the right and middlemost columns and carry the other digits to the leftmost column.

EXAMPLE-27. Find 64^2

SOLUTION : 64^2 will be solved using $(a \mid b)^2 = a^2 \mid 2ab \mid b^2$

(1) $b^2 = 4^2 = 16$

(2) $2ab = 2 \times 6 \times 4 = 48$

$48 + 1$ (carried from 16) $= 49$

(3) $a^2 = 6^2 = 36$

$36 + 4$ (carried from 49) $= 40$

$$64^2 = 36 \mid 48 \mid 16$$
$$= 36 \quad 49 \quad 6$$
$$\overline{40 \quad 9 \quad 6} \text{ Answer}$$

EXAMPLE-28. Find 48^2.

SOLUTION : We will use $(a \mid b)^2 = a^2 \mid 2ab \mid b^2$

$$48^2 = 4^2 \mid 2 \times 4 \times 8 \mid 8^2$$
$$= 16 \mid 64 \mid 64$$
$$= 16 \quad 70 \quad 4$$
$$\overline{= 23 \quad 0 \quad 4} \text{ Answer}$$

Exercise - 1.8

Use $(a + b)^2 = a^2 + 2ab + b^2$ ($(a | b)^2 = a^2 | 2ab | b^2$) to find the following squares:

(1) 34^2 (2) 19^2 (3) 54^2 (4) 64^2 (5) 92^2

(3) Yaavat oonam taavat oonee krutya vargam ch yojayet Sutra- In this method, we find the deviation of the number (whose square one has to find) from its base. If the deviation is less, we reduce that from the number and if it is more, we add it in the number to get the left side of the answer. The right side is obtained by squaring the deviation.

We can take help of the following formula to get the square:

$\text{Number}^2 = \text{number} \pm \text{deviation} \mid (\pm\text{deviation})^2$

Example: $13^2 = 13 + 3 \mid 3^2$ (13 is 3 more than base 10)

$= 169$ Answer.

Example: $7^2 = 7 - 3 \mid 3^2$ (7 is 3 less than base 10)

$= 49$ Answer

Example: $98^2 = 98 - 2 \mid (02)^2$

$= 96\ 04$ Answer (04 because the base has two zeros)

Example: $106^2 = 106 + 6 \mid 6^2$

$= 112\ \ 36$ Answer

Exercise - 1.9

Solve using the Formula: यावत=ऊनं=तावत्=ऊनी=कृत्य=चर्गं=च=योजयेत ...... :

12^2, 14^2, 102^2, 105^2, 108^2, 94^2, 996^2,

Square Root

We already know the methods of finding square root of a number using prime factorization and division method. We shall now see how we can find the square root using Vilokanam Vidhi.

Vidhi -Vilokanam: We can find the square root of a four or five digits perfect square by inspection. Observe the following table:

Number	1	2	3	4	5	6	7	8	9	10
Number^2	1	4	9	16	25	36	49	64	81	100
Digit sum	1	4	9	7	7	9	4	1	9	1

Memorize the following:

Unit place of square number	-	1	4	5	6	9	0
Unit place of square root	-	1 or 9	2 or 8	5	4 or 6	3 or 7	0

Note : (1) If the units place of a square number is 2,3,7 or 8, they are not perfect squares.

(2) The number of pairs that you can make in the square number, those many digits will be there in the square root of the number.

(3) Numbers whose digit sum is 2,3,5,6 or 8 are not perfect squares.

Example-29. Find the square root of the perfect square 6889 using Vilokanam vidhi.

Solution : $\sqrt{6889}$ sutra vilokanam

$\sqrt{6889}$ = 83 or 87

(1) You can make two pairs in the square number, hence square root will have two digits.

(2) The right pair (89) decides the units place and the left pair(68) decides the tens digit.

(3) Since units place of square is 9, hence the units place of square root will be 3 or 7.

(4) The left pair(68)helps decide the tens place. The closest square root to that is 8 as $8^2 = 64$ and $9^2 = 81$, which is definitely more than 68. So we take tens place digit as 8.

(5) The answer could be 83 or 87.

(6) As 6889 is smaller than 7225 which is the square of 85, hence the square root is 83. $\sqrt{6889} = 83$

Exercise - 1.10

Find the square root of the following using Vilokanam Vidhi:

(1) 9409 (2) 7569 (3) 8281 (4) 3249

Algebra

Multiplication (by Urdhwatiryak Vidhi): This formula is used in Arithmetic but works equally well with multiplication of algebraic expressions.

EXAMPLE-30. Multiply $3x + 1$ with $2x + 4$ and check the product.

SOLUTION : Urdhwatiryak Vidhi

$$\begin{array}{r} 3x + 1 \\ \times 2x + 4 \\ \hline 6x^2 + 14x + 4 \end{array}$$

(1) First Column

$$\begin{array}{r} +1 \\ \times\ +4 \\ \hline +4 \end{array}$$

Verticle multiplication

(2) First and second column

$$\begin{array}{r} 3x + 1 \\ \times\ 2x + 4 \\ \hline \end{array}$$

Diagonal multiplication

adding $(3x \times 4) + (2x \times 1)$

$= 12x + 2x = 14x$

(3) Third column

$$\begin{array}{r} 3x \\ \times 2x \\ \hline 6x^2 \end{array}$$

Verticle multiplication

Answer is $6x^2 + 14x + 4$

Check: Digit sum of (digit sum of numerical coefficients of the first expression x digit sum of numerical coefficients of the second expressions) = The digit sum of the numerical coefficients of the answer.

(1) $4 \times 6 = 24$, whose digit sum is 6

(2) $6 + 14 + 4 = 24$, whose digit sum is 6

As (1) and (2) give the same digit sum, our answer is correct.

EXAMPLE-31. Solve using Urdhwa tiryak Vidhi $(2x + y) \times (3x + 5y)$

SOLUTION :

$$\begin{array}{r} 2x + y \\ \times\ 3x – 5y \\ \hline +\ 6x^2 – 7xy – 5y^2 \end{array}$$

$$\begin{array}{cccc} 2x & 2x & +y & +y \\ 3x & 3x & -5y & -5y \end{array}$$

EXAMPLE-32. Find the product of polynomials $x^2 + 3x + 2$ and $5x^2 + x + 1$ using Urdhwa tiryak Vidhi.

SOLUTION :

$$\begin{array}{r} x^2 + 3x + 2 \\ \times\ 5x^2 + x + 1 \\ \hline 5x^4 + 16x^3 + 14x^2 + 5x + 2 \end{array}$$

formula Urdhwa tiryagbhyaam

(1) First column

$$\begin{array}{r} +2 \\ +1 \\ \hline +2 \end{array}$$

Vertical multiplication

(2) First and second column

$$\begin{array}{r} +3x + 2 \\ \times\ \ +x + 1 \\ \hline \end{array}$$

Multiply diagonally and add

$(3x \times 1) + (x \times 2)$

$3x + 2x = 5x$

(3) First, second and third column

$$\begin{array}{c} x^2\ +3x+2 \\ 5x^2+x+1 \\ \hline (x^2 \text{ x} 1) + (5x^2 \text{ x } 2) + (3x \text{ x } x) \\ = x^2 + 10x^2 + 3x^2 = 14x^2 \end{array}$$

as per signs,multiply first with third terms diagonally and second by second vertically and add.

(4) Third and second column

$$\begin{array}{r} x^2 + 3x \\ \times\ 5x^2 + x \\ \hline \end{array}$$

multiply diagonally and add

$(x^2 \times x) + (5x^2 \times 3x)$

$x^3 + 15x^3 = 16x^3$

(5) Third column

$$\begin{array}{r} x^2 \\ \times\ 5x^2 \\ \hline 5x^4 \end{array}$$

Vertical multiplication

Exercise - 1.11

Solve using Urdhwatiryak Vidhi and check the answers:

(1) $\begin{array}{r} 4x+1 \\ \times\ x+5 \\ \hline \end{array}$ (2) $\begin{array}{r} 4x+2y \\ \times\ 3x+3y \\ \hline \end{array}$ (3) $\begin{array}{r} x-3y \\ \times\ x+3y \\ \hline \end{array}$ (4) $\begin{array}{r} x+4 \\ \times\ x+4 \\ \hline \end{array}$

(5) $\begin{array}{r} x^2+2x+1 \\ \times\ x^2+3x+4 \\ \hline \end{array}$ (6) $\begin{array}{r} 2x^2+3y-4 \\ \times\ 3x^2+4y+5 \\ \hline \end{array}$

Division

Paraavartya Vidhi: Division in Arithmetic and Algebra is done using Paraavartya Vidhi.

EXAMPLE-33. Divide $7x^2 - 5x + 3$ by $x + 1$

SOLUTION : Divisor $x + 1$

Modified divisor is –1

	$7x^2$	$-5x$	$+3$
	$+7$	-5	$+3$
		-7	$+12$
	$+7$	-12	$+15$

(1) Write the dividend and divisor in their respective places. The deviation of the divisor x + 1 is +1 and the inverse (revised value) of this is -1, so the modified divisor is -1

(2) Write the coefficients of the terms of the dividend with its signs.

(3) Since modified divisor has one digit, hence we leave one digit in the units place of the dividend and make the line for division.

(4) The first digit of the dividend that is 7 is the first digit of the answer. Write it down as it is.

(5) (First digit of answer x modified divisor), the product of these two is written below –5.

$+7 \times (-1) = -7$

(6) $(-5) + (-7) = -12$ is the second digit of the answer.

(7) The second digit of the answer x modified divisor, the product of these two is written below +3 of the dividend.

$-12 \times (-1) = +12$

(8) Now we have been crossed the division line. Hence +3 +12 = +15

(9) The quotient is 7x - 12 and the remainder is +15

EXAMPLE-34. Divide $x^3 + 2x + 12$ by $x + 2$.

SOLUTION : Divisor $x + 2$

Modified divisor –2

	$x^3 + 0x^2$	$+2x$	$+12$
	$+1 + 0$	$+2$	$+12$
	-2		
		$+4$	-12
	$+1 - 2$	$+6$	0

(1) The divisor is x + 2 and the deviation is +2, hence the modified divisor is the inverse of this, that is -2.

(2) Write the dividend in the reducing power of x. it does not have the term x^2, so we write the coefficient of this as 0.

(3) Modified divisor has one digit, so we draw the division line leaving a digit from the units place in the dividend.

(4) The first coefficient of the dividend, +1, is the first digit of the answer.

(5) First digit of the answer × modified divisor = +1 × (–2) = –2 is written below 0.

(6) +0 + (–2) = -2, is the second digit of the answer.

(7) Second digit of the answer x modified divisor = –2 × (–2) = + 4, is written below +2.

(8) +2 + (+4) = +6 is the third digit of the answer.

(9) Third digit of the answer x modified divisor = +6 × (-2) = –12, is written after the line for division below +12. –12 + 12 = 0 is the remainder after division line.

(10) Hence, the quotient +1 –2 +6 is written in the increasing order of the powers of x from units place.

Quotient $= x^2 - 2x + 6$

Remainder = 0

EXAMPLE-35. Divide $4x^3 - 5x - 9$ by $2x + 1$.

SOLUTION :

divisor	dividend	dividend in the reducing power of x
$2x + 1$	$4x^3 - 5x - 9$	$4x^3 + 0x^2 - 5x - 9$

(1) Divide the divisor by 2, so we get 1 as the coefficient of x, as the maximum power of x in divisor needs to have a coefficient 1 for this method.

Hence $\dfrac{2x+1}{2} = x + \dfrac{1}{2}$

Dividend written with the signs of the coefficients in the reducing power of x

New divisor $x+\frac{1}{2}$	+4	+0	–5	–9
Modified divisor $-\frac{1}{2}$		–2	+1	+2
	+4	–2	–4	–7

(1) First digit of answer is +4

(2) First digit of the answer × modified divisor = –½ × 4 = –2, is written below 0.

(3) +0 – 2 = –2, is the second digit of the answer.

(4) Second digit of the answer x modified divisor = –½ × –2 = 1, is written below -5.

(5) –5 + 1 = -4 is the third digit of the answer.

(6) Third digit of the answer × modified divisor = –4 × (–½) = +2, is written after the division line, below -9

(7) –9 + 2 = –7 is the remainder.

(8) In the quotient, +4 – 2 – 4, we have to divide by 2 because we have divided the divisor by 2, hence the quotient is +2 – 1 – 2 which when written with x is, $2x^2 - x - 2$

Remainder is –7.

EXAMPLE-36. Divide p(x) by g(x) when p(x) = $x^4 + 1$ and g(x) = $x + 1$

SOLUTION : Divisor dividend

$x + 1$ $x^4 + 1$

(1) We write the dividend in the reducing power of x, and write the coefficient of those terms that are not there as 0.

Divisor Dividend

$x + 1$ $x^4 + 0x^3 + 0x^2 + 0x + 1$

Write the coefficients of the dividend with the signs

Divisor $x + 1$	+1	+0	+0	+0	+1
Modified divisor –1		–1	+1	–1	
					+1
	+1	–1	+1	–1	+2

Quotient $x^3 - x^2 + x - 1$ (we write the quotient along with x as x^0 in units, x^1 in the tens and so on in the increasing power)

Remainder = 2

Check of the answer:

Digit sum of the dividend should be = Digit sum of (product of the digit sum of the divisor x digit sum of the quotient) + digit sum of remainder

LHS = 2

RHS = $(2 \times 0) + 2 = 2$ Since LHS = RHS, the answer is correct.

Wl ecYrW

ung BG

Let us know the history of arithmetic and algebra...

If you look carefully at the things, events and phenomena around you and think about them, you will discover that all of these are in some way related to mathematics. Whether we are buying or selling building something, setting our daily routine or planning something big- we are using mathematics every where.

Relationship of mathematics and the world around us is not something new. Along with evolution of human civilization journey of maths has also evolved. In this process of evolution human has tried to find solutions by using numbers, along with a whole range of symbols. As a result of these attempts only this branch of mathematics, which we call algebra, was born.

Algebra is that branch of mathematics where numbers are represented using alphabets. However, the signs of mathematical operations remain as they are in arithmetic.

Although modern symbolism of algebra was introduced only a few centuries ago, but the quest of solving equations is ancient. Even 2000 years before the Chrishian Era also people guessed at how to solve equations.

The tradition of expressing algebraic equations in symbols started in the following order. It is estimated that around 300 to 250 CE term algebra was used in popular conversations. for example, $x + 1 = 2$ was said as “By adding one to something, we get two.” In around 250 CE, we see shorthand symbols being used in Diophantus’s ‘Arithmetica’. Brahmgupta’s Brahmasfut Principle also enumerates similar things. This would have been written as- “when 1 is added to x, we would get 2.”

It was in the 7^{th} century that we started expressing variables and constant by way of symbols. Such as- $x + 1 = 2$ can be written in expanded form as : $ax + b = c$ where x is variable and a, b and c are constant numbers.

To learn how this must have happened, let us try to solve a riddle that is often asked by the elderly in villages. Look at this-

सौ. गोड़. आउ. बहत्तर. आँखी
कतका. हाथी,. कतका. पाँखी

It means- In a group of elephants and birds, we have a total of 72 eyes and 100 feet. Now say how many of them are elephants and how many of them are birds?

How will be solve this? We can solve this by hit and trial method, like by guessing if there are 20 elephants, there would be 80 feet. In that case we will have 10 birds to give us the remaining 20 feet. So, in all we have (20 + 10) 30 animals, which will give us 50 eyes. But, eyes we have 72! So, there must be some other solution.

Clearly, we will have to reduce the number of elephants. Think why.

We can also start from this end- if there are 72 eyes, there must be 36 animals. If number of elephants is ‘E’, then number of birds will be 36-E. Now, if we are to add the feet of ‘E’ elephants and (36-E) birds we know we will get 100. That means, Ex4 + (36 - E) ´ 2 = 100. That is how we get a simple equation which will help us get the value of E.

We find evidences according to which between 800-500 BC, shulv- sutras (शतवः सूत्र) were created in India. These shulv sutras were used in creating vedis of different kinds. These sutras hold solution to the problems like how to make vedis of different designs and shapes, keeping the area the same. Look at one example- how can we make a rectangle that has an area equal to a square? Although this looks like a problem of geometry, we can also solve it with the help of algebraic expressions. Suppose we have a square with side of length ‘a’. If we make a rectangle with a fixed length ‘l’ and we also know that its area is equal to that square. Then, what will be breadth of that rectangle? To solve this, we get an equation:

$$a \times a = \mathrm{l} \times b.$$

It is not the case that this was solely developed in India. There were others in all parts of the world who studied this new branch in mathematics. Around 500-300 BC. Archimedes had figured out sum of the square of natural numbers.

A lot of work related to writing of arithmetic and number groups in expanded form was done in India in the 6th century BC. Similarly in 5th and 6th century AD Aryabhatt and Brahmgupta discovered different types of general sums of number series.

They also figured out many different types of solution of general binomial and simple equations.

In India we get many examples of playing with numbers and discovering many general and specific ways in which numbers work.

There was a lot of exchange of ideas related to mathematics of Indians with Greek and Arabic mathematics too.

In 12th century India again a lot of work was done in the field of derivatives, micro, mean value theroem etc. Later on in 14th century further work was done in infinite series for sine and cosine. But this work could not be presented systematically and other people could not understand its importance.

Today the algebra we have is a result of a combination of a varied efforts made in different countries.

This is not the complete history of airthmetic and algebra. The information presented here is collected from various texts. Students and teachers can get more information about algebra from other sources too.

---

r n

4r g n 5

02

In a class, students came up with these numbers when asked to think of all possible types of numbers.

8,	-108,	12	205
0.15,	-0.37,	0	$\frac{22}{7}$,
$\frac{3}{7}$, 1,	$\frac{10}{9}$,	$\frac{17}{19}$,	-55
-3	105	-7.5	3.2323

Does it cover all kinds of numbers?

Can you give examples of any other kind of number?

Discuss with each other and give examples.

How will we classify these numbers on the basis of their properties? Which properties do we take? Can we classified these numbers as even-odd numbers? Can they be classified by any other method? How many different ways of classifying them, can there be?

From Natural to Rational

Manisha and Salma classified these numbers as N (Natural numbers), W (Whole numbers), Z (Integers) and Q (Rational numbers) as follows-

N	W	Z	Q
8, 105, 12, 205, 1	0, 8, 105, 12, 205, 1	0, 8, 105, 12, 205, 1, -108, -55, -3	0, 8, 105, 12, 205, 1, -3, -108, -55, $\frac{22}{7}$, $\frac{3}{7}$, -0.37, 3.2323, -7.5, $\frac{10}{9}$, $\frac{17}{19}$, 0.15

You write three more numbers in all these boxes.

Are there some numbers which are there in all the boxes?

Try This

1. Write those numbers which you have in Q but not in N.
2. Write those numbes which you have in Z but not in N.
3. Write those numbers which you have in W but not in N.

All numbers that are there in box Q, can be written in the form $\frac{p}{q}$. In this p and q are integers and $q \neq 0$, that means p can be any integer and q can be any integer except zero. These are all rational numbers.

Rashmi said that natural numbers, whole numbers, Integers are also included in rational numbers because they can be written in the form $\frac{p}{q}$.

Do you agree? Reshma gave some examples like as follows-

$$8 = \frac{8}{1}, \qquad -3 = \frac{-3}{1}, \qquad 0 = \frac{0}{1}$$

Can you look for such integers which can not be written in form of rational numbers? So we can say that the group of natural numbers are included in integers and the group of integers are included in rational numbers.

Natural numbers (N) and Integers (Z) follows one rule. Each number is one more than the previous number or the number on its left side; or one less than the next number or the number on its right side. That means distance between any two consecutive numbers is always same, that is one unit.

Number Line

To show whole number on number line, draw a line and mark points at equal intervals.

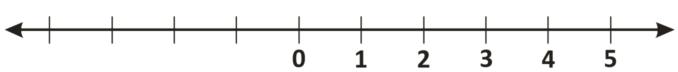


Fig. 1

Assuming any one point O, write 1, 2, 3, ... at equal intervals on the right hand side.

Number Line for Integers

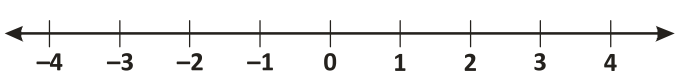


Fig. 2

To show integers on number line, write -1,-2,-3,......... on the left hand side of (*Fig.*1) number line as shown in *Fig.*2.

We can see that as we move to the right side, the number exceeds by one and as we move to the left side the number decreases by one.

Try This

1. Show on number line: -2, $+3$, $+4$ 2. How many steps ahead +2 is form -3?

Showing Rational Numbers on Number Line.

Can you show rational number $\frac{p}{q}$ on number line?

Can you show $\frac{1}{3}, \frac{5}{3}, \frac{8}{3}$ etc. on a number line?

Rashmi showed them on number line as follows-

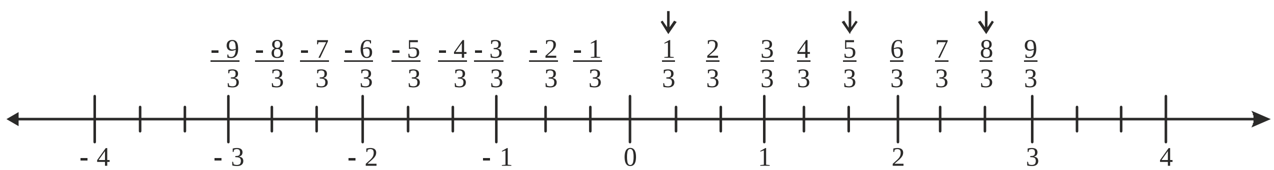


Fig. 3

In this, she divided each unit in three equal parts and then showed the numbers.

Similarly, you also try and show $\frac{7}{3}, \frac{11}{3}, -\frac{2}{3}$.

Think and discuss

What will you have to do to show $\frac{-3}{5}, \frac{2}{5}, \frac{8}{5}, \frac{12}{5}$ etc. on number line.

Equivalent Rational Numbers & Number line

Just as $\frac{1}{2}$ is rational number, similarly $\frac{2}{4}, \frac{4}{8}$, ... are also rational numbers. How to we shows them on number line? Let us try and see.

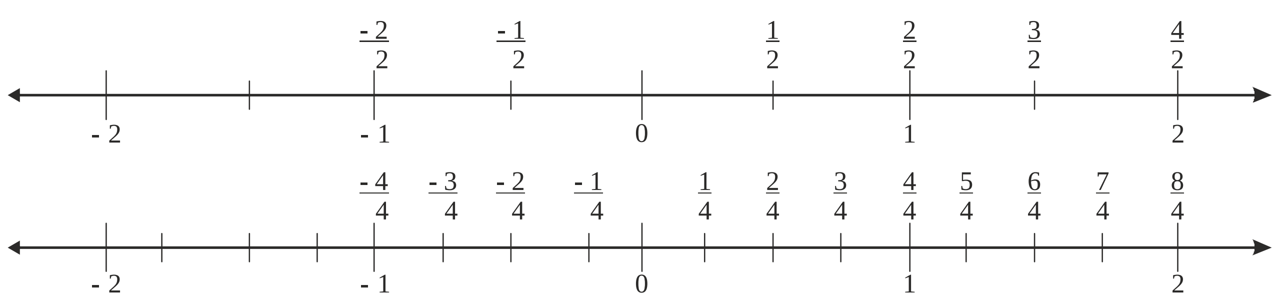


Fig. 4 (i), (ii)

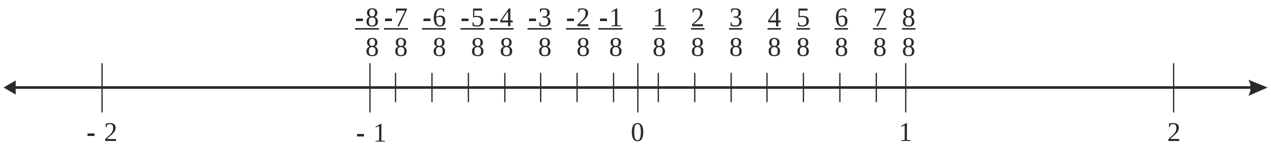


Fig. 4 (iii)

The place of $\frac{1}{2}$ on number line is also the same for $\frac{2}{4}$ and $\frac{4}{8}$. So $\frac{1}{2}, \frac{2}{4}, \frac{4}{8}$ which are equivalent rational numbers, occur at the same point on the number line.

How many rational number exist between two rational numbers– When we take any two integers, we can find the number of integers that lie between them.

Try This

1. How many integers lie between 5 & 15.
2. How many integers lie between -3 & 8.

Can we do similar counting for the rational numbers?

How many rational numbers lie between $\frac{1}{2}$ and $\frac{1}{4}$? Discuss with each other.

Reshma said– Looking at the number line, $\frac{3}{8}$ lies between $\frac{1}{2}$ and $\frac{1}{4}$.

$\frac{3}{8}$ means half of $\left(\frac{1}{2}+\frac{1}{4}\right)$

Salma said that between the rational numbers $\frac{3}{8}$ and $\frac{1}{2}$, will lie half of $\left(\frac{3}{8}+\frac{1}{2}\right)$ which means $\frac{7}{16}$. Again the mid point of $\frac{7}{16}$ and $\frac{1}{2}$ is $\frac{15}{32}$.

All these rational numbers lie between $\frac{1}{4}$ and $\frac{1}{2}$. Similarly we can find more rational numbers between these two numbers.

Can you think that how many rational numbers will lie between these two?

Discuss with each other and find some more rational numbers.

We see that as many times as we try, we can find new rational numbers lying between these rational numbers.

Now we can say that there are so many rational numbers are between $\frac{1}{4}$ *and* $\frac{1}{2}$ *that we can not count them.*

This is true for any two rational numbers.

Innumerable Numbers between Rational Numbers

a, b are any two rational number in which $a < b$

then $a + a < b + a$

$2a < b+a$

or $a < \frac{b+a}{2}$ ....(i)

Again $a < b$

$a + b < b + b$

$a + b < 2b$

$\frac{a+b}{2} < b$ ....(ii)

Hence $\frac{a+b}{2}$, is between a and b. i.e $a < \frac{a+b}{2} < b$

If we take any two rational numbers a and b, then there always is one rational number $\frac{a+b}{2}$ in between them and there is also a rational number between $\frac{a+b}{2}$ and a. So we can say that there are innumerable numbers between any two rational numbers.

Finding Rational Numbers between Two Given Rational Numbers.

We take two rational number $a = \frac{5}{1}$ and $b = \frac{6}{1}$.

Although there are innumerable numbers between them, we can find some numbers using this method.

A rational number which lies between $\frac{5}{1}$ and $\frac{6}{1} = \frac{a+b}{2} = \frac{5+6}{2} = \frac{11}{2}$

$$\frac{5}{1}=\frac{5\times 2}{1\times 2}=\frac{10}{2}$$;: (equivalent rational numbers)

$$\frac{6}{1}=\frac{6\times 2}{1\times 2}=\frac{12}{2},$$

$$\frac{10}{2}<\frac{11}{2}<\frac{12}{2}$$

We can see that the rational number $\frac{11}{2}$ lies between $\frac{10}{2}$ and $\frac{12}{2}$.

Similarly

$$\frac{5}{1}=\frac{5\times 3}{1\times 3}=\frac{15}{3},$$

$$\frac{6}{1}=\frac{6\times 3}{1\times 3}=\frac{18}{3},$$

i.e. Rational numbers $\frac{16}{3},\frac{17}{3}$ lie between $\frac{15}{3}$ and $\frac{18}{3}$.

Thus, we can find several numbers between two given rational numbers by using equivalent rational numbers. For example when we multiply $\frac{5}{1}$ and $\frac{6}{1}$ by 11, we find 10 new rational numbers between these two.

If we have to find 3 rational numbers between $\frac{1}{5}$ and $\frac{2}{5}$, we will multiply the denominator and numerator of both rational numbers by one more than three i.e.: 3+1:=:4.

$$\frac{1}{5}=\frac{1\times 4}{5\times 4}=\frac{4}{20}$$: (equivalent rational number)

$$\frac{2}{5}=\frac{2\times 4}{5\times 4}=\frac{8}{20}$$

$$\frac{1}{5}=\frac{4}{20}<\frac{5}{20}<\frac{6}{20}<\frac{7}{20}<\frac{8}{20}=\frac{2}{5}$$

So three rational number $\frac{5}{20},\ \frac{6}{20},\ \frac{7}{20}$ lie between $\frac{1}{5}$ and $\frac{2}{5}$.

Try This

1. Find any 5 rational numbers between $\frac{2}{7}$ and $\frac{4}{7}$.
2. Find 3 rational numbers between $\frac{1}{5}$ and $\frac{1}{7}$.
3. Find 11 rational numbers between $\frac{-1}{3}$ and $\frac{1}{2}$.

Properties of Rational Numbers

(i) Whole Numbers and Integers

Once again discuss about properties of numbers in brief. Start with closure property.

Complete the table given below by discussion. Please give relevant example in this table.

Numbers	Operations			
	Addition	Subtraction	Multiplication	Division
Whole numbers	a + b is whole numbers for any two whole numbers a & b, which means it is closed under addition e.g.:	It is not closed because 5-7 = -2 is not a whole number.	It is closed ––––––– –––––––	It is not closed because $5 \div 8 = \frac{5}{8}$ is not a whole number
Integers	$-6+4=-2$ is a integer. Integers are closed under the addition. e.g..............	It is closed because a-b is also an integer for any two integers a–b. e.g................	$5 \times 9 = 45$ $-5 \times 9 = -45$ and $-5 \times (-9) = 45$ are all integers. Generally we can say that a × b is an integer for any two integers a & b. e.g.....	It is not closed because ––––––– –––––––

(ii) **Rational numbers - Closure Property**

(a) **Addition**

Assume two rational number $\frac{2}{7}, \frac{5}{8}$.

$$\frac{2}{7}+\frac{5}{8} = \frac{16+35}{56} = \frac{51}{56}$$

The result $\frac{51}{56}$ is again a rational number.

$8+\left(\frac{-19}{2}\right) =$ ____________ Is it rational number?

$\frac{2}{7} + \frac{-2}{7} =$ ____________ Will the result be a rational number?

Check this property with other numbers.

$3+\frac{5}{7}$, $\quad 0+\frac{1}{2}$, $\quad \frac{7}{2}+\frac{2}{7}$

We see that by adding any two rational numbers, the result is also a rational number. If a & b are rational numbers then $a+b$ will also be a rational number. So, rational numbers are closed with respect to addition.

(b) **Subtraction**

Assume any two rational number $\frac{5}{9}$ and $\frac{3}{4}$.

than $\frac{5}{9}-\frac{3}{4} = \frac{5\times4-3\times9}{36} = \frac{20-27}{36} = \frac{-7}{36}$

$\frac{-7}{36}$ is rational number: (because: $-7, 36$ are integers and 36 is non zero, so $\frac{-7}{36}$ also a rational number.)

Examine it with respect to the following rational numbers too.

(i) $\frac{2}{3} - \frac{3}{7} = \frac{14-9}{21} =$ ____________ Is it a rational number?

(ii) $\left(\frac{48}{9}\right) - \frac{11}{18}$ = ______________ Is this a rational number?

We found that for any two rational numbers, their difference is also rational number. So rational numbers are closed with respect to subtraction.

For any two rational numbers a and b, a–b is also a rational number.

(c) Multiplication

Please consider the following

(i) $3 \times \frac{1}{2} = \frac{3}{2}$

(ii) $\frac{6}{5} \times \frac{-11}{2} = \frac{-66}{10} = \frac{-33}{5}$

(iii) $\frac{3}{7} \times \frac{5}{2}$ = __________

(iv) $\frac{2}{1} \times \frac{19}{13}$ = __________

We see in all examples that the product of any two rational numbers is a rational number. Please multiply some more pairs of rational numbers. Check whether their product is a rational number or not? Can you give such rational numbers whose product is not a rational number? Therefore, it shows us that rational number are closed with respect to the multiplication.

For any two rational numbers a and b, a × b is also a rational number.

(d) Division

Take two rational numbers $\frac{2}{3}$ and $\frac{7}{8}$.

then $\frac{2}{3} \div \frac{7}{8} = \frac{2}{3} \times \frac{8}{7} = \frac{16}{21}$ which is a rational number?

Checkout it with some other examples.

(i) $\frac{5}{7} \div 2 = \frac{5}{7} \div \frac{2}{1} = \frac{5}{7} \times \frac{1}{2} = \frac{5}{14}$

(ii) $-\frac{2}{3} \div \frac{6}{11}$ = __________ = __________ = __________

(iii) $3 \div \frac{17}{13} = \frac{3}{1} \div \frac{17}{13} = \_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_$

We see in all above examples that when we divide two rational numbers, the result is a rational number. Now can we say that closure property is true for the division of rational numbers?

Come, let us examine this : 0, 5 are rational numbers but $\frac{5}{0}$ is undefined. So that group of rational numbers 'Q' is not closed with respect to division.

If we remove zero from the Q than this group becomes closed with respect to division.

Try This

If we remove zero from the set of integers, is it closed with respect to division? Similarly checkout for whole numbers.

Complete the table.

Numbers	Closed under			
	Addition	**Subtraction**	**Multiplication**	**Division**
Natural number	Yes	....	....	....
Whole numbers	....	....	....	No
Integers	....	Yes	....	....
Rational number	....	....	Yes	....

Commutative Property

Come, Let us recall the commutative property with different operation for both whole numbers and Integers.

Complete the following table :

(i) Whole numbers

Operation	Examples	Comments
Addition	2,:3 are whole numbers 2+3 = 5 and 3 + 2 = 5	Whole numbers are commutative with respect to addition $\therefore$ 2 + 3 = 3 + 2

Subtraction	Is 3 – 2 and 2 –3 the same	It is not commutative.
Multiplication	------	------
Division	4 ÷ 2 = ? 2 ÷ 4 = ? Is 4 ÷ 2 = 2 ÷ 4 ?	-------

(ii) Integers

Operations	Examples	Comments
Addition	---	Addition is commutative in integers
Subtraction	2, 3 are integers. 2 – (3) = ? (3) – 2 = ? Is 2 – (3) = (3) – 2 = ?	....
Multiplication	....	....
Division	....	Division is not commutative in intergers

(iii) Rational numbers

(a) **Addition**

Take rational numbers $\frac{5}{2}$, $\frac{-3}{4}$. Add them.

$$\frac{5}{2} + \frac{(-3)}{4} = \frac{2\times5+1\times(-3)}{4} = \frac{10-3}{4} = \frac{7}{4}$$

and $$\frac{(-3)}{4} + \frac{5}{2} = \frac{1\times(-3)+2\times5}{4} = \frac{-3+10}{4} = \frac{7}{4}$$

So, $$\frac{5}{2} + \left(\frac{-3}{4}\right) = \frac{-3}{4} + \frac{5}{2}$$

Now check this property with some more pair of rational numbers.

Take two rational numbers $\frac{1}{2}$ and $\frac{5}{7}$. Is $\frac{1}{2} + \frac{5}{7} = \frac{5}{7} + \frac{1}{2}$?

Is $\frac{-2}{3} + \left(\frac{-4}{5}\right) = \frac{(-4)}{5} + \left(\frac{-2}{3}\right)$?

Can you suggest any pairs of rational numbers for which this rule is not true?

We can say that for any two rational numbers a and b, a + b = b + a . Thus, rational numbers are commutative under addition.

(b) **Subtraction :** Take two rational numbers $\frac{2}{3}$ and $\frac{7}{8}$.

$$\frac{2}{3} - \frac{7}{8} = \frac{16-21}{24} = \frac{-5}{24} \quad \text{and} \quad \frac{7}{8} - \frac{2}{3} = \frac{21-16}{24} = \frac{5}{24}$$

So $\frac{2}{3} - \frac{7}{8} \neq \frac{7}{8} - \frac{2}{3}$

Check these

Is $2 - \frac{5}{4} = \frac{5}{4} - 2$?

Is $\frac{1}{2} - \frac{3}{5} = \frac{3}{5} - \frac{1}{2}$?

Thus, we can say that subtraction is not commutative for rational numbers.

Hence for any two rational numbers a and b, $a - b \neq b - a$.

(c) Multiplication : Take two rational numbers: 2 and $\frac{-5}{7}$.

$$2 \times \frac{-5}{7} = \frac{-10}{7} \quad ; \quad \frac{-5}{7} \times 2 = \frac{-10}{7} \quad \text{Therefore} \quad 2 \times \frac{-5}{7} = \frac{-5}{7} \times 2$$

Is $\frac{-1}{2} \times \left(\frac{-3}{4}\right) = \left(\frac{-3}{4}\right) \times \left(\frac{-1}{2}\right)$?

Check out it for some more rational numbers.

We conclude that rational numbers are commutative under multiplication.

i.e. $a \times b = b \times a$ for any two rational numbers a & b.

(d) Division

is $\frac{7}{3} \div \frac{14}{9} = \frac{14}{9} \div \frac{7}{3}$?

$$\frac{7}{3} \div \frac{14}{9} = \frac{\cancel{7}^{1}}{\cancel{3}_{1}} \times \frac{\cancel{9}^{3}}{\cancel{14}_{2}} = \frac{3}{2} \quad \text{and} \quad \frac{14}{9} \div \frac{7}{3} = \frac{\cancel{14}^{2}}{\cancel{9}_{3}} \times \frac{\cancel{3}^{1}}{\cancel{7}_{1}} = \frac{2}{3}$$

$$\frac{7}{3} \div \frac{14}{9} \neq \frac{14}{9} \div \frac{7}{3}$$

Thus, we can say that division is not commutative for rational numbers.

Try This

Complete the table.

Numbers	Commutativity under			
	Addition	Subtraction	Multiplication	Division
Natural numbers	Yes	No	Yes	----
Whole numbers	----	----	----	No
Integers	----	----	----	----
Rational numbers	----	----	----	No

Associative Property

Recall the associativity of whole numbers relative to various operations i.e. Addition, Subtraction, Multiplication and Division.

(i) Whole numbers

Complete the table by giving necessary examples and comments.

Operation	Examples of Whole numbers	Comments
Addition	$2 + (3 + 0) = 2 + 3 = 5$ $(2 + 3) + 0 = 5 + 0 = 5$ $\Rightarrow 2 + (3 + 0) = (2 + 3) + 0$ $a + (b + c) = (a + b) + c$ For any whole numbers a, b, c	------- -------

Subtraction	$(4-3)-2=?\ \ 4-(3-2)=?$ Is $(4-3)-2 = 4-(3-2)$?	Subtraction is not associative
Multiplication	----------------------------	Multiplication is associative
Division	$2\div(3\div5)=2\div\frac{3}{5}=2\times\frac{5}{3}=\frac{10}{3}$ $(2\div3)\div5=\frac{2}{3}\div5=\frac{2}{3}\times\frac{1}{5}=\frac{2}{15}$ $2\div(3\div5)\neq(2\div3)\div5$	Division is not associative

(ii) Integers

Recall the associativity of integers relative to all operations.

Complete the following table with necessary comments.

Operation	Integers with examples	Comments
Addition	$2+[(-3)+5]=2+[-3+5]=2+2=4$ $[2+(-3)]+5=[2-3]+5=-1+5=4$ For any three integers a,b,c $a+(b+c)=(a+b)+c$	____________
Subtraction	Is $6-(9-5)=(6-9)-5$?	____________
Multiplication	Is $2\times[7\times(-3)]=(2\times7)\times(-3)$?	____________
Division	$10\div[2\div(-5)]=10\div\frac{-2}{5}=10\times\frac{-5}{2}=-25$ Now $(10\div2)\div(-5)=\frac{10}{2}\div(-5)=5\div(-5)=\frac{5}{-5}=-1$ Thus $10\div[2\div(-5)]\neq(10\div2)\div(-5)$	____________

(iii) Rationals numbers - Associativity

(a) **Addition**

Assume that three rational numbers are $\frac{2}{7}$, 5, $\frac{1}{2}$. Examine if

$$\frac{2}{7}+\left[5+\left(\frac{1}{2}\right)\right]=\left[\frac{2}{7}+5\right]+\left(\frac{1}{2}\right)$$

$$\text{L.H.S.} = \frac{2}{7}+\left[5+\left(\frac{1}{2}\right)\right]=\frac{2}{7}+\left[5+\frac{1}{2}\right]=\frac{2}{7}+\left[\frac{10+1}{2}\right]=\frac{4+77}{14}=\frac{81}{14}$$

$$\text{R.H.S.} = \left[\frac{2}{7}+5\right]+\left(\frac{1}{2}\right)=\left[\left(\frac{2+35}{7}\right)\right]+\frac{1}{2}=\frac{37}{7}+\frac{1}{2}=\frac{74+7}{14}=\frac{81}{14}$$

$\therefore$ L.H.S. = R.H.S.

Find out $\frac{1}{2}+\left[\frac{3}{7}+\left(\frac{4}{3}\right)\right]$ and $\left[\frac{1}{2}+\frac{3}{7}\right]+\left(\frac{4}{3}\right)$

Are the two additions equal?

Check out the associativity of rational numbers with some more examples.

We have seen that rational numbers are associative under addition.

Therefore, for any three rational numbers a, b and c, a + (b + c) = (a + b) + c.

(b) **Subtraction**

Take three rational numbers $\frac{1}{2}, \frac{3}{4}$ and $\frac{-5}{4}$.

Check $\frac{1}{2}-\left[\frac{3}{4}-\left(\frac{-5}{4}\right)\right]=\left[\frac{1}{2}-\frac{3}{4}\right]-\left(\frac{-5}{4}\right)$

$$\text{L.H.S.} = \frac{1}{2}-\left[\frac{3}{4}-\left(\frac{-5}{4}\right)\right] = \frac{1}{2}-\left(\frac{3}{4}+\frac{5}{4}\right)=\frac{1}{2}-\left(\frac{8}{4}\right)$$

$$= \frac{1}{2}-2=\frac{1-4}{2}=\frac{-3}{2}$$

$$\text{R.H.S.} = \left(\frac{1}{2}-\frac{3}{4}\right)-\left(\frac{-5}{4}\right) = \left(\frac{1\times 2-3}{4}\right)+\frac{5}{4}=\left(\frac{-1}{4}\right)+\frac{5}{4}$$

$$= \frac{-1+5}{4}=\frac{4}{4}=1$$

$$\therefore \frac{1}{2} - \left[\frac{3}{4} - \left(\frac{-5}{4}\right)\right] \neq \left(\frac{1}{2} - \frac{3}{4}\right) - \left(\frac{-5}{4}\right)$$

L.H.S. $\neq$ R.H.S.

We find out that subtraction does not follow the rule of associativity in rational numbers. Therefore, for three rational numbers a, b and c ; $a - (b - c) \neq (a - b) - c$

(c) **Mulitiplication**

Take three rational numers $\frac{2}{3}, \frac{4}{7}, \frac{-5}{7}$.

Is $\frac{2}{3} \times \left[\frac{4}{7} \times \left(\frac{-5}{7}\right)\right] = \left(\frac{2}{3} \times \frac{4}{7}\right) \times \left(\frac{-5}{7}\right)$

L.H.S. $= \frac{2}{3} \times \left[\frac{4}{7} \times \left(\frac{-5}{7}\right)\right] = \frac{2}{3}\left[\frac{-20}{49}\right] = \frac{-40}{147}$

R.H.S. $= \left(\frac{2}{3} \times \frac{4}{7}\right) \times \left(\frac{-5}{7}\right) = \left(\frac{8}{21}\right) \times \left(\frac{-5}{7}\right) = \frac{-40}{147}$

L.H.S. = R.H.S.

Check this.

Find $2 \times \left(\frac{1}{2} \times 3\right)$ and $\left(2 \times \frac{1}{2}\right) \times 3$

Is $2 \times \left(\frac{1}{2} \times 3\right) = \left(2 \times \frac{1}{2}\right) \times 3$

Find $\frac{5}{3} \times \left(\frac{3}{7} \times \frac{7}{5}\right)$ and $\left(\frac{5}{3} \times \frac{3}{7}\right) \times \frac{7}{5}$

Is $\frac{5}{3} \times \left(\frac{3}{7} \times \frac{7}{5}\right) = \left(\frac{5}{3} \times \frac{3}{7}\right) \times \frac{7}{5}$

In all above situations, we get that L.H.S. = R.H.S.

Thus, multiplication is associative in rational numbers.

Hence, for any rational numbers a,b,c; $a \times (b \times c) = (a \times b) \times c$.

(d) Division

Take three rational number like– $\frac{2}{3}, \frac{3}{4}$ and $\frac{1}{7}$.

Is $\frac{2}{3} \div \left(\frac{3}{4} \div \frac{1}{7}\right) = \left(\frac{2}{3} \div \frac{3}{4}\right) \div \frac{1}{7}$ **?**

$$\text{L.H.S.} = \frac{2}{3} \div \left(\frac{3}{4} \div \frac{1}{7}\right) = \frac{2}{3} \div \left(\frac{3}{4} \times \frac{7}{1}\right) = \frac{2}{3} \div \frac{21}{4} = \frac{2}{3} \times \frac{4}{21} = \frac{8}{63}$$

$$\text{R.H.S.} = \left(\frac{2}{3} \div \frac{3}{4}\right) \div \frac{1}{7} = \left(\frac{2}{3} \times \frac{4}{3}\right) \div \frac{1}{7} = \left(\frac{8}{9}\right) \div \frac{1}{7} = \frac{8}{9} \times \frac{7}{1} = \frac{56}{9}$$

$$\frac{2}{3} \div \left(\frac{3}{4} \div \frac{1}{7}\right) \neq \left(\frac{2}{3} \div \frac{3}{4}\right) \div \frac{1}{7}$$

$\therefore$ L.H.S. $\neq$ R.H.S.

Therefore, for any three rational number a, b, c ; $a \div (b \div c) \neq (a \div b) \div c$.

Thus, division is not associative in rational numbers.

Try This

Complete the table.

Numbers	Associativity under			
	Addition	Subtraction	Multiplication	Division
Natural numbers	Yes	No	....	....
Whole numbers	....	....	....	No
Integers	....	No	Yes	....
Rational numbers	....	....	....	....

Role of Zero

Can you think of a number, which added to any number, gives the same number? When zero is added to any rational number, we get the same rational number.

For example–

$1 + 0 = 1$ and $0 + 1 = 1$

$-2 + 0 = -2$ and $0 + (-2) = -2$

$$\frac{11}{3} + 0 = \frac{11}{3} \text{ and } 0 + \frac{11}{3} = \frac{11}{3}$$

Hence, we call zero as the additive identity. The statment of this property is presented below.

If a represents any rational number, then $a + 0 = a$ and $0 + a = a$

Is there an additive identity in natural numbers?

Role of 1

Fill in the blank boxes given below:

$\boxed{} \times 3 = 3$ and $3 \times \boxed{} = 3$

$-2 \times \boxed{} = -2$ and $\boxed{} \times -2 = -2$

$\frac{7}{8} \times \boxed{} = \frac{7}{8}$ and $\boxed{} \times \frac{7}{8} = \frac{7}{8}$

Have you seen some thing special in the above products?

We can see that any rational number when multiplied by '1', we get the same rational number as the product.

We can tell that '1' is the multiplicative identity for rational numbers.

For example, we do the following when we write $\frac{15}{50}$ in a simplified form.

$$\frac{15}{50} = \frac{3\times5}{10\times5} = \frac{3}{10} \times \frac{5}{5} = \frac{3}{10} \times 1 = \frac{3}{10}$$

Where we are write that $\frac{3}{10} \times 1 = \frac{3}{10}$, there we use the identity property of multiplication.

The Existence of Inverse

(i) Additive inverse

$3 + (-3) = 0$ and $-3 + 3 = 0$

$-5 + 5 = 0$ and $5 + (-5) =$ ______

$$\frac{2}{3} + ? = 0 \quad \text{and} \quad ? + \frac{2}{3} = 0$$

$$\left(-\frac{1}{2}\right) + ? = 0 \quad \text{and} \quad ? + \left(-\frac{1}{2}\right) = 0$$

In the first example -3 and 3 are additive inverses to each other, because we get '0' by adding those. Any two number whose sum is '0' are called additive inverses of each other. Generally If a is rational number $a + (-a) = 0$ and $(-a) + a = 0$.

Then 'a', '$-a$' are additive inverses of each other.

The additive inverse of 0, remains 0 because $0 + 0 = 0$

(ii) Multiplicative inverse

Which number should be multiplied to a rational number $\frac{2}{7}$ to give a product of 1?

We see $\frac{2}{7} \times \frac{7}{2} = 1$ and $\frac{7}{2} \times \frac{2}{7} = 1$

Fill in the following blank boxes.

$2 \times \square = 1$ and $\square \times 2 = 1$

$-5 \times \square = 1$ and $\square \times 5 = 1$

$\frac{-17}{19} \times \square = 1$ and $\square \times \frac{-17}{19} = 1$

$1 \times ? = 1$

$-1 \times ? = 1$

Any two numbers whose product is '1' are called multiplicative inverses of each other.

For example, $4 \times \frac{1}{4} = 1$ and $\frac{1}{4} \times 4 = 1$, therefore number 4 and $\frac{1}{4}$ are multiplicative inverses of each other.

We can say that if $\frac{a}{b} \times \frac{c}{d} = 1$ and $\frac{c}{d} \times \frac{a}{b} = 1$, the rational number $\frac{c}{d}$ is called multiplicative inverse of the second rational number $\frac{a}{b}$.

Rational Number and their Decimal Form

If we want to write decimal form of $\frac{7}{4}, \frac{9}{11}, \frac{7}{6}$ etc. we will divide 7 by 4, 9 by 11 and 7 by 6.

$$\begin{array}{r|l} & 1.75 \\ \hline 4 & 7 \\ & 4 \\ \hline & 30 \\ & 28 \\ \hline & \;\;20 \\ & \;\;20 \\ \hline & \;\;\;0 \end{array} \qquad \begin{array}{r|l} & 0.8181 \\ \hline 11 & 9.0 \\ & 88 \\ \hline & \;\;20 \\ & \;\;11 \\ \hline & \;\;\;90 \\ & \;\;\;88 \\ \hline & \;\;\;\;20 \\ & \;\;\;\;11 \\ \hline & \;\;\;\;\;9 \end{array} \qquad \begin{array}{r|l} & 1.1666 \\ \hline 6 & 7 \\ & 6 \\ \hline & 10 \\ & \;\;6 \\ \hline & \;\;40 \\ & \;\;36 \\ \hline & \;\;\;40 \\ & \;\;\;36 \\ \hline & \;\;\;\;40 \\ & \;\;\;\;36 \\ \hline & \;\;\;\;\;4 \end{array}$$

Do the calculation and match with the table.

1. Remainder = 3, 2, 0 Divisor 4
2. Remainder = 2, 9, 2, 9, 2, 9, ... Divisor 11
3. Remainder = 1,4,4,4,4....... Divisor 6

Look at these remainders. Can you see anything special?

1. On dividing by 4, remainder is zero after one point.
2. On dividing by 11, 2 and 9 keep recurring as reminder turn by turn.
3. On dividing by 6, the remainder is 4 repeatedly after the first time.
4. When there is a repetition of the remainder there is also a repetition observed in quotient.

To understand this, let us make a table.

Number	Quotient	Analysis	Conclusion
$\frac{7}{4}$	1.75	Remainder is zero after the two places of decimal	Terminating decimal
$\frac{9}{11}$	0.8181.....	There is a repetition of the digits after first two places of decimal	Non- terminating recurring decimal
$\frac{7}{6}$	1.166.........	There is the repetition of the number which comes after the first decimal place.	Non-terminating recurring decimal

We find from this table-

Reminder is zero after a few decimal places in number $\frac{7}{4}$. Some other example of this are $\frac{1}{2} = 0.5$, $\frac{7}{8} = 0.875$ etc. Such decimals are called terminating decimals, in which numbers after decimal point are limited.

The repetition occurs in the quotient after a fixed place in numbers like $\frac{9}{11}, \frac{7}{6}$. We also write $\frac{7}{6} = 1.1666...$ or as $1.1\overline{6}$. Similarly, write $\frac{9}{11} = 0.8181...$ as $0.\overline{81}$. This is non-terminating recurring i.e. unlimited and some groups of digits are repeated after some places of decimal.

Try This

1. Write two such rational numbers whose decimal form is non-terminating recurring.
2. Write two such rational numbers whose decimal form is terminating.

Writing the Decimal Form of Rational Number in its General Form

When we write decimal form of a number as form $\frac{p}{q}$ than we understood the number better. Let us take some recurring decimal numbers-

EXAMPLE-1. Express 1.555.........= $1.\overline{5}$ in form of $\frac{p}{q}$.

SOLUTION : Assume $\frac{p}{q} = 1.55555$.................. (i)

multiply both sides by 10.

$10\frac{p}{q} = 15.5555$.................. (ii)

subtract equation (i) from (ii)

$10\frac{p}{q} - \frac{p}{q} = (15.55555 .....) - (1.555 ......)$

$9\frac{p}{q} = 14$

$$\frac{p}{q} = \frac{14}{9}$$

EXAMPLE-2. To express $7.3\overline{456}$ in the form of $\frac{p}{q}$.

SOLUTION : Assume $\frac{p}{q} = 7.3\overline{456} = 7.3456456.................$

Multiply both sides by 10.

$10\frac{p}{q} = 73.456456................$ .....(i)

Multiply both sides by 1000 in equation (i), we get

$10000\frac{p}{q} = 73456.456456.................$ .....(ii)

subtract equation (i) from equation (ii).

$$9990\frac{p}{q} = 73383$$

$$\frac{p}{q} = \frac{73383}{9990}$$

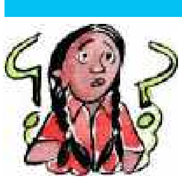
$$\therefore \frac{p}{q} = \frac{24461}{3330}$$

Try This

Change the following decimals in the form $\frac{p}{q}$.

(i) 1.2333....., (ii) 3.88...... (iii) 3.204343........

Exercise-2.1



1. Give examples for the following statements.
   (i) A number which is natural number, whole number and integer.
   (ii) A number which is whole number but not the natural number.
   (iii) A number which is rational number but not the natural number.

2. Find 3 rational numbers between the 6 & 7.

3. Find 5 rational numbers between $\frac{4}{7}$ and $\frac{5}{7}$.

4. Find 3 rational numbers between $\frac{2}{3}$ and $\frac{3}{4}$.

5. Find any 4 rational numbers between -1 and: 2.

6. Write any 3 rational numbers between $\frac{5}{10}$ and $\frac{9}{10}$.

7. Represent the $\frac{7}{3}$ and $-\frac{7}{3}$ on a number line.

8. Change the following rational number to decimal form and find what kind of decimal form they are?

   (i) $\frac{126}{5}$ (ii) $\frac{335}{16}$ (iii) $\frac{22}{7}$ (iv) $-\frac{118}{3}$

9. Express the following decimal numbers in $\frac{p}{q}$ form.

   (i) 0.53 (ii) 16.8 (iii) 105.25 (iv) 7.36

10. Change the following decimal numbers in $\frac{p}{q}$ form.

   (i) $0.\overline{70}$ (ii) $8.\overline{39}$ (iii) $3.12\overline{7}$ (iv) $5.\overline{125}$

Irrational Numbers

So far we have seen the decimal conversion of rational numbers, for example, $\frac{9}{18} = 0.5$, $\frac{7}{6} = 1.166\ldots\ \ldots. = 1.1\overline{6}$. It is obvious from these examples that their decimal forms are either terminating or non-terminating but recurring. Can you think of such numbers which are non-terminating but not recurring?

1.414213 ....

1.7320508 ....

2.2360679774 ....

As you can see in the above examples, neither do the decimals end nor do they repeat. These are called infinite non-recurring. See more such numbers. It can not be written in the form of $\frac{p}{q}$.

Similarly, $\sqrt{3} = 1.7320508075\ldots \ldots \ldots \ldots$

$\sqrt{5} = 2.2360679\ldots \ldots \ldots \ldots$

These too can not be written in the form of $\frac{p}{q}$. i.e. those numbers which are not the perfect squares, their root is an irrational number. It means $\sqrt{4}$, $\sqrt{36}$ is rational number because $\sqrt{4} = 2 = \frac{2}{1}$ and $\sqrt{36} = 6 = \frac{6}{1}$ but $\sqrt{3}$ and $\sqrt{7}$ is not a rational number.

Try This

Identify the rational number, irrational number in following numbers.

(i) $\sqrt{6}$ (ii) $\sqrt{7}$ (iii) $-\sqrt{25}$ (iv) $-\sqrt{8}$ (v) $\sqrt{9}$ (vi) $\sqrt{\frac{9}{16}}$

® **(Pi)**

π = 3.14159265389.............

π (Pi) is an irrational number which is approximated as the ratio of perimeter to the diameter of circle.

$$\pi = \frac{Perimeter}{Diameter} = \frac{c}{d}$$

Does this mean that at least one of the two numbers in the ratio is irrational or is there no such rational number which when multiplied to diameter to give us the perimeter. Often we assume the value of π as $\frac{22}{7}$ but this is just an approximate value of π.

Place Determination of Irrational Number on Number Line.

We have see that several rational numbers are found between two given rational numbers and these can be shown on the number line. Is there any place is blank between these rational numbers on number line? If the irrational numbers can also shown on number line then we can surely say that rational numbers do not cover the whole number line. Let us see how we can determine the place of $\sqrt{2}$ on number line.

EXAMPLE-3. Locate the place of $\sqrt{2}$ on number line.

Draw a perpendicular AB on point A (*Fig*.5). Take the length of $AB = 1$ unit. With the help of compass draw an arc of radius OB with O as the centre. It cuts the number line at a point P. Because the value of OB is $\sqrt{2}$, so OP is also $\sqrt{2}$. It means P is the point on number line which represents $\sqrt{2}$. Hence, there can not be any rational number at this point.

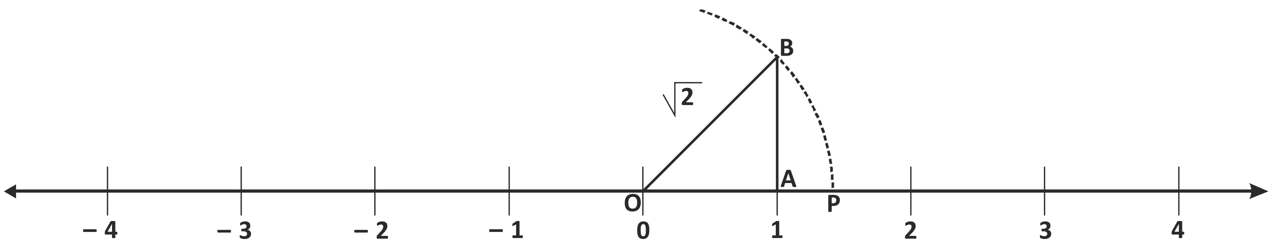


Fig. 5

Similarly, We may locate the place of $\sqrt{3}$ on number line.

Drawn a perpendicular AB at point A on number line (*Fig*.6), Where we take $AB = 1$ unit. Join O to B, then $OB = \sqrt{2}$.

We assume the centre 'O' and cut the arc with OB as radius.

This arc is cuts the number line at a point P. Draw a perpendicular CP at P where CP= 1 unit. Join O to C. Assume the centre O and cut the arc with OC as radius, which cuts the number line at the point Q. The distance OQ shows $\sqrt{3}$.

Point Q corresponds to $\sqrt{3}$.

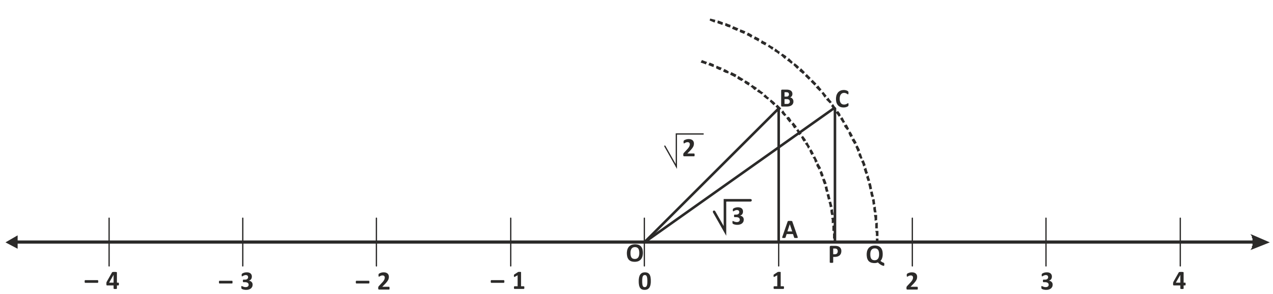


Fig. 6

Try This

1. Following numbers are rational or irrational?

(i) $\sqrt{81}$ (ii) $-\sqrt{625}$ (iii) $\sqrt{11}$

(iv) $-\sqrt{\dfrac{81}{25}}$ (v) $3.232323\ldots\ldots\ldots$ (vi) $5.7070070007\ldots$

2. Locate $\sqrt{5}, \sqrt{7}$ on number line.

3. Locate $-\sqrt{2}, -\sqrt{5}$ on number line.

Real Numbers

If we show all the rational and irrational numbers on the number line then will any number remain on number line? No, the collection which we get by combining the rational and irrational number, will cover the all points of number line. This big collection (set) is known as real numbers.

Operations on Real Numbers

We are doing operations of addition, subtraction, multiplication, division of rational numbers. We have seen that the addition and multiplication of rational number is always a rational number. Do we always get an irrational number when we do operations on irrational numbers? See the following examples–

$$\sqrt{5} + \left(-\sqrt{5}\right) = 0$$

$$\sqrt{3} \times \sqrt{3} = \sqrt{9} = 3$$

$$\frac{\sqrt{5}}{\sqrt{5}} = 1$$

In all these, the numbers which we got after operation, are not irrational. It means the addition, subtraction, multiplication & division of irrational numbers does not always give an irrational number.

Try This

1. You take some irrational numbers and check by doing operations.
2. Find atleast 5 examples where we don't get irrational number after doing some operations on them. Also think of such examples in which we get irrational number after doing operation.

Identifying Irrational Numbers

We know that $\sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{8}, \sqrt{10}$ etc. are irrational numbers. However $\sqrt{1}, \sqrt{4}, \sqrt{9}$ are rational numbers, but are $5+\sqrt{2}$, $\sqrt{3}-1$, $\frac{\sqrt{5}}{2}$ and other such numbers irrational numbers? Let us see-

We know $\sqrt{2}$ = 1.41421....

$\sqrt{3}$ = 1.73205

$\sqrt{5}$ = 2.23606....

(i) Therefore, $5+\sqrt{2}=6.41421......$ This representation is non terminating and non recurring. So $\left(5+\sqrt{2}\right)$ is not rational number.

(ii) $\sqrt{3}-1=(1.73205.........)-1$

$=$ 0.73205................ is non terminating and non recurring.

Therefore $\sqrt{3}-1$ is an irrational number.

(iii) $3\sqrt{2}=3\times(1.41421\ \ldots\ \ldots\ \ldots\ \ldots)$

$=$ 4.24263: .... is non terminating and non recurring decimal number.

Therefore $3\sqrt{2}$ is an irrational number.

(iv) $\frac{\sqrt{5}}{2}=\frac{2.23606.....}{2}$

$=$ 1.11803............... is non terminating and non recurring decimal number.

Therefore $\frac{\sqrt{5}}{2}$ is an irrational number.

What can be conclude from this?

By above examples we can say that addition, subtraction, multiplication and division between one rational and one irrational number is always irrational number.

Suppose a and b are two positive real numbers. Then

(i) $a\sqrt{b}+c\sqrt{b}=(a+c)\sqrt{b}$

(ii) $a\sqrt{b} - c\sqrt{b} = (a-c)\sqrt{b}$

(iii) $\sqrt{a}\ \sqrt{b} = \sqrt{ab}$

(iv) $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$

(v) $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{c}+\sqrt{d}\right) = \sqrt{ac}+\sqrt{ad}+\sqrt{bc}+\sqrt{bd}$

(Where c and d are positive real numbers)

(vi) $\left(\sqrt{a}+\sqrt{b}\right)\ \left(\sqrt{a}-\sqrt{b}\right) = a-b$

(vii) $\left(a+\sqrt{b}\right)\ \left(a-\sqrt{b}\right) = a^2-b$

Try This

1. Solve these-

   (i) $3\sqrt{2}+3\sqrt{2}$ (ii) $3\sqrt{6}.2\sqrt{3}$

2. Subtract $5\sqrt{3}+7\sqrt{5}$ from $3\sqrt{5}+7\sqrt{3}$.

3. Multiply $6\sqrt{3}$ by $13\sqrt{3}$.

4. Solve these-

   (i) $\left(3+\sqrt{3}\right)\left(2+\sqrt{2}\right)$ (ii) $3\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)$

   (iii) $\left(\sqrt{5}+\sqrt{2}\right)^2$ (iv) $\left(\sqrt{5}-\sqrt{2}\right)\left(\sqrt{5}+\sqrt{2}\right)$

Some other Operations

When there are several parts in a real number, then how do we add?

How can we add $a\sqrt{b}+c\sqrt{b}+e\sqrt{b}$? This addition: $= (a+c+e)\sqrt{b}$

Similarly other real numbers can be added.

EXAMPLE-4. Add $7\sqrt{3}-5\sqrt{5}$ to $9\sqrt{3}+7\sqrt{5}$.

SOLUTION : $9\sqrt{3}+7\sqrt{5}+7\sqrt{3}-5\sqrt{5}$

$= 9\sqrt{3}+7\sqrt{3}+7\sqrt{5}-5\sqrt{5}$

$= (9+7)\sqrt{3} + (7-5)\sqrt{5}$

$= 16\sqrt{3} + 2\sqrt{5}$

EXAMPLE-5. Subtract $3\sqrt{2} - 2\sqrt{3}$ from $3\sqrt{3} - 2\sqrt{5}$.

SOLUTION : $3\sqrt{3} - 2\sqrt{5} - \left(3\sqrt{2} - 2\sqrt{3}\right)$

$= 3\sqrt{3} - 2\sqrt{5} - 3\sqrt{2} + 2\sqrt{3}$

$= 3\sqrt{3} + 2\sqrt{3} - 2\sqrt{5} - 3\sqrt{2}$

$= (3+2)\sqrt{3} - 2\sqrt{5} - 3\sqrt{2}$

$= 5\sqrt{3} - 2\sqrt{5} - 3\sqrt{2}$

EXAMPLE-6. Solve the following-

(i) $\sqrt{3} \times \sqrt{5}$ (ii) $2\sqrt{3} \times 7\sqrt{2}$ (iii) $\left(2+\sqrt{5}\right)\left(2-\sqrt{5}\right)$

(iv) $\left(\sqrt{7}+\sqrt{3}\right)\left(\sqrt{2}+\sqrt{5}\right)$

SOLUTION : (i) $\sqrt{3} \times \sqrt{5} = \sqrt{3 \times 5} = \sqrt{15}$

(ii) $2\sqrt{3} \times 7\sqrt{2} = 2 \times 7\sqrt{3 \times 2} = 14\sqrt{6}$

(iii) $\left(2+\sqrt{5}\right)\left(2-\sqrt{5}\right) = 4 - 5 = -1$

(iv) $\left(\sqrt{7}+\sqrt{3}\right)\left(\sqrt{2}+\sqrt{5}\right) = \sqrt{14} + \sqrt{35} + \sqrt{6} + \sqrt{15}$

Rationalising the Denominator

We have shown $\sqrt{2}$ on number line, then can we show $\frac{1}{\sqrt{2}}$ on number line?

What is the value of $\frac{1}{\sqrt{2}}$.

$\frac{1}{\sqrt{2}} = \frac{1}{1.414213}....$

Making a rational number of the denominator means 'rationalisation'

Can we divide 1 by $\sqrt{2} = 1.414213...$? This is not easy. Because $\sqrt{2}$ is non terminating and non recurring decimal number. In this situation we need to rationalise the denominator.

To rationalise the denominator we multiply both numerator and denominator of $\frac{1}{\sqrt{2}}$ by $\sqrt{2}$.

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{1}{2}\sqrt{2} = \frac{\sqrt{2}}{2}$$ this is half of $\sqrt{2}$.

$\frac{\sqrt{2}}{2}$ on the number line?

$\frac{\sqrt{2}}{2}$ is half of $\sqrt{2}$, This is between 0 and 1.

EXAMPLE-7. Rationalise the denominator of $\frac{1}{4-\sqrt{7}}$.

SOLUTION: $\frac{1}{4-\sqrt{7}} = \frac{1}{4-\sqrt{7}} \times \frac{4+\sqrt{7}}{4+\sqrt{7}}$ [the rationalising factor of $\left(a-\sqrt{b}\right)$ is $\left(a+\sqrt{b}\right)$]

$$= \frac{4+\sqrt{7}}{(4)^2 - \left(\sqrt{7}\right)^2}$$

$$= \frac{4+\sqrt{7}}{16-7}$$

$$= \frac{4+\sqrt{7}}{9}$$

EXAMPLE-8. Rationalise the denominator of $\frac{3-\sqrt{2}}{3+\sqrt{2}}$.

SOLUTION : $\frac{3-\sqrt{2}}{3+\sqrt{2}} = \frac{3-\sqrt{2}}{3+\sqrt{2}} \times \frac{3-\sqrt{2}}{3-\sqrt{2}}$ [the rationalising factor of $\left(a+\sqrt{b}\right)$ is $\left(a-\sqrt{b}\right)$]

$$= \frac{3\times3 - 3\times\sqrt{2} - \sqrt{2}\times3 + \sqrt{2}\times\sqrt{2}}{(3)^2 - \left(\sqrt{2}\right)^2}$$

$$= \frac{9-6\sqrt{2}+2}{9-2}$$

$$= \frac{11-6\sqrt{2}}{7}$$

Try This

1. Rationalise the denominator.

(i) $\dfrac{1}{4+\sqrt{5}}$ (ii) $\dfrac{1}{7+4\sqrt{3}}$ (iii) $\dfrac{1}{7+4\sqrt{3}}+\dfrac{1}{2+\sqrt{5}}$

Exercise 2.2

1. Simplify:–

(i) $7\sqrt{3}+11\sqrt{3}$ (ii) $5\sqrt{7}-2\sqrt{7}$

2. Find the sum of $2\sqrt{2}+3\sqrt{5}$ and $5\sqrt{2}-2\sqrt{5}$.

3. Subtract $3\sqrt{5}+5\sqrt{7}$ from $8\sqrt{7}-5\sqrt{5}$.

4. Simplify.

(i) $\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)$ (ii) $\left(5+\sqrt{5}\right)\left(5-\sqrt{5}\right)$

(iii) $\left(\sqrt{2}+\sqrt{3}\right)\left(\sqrt{3}+\sqrt{8}\right)$ (iv) $\left(\sqrt{7}-\sqrt{3}\right)^2$

5. Rationalise the denominator.

(i) $\dfrac{1}{\sqrt{5}}$ (ii) $\dfrac{2}{\sqrt{6}}$ (iii) $\dfrac{\sqrt{21}}{\sqrt{3}}$

6. If a and b are two rational numbers then find the value of a and b in following equations

(i) $\dfrac{6+\sqrt{3}}{6-\sqrt{3}}=a+b\sqrt{3}$ (ii) $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}=a+b\sqrt{15}$

7. Simplify

(i) $\dfrac{5+\sqrt{3}}{2-\sqrt{3}}+\dfrac{2-\sqrt{3}}{2+\sqrt{3}}$ (ii) $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}+\dfrac{2-\sqrt{3}}{2+\sqrt{3}}+\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

8. If $x=3-2\sqrt{2}$ then find the value of $x+\dfrac{1}{x}$.

What Have We Learnt

1. Any number is said to be a rational number, if it can be written in form $\frac{p}{q}$, $(q \neq 0)$ where p and q are integers.
2. Any number is said to be an irrational number, if it can not be written in form $\frac{p}{q} (q \neq 0)$ when p and q are integers.
3. To find a rational number between two rational numbers, take average of both rational numbers.
4. There are so many rational numbers lying between two rational numbers that they are not countable.
5. Decimal expansion of a rational number is either terminating or non terminating and recurring.
6. Decimal expansion of irrational number is non terminating and non-recurring.
7. $\sqrt{2}, \sqrt{3}, \sqrt{5}, \pi$ are irrational numbers.
8. We get a set of real numbers by combining all rational and irrational numbers.
9. Every point on the number line corrosponds to a unique real number and every real number corresponds to a unique point on the number line.
10. By adding, subtracting, multiplying or dividing a rational and an irrational number gives us an irrational number.

11 To rationalization of the denominator of $\frac{1}{\sqrt{a} - b}$, We multiply it by $\frac{\sqrt{a}+b}{\sqrt{a}+b}$ when a and b are integers.

C

03

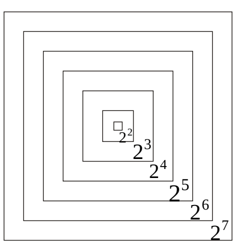
Nidhi, Mayank and Reshmi were asking each other questions about numbers-

Nidhi- What will we get when we multiply ten thousand by one lakh?

Reshmi- Hundred million which is the same as one billion. But do you know how many zeroes are there in this number?

Mayank- These will be 9 zeroes because ten thousand is10^4 and one lakh is10^5 and $10^4 \times 10^5 = 10^9$.

Mayank- My turn now. There are seven boxes. Each box is inside a bigger one. The smallest one has 2 beads and the next bigger box has twice as many and so on for every next box. Then how many beads are there in the seventh box? It is very difficult, it will take you a long time.



Nidhi- Why? 7 box and the numbers is doubled each time that means $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$ means $2^7 = 128$.

Reshmi- But let us try and also find out the total numbers of beads.

Discuss among yourselves and find out the total number of beads.

Try This

1. There are four rice grains in one of the square of the chess board. In the second square next to it, the number of grains is four times of that, in the third square number is four times of second and so on. Tell me how many grains of rice will be there in the fourth square. Write it in exponent form?
2. Solve these-

   (i) $3^5 \times 3^7$ (ii) $17 \times 17 \times 17 \times 17$ (iii) $5 \times 3 \times 3 \times 3 \times 3$

   (iv) $\left(\frac{2}{3}\right)^3 \times \left(\frac{2}{3}\right)^5$ (v) $\frac{2}{5} \times \frac{2}{5} \times \frac{2}{5}$
3. Distance between Sun-Earth is approximately 150000000 kilometer. Write it as an exponent?

   You can also prepare three question of this kind and give friends to attempt to solve them.

Laws of Exponent

If a is a real number and m, n are integers then,

1. Rule of multiplication is $a^m \times a^n = a^{m+n}$

2. Rule of division is $\dfrac{a^m}{a^n} = a^{m-n} (a \neq 0)$

3. Rule of powers of exponent is $\left(a^m\right)^n = a^{mn}$

4. $(ab)^m = a^m \times b^m$

5. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$

6. Meaning of a^0

$\dfrac{a^m}{a^n} = a^{m-n}$ where $a \neq 0$

if $m = n$, then

$\dfrac{a^m}{a^m} = a^{m-m}$

$1 = a^0$

Therefore $a^0 = 1$

7. $a^m = \dfrac{1}{a^{-m}}$; or $a^{-m} = \dfrac{1}{a^m}$

EXAMPLE-1. Simplify these-

(i) $3^5 \times 9^4$ (ii) $(4 \times 3)^4 \times \left(\dfrac{2}{4}\right)^6$ (iii) $\dfrac{(6^2)^4 \times 6^5}{6^4}$

SOLUTION : (i) $3^5 \times 9^4$

$= 3^5 \times (3^2)^4 = 3^5 \times 3^{2\times 4}$ $\quad \left[(a^m)^n = a^{m\times n}\right]$

$= 3^5 \times 3^8 = 3^{(5+8)}$ $\quad \left[a^m \times a^n = a^{m+n}\right]$

$= 3^{13}$

(ii) $(4\times 3)^4 \times \left(\dfrac{2}{4}\right)^6$

$= \left(4^4 \times 3^4\right)\times \dfrac{2^6}{4^6}$ $\qquad \because \left[\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}\right]$ or $\left[(ab)^m = a^m \times b^m\right]$

$= \left((2^2)^4 \times 3^4\right)\times \dfrac{2^6}{(2^2)^6}$

$= (2^{2\times 4} \times 3^4)\times \dfrac{2^6}{2^{2\times 6}}$ $\qquad \left[(a^m)^n = a^{m\times n}\right]$

$= 2^8 \times 3^4 \times \dfrac{2^6}{2^{12}} = 2^8 \times 3^4 \times 2^{6-12}$ $\qquad \left[\dfrac{a^m}{a^n} = a^{m-n}\right]$

$= 2^8 \times 3^4 \times 2^{-6}$

$= 2^{8-6} \times 3^4 = 2^2 \times 3^4$ $\qquad \left[a^m \times a^n = a^{m+n}\right]$

(iii) $\dfrac{\left(6^2\right)^4 \times 6^5}{6^4} = \dfrac{6^{2\times 4} \times 6^5}{6^4}$ $\qquad \left[(a^m)^n = a^{m\times n}\right]$

$= \dfrac{6^8 \times 6^5}{6^4} = \dfrac{6^{8+5}}{6^4} = \dfrac{6^{13}}{6^4}$ $\qquad \left[a^m \times a^n = a^{m+n}\right]$

$= 6^{13-4} = 6^9$ $\qquad \left[\dfrac{a^m}{a^n} = a^{m-n}\right]$

Try This

Simplify the following-

(i) $\dfrac{5^4 \times 5^6}{5^3}$ $\qquad$ (ii) $\dfrac{(2^2)^3 \times 8^7}{4^4}$

(iii) $\dfrac{(9\times 3)^8}{(3)^5}$ $\qquad$ (iv) $(3\times 2)^4 \times \left(\dfrac{2}{3}\right)^6$

Negative Exponents

We know that numbers can be written in the form of other numbers and their exponents-

Like-

1 kilometer	= 1000 meter	= 10^3 meter
1 hectometer	= 100 meter	= 10^2 meter
1 decameter	= 10 meter	= 10^1 meter
1 meter	= 1 meter	= ?

Here the number are expressed in the standard form of powers of 10.

If we have a number smaller than 1, then how do we write that -

To do that let us look at the following pattern-

1 decimeter $= \frac{1}{10}$ meter $= 10^{-1}$ meter

1 centimeter $= \frac{1}{100}$ meter $= 10^{-2}$ meter

1 milimeter $= \frac{1}{1000}$ meter $= 10^{-3}$ meter

As shown above, we can write $\frac{1}{10} = \frac{1}{10^1} = 10^{-1}$

$$\frac{1}{100} = \frac{1}{10 \times 10} = \frac{1}{10^2} = 10^{-2}$$

$$\frac{1}{1000} = \frac{1}{10 \times 10 \times 10} = \frac{1}{10^3} = 10^{-3}$$

This means $\frac{1}{10^n} = 10^{-n}$

Then is $\frac{1}{10^{-n}} = 10^n$? Discuss.

We write 4^{-3} as $\frac{1}{4^3}$. Similarly, we can write-

$$5^{-6} = \frac{1}{5^6}$$

Let us consider some more examples-

$$\frac{1}{6^{-3}} = 6^3$$

$$\frac{1}{9^2} = 9^{-2}$$

$$3^3 = \frac{1}{3^{-3}}$$

From the above examples we can see-

$1 = 6^3 \times 6^{-3}$

$1 = 9^2 \times 9^{-2}$

$3^3 \times 3^{-3} = 1$

Using these we can say that for any rational number 'a' (other than '0') $a^{-m} = \frac{1}{a^m}$, which is the multiplicative inverse of a^m.

because $a^m \times a^{-m} = a^{m+(-m)} = a^0 = 1$, where 'm' is an integer.

Try This

1. Express the following numbers in exponent form-

   (i) $\frac{1}{8}$ (ii) $\frac{1}{243}$ (iii) $\frac{1}{196}$

2. Write the multiplicative inverse of-

   (i) 10^{-5} (ii) $\frac{1}{2^3}$ (iii) P^{-n} (iv) 5^{-7}

3. Simplify each of the following-

   (i) $\left((5^2)^3 \times 5^4\right) \div 5^6$ (ii) $2^2 \times \frac{3^2}{2^{-2}} \times 3^{-1}$ (iii) $(14^{-2} \times 13^{-2}) \div 6^{-1}$

EXAMPLE-2. Simplify each of the following-

(i) $3^4 \times 3^{-8}$ (ii) $(-2)^{-3} \times (-2)^{-4}$

SOLUTION : (i) We know that- $3^{-8} = \frac{1}{3^8}$ $\left[a^{-m} = \frac{1}{a^m}\right]$

Hence, $3^4 \times 3^{-8} = 3^4 \times \frac{1}{3^8} = \frac{3^4}{3^8}$

$= 3^{4-8} = 3^{-4}$

(ii) $(-2)^{-3} \times (-2)^{-4} = \frac{1}{(-2)^3} \times \frac{1}{(-2)^4} = \frac{1}{(-2)^{3+4}}$

$= \frac{1}{(-2)^7} = (-2)^{-7}$

EXAMPLE-3. Find the values of -

(i) 5^{-2} (ii) $\frac{1}{2^{-5}}$ (iii) $\frac{4^7}{4^4}$

SOLUTION : (i) $5^{-2} = \frac{1}{(5)^2} = \frac{1}{5 \times 5} = \frac{1}{25}$

(ii) $\frac{1}{2^{-5}} = 2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$

(iii) $\frac{4^7}{4^4} = 4^{7-4} = 4^3 = 4 \times 4 \times 4 = 64$

EXAMPLE-4. Find the values of -

(i) $\left(\frac{4}{7}\right)^{-3}$ (ii) $4^4 \times 16^{-2} \times 4^0$

SOLUTION : (i) $\left(\frac{4}{7}\right)^{-3} = \frac{(4)^{-3}}{(7)^{-3}} = 4^{-3} \times \frac{1}{7^{-3}}$ $\left[\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}\right]$

$$= \frac{7^3}{4^3} \quad \left[a^{-m} = \frac{1}{a^m} \text{ or } a^m = \frac{1}{a^{-m}}\right]$$

$$= \frac{7 \times 7 \times 7}{4 \times 4 \times 4} = \frac{343}{64}$$

(ii) $4^4 \times 16^{-2} \times 4^0$

$$= \frac{4^4 \times 4^0}{16^2} \qquad \left[a^{-m} = \frac{1}{a^m}\right]$$

$$= \frac{4^4 \times 4^0}{\left(4^2\right)^2}$$

$$= \frac{4^4 \times 4^0}{4^4} \qquad \left[\left(a^m\right)^n = a^{m \times n}\right]$$

$= 4^{4+0-4} = 4^0$

$= 1$

EXAMPLE-5. Simplify- $\left[\left\{\left(\frac{1}{3}\right)^{-3} - \left(\frac{1}{2}\right)^{-3}\right\} \div \left(\frac{1}{5}\right)^{-2}\right]$

SOLUTION : $\left[\left\{\left(\frac{1}{3}\right)^{-3} - \left(\frac{1}{2}\right)^{-3}\right\} \div \left(\frac{1}{5}\right)^{-2}\right] \qquad \left[\text{we know that } \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}\right]$

$$= \left[\left(\frac{1^{-3}}{3^{-3}} - \frac{1^{-3}}{2^{-3}}\right) \div \frac{1^{-2}}{5^{-2}}\right] \qquad \left[\text{we know that } a^{-m} = \frac{1}{a^m} \text{ and } a^m = \frac{1}{a^{-m}}\right]$$

$$= \left[\left(\frac{3^3}{1^3} - \frac{2^3}{1^3}\right) \div \frac{5^2}{1^2}\right] = \left(\frac{27}{1} - \frac{8}{1}\right) \div 25$$

$$= (27 - 8) \div 25 = \frac{19}{25}$$

EXAMPLE-6. Simplify- $\left(\frac{2}{5}\right)^{-3} \times \left(\frac{25}{4}\right)^{-2}$

SOLUTION : $\left(\frac{2}{5}\right)^{-3} \times \left(\frac{25}{4}\right)^{-2}$

$$= \left(\frac{2}{5}\right)^{-3} \times \left(\frac{5^2}{2^2}\right)^{-2} \quad \left\{\because \frac{25}{4} = \frac{5\times5}{2\times2} = \frac{5^2}{2^2}\right\} \quad \left[\because (a^m)^n = a^{mn}\right]$$

$$= \frac{5^3}{2^3} \times \frac{2^4}{5^4} = 5^{3-4} \times 2^{4-3} \qquad \left[like \frac{1}{a^m} = a^{-m} \text{ and } \frac{1}{a^{-m}} = a^m\right]$$

$$= 5^{-1} \times 2^1 = \frac{2}{5}$$

EXAMPLE-7. If $x = \left(\frac{3}{2}\right)^2 \times \left(\frac{2}{3}\right)^{-4}$ then find the value of x^{-2}.

SOLUTION : $x = \left(\frac{3}{2}\right)^2 \times \left(\frac{2}{3}\right)^{-4}$

$$x = \frac{3^2}{2^2} \times \frac{2^{-4}}{3^{-4}} \quad \left[we\ know \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}\right]$$

$$x = \frac{3^2}{2^2} \times \frac{3^4}{2^4} = \frac{3^{2+4}}{2^{2+4}} = \frac{3^6}{2^6} = \left(\frac{3}{2}\right)^6$$

$$x^{-2} = \left[\left(\frac{3}{2}\right)^6\right]^{-2} = \left(\frac{3}{2}\right)^{-12}$$

$$x^{-2} = \frac{3^{-12}}{2^{-12}} = \frac{2^{12}}{3^{12}} = \left(\frac{2}{3}\right)^{12}$$

EXAMPLE-8. Simplify-

(i) $\frac{3^{2m+1} \times 9^{3m}}{3^{4m+1}}$ (ii) $\frac{x^{a+b} y^{a-b}}{x^{a+2b} y^{a-2b}}$

SOLUTION : (i) $\dfrac{3^{2m+1}\times 9^{3m}}{3^{4m+1}}$

$= \dfrac{3^{2m+1}\times (3^2)^{3m}}{3^{4m+1}}$

$= \dfrac{3^{2m+1}\times 3^{6m}}{3^{4m+1}} \quad = \dfrac{3^{2m+1+6m}}{3^{4m+1}}$

$= \dfrac{3^{8m+1}}{3^{4m+1}} \quad = 3^{8m+1-4m-1}$

$= 3^{4m}$

(ii) $\dfrac{x^{a+b}y^{a-b}}{x^{a+2b}y^{a-2b}}$

$= x^{a+b}\times y^{a-b}\times x^{-(a+2b)}\times y^{-(a-2b)}$

$= x^{a+b-a-2b}\times y^{a-b-a+2b}$

$= x^{-b}\times y^{b}$

$= \dfrac{1}{x^b}\times y^b \quad = \left(\dfrac{y}{x}\right)^b$

Expanded Form of Decimal Numbers

Expanded form of 328 is-

$$328 = 300+20+8$$

$$= 3\times 100+2\times 10+8\times 1$$

$$= 3\times 10^2+2\times 10^1+8\times 10^0$$

Similarly, $4158 = 4\times 1000+1\times 100+5\times 10+8\times 1$

$$= 4\times 10^3+1\times 10^2+5\times 10^1+8\times 10^0$$

Similarly, if we expand 132.28 we get the number in the form of exponents-

$$132.28 = 1 \times 100 + 3 \times 10 + 2 \times 10^0 + 2 \times \frac{1}{10} + 8 \times \frac{1}{100}$$

$$= 1 \times 10^2 + 3 \times 10^1 + 2 \times 10^0 + 2 \times 10^{-1} + 8 \times 10^{-2}$$

Try This

Write the following numbers in expanded form of exponents-

(i) **15.1** **(ii)** **512.23**

(iii) **537.204** **(iv)** **205.003**

Standard Representations of Very Large and Very Small Numbers

Large number such as- The diameter of the Sun is approximately 1400000000 meters, can be written in the standard form as 1.4×10^9 meter. The number will be written as powers of 10. Similarly, 6.2×10^5 meter will be expressed in the common form as- 6.2×100000 meter = 620000 meter.

Similarly, a very small number like- The charge of an electron is 0.00000000000000000016 coulomb, can be written as 1.6×10^{-19} coulomb.

Here we write numbers in the form of powers of 10 and represent both large and small numbers, in the standard form.

Comparison between Very Large and Very Small Numbers

The distance between the Sun and the Earth is 1.496×10^{11}m and the distance between the Earth and the Moon in 3.703×10^8m. During a solar eclipse, the Moon comes between the Sun and the Earth. What will be the distance between the Moon and Sun at this time?

$= 1.496 \times 10^{11} - 3.703 \times 10^8$

$= 1.496 \times 1000 \times 10^8 - 3.703 \times 10^8$

$= (1496 - 3.703) \times 10^8$

$= 1492.297 \times 10^8$m

EXAMPLE-9. Express the following in standard form -

(i) 40600000000 (ii) 2150000000000

SOLUTION : (i) 4.06×10^{10}

(ii) 2.15×10^{12}

EXAMPLE-10. Express the following in the decimal form-

(i) 3×10^{-8} (ii) 4.37×10^{-5}

SOLUTION : (i) 3×10^{-8}

$$=\frac{3}{10^8}=\frac{3}{100000000}=0.00000003$$

(ii) 4.37×10^{-5}

$$=\frac{4.37}{10^5}=\frac{4.37}{100000}=0.0000437$$

Exercise - 3.1

1. Find the values of -

(i) $\left(\frac{1}{2}\right)^{-5}$ (ii) $\frac{1}{3^{-4}}$ (iii) $\frac{6^7}{2^3\times3^7}$

2. Simplify each of the following-

(i) $(-4)^3\times(-2)^{-3}$ (ii) $(-3)^4\times\left(\frac{5}{3}\right)^4$ (iii) $(-5)^3\div(5)^{-7}$

3. Simplify-

(i) $\frac{16\times t^{-3}}{4^{-3}\times8\times t^{-6}}$ $(t\neq0)$ (ii) $\left\{\left(\frac{1}{3}\right)^{-1}-\left(\frac{1}{4}\right)^{-1}\right\}^{-1}$

4. Prove-

(i) $\left(\frac{x^a}{x^b}\right)^{a+b}\times\left(\frac{x^b}{x^c}\right)^{b+c}\times\left(\frac{x^c}{x^a}\right)^{c+a}=1$

(ii) $\frac{1}{1+x^{m-n}}+\frac{1}{1+x^{n-m}}=1$

5. Write the following numbers in the standard form-
   (i) 0.00000000000852 (ii) 8020000000000000
   (iii) 41960000000
6. Express the following numbers in the decimal form-
   (i) 5.02×10^{-6} (ii) 7×10^{-8}
   (iii) 1.00001×10^{9}
7. Write numbers in the following statements in standard form-
   (i) Size of red blood cells is 0.000007 m.
   (ii) The diameter of the Earth is 12756000 m.
   (iii) Thickness of a paper is 0.08 m.

Positive rational Exponent

We know that $2^3 = 8$

We may express this relation as $8^{\frac{1}{3}} = 2$

Similarly, $5^3 = 125$ can also be expressed as $(125)^{\frac{1}{3}} = 5$.

In general, if x and y are non-zero rational numbers and m is a positive integer such that $x^m = y$, then we may write $y^{\frac{1}{m}} = x$. We may also write $y^{\frac{1}{m}}$ as $\sqrt[m]{y}$ and call it the mth root of y.

For example the second root of $9 = \sqrt{9} = \sqrt[2]{9} = 3$

Some other example are, the volume of a cube is 64 units so its side will be $64^{\frac{1}{3}}$ units means-

Third root of $64 = \sqrt[3]{64} = 4$ units

That is the side of cube will be 4 units.

> The second root is called 'square root' and the third root is the 'cube root'.

Fourth root of $625 = \sqrt[4]{625} = 5$

Thus we can define x^m as a positive rational exponent m.

If x is a positive rational number and $m = \frac{p}{q}$ is a positive rational exponent, then we define $x^{\frac{p}{q}}$ as the q^{th} root of x^p.

For example the volume of a sphere is $\frac{4}{3}\pi \times (125)$ implies $r^3 = 125$

So its radius will be-

$$r = \sqrt[3]{125} = 125^{\frac{1}{3}} = (5^3)^{\frac{1}{3}}$$

Radius $r = 5^{3/3} = 5^1 = 5$

That is, $x^{\frac{p}{q}} = \left(x^p\right)^{\frac{1}{q}}$

For example, $(8)^{\frac{5}{3}}$ can be expressed in various ways-

$(8)^{\frac{5}{3}}$ or $(8^5)^{\frac{1}{3}}$ or $\left(8^{\frac{1}{3}}\right)^5$ or $\left(2^{3\times\frac{1}{3}}\right)^5$

or 2^5 or 32

If x is a positive rational number then for positive rational exponent $\frac{p}{q}$,

$$\left(x^p\right)^{\frac{1}{q}} = \left(x^{\frac{1}{q}}\right)^p$$

EXAMPLE-11. Find the value of-

(i) $(27)^{\frac{2}{3}}$ (ii) $\left(\frac{32}{243}\right)^{\frac{4}{5}}$

SOLUTION : (i) $(27)^{\frac{2}{3}} = \left(27^2\right)^{\frac{1}{3}} = (729)^{\frac{1}{3}}$

$= (3\times3\times3\times3\times3\times3)^{\frac{1}{3}}$

$= 3^{6\times\frac{1}{3}} = 3^2 = 9$

or $\quad (27)^{\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^2$

$$= \left(3^{3\times\frac{1}{3}}\right)^2 = (3)^2 = 9$$

(ii) $\quad \left(\dfrac{32}{243}\right)^{\frac{4}{5}} = \left\{\left(\dfrac{2\times2\times2\times2\times2}{3\times3\times3\times3\times3}\right)^{\frac{1}{5}}\right\}^4$

$$= \left\{\left(\frac{2^5}{3^5}\right)^{\frac{1}{5}}\right\}^4 = \left\{\left(\frac{2}{3}\right)^{5\times\frac{1}{5}}\right\}^4 = \left(\frac{2}{3}\right)^4$$

$$= \frac{2\times2\times2\times2}{3\times3\times3\times3} = \frac{16}{81}$$

or $\quad \left(\dfrac{32}{243}\right)^{\frac{4}{5}} = \left[\left(\dfrac{32}{243}\right)^4\right]^{\frac{1}{5}}$

$$= \left[\left(\frac{2^5}{3^5}\right)^4\right]^{\frac{1}{5}} = \left[\frac{2^{5\times4}}{3^{5\times4}}\right]^{\frac{1}{5}}$$

$$= \left(\frac{2^{20}}{3^{20}}\right)^{\frac{1}{5}} = \frac{2^4}{3^4}$$

$$= \left(\frac{2}{3}\right)^4 = \frac{16}{81}$$

In the above example, we have used both the forms. Which form do you think is easier for the purpose of calculation?

The exponent rules also apply to rational exponents. Let us see.

The first method $\quad \left(\dfrac{4}{25}\right)^{\frac{1}{2}} \times \left(\dfrac{4}{25}\right)^{\frac{3}{2}}$

$$\left\{\left(\frac{2}{5}\right)^2\right\}^{\frac{1}{2}} \times \left\{\left(\frac{2}{5}\right)^2\right\}^{\frac{3}{2}}$$

or $$\left\{\frac{2}{5}\right\}^{2\times\frac{1}{2}} \times \left(\frac{2}{5}\right)^{\cancel{2}\times\frac{3}{\cancel{2}}}$$

$$\left(\frac{2}{5}\right)^1 \times \left(\frac{2}{5}\right)^3 = \frac{2}{5}\times\frac{2}{5}\times\frac{2}{5}\times\frac{2}{5} = \frac{16}{625}$$

The second method $$\left(\frac{4}{25}\right)^{\frac{1}{2}} \times \left(\frac{4}{25}\right)^{\frac{3}{2}} = \left(\frac{4}{25}\right)^{\frac{1}{2}+\frac{3}{2}}$$

$$\left(\frac{4}{25}\right)^{\frac{1+3}{2}} = \left(\frac{4}{25}\right)^{\frac{\cancel{4}^{2}}{\cancel{2}}} = \left(\frac{4}{25}\right)^2$$

$$= \frac{4\times4}{25\times25} = \frac{16}{625}$$

We get the same value by both the methods. Hence $\left(\frac{4}{25}\right)^{\frac{1}{2}} \times \left(\frac{4}{25}\right)^{\frac{3}{2}} = \left(\frac{4}{25}\right)^{\frac{1}{2}+\frac{3}{2}}$.

Rational exponent also follow the rules of exponents $x^m \times x^n = x^{m+n}$.

Is $\left(\frac{16}{81}\right)^{\frac{5}{4}} \div \left(\frac{16}{81}\right)^{\frac{3}{4}} = \left(\frac{16}{81}\right)^{\frac{5}{4}-\frac{3}{4}}$, let's check this-

$$\left(\frac{16}{81}\right)^{\frac{5}{4}} \div \left(\frac{16}{81}\right)^{\frac{3}{4}} = \left(\frac{16}{81}\right)^{\frac{5}{4}-\frac{3}{4}}$$

$$\left[\left(\frac{2}{3}\right)^4\right]^{\frac{5}{4}} \div \left[\left(\frac{2}{3}\right)^4\right]^{\frac{3}{4}} = \left[\left(\frac{2}{3}\right)^4\right]^{\frac{5}{4}-\frac{3}{4}}$$

$$\left(\frac{2}{3}\right)^{\cancel{4}\times\frac{5}{\cancel{4}}} \div \left(\frac{2}{3}\right)^{\cancel{4}\times\frac{3}{\cancel{4}}} = \left(\frac{2}{3}\right)^{\cancel{4}\times\frac{2}{\cancel{4}}}$$

$$\left(\frac{2^5}{3^5}\right) \div \left(\frac{2^3}{3^3}\right) = \left(\frac{2}{3}\right)^2$$

$$\left(\frac{2^5}{3^5}\times\frac{3^3}{2^3}\right)=\left(\frac{2^2}{3^2}\right)$$

$$\left(\frac{2^{5-3}}{3^{5-3}}\right)=\left(\frac{2^2}{3^2}\right)$$

$$\left(\frac{2^2}{3^2}\right)=\left(\frac{2^2}{3^2}\right) \qquad \text{R.H.S.} = \text{L.H.S.}$$

$$\left(\frac{16}{81}\right)^{\frac{5}{4}}\div\left(\frac{16}{81}\right)^{\frac{3}{4}}=\left(\frac{16}{81}\right)^{\frac{5}{4}-\frac{3}{4}}$$

Therefore, this follows the rule $x^m \div x^n = x^{m-n}$

EXAMPLE-12. Find the value of each of following-

(i) $\left(\frac{8}{125}\right)^{\frac{4}{3}}\times\left(\frac{8}{125}\right)^{\frac{2}{3}}$ (ii) $\left(\frac{27}{64}\right)^{\frac{7}{3}}\div\left(\frac{27}{64}\right)^{\frac{5}{3}}$

SOLUTION : (i) Rule of rational numbers $a^m \times a^n = a^{m+n}$

$$=\left(\frac{8}{125}\right)^{\frac{4}{3}+\frac{2}{3}}$$

$$=\left(\frac{8}{125}\right)^{\frac{\cancel{6}\,2}{\cancel{3}}}=\left(\frac{8}{125}\right)^{2}$$

$$=\frac{8}{125}\times\frac{8}{125}=\frac{64}{15625}$$

(ii) $\left(\frac{27}{64}\right)^{\frac{7}{3}}\div\left(\frac{27}{64}\right)^{\frac{5}{3}}$

$$=\left(\frac{27}{64}\right)^{\frac{7-5}{3}}=\left(\frac{27}{64}\right)^{\frac{2}{3}}=\left(\frac{3^3}{4^3}\right)^{\frac{2}{3}}$$

$$=\left(\frac{3}{4}\right)^{3\times\frac{2}{3}}=\left(\frac{3}{4}\right)^{2}=\frac{3\times3}{4\times4}=\frac{9}{16}$$

Try This

Find the values of-

(i) $\left(\frac{16}{81}\right)^{\frac{3}{4}} \times \left(\frac{16}{81}\right)^{\frac{5}{4}}$ (ii) $\left(\frac{16}{81}\right)^{\frac{3}{4}+\frac{5}{4}}$ (iii) $\left(\frac{16}{81}\right)^{\frac{5}{4}-\frac{3}{4}}$ (iv) $8^{\frac{2}{3}}$

Which is the Bigger Number?

From 27 and 16, 27 is bigger, but which one is bigger from $\sqrt{16}$ and $\sqrt[3]{27}$?

$$\sqrt{16} = \sqrt{\underline{2\times2}\times\underline{2\times2}}$$

Means $\sqrt{16} = 2 \times 2$

$= 4$

And $\sqrt[3]{27} = \sqrt[3]{\underline{3\times3\times3}}$

Means $\sqrt[3]{27} = 3$

Therefore $\sqrt{16} > \sqrt[3]{27}$

Similarly,

Is $\sqrt[6]{64}$ bigger or $\sqrt[3]{125}$

$$\sqrt[6]{64} = \sqrt[6]{\underline{2\times2\times2\times2\times2\times2}}$$

$= 2$

$$\sqrt[3]{125} = \sqrt[3]{\underline{5\times5\times5}}$$

$= 5$

Thus, $\sqrt[3]{125} > \sqrt[6]{64}$

Try This

1. Which is the bigger number out of the pair?

   (i) $\sqrt[3]{125}, \sqrt{36}$ (ii) $\sqrt{121}, \sqrt[3]{729}$

   (iii) $\sqrt[4]{625}, \sqrt[5]{1024}$

2. Write in descending order- $\sqrt[3]{125}, \sqrt[3]{729}, \sqrt[5]{1024}, \sqrt{36}$

Surds

An irrational number $\sqrt[p]{a}$ is called a surd, here 'a' is a positive rational number. The sign '$\sqrt{\ }$' is called the sign of the surd (radix). The number p is called the order of the surd and 'a' is called the radicand (quantity under the radix) of the surd.

$\sqrt{5}, \sqrt{3}, \sqrt{2}$ are irrational numbers, where as $\sqrt[3]{8}$ is a rational number because $\sqrt[3]{8} = 2$. In $\sqrt{5}, \sqrt{3}, \sqrt{2}$ the radicands 5, 3, 2 are positive rational numbers so $\sqrt{5}, \sqrt{3}, \sqrt{2}$ are surds. In $\sqrt[3]{8}$ the radicand 8, is a positive rational number, but $\sqrt[3]{8}$ is not irrational and hence $\sqrt[3]{8}$ is not a surd.

Is $\sqrt{2+\sqrt{3}}$ a surd or not?

Since $\sqrt{3}$ is an irrational number and 2 is a rational number and since the sum of a rational and an irrational number is an irrational numbers. Hence $2+\sqrt{3}$ is also an irrational number. Radicand of the expression $\sqrt{2+\sqrt{3}}$ is irrational therefore this is not a surd.

EXAMPLE-13. Out of these which one are surd?

(i) $\sqrt[3]{5+\sqrt{9}}$ (ii) $\sqrt{\sqrt{3}}$

SOLUTION : (i) $\sqrt[3]{5+\sqrt{9}}$

$= \sqrt[3]{5+3}$

$= \sqrt[3]{8}$

$= \sqrt[3]{2^3}$

$= 2^{3\times\frac{1}{3}}$ $\left[\sqrt[m]{y} = y^{\frac{1}{m}}\right]$

$= 2$

$\because \sqrt[3]{5+\sqrt{9}} = 2$ is a rational number, so it is not a surd.

(ii) $\sqrt{\sqrt{3}}$

$= \sqrt{3^{\frac{1}{2}}}$

$$= \left(3^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

$$= 3^{\frac{1}{4}}$$

$$= \sqrt[4]{3}$$

$$\therefore \sqrt{\sqrt{3}} = \sqrt[4]{3}$$

So this is a surd.

Try This

1. Identify surd from $\sqrt{3+\sqrt{16}}$, $\sqrt{\sqrt{16}}$, $\sqrt{3+\sqrt{2}}$ and write the reasons for your choice.
2. Write 3 numbers that are irrational but are not surds.

Exercise - 3.2

1. Find the value of each of the following-

   (i) $(16)^{\frac{1}{2}}$ (ii) $(243)^{\frac{1}{5}}$ (iii) $(15625)^{\frac{1}{6}}$

2. Simplify each of the following-

   (i) $23^{\frac{1}{2}} \times 23^{\frac{3}{2}}$ (ii) $11^{\frac{4}{3}} \times 11^{\frac{5}{3}}$

   (iii) $21^{\frac{7}{3}} \div 21^{\frac{1}{3}}$ (iv) $15^{\frac{3}{2}} \div 15^{\frac{5}{2}}$

3. Find the values of-

   (i) $\left(\frac{625}{81}\right)^{\frac{4}{3}} \div \left(\frac{625}{81}\right)^{\frac{2}{3}}$ (ii) $\left(\frac{2}{13}\right)^{\frac{4}{3}} \div \left(\frac{2}{13}\right)^{\frac{5}{3}}$

   (iii) $3 \times 9^{\frac{1}{2}} \div 9^{\frac{3}{2}}$ (iv) $27^{\frac{2}{3}} \div 27^{\frac{1}{3}} \times 27^{\frac{4}{3}}$

4. Write in an ascending order the surds given in each of the following sets-

(i) $\sqrt{81},\ \sqrt[3]{64},\ \sqrt[9]{512}$ (ii) $\sqrt[4]{625},\ \sqrt{100},\ \sqrt[3]{343}$

(iii) $\sqrt[3]{216},\ \sqrt[5]{243},\ \sqrt{64}$ (iv) $\sqrt[4]{256},\ \sqrt[7]{128},\ \sqrt[3]{1000}$

5. In the following , find out which are surds and which are not-

(i) $\sqrt{8}$ (ii) $\sqrt[3]{64}$ (iii) $\sqrt{90}$

(iv) $\sqrt{5+\sqrt{2}}$ (v) $\sqrt[5]{2+\sqrt{4}}$

What Have We Learnt

1. When we multiply a number by itself many times, we write that in the exponent form. The number is called the base, for example in 3^6, 6 is the exponent and 3 is the base.

2. The exponent is a way to write very small and very large numbers in a brief and standard form.

3. Multiplicative inverse - The multiplicative inverse of 2^3 is $\frac{1}{2^3}$.

4. Rules of exponents-

$x^m \times x^n = x^{m+n}$ $x^m \div x^n = x^{m-n}$

$(x^m)^n = x^{mn}$ $x^m \times y^m = (xy)^m$

Here x and y are non-zero rational numbers and m and n are rational exponents.

5. Rules for rational exponent are the same as for integer exponents.

p

08

Digit and Numbers

Consider the following statements:

You are reading chapter no. 4. Gomti's age is 14 years. The night temperature was 10° celsius. The population of Aamgaav is 6000. There is 30 *kg* of rice in the sack. The distance to mars from Earth is 54.6 million kilometers. The speed of light is 186000 miles per second etc. Several such statements are part of our conversations.

In all these statements 4, 14, 10, 6000, etc. are numbers that are written using the digits 0, 1, 2, ..., 5, ..., 9. These can also be written using other kind of digits as symbols (I, II, ..., IX, X, ...) or written in some other number system. Can you give some other examples of numbers written using a different system?

Some more Mathematical Statements

In all the above statements we expressed distance, age, temperature, population, speed etc. using digits and numeral systems.

We make use of some other types of statements as well. For instance-

- If the side of a square is a, then its perimeter will be $4a$.
- The length of a rectangle is l and its breadth b, find its area and perimeter.
- The age of the father is 6 years more than twice the age of his son's.This statement can be written as $x = 2y + 6$.
- The relation between principal, interest etc. is simple interest = $\frac{p \times r \times t}{100}$
- Two lines make an angle θ with each other.

In all these examples a, l, b, x, y, r, t, θ etc. all signify some number. These are known as 'literal numbers.'

Do you see any difference between the two types of numbers mentioned above, i.e. numbers written using digits and literal numbers?

Are there any such literal numbers whose numerical values is fixed?

One such number is π.

If the diameter of a circle is D, than its circumference P will be written as P = πD. Here the values of D will be different for different circles and hence the values of their circumference P. But the values of π will remain $\frac{22}{7}$ or 3.14.

Do you know any other such literal number?

One difference is that 5, 14, 10 etc. are written using digits while a, l, b, x, y, r, t, θ etc. are expressed using alphabets.

Another difference is that the first type of numbers are a fixed value while the literal numbers have different values in different situation.

Do they also have some similarity?

Think about the situations where you use literal numbers.

Can we perform operations like additon, subtraction, multiplication or division on literal numbers as well?

You can discuss such question with your friends.

Algebraic Expression and their Terms

In previous classes, you may have seen several examples of numbers in which literal numbers were also used:-

like, $4a$, $\frac{\sqrt{3}a^2}{4}$, $a+b+c$, $\frac{4}{3}$®r^3, x^2+2x+3, $(x+3)$, $\sqrt{\frac{x}{y}}$, $m-9$, $2p$ + q

We know each one of these as an algebraic expression.

In these algebraic expressions, some have only one term, some have two terms while others may have three or more terms.

For example, $4a$, $\frac{\sqrt{3}a^2}{4}$, $\frac{4}{3}$®r^3, $5ax^2yz$ etc.

are all algebraic expressions having only one term.

$(x + 3)$, $m - 9$, $2p + q$ etc. are algebraic expressions having two terms.

In $m - 9$, m and -9 are the two terms and in $2p + q$, $2p$ and q are the two terms, similarly $a + b + c$, $x^2 + 2x + 3$ are algebraic expressions having three terms. In $a + b + c$, a is the first term, b is second and c is the third term. In the second example x^2 is the first term, $2x$ is second and 3 is the third term.

Can you think of why these have two or three terms while several expressions that seem longer contain only one term?

The term of an expression is a number, or an literal number or the product of one or more numbers and literal numbers. The expression can be long with many numbers and literal numbers multiplied to each other. Any expression is made up of one or more than one term. The terms of an expression are decided based on '+' or '–' and not on '×' or '÷' signs.

Look at the following :

1. $\underbrace{3x}_{\text{One term}}$

2. $\underbrace{2x}+\underbrace{3y}$ — Two terms

3. $\underbrace{-xy}-\underbrace{4x}+\underbrace{35}$ — Three terms

4. $\underbrace{\dfrac{x}{yz}}_{\text{One term}}$

5. $\underbrace{xy}_{\text{One term}}$

How many terms will be there in the expression 2P + P, $(x + 3)^2$, $(x–y)^2$? Can you say?

By just looking at all the three expressions, it may seem that each one of these has two terms since these terms are seperated by '+' or '–' signs, but it is not like that. We can replace $2p + p$ by $3p$ and thus, it is clear that $3p$ is just one term. Similarly observe the following:-

$(x+3)^2$ can be written as $x^2 + 6x + 9$, where we can see three terms.

$(x–y)^2$ can be written as $x^2 – 2xy + y^2$, here also there are three terms.

You saw that when we simplify the expressions we find that the number of terms can be different from what appears to be before simplification.

Therefore, we can say that if it is possible to write an expression in its simplified form then we must count its terms only after writing it in simplified form.

Polynomials (Special Kinds of Algebraic Expression)

Some algebraic expressions are given below:-

$x^2 + 5x$ $\qquad$ $p – 1$ $\qquad$ $x^3 – 2x^2 + 3x – 7$

All these are called polynomials. Is there any special property that is common to all these?

Look at some algebraic expressions that are not polynomial.

$x + \dfrac{1}{x}$, $\qquad$ $y^2 + y^{½} + 3$, $\qquad$ $p^3 – 2p + \sqrt[3]{p}$, $\qquad$ $3x^{–1}$

Can you find any property in these four examples that makes them different from the previous examples? What could be the reason for these not being polynomials?

Compare all the expressions those that are polynomials and those that are not polynomials. Discuss with your teacher and friends about when can an expression be called as a polynomial and when it cannot be?

0, 1, 2, 3, 4, ...
are whole numbers

You would have reached the conclusion that only these algebraic expressions that have non negative integral powers of literal numbers, are polynomials.

Try This

1. Which of the following are polynomials-

(i) S-3, (ii) $5y^{-3}$, (iii) $p+\frac{1}{p}$,

(iv) $ax^2 + b$, (v) $x^{\frac{1}{2}}+1$, (vi) $5p^2 + 2p + 1$

2. Make 5 new polynomials.

Terms of Polynomials

We need to learn how to count the numbers of terms of a polynomial, to find the coefficient of the terms of a polynomial and to find the powers of polynomials. We will need them later. In the polynomial $x^2 + 3x$, these are two terms, first term is x^2 and the second is $3x$. Similarly there are four terms in $m^3 - 2m^2 + 9m + 1$, three terms in $x^2 - x + 1$ and one term in $3y$.

The polynomials are named according to the number of terms in them. As polynomials are algebraic expressions therefore their terms are counted just as an algebraic expresions.

In the above examples:-

$3y$ is a monomial,

$x^2 + 3x$ is a binomial, $x^2 - x + 1$ is a trinomial,

$m^3 – 2m^2 + 9m + 1$ is a polynomial.

Think and Discuss

1. How many terms can be there in a polynomial? Would they be finite or infinite?
2. Is $2p + p$ a monomial or a binomial?
3. How many terms does the expression $(x + 2)^2$ has?

Try This

Pick out the following expressions based on the number of terms:-

(i) 9C (ii) $\frac{1}{2}t+\frac{a}{2}$ (iii) $a^2+2ab+b^2$ (iv) $\frac{p}{q}$

(v) $4x-y$ (vi) $2m+c$ (vii) x^4+3x^2+1

Degree of Polynomials

In a polynomial there are some other numbers that are written in the terms of a polynomial in the form of literal numbers. For example -

$3x^5 - 2x^4 + 3x^3 + 9$

In this polynomial, the powers of the literal number x are 5, 4 and 3 respectively. The largest power among these is 5. In this situation, the degree of polynomial $3x^5 - 2x^4 + 3x^3 + 9$ is five.

> Polynomials of degree one are also called linear polynomials.

Look at some more examples.

In $x^2 + 3$, the degree of the polynomial is 2.

In $x^2 - 2x^7 + 3x - 1$, the degree of polynomial is 7.

In $5y$, the degree of polynomial is 1.

What will be the degree of the polynomial $x^2y + xy$?

In the polynomial $x^2y + xy$, 2 is the degree for x in first term and 1 is the degree for y. Therefore the degree for x^2y is 3 and similarly 2 is the degree for xy. So the degree of $x^2y + xy$ is 3.

> What will be the degree of polynomial xy? In the polynomial xy, there are two variables; x and y. Both have degrees1. The degree of polynomial 'xy' is equal to the sum of these degree i.e., 2.

> Sometimes, numbers are also represented by letters. In this case, literal number are fixed or are constant. For example in $ax + b$, if a and b are constant, then literal number x is a variable but a, b will be constants.

Try This

1. In the following polynomials find the degree of the polynomial.
   (i) 8 (ii) xyz (iii) $uv + uv^2 + v^3$
2. Write some polynomials, and with the help of discussions with your friends, find out their degrees.

Constant Polynomials

Now think about this question:-

Is 6 an algebraic expression?

Is 6 also a polynomial?

It seems that because there is no literal number with 6 and in the standard example of expressions we always find an literal number, so 6 not an algebraic expression. But can we not write 6 as $6x^\circ$? ($x^\circ = 1$).

In this the power of literal number x is zero. Zero is a whole number. Therefore $6x^{\circ}$ or 6 is also an algebraic expression and a mononomial too.

So, terms that are just a number are also algebraic expressions as well as a polynomial.

Such polynomials are called constant polynomials. That means 2, 7, –6, $\frac{3}{2}$, 122 are constant polynomials. Can you make a few more examples of constant polynomials?

Representation of Polynomials

Sometimes we require that a polynomial be written several times. In such situations we will have to write the big long polynomial several times. There is one more way to express a polynomial. In this we only know the literal number of the polynomial but nothing else. However in the context of the question, we know which polynomial does the expression relate to.

If the literal number of the polynomial is x then we express it as $p(x)$, $q(x)$, $r(x)$ etc. If the literal number of the polynomial is y then it is expressed as $p(y)$, $q(y)$, $s(y)$, $t(y)$. See few examples:-

$p(x) = 3x^5 - 2x^4 + 3x^3 + 9$ $\qquad$ $t(x) = x^6 - 2x^7 + 3x - 1$

$q(y) = 5y$ $\qquad$ $s(u) = u^2 + 3u^3$ $\qquad$ $r(b) = b^4 - b^2 + 6$

In this, there is no basis for choice p, t, q. But if we choose them, we must use the same form for that particular question.

General form of Polynomials

Observe the polynomials of degree one given below carefully.

$p(x) = 2x + 3$ $\qquad$ $q(x) = \sqrt{2} - x$ $\qquad$ $(x) = \frac{1}{2}x + \frac{3}{2}$

$r(x) = x$ $\qquad$ $p(x) = \sqrt{7}\,x - 4$ $\qquad$ $p(x) = 2(3x + 8)$

The maximum number of terms in these are two.

Think and Discuss

Can you make a polynomial of a single degree with more than two terms. Keep in mind that while making a polynomial, similar terms must be collected and written together.

In the above examples, every polynomial has atleast one literal number. In other words we can also say that in these polynomials the coefficient of the literal numbers is never zero. Besides, other real numbers (+3, –4, $\frac{1}{2}$, $\sqrt{2}$, 0 etc.) are also included in the polynomials.

Can we express these polynomials in the form of $ax + b$ where a and b are some real constant numbers and $a \neq 0$?

Compairing the second example to $ax + b$:-

$$\sqrt{2} - x = -x + \sqrt{2}$$

$$= (-1)\,x + \sqrt{2}$$

$$= a\,x + b \qquad \text{(Compare)}$$

Here, $a = (-1)$ where a is not zero.

$b = \sqrt{2}$ where $\sqrt{2}$ is a real number.

We can see that polynomial $\sqrt{2} - x$ is similar to $ax + b$.

See one more polynomial:-

$$x = 1 \times x$$

$$= 1 \times x + 0$$

$$= ax + b$$

$$a = 1,\ b = 0$$

The number that multiplies the literal number is called as the coefficient. For example:- $m^3 - 3m^2 + 1$, the coefficient of m^3 is 1 and coefficient of m^2 is –3. Think, what will be the coefficient of m^2 in $m^2 + 1$?

Real numbers comprise of all integers, rational and irrational numbers.

$-2, -1, 0, 1, 2, \ldots, -\frac{1}{2}, -\frac{4}{5}, -\frac{2}{7}, \sqrt{2}, \sqrt{7}, \ldots$

Therefore polynomial x can also be written in the form of $ax + b$. Now write the remaining four polynomials in the form of $ax + b$.

$ax + b$ is a linear polynomial of x with degree 1, in this a and b as real constant numbers and $a \neq 0$.

Now consider the following polynomials:-

$$4x^2 + 3x, \qquad -y^2 + 2, \qquad x^2 - 4x - 9, \qquad \sqrt{2}\,m^2 - \frac{3}{2}m - 9$$

These polynomials are binomials each with one literal number. These are called as quadratic polynomials. We can write these in the form of $ax^2 + bx + c$. Where a, b and c are real constants and $a \neq 0$.

Try This

1. Make five new quadratic polynomials.
2. What is the largest number of terms possible in a quadratic polynomial?
3. What is the minimum number of terms possible in a quadratic polynomial?

General form of Polynomials with higher degree

Consider $4x^3+2x^2+5x-7$, $y^4+3y^3-5y^2+7y$, m^5-3m^2+2, $z^6-5z^5-3z^2+2z$

These are all polynomials. We can see that in these, the degree of polynomials is increasing and the number of maximum possible terms may also increase. In all these, apart from various powers of x, y, m, z etc. we have numbers that are real numbers. The degree of a polynomial depends upon the maximum degree of the literal numbers (x, y, m, z) contained in it. Therefore the coefficient of the literal with the maximum power can not be zero. Hence, we gave the condition of $a \neq 0$ for polynomials with degree two.

We will study more about polynomials and will also get to know about different uses of literal numbers. We will see the more generalized form of the polynomials for all the degrees. All this will however, be done in further classes. Can you write the polynomials with higher degree in a forms like $ax + b$, $ax^2 + bx + c$.

Zero Polynomials

If all the coefficients of a polynomial are zero; for example in $ax^2 + bx + c$ with $a = b = c = 0$ then we will get 0 as the result. This is called as zero polynomial. The degree of this is undefined. Therefore it can be written as a polynomial of any degree.

Exercise - 4.1

1. Which of the following expressions are polynomials and which are not? Give reasons for your answer.

   (i) $4x^2 - 3x + 5$ (ii) $z + \frac{3}{z}$ (iii) $\sqrt{y} + 2y + 3$

   (iv) $x^2 + \frac{3}{2}$ (v) $x^{10} + y^3 + t^{50}$

2. Write the coefficient of x^2 in the following polynomials :-

   (i) $3x^2 + 2x^2 + 3x + 2$ (ii) $3x^2 + 1$ (iii) $2 - 5x^2 + \frac{1}{2}x^3$

   (iv) $\frac{x^2}{2} + 1$ (v) $x^4 + x^3 + \frac{1}{4}x^2$

3. Write the coefficient of x and constant terms for the following:-

(i) $x^2 + \frac{1}{5}x + 5$ (ii) $\sqrt{2}\,x + 7$ (iii) $x^2 + 2$

4. Write an example for each of the following:-

(i) binomial of degree 4 (ii) trinomial of degree 6

(iii) monomial of degree 5

5. In the following polynomials write the degree of each.

(i) $x^3 - 6x^2 + x + 1$ (ii) $y^9 - 3y^7 + \frac{3}{2}y^2 + 4$ (iii) $3 - y^3z$

(iv) $x^2y - 2x + 1$ (v) $5t - \sqrt{11}$ (vi) 7

6. From the following, pick out the constant, linear, binomials and trinomials.

(i) $x^3 + x^2 + x + 1$ (ii) $9x^3$ (iii) $y + y^2 + \frac{3}{4}$

(iv) $t + 3$ (v) $y - y^3$ (vi) 8

(vii) $2x^2 + 3$ (viii) $P^2 - P + 5$ (ix) $x + \frac{2}{3}$

(x) 4 (xi) $-\frac{u}{4} + \frac{3}{2}$ (xii) $-\frac{3}{7}$

Zeroes of a Polynomial

By putting values $x = 1, 2, 0$ etc. in the polynomial $p(x) = x^2 + x - 6$ we get,

at $x = 1$, $p(1) = 1 + 1 - 6 = -4$

$p(1) = -4$

We say that, for $x = 1$, $p(x)$ has the value -4.

at $x = 2$

$p(2) = 4 + 2 - 6$

$p(2) = 0$

For $x = 2$, the value of $p(x)$ becomes 0. We say that 2 is a zero of the polynomial $p(x)$.

We can say that if the value of a polynomial is zero for some value of the variable than that value of the variable is a zero of the polynomial.

EXAMPLE-1. Find the zero of the polynomial $p(x) = 2x + 1$.

SOLUTION : Finding the zero of a polynomial is similar to finding the solution of an equation.

For $p(x) = 0$ we need the value of x which satisfies it.

Therefore, it is the value of x, when $2x + 1 = 0$

$$2x = -1 \qquad \text{or} \qquad x = \frac{-1}{2}$$

Clearly, putting the value $x = \frac{-1}{2}$ in $p(x)$ we get zero. Therefore $\frac{-1}{2}$ is a zero of the polynomial $p(x)$.

EXAMPLE-2. Check if 0 or 2 are zero of the polynomial $x^2 - 2x$.

SOLUTION : $p(x) = x^2 - 2x$

then putting $x = 0$

$p(0) = (0) - 2 \times 0 = 0$

that means 0, is a zero of $p(x)$.

putting $x=2$

$p(2) = 2^2 - 2 \times 2$

$= 4 - 4 = 0$

another zero of $p(x)$ is 2.

Exercise - 4.2

1. Find the value of polynomial $5x^3 - 2x^2 + 3x - 2$, for

   (i) $x = 0$ (ii) $x = 1$ (iii) $x = -2$

2. For all the following polynomials, find the values of $p(0)$, $p(-1)$, $p(2)$, $p(3)$.

   (i) $p(x) = 4x^3 + 2x^2 - 3x + 2$ (ii) $p(r) = (r - 1)(r + 1)$

   (iii) $p(t) = \frac{2}{3}t^2 - \frac{1}{3}t + \frac{1}{3}$ (iv) $p(y) = (y^2 - y + 1)(y + 1)$

   (v) $p(x) = x + 2$

3. Find out if the values written besides the polynomial, are their zeroes.

(i) $p(x) = 3x + 1 \; ; \; x = \frac{-1}{3}$ (ii) $p(x) = x + 2 \; ; \; x = -2$

(iii) $p(x) = 5x - 4 \; ; \; x = \frac{5}{4}$ (iv) $p(y) = y^2 - 1 \; ; \; y = 1, -1$

(v) $p(t) = (t + 1)(t - 2) \; ; \; t = +1, -2$ (vi) $p(x) = lx + m \; ; \; x = \frac{-m}{l}$

(vii) $p(r) = 3r^2 - 1 \; ; \; r = \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

4. Find the zeroes of the following polynomials

(i) $p(x) = x + 6$ (ii) $p(x) = x - 6$ (iii) $p(y) = 5y$

(iv) $p(t) = at, a \neq 0$, a is a real constant number.

(v) $p(r)=cr+d$, $c\neq0$, c, d are real constant numbers.

(vi) $p(u) = 3u - 6$ (vii) $r(s) = 2s + 3$

(viii) $p(x) = \sqrt{5} - x$ (ix) $q(t) = \frac{1}{2}t - \frac{2}{3}$

Addition and Subtraction of Polynomials

We have practiced the addition and subtraction of algebraic expressions. We have also learnt that all polynomials are algebraic expressions. Therefore the addition and subtraction of polynomials is similar to that of algebraic expressions.

We can perform addition/subtraction of the coefficients after observing all the terms and collecting similar ones.

Consider the following examples and tell what things you need to keep in mind while solving such questions.

EXAMPLE-3. Add the polynomial $3x^3 - x^2 + 5x - 4$ and $3x^2 - 7x + 8$.

SOLUTION : $3x^3 - x^2 + 5x - 4$ (We will write terms with same degrees together).

$$\begin{array}{l} \quad 3x^3 - x^2 + 5x - 4 \\ + \qquad\quad 3x^2 - 7x + 8 \\ \hline 3x^3 + (-1+3)x^2 + (5-7)x + (-4+8) \\ \hline \end{array} \quad = \quad 3x^3 + 2x^2 - 2x + 4$$

EXAMPLE-4. Find the sum of the polynomials $\frac{3}{2}y^3 + y^2 + y + 1$ and $y^4 - \frac{1}{2}y^3 - 3y + 1$.

SOLUTION :

$$\begin{array}{l} \qquad\qquad \frac{3}{2}y^3 + y^2 + y + 1 \\ +y^4 - \frac{1}{2}y^3 \qquad\qquad -3y + 1 \\ \hline y^4 + \left(\frac{3}{2} - \frac{1}{2}\right)y^3 + y^2 + (1-3)y + (1+1) \\ \hline \end{array} = y^4 + y^3 + y^2 - 2y + 2$$

EXAMPLE-5. Subtract $4x^2 + 3x - 2$ from the polynomial $9x^2 - 3x - 7$.

SOLUTION :

$$\begin{array}{l} 9x^2 - 3x - 7 \\ 4x^2 + 3x - 2 \\ - \quad - \quad + \quad \text{(sign changes on subtracting)} \\ \hline (9-4)x^2 + (-3-3)x + (-7+2) \\ \hline \end{array} = 5x^2 - 6x - 5$$

EXAMPLE-6. Subtract $s(z) = 3z - 5z^2 + 7 + 3z^3$ from $x(z) = 2z^2 - 5 + 11z - z^3$

SOLUTION : First write both the polynomials in the decreasing order of the power of literal numbers.

$x(z) = 2z^2 - 5 + 11z - z^3 = -z^3 + 2z^2 + 11z - 5$

and $s(z) = 3z - 5z^2 + 7 + 3z^3 = 3z^3 - 5z^2 + 3z + 7$

now $x(z) - s(z) = -z^3 + 2z^2 + 11z - 5$

$$\begin{array}{l} 3z^3 - 5z^2 + 3z + 7 \\ - \quad + \quad - \quad - \\ \hline (-1-3)z^3 + (2+5)z^2 + (11-3)z + (-5-7) \\ \hline \end{array}$$

$= -4z^3 + 7z^2 + 8z - 12$

EXAMPLE-7. Find the sum of the polynomials $3x + 4 - 5x^2$, $5 + 9x$ and $4x - 17 - 5x^2$. What is the degree of the polynomial obtained after the addition.

SOLUTION : $-5x^2 + 3x + 4$ (Writing all in decreasing degree of x).

$$\begin{array}{rl} & -5x^2 + 3x + 4 \\ & \quad\;\; + 9x + 5 \\ + & -5x^2 + 4x - 17 \\ \hline & (-5-5)x^2 + (3+9+4)x + (4+5-17) = -10x^2 + 16x - 8 \end{array}$$

The resultant, polynomial $-10x^2 + 16x - 8$ has the degree 2.

EXAMPLE-8. Subtract $r(x) = 2x^2 - 3x - 1$ from the sum of $p(x) = 3x^2 - 8x + 11$ and $q(x) = -4x^2 + 15$. Find the degree of the resultant polynomial.

SOLUTION : First, we add $p(x)$ and $q(x)$.

$$\begin{array}{rl} p(x) + q(x) = & 3x^2 - 8x + 11 \\ & -4x^2 \quad\;\; + 15 \\ \hline & (3-4)x^2 - 8x + (11+15) = -x^2 - 8x + 26 \end{array}$$

Now from this sum, subtract $r(x)$

$$\begin{array}{rl} p(x) + q(x) - r(x) = & -x^2 - 8x + 26 \\ & 2x^2 - 3x - 1 \\ & - \quad + \quad + \qquad \text{(On changing the signs)} \\ \hline & (-1-2)x^2 + (-8+3)x + (26+1) = -3x^2 - 5x + 27 \end{array}$$

$p(x) + q(x) - r(x) = -3x^2 - 5x + 27$

Therefore, the degree of resultant polynomial $-3x^2 - 5x + 27$ is 2.

EXAMPLE-9. Find the sum and difference of $p(x) = 4x^3 + 3x^2 + 2x - 1$ and $q(x) = 4x^3 + 2x^2 - 2x + 5$.

SOLUTION :

$$\begin{aligned} p(x) + q(x) &= (4x^3 + 3x^2 + 2x - 1) + (4x^3 + 2x^2 - 2x + 5) \\ &= (4 + 4)\,x^3 + (3 + 2)\,x^2 + (2 - 2)\,x + (-1 + 5) \\ &= 8x^3 + 5x^2 + 0x + 4 \\ &= 8x^3 + 5x^2 + 4 \end{aligned}$$

Similarly, the difference is

$$\begin{aligned} p(x) - q(x) &= (4x^3 + 3x^2 + 2x - 1) - (4x^3 + 2x^2 - 2x + 5) \\ &= (4 - 4)\,x^3 + (3 - 2)\,x^2 + (2 + 2)\,x + (-1 - 5) \\ &= x^2 + 4x - 6 \end{aligned}$$

Try This

Find the sum and difference of the following polynomials and give the degree of the resultant.

(i) $p(y) = y^2 + 5y$, $q(y) = 3y - 5$

(ii) $p(r) = 5r^2 - 9$, $s(r) = 9r^2 - 4$

(iii) $p(y) = 15y^4 - 5y^2 + 27$, $s(y) = 15y^4 - 9$

Sometimes when we know the sum or difference of two polynomials and know one of the polynomials, then we can easily find the polynomial.

EXAMPLE-10. What should be added to $2u^2 - 4u + 3$ so that we get the sum as $4u^3 - 5u^2 + 1$?

SOLUTION : Suppose that on adding $q(u)$ in $p(u) = 2u^2 - 4u + 3$ we get $r(u)=4u^3-5u^2+1$.

That means $p(u) + q(u) = r(u)$

$$q(u) = r(u) - p(u)$$

$$\begin{aligned} q(u) &= (4u^3 - 5u^2 + 1) - (2u^2 - 4u + 3) \\ &= 4u^3 - 5u^2 + 1 - 2u^2 + 4u - 3 \\ &= 4u^3 + (-5 - 2)\, u^2 + 4u + (1 - 3) \\ &= 4u^3 - 7u^2 + 4u - 2 \end{aligned}$$

EXAMPLE-11. What should be subtracted from $2y^3 - 3y^2 + 4$ to get $y^3 - 1$ as the resultant.

SOULUTION : Suppose on subtracting $q(y)$ from $p(y)=2y^3-3y^2+4$ we get the difference $r(y)=y^3-1$.

That means $p(y) - q(y) = r(y)$

or $q(y) = p(y) - r(y)$

$$\begin{aligned} q(y) &= (2y^3 - 3y^2 + 4) - (y^3 - 1) \\ &= 2y^3 - 3y^2 + 4 - y^3 + 1 \\ &= (2 - 1)\, y^3 - 3y^2 + (4 + 1) \\ &= y^3 - 3y^2 + 5 \end{aligned}$$

Try This

1. What should be subtracted from $5t^2 - 3t + 4$ to get $2t^3 - 4$.
2. What should be added to $6r^2 + 4r - 2$ to get $15r^2 + 4$.
3. Make 5 more questions of this kind and solve them.

Multiplication of Polynomials

Like addition and subtraction, the multiplication of polynomials is also similar to that for algebraic expressions;

EXAMPLE-12. Multiply the polynomial $p(x) = 2x^2 + 3x + 4$ by 3.

SOLUTION : $3\,p(x) = 3 \times (2x^2 + 3x + 4)$

$= 6x^2 + 9x + 12$

EXAMPLE-13. $(2x + 5) \times (4x + 3)$

SOLUTION : $= 2x(4x + 3) + 5(4x + 3)$

$= [(2x \times 4x) + (2x \times 3)] + [(5 \times 4x) + (5 \times 3)]$

$= 8x^2 + 6x + 20x + 15$

$= 8x^2 + 26x + 15$

EXAMPLE-14. $(2x + 5) \times (3x^2 + 4x + 6)$, find the degree of the polynomial?

SOLUTION : $= [2x \times (3x^2 + 4x + 6)] + [5 \times (3x^2 + 4x + 6)]$

$= 2x \times 3x^2 + 2x \times 4x + 2x \times 6 + 5 \times 3x^2 + 5 \times 4x + 5 \times 6$

$= 6x^3 + 8x^2 + 12x + 15x^2 + 20x + 30$

$= 6x^3 + (8 + 15)\,x^2 + (12 + 20)\,x + 30$

$= 6x^3 + 23x^2 + 32x + 30$

Here, the degree of the polynomial is 3.

EXAMPLE-15. If $p(x) = 2x + 3$

$q(x) = x^2 + x - 2$

SOLUTION : Then $p(x) \,.\, q(x) = (2x + 3)(x^2 + x - 2)$

$= 2x(x^2 + x - 2) + 3(x^2 + x - 2)$

$= 2x \times x^2 + 2x \times x + 2x \times (-2) + 3 \times x^2 + 3x + 3 \times (-2)$

$= 2x^3 + 2x^2 - 4x + 3x^2 + 3x - 6$

$= 2x^3 + (2 + 3)\,x^2 + (-4 + 3)\,x - 6$

$= 2x^3 + 5x^2 - x - 6$

Try This

Multiply the following polynomials, find the degree of the product polynomial:-

(i) $p(x) = x^2 + 3x + 2$; $q(x) = x^2 + 3x + 1$

(ii) $p(v) = v^2 - 3v + 2$; $q(v) = v + 1$

(iii) $p(x) = 2x^2 + 7x + 3$; $q(x) = 5x^2 - 3x$

(iv) $p(y) = y^3 - y^2 + y - 1$; $q(y) = y + 1$

(v) $p(u) = 3u^2 - 12u + 4$; $q(u) = u^2 - 2u + 1$

Exercise - 4.3

1. Add the following polynomials:-

   (i) $2x^2 + x + 1$ and $3x^2 + 4x + 5$ (ii) $8p^2 - 3p + 4$ and $3p^3 - 4p + 7$

   (iii) $-5x^3 + 9x^2 - 5x + 7$ and $-2x^2 + 7x^3 - 3x - 8$

2. Add the following polynomials. Find the degree of the resultant polynomial.

   (i) $3y^2 + 2y - 5$; $2y^2 + 5 + 8y$ and $-y^2 - y$

   (ii) $5 + 7r - 3r^2$; $r^2 + 7$ and $r^2 - 3r + 5$

   (iii) $4x + 7 - 3x^2 + 5x^3$; $7x^2 - 2x + 1$ and $-2x^3 - 2x$

3. Subtract

   (i) $t^2 - 5t + 2$ from $7t^3 - 3t^2 + 2$

   (ii) $3p - 5p^2 + 7 + 3p^3$ from $2p^2 - 5 + 11p - p^3$

   (iii) $5z^3 + 7z^2 + 2z - 4$ from $-3z^2 + 11z + 12z^3 + 13$

4. From the sum of $x^4 + 3x^3 + 2x + 6$ and $x^4 - 3x^2 + 6x + 2$ subtract $x^3 - 3x + 4$.

5. If $p(u) = u^7 - u^5 + 2u^2 + 1$ and $q(u) = -u^7 + u - 2$, find the degree of $p(u) + q(u)$.

6. What should be added to $x^3 - 3x^2 + 6$ to get the sum as $x^2 - x + 4$?

7. What should be added to $u^7 - 3u^6 + 4u^2 + 2$ to get the sum as $u^6 - u - 4$?

8. What should be subtracted from $y^3 - 3y^2 + y + 2$ to get the difference as $y^3 + 2y + 1$?

9. What should be subtracted from $t^2 + t - 7$ to get the difference as $t^3 + t^2 + 3t + 4$?

10. Multiply the following

    (i) $3x + 4$ by $7x^2 + 2x + 1$ (ii) $5x^3 + 2x$ by $3x^2 - 9x + 6$

    (iii) $p^4 - 5p^2 + 3$ by $p^3 + 1$

11. If $p(x) = x^3 + 7x + 3$ and $q(x) = 2x^3 - 3$, then find the value of $p(x)\,q(x)$

12. If $p(u) = u^2 + 3u + 4$, $q(u) = u^2 + u - 12$ and $r(u) = u - 2$, then find the degree of $p(u)q(u)r(u)$.

What Have We Learnt

1. Any algebraic expression, in which the degree of literal number (variable) is an integer is called a polynomial.
2. The terms of a polynomial are identified by the '+' or '–' sign.
3. The coefficient of a polynomial is a number that multiplies the literal number or variable. Sometimes coefficients are expressed as literals and so the coefficients is not written as a number but as a literal. for example, the coefficient of x in $ax + b$ is a.
4. When the coefficients of all the literal number variables in a polynomial are zero, then it is called a zero polynomial.
5. Any number such as 3, 5, 6, ... etc. is an algebraic expression as well as a polynomial.
6. The polynomials with only numbers (having only numerals or constant literal numbers) are constant polynomials.
7. In a polynomial, the maximum degree of the variables (literal numbers) is the degree of the polynomial. Example- the degree of polynomial $x^5 + 3x^3 + 2x$ is 5.
8. Polynomials can be written as $p(x)$, $q(x)$, $r(x)$ etc., where the literal number written within the brackets represents the variable of the polynomial.
9. $ax + b$ is a one variable polynomial of degree 1 where a, b are constant real numbers and $a \neq 0$.
10. The polynomial in one variable having degree 2 is a quadratic polynomial.
11. When the value of a polynomial becomes zero for some value of the variable , then that value of the variable is a zero of the polynomial.
12. A general linear polynomial is $ax + b$, where a, b are constant real numbers and $a \neq 0$.
13. A general quadratic polynomial is $ax^2 + bx + c$ where a, b, c are constant real numbers and $a \neq 0$.
14. A general form of a cubic polynomial is $ax^3 + bx^2 + cx + d$ where a, b, c, d are constant real numbers and $a \neq 0$.

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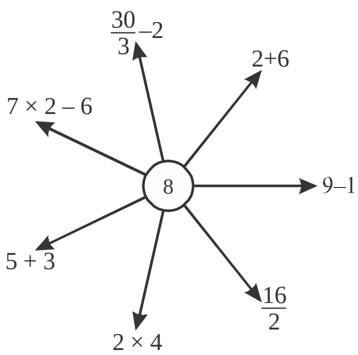


Fig. 1

A numeric expression can be expressed in many ways. (*Fig.*1)

Similarly, can we express $5x$ in different ways? (*Fig.*2)

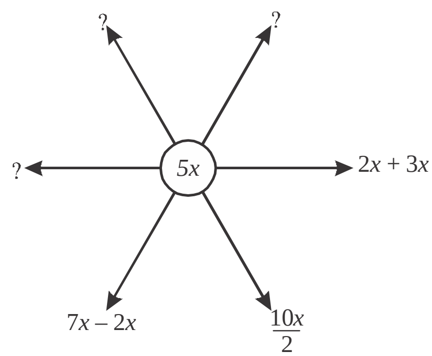
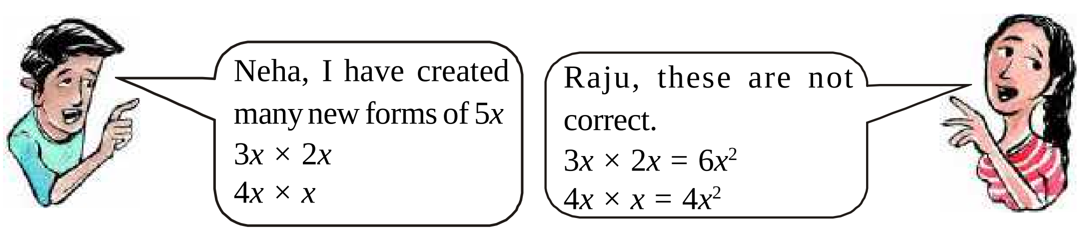


Fig. 2

Try to write some other expressions for $5x$.

Are your solutions different from the solutions of your friends?



What do you think? Who is right? Discuss among your friends.

Therefore, like numbers, algebraic expression can also be represented in many ways.

We will have to take care of the rules of operations for algebraic expressions.

Try This

Take some other algebraic expressions and write them in different ways.

Equation in One Variable

Rinku : Jitu there are five coins of same value in my pocket, can you tell, how much money I have?

Jitu : How can I tell? Show me the coins and then I can tell you.

Reshma: If you have one rupee coin then you will have five rupees in your pocket and if you have two rupees coin then you will have ten rupees. That is you have $5x$ rupees in your pocket.

Rinku : There are 25 rupees in my pocket, so which coins do I have?

Jitu : Now that's easy. Since you have $5x$ rupees, $5x = 25$

I will check the value of $5x$ for $x = 1, 2, 5, 10$ and for whatever value both sides of "=" are same, will be the value of your coins.

Think any two numbers. Like 9 and 21, now take any other two numbers smaller than ten and make relationship using the four operations:-

(i) If we add 3 to double of nine we get 21 $(9 \times 2) + 3 = 21$

(ii) If 6 is subtracted from three times of 9 we get 21 $(\quad \times \quad) - \quad =$

(iii) If we divide 21 by 7 and then add six to it, we get 9 $\frac{21}{7} + 6 = 9$

(iv) If we subtract 3 from 21 and then divide it by 2, we get 9 - ....

Similarly write 5 statements of different equations using 9 and 21 with different operations and numbers.

Obviously this can be done with any two numbers.

Try This

Create three relation for each of these-

(i) 2 and 5 (ii) 11 and 25 (iii) 40 and 123

See the statements given below:-

(i) The age of Raju is 9 years and his mother's age is 35 years.

(ii) There are 25 children in class-4 and 45 children in class-9.

(iii) Reema has 10 rupees and Meera has 7 rupees.

In each of these we can find some relationship like the sum of Raju's mother's age and his age is 44 years.

Raju's age + Mother's age = 44 years.

Children in class-9 – children in class 4 = 20

Many relationships can be developed for all such situations.

Create three relationships for each.

Let us Now Learn to Solve One Variable Equation

Ramola : I have thought of one number, this number is 5 less than twice of 25, what is the number?

Malini : The number $= (25 \times 2) - 5$

$= 50 - 5 = 45$

Therefore the number is 45.

Let us see an other example- 5 is 10 less than any number?

Then its equation will be $x - 10 = 5$ (Suppose that number is x)

Try This

1. 4 is 2 less than any number.
2. Addition of 8 and any number is 12.

Making Equations

An equation shows the equality of both sides. Let us see an example to understand how it is formed.

Example-1. 4 more than 3 times of a number is equal to 13.

Solution : $3x + 4 = 13$

Example-2. The current age of a boy is twice his age four years ago.

Solution : Suppose the current age of the boy = b years

His age four years ago = $(b - 4)$

then $b = 2\,(b - 4)$

In the previous chapter you saw that $ax + b$ represents a linear algebraic expression with one variable. An equation of one variable is represented as $ax + b = c$. Where a, b, c are . . . . $a \neq 0$. Similarly you can create equation from polynomial $ax^2 + bx + c$. Obviously here $ax + b = c$, $ax^2 + bx + c = d$ are respectively linear equation with one variable and quadratic equation in one variable where a, b, c, d are real constant numbers and $a \neq 0$.

Try This

(i)

5	5	5	5
Add 5	Square 5	Multiply by 2	Subtract 4
Square the sum	Add 5	Subtract 4	Multiply by 2
$(5 + 5)^2 = 100$			

(ii)

x	x	x	x
Add 5	Square x	Multiply by 2	Subtract 4
Square the sum	Add 5	Subtract 4	Multiply by 2
$(x + 5)^2$			

(iii)

x	x	x	x
Multiply by 3	Subtract 6	Add 6	Divide by 3
Add 6	Multiply by 3	Divide by 3	Subtract 6
			$\frac{x}{3} - 6 = 18$

Make Statement from Given Algebraic Equation

Consider an algebraic equation $5x - 2 = 10$. A possible statement for this may be as follows:-

If 2 is subtracted from 5 times of any number then it is equal to 10. What is the number?

EXAMPLE-3. Write the mathematical statement for $\frac{x}{4} + x = 10$.

SOLUTION : Adding $\frac{1}{4}$ of a number to itself gives 10. What is the number?

Try This

Make statements for the following algebraic equations:-

(i) $2x-3=42$ (ii) $12x-3 = 105$ (iii) $\frac{x}{x+4} = 28$

(iv) $x+6=28$ (v) $3x+3=15$

Solution of Equation

To understand what is the solution of an equation see the following example:-

In $x+2=7$

If we place the value of $x=3$

$3+2\neq 7$ the equation is not valid

If we place the value of $x=4$

$4+2\neq 7$ the equation is not valid

If we place the value of $x=5$

$5 + 2 = 7$ the equation is true because both sides are equal

Therefore, we say that $x=5$ is a solution of the equation, because equation is valid for this value of x.

We can say that the solution of an equation is that value of variable for which the equation holds true.

Properties of Equation

1. In an equation there are two sides of the symbol '='.

   Left hand side = Right hand side.

2. The four rules of equality – for real numbers a, b, c

   (i) If $a = b$ then $a + c = b + c$
   that is if two expressions are equal then the sum will remain equal if we add the same number on both the sides of the equation.

   (ii) If $a = b$ then $a - c = b - c$
   that is if two expressions are equal then the difference will remain equal when we subtract the same number from both side the of the equation.

   (iii) If $a = b$ then $ac = bc$
   that is if two expressions are equal then the product will remain equal if we multiply the same number on both sides of the equation.

   (iv) If $a = b$ then $\frac{a}{c} = \frac{b}{c}$
   that is if two expressions are equal then the quotient will remain equal when we divide the expressions by the same number on both the sides of the equation.

EXAMPLE-4. Solve the equation $x - 7 = 3$

SOLUTION : $x - 7 = 3$

$x - 7 + 7 = 3 + 7$

$x = 10$ (using equality rule (i) we add 7 to both sides and thus cancels (–7))

EXAMPLE-5. Solve the equation $3x + 5 = 14$

SOLUTION : $3x + 5 = 14$

$3x + 5 - 5 = 14 - 5$ (using equality rule (ii) subtract 5 from both sides to cancel 5).

$3x = 9$

$\frac{3x}{3} = \frac{9}{3}$ (using equality rule (iv) divide both sides by 3 to get rid of 3)

$x = 3$

EXAMPLE-6. Solve the equation $\frac{4x+5}{3} = 15$.

SOLUTION : $\frac{4x+5}{3} = 15$

$\frac{(4x+5)}{3} \times 3 = 15 \times 3$ (using equality rule (iii))

$4x + 5 = 45$

$4x + 5 - 5 = 45 - 5$ (using equality rule (ii))

$4x = 40$

$\frac{4x}{4} = \frac{40}{4}$ (using equality rule (iv))

$x = 10$

EXAMPLE-7. Solve the equation $4x - 2x - 5 = 4 + 6x + 3$.

SOLUTION : $4x - 2x - 5 = 4 + 6x + 3$

$2x - 5 = 7 + 6x$ (Adding of similar terms)

$2x - 5 - 6x = 7 + 6x - 6x$ (using equality rule (ii))

$-4x - 5 = 7$

$-4x - 5 + 5 = 7 + 5$ (using equality rule (i))

$-4x = 12$

$\frac{-4x}{-4} = \frac{12}{-4}$ (using equality rule (iv))

$x = -3$

Exercise - 5.1

1. Make equations from the statements given below:-
   (i) The sum of two consecutive numbers is 11.
   (ii) The sum of Tikendra and Tejaram's age is 30 whereas Tikendra's age is twice of Tejaram's age.
   (iii) One side of triangle is twice of the second side and equal to third side. Sum of the sides is 40.
   (iv) The length of the rectangle is 3 units more than its breadth. The perimeter of that rectangle is 15 units.
   (v) Ramesh, Dinesh and Satish have pencil in ratio of 2 : 3 : 4. The total number of pencils are 18.

2. Solve the equations given below:-
   (i) $5x + 2 = 17$ (ii) $5p + 1 = 24$
   (iii) $4x + 8x = 17x - 9 - 1$ (iv) $-7 + 3t - 9t = 12t - 5$
   (v) $3(z-2) + 5z = 2$ (vi) $-2 + (x + 4) = 8x$

3. Make the statement for the following equations:-
   (i) $x+3 = 27$ (ii) $\frac{x}{2} + x = 18$ (iii) $\frac{x}{x+2} = 30$

4. Solve the equation:-
   (i) $6 + (4 - m) = 8(3m + 5)$ (ii) $2(k - 5) + 3k = k + 6$
   (iii) $5p + 4(3 - 2p) = 2 + p - 10$ (iv) $\frac{x}{3} + 1 = \frac{7}{75}$
   (v) $m - \frac{m-1}{2} = 1 - \frac{m-2}{3}$ (vi) $\frac{3t-2}{4} - \frac{2t+3}{3} = \frac{2}{3} - t$
   (vii) $\frac{x}{2} - \frac{1}{5} = \frac{x}{3} + \frac{1}{4}$ (viii) $\frac{9x}{7-6x} = 15$
   (ix) $\frac{3y+4}{2-6y} = \frac{-2}{5}$

Application of Equation

We use equations to solve many questions in mathematics like- the length of the rectangle is 11 *cm* more than its breadth. If the perimeter of the rectangle is 110 *cm* then find length and breadth of the rectangle. In this we have been given the perimeter but to calculate the length and breadth of the rectangle we need to establish a relation with the given information. We will learn to find solution to such qeustions with the help of equations.

EXAMPLE-8. The length of a rectangle is 1 *cm* more than twice of its breadth. If perimeter of the rectangle is 110 *cm*. Then find the length and breadth of the rectangle.

SOLUTION : Suppose breadth of the rectangle is w *cm*.

Then according to the question, length of the rectangle = $(2w + 1)$ *cm*.

Given perimeter of the rectangle = 2 (length + breadth)

Putting the values $110 = 2\,(2w + 1 + w)$

$$110 = 4w + 2 + 2w$$

$$110 = 6w + 2$$

$$110 - 2 = 6w + 2 - 2$$

$$108 = 6w$$

$$w = \frac{108}{6} = 18\ cm$$

Length of rectangle (L) = $2w + 1$

$$L = 2 \times 18 + 1 = 37\ cm$$

EXAMPLE-9. Find four consecutive even integer numbers, if sum of first three number is more than 8 from the fourth number.

SOLUTION : Suppose $2x$ is an even integer number than four consecutive numbers

$2x,\ 2x + 2,\ 2x + 4,\ 2x + 6$

According to the question:-

Sum of first three numbers = fourth number + 8

$$2x + (2x + 2) + (2x + 4) = (2x + 6) + 8$$

$$2x + 2x + 2 + 2x + 4 = 2x + 6 + 8$$

$$6x + 6 = 2x + 14$$

$$6x + 6 - 2x = 2x + 14 - 2x$$

$$4x + 6 = 14$$

$$4x + 6 - 6 = 14 - 6$$

$$4x = 8$$

$$\frac{4x}{4} = \frac{8}{4}$$

$$x = 2$$

EXAMPLE-10. What must be added to the double of rational number $-\frac{7}{3}$ to get $\frac{3}{7}$.

SOLUTION : Twice of rational numbers is $-\frac{7}{3}$, $2 \times \left(-\frac{7}{3}\right) = -\frac{14}{3}$

Suppose we add x to it $-\frac{14}{3}$ to get $\frac{3}{7}$. Then

$$-\frac{14}{3} + x = \frac{3}{7}$$

$$\frac{-14+3x}{3} = \frac{3}{7}$$

$$7\ (-14 + 3x) = 3 \times 3$$

$$[7 \times (-14)] + (7 \times 3x) = 9$$

$$(-98) + 21x = 9$$

$$-98 + 21x = 9$$

$$21x = 9 + 98$$

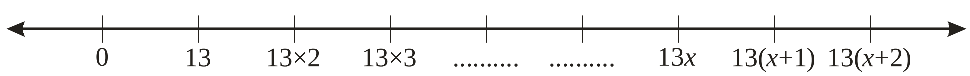
$$21x = 107$$

$$x = \frac{107}{21}$$

Thus, if we add $\frac{107}{21}$ to double of rational number $-\frac{7}{3}$ we will get $\frac{3}{7}$.

EXAMPLE-11. If sum of three continuous multiples of 13 is 390, then what are they?

SOLUTION : Suppose a multiple of 13 is $13x$, then the next multiples will be $13\ (x+1)$ and $13\ (x+2)$.



Sum of three continuously multiples of 13 is 390, therefore

$$13x + 13(x + 1) + 13(x + 2) = 390$$

$$13x + 13x + 13 + 13x + 26 = 390$$

$$39x + 39 = 390$$

$$39x = 390 - 39$$

$$39x = 351$$

$$x = \frac{351}{39}$$

$$x = 9$$

so the required multiples of 13 are $13 \times 9 = 117$, $13 \times (9+1) = 130$ and $13 \times (9+2) = 143$.

EXAMPLE-12. In a two digit number the difference between its digits is 3. If we add this number to the number obtained by interchanging the places of its digit, we get 143. A digit at tens place is bigger. Find the number.

SOLUTION : Suppose that the digit at units place is x. The difference between the digit at the unit and tens place is 3, therefore the digit at tens place is $= x + 3$.

Therefore, the two digit number is:-

$$= 10\ (x + 3) + x$$
$$= 10x + 30 + x$$
$$= 11x + 30$$

If the two digit number is 43 then we can write it as $43 = (10 \times 4) + 3$

Now, let us interchange the places of the digits of this number, that means write the units digit at the tens place and the tens digit at the unit place, the number thus obtained is:-

$$= 10x + (x + 3)$$
$$= 10x + x + 3$$
$$= 11x + 3$$

On interchanging the digits of 43 we get the number 34, and we can write it as $34 = (10\times3) + 4$

On adding both these numbers we get 143, so:-

$$(11x + 30) + (11x + 3) = 143$$
$$11x + 30 + 11x + 3 = 143$$
$$22x + 33 = 143$$
$$22x = 143 - 33$$
$$x = \frac{110}{22}$$
$$x = 5$$

therefore digit in units place $= 5$

the digit in tens place $= 5 + 3$

$= 8$

So the number is $= 85$

Verification of the solution:- On interchanging the digit of 85 we get 58. On adding 85 and 58 we get 143.

EXAMPLE-13. The ratio between Indramani and Sohan's current age is 4 : 5. After 8 years the ratio of their ages would be 5 : 6. Find their present age.

SOLUTION : Suppose the present age of Indramani is $4x$ years and the present age of Sohan is $5x$ years.

Age of Indramani after 8 year $= (4x + 8)$ year

Age of Sohan after 8 year $= (5x + 8)$ year

According to the question, the ratio of their ages after 8 years will be 5 : 6, therefore:-

$$\begin{aligned}
\frac{4x+8}{5x+8} &= \frac{5}{6} \\
6\,(4x + 8) &= 5\,(5x + 8) \\
(6\times4x) + (6\times8) &= (5\times5x) + (5\times8) \\
24x + 48 &= 25x + 40 \\
24x + 48 - 40 &= 25x \\
24x + 8 &= 25x \\
8 &= 25x - 24x \\
8 &= x
\end{aligned}$$

So, present age of Indramani $= 4x$

$= 4 \times 8$

$= 32$ year

And, present age of Sohan $= 5x$

$= 5 \times 8$

$= 40$ year

Checking the solution: Indramani's age after 8 years $= 32 + 8 = 40$ year

Sohan's age after 8 years $= 40 + 8 = 48$ year

The ratio of their age $= \dfrac{40}{48} = \dfrac{5}{6}$

Exercise - 5.2

1. Area of a triangle is 36 *sq m* and the length of its base is 12 meter then find the height of the triangle.
2. The length of the rectangle is 5 *cm* more than its breadth. Its perimeter is five times its breadth. Find the length and breadth of the rectangle.

3. If one angle of triangle is 15° more than its second angle. The third angle is 25° more than double the second angle. Find the three angles of the triangle.
4. Find three consecutive odd numbers when three times of their sum is 5 more than 8 times the middle number.
5. If one side of triangle is $\frac{1}{4}$ of its perimeter, other side is 7 *cm* and third side is $\frac{2}{5}$ of its perimeter, then find the perimeter of the triangle.
6. Sum of the digits of a two-digit number is 8. The number obtained by interchanging the digits exceeds the original number by 18. Find the number.
7. The ages of Vimla and Sarita are in the ratio of 7 : 5. Four year later, their ages will be in the ratio 4 : 3. Find their ages.
8. Alka thinks of a number, she adds 5 to it. To this she adds double of the original number and then subtracts 10 from it to get 40. Find the number.
9. Difference between two positive integers is 40. The integers are in the ratio 2:3. Find the integers.
10. Sum of three consecutive multiples of 5 is 555. Find these multiples.
11. Age of Rohit is 5 more than twice of Pradeep's age. 6 year ago, age of Pradeep was $\frac{1}{3}$ of Rohit's age. Find their ages.
12. A motor boat goes downstream in a river, covers the distance between two coastal towns in 5 hours. It covers same distance upstream in six hours. The speed of water is 2 *km*/*h* find the speed of the motor boat in still water.

What Have We Learnt

1. Only one variable is used in these equations and these are linear that is the power of the variable is 1.
2. Both sides of the equation can be a linear expression.
3. Linear equation with one variable is $ax + b = c$ (a, b, c real finite numbers and $a \neq 0$)
4. A quadratic equation of one variable is- $ax^2 + bx + c = d$ (a, b, c, d are real finite numbers and $a \neq 0$).
5. First we simplify the expressions before solving any equation.
6. We solve the equations using the rules of equality.

p n

०६

Guessing the Numbers

Often when we do operations like addition, multiplication on numbers, we know about the values of number and its digits. What happened if we don't know the value of numbers and its digits.

चित्र-१

If we have numbers like A, M, P, N, PQ, MN, ABC each of whose letter represents any one digit then how do we think about the possibilities to get to know the value of these digits?

If we think about number with one digit A, M, P, N as given above then we know that these digits can be any one from 0 to 9.

If we think about two digit numbers PQ and MN then P, Q, M and N are also between 0 to 9. In PQ, Q is the digit in the units place and P is the digit in the tens place. Therefore this number is actually 10P + Q. Similarly, number MN is 10M + N.

Similarly a three digit number like ABC is actually 100A + 10B + C.

Now, try to representing such number like ML, XY, AB, PQM, XYZ according to their place values. Think also of some four digit numbers.

Try This

1. If A = 3, B = 4, C = 5, D = 0 and each digit has to be used only one time then using these digits:-

   (i) What will be the largest number.

   (ii) Which number is bigger in ABCD and ADBC.

   (iii) What is the smallest number? Is it a three digit or four digit number?

   (iv) What is the value of DBAC? How many digits will be there in this number?

2. If A = 1, B = 2, C = 3, D = 4 then

   (i) Find the value of AB × CD. (ii) What will be the value of AB + CD.

Guessing the Numbers with Operation

Add these and see

Can you find value of P and Q here?

$$\begin{array}{r} P \\ P \\ +P \\ \hline QP \end{array}$$

We know that the value of P and Q can be between 0 to 9.

Now P + P + P = QP

3P = 10Q + P (According to the expanded form of the number)

2 P = 10 Q

$$\frac{P}{5} = Q$$

It means that P is such number which is divisible by 5.

i.e. P can only be 5, so P = 5

If P = 5 then $\frac{5}{5}$ = Q or Q = 1

Now on checking P + P + P = QP then 5 + 5 + 5 = 15

EXAMPLE-1. PQ – QP = 27 then what will be P and Q.

SOLUTION : (10P + Q) – (10Q + P) = 27

10P + Q – 10Q – P = 27

9P – 9Q = 27

9 (P – Q) = 27

P – Q = 3

Therefore, the possible answers of P and Q are:-

If (i) P = 9 then Q = 6 (ii) P = 8 then Q = 5

(iii) P = 7 then Q = 4 (iv) P = 6 then Q = 3 .....etc.

Thus we get 7 different values of P and Q.

Try This

1. $\begin{array}{r} AB \\ \underline{+BA} \\ 77 \end{array}$ than what is the value of A, B?

2. Similarly make same more questions which can have more than one answer.

Guessing the Three Digit Number

What will be the value of 'Y' in 5Y1 – 23Y = 325.

Here if we compare the digit in units place then $1 - Y = 5$

Therefore $11 - Y = 5$ or $Y = 6$

Another method:- $(500 + 10Y + 1) - (200 + 30 + Y) = 325$

$501 - 230 + 10Y - Y = 325$

$271 + 9Y = 325$

$9Y = 54$ Therefore $Y = 6$

Guessing the Numbers when Multiplying and Dividing

EXAMPLE-2.

$$\begin{array}{r} AB \\ \underline{\times\, AB} \\ ACC \end{array}$$

SOLUTION : $(10A + B)(10A+B) = 100A^2 + 20\,AB + B^2$

It's obvious that $A = 1$

Since units digit is equal to tens digit in the answer,

$2\,AB = B^2$

Therefore $2\,B = B^2$ (Putting $A = 1$)

That is, $B = 2$, $C = 4$

$$\begin{array}{r} 12 \\ \underline{\times\, 12} \\ 144 \end{array}$$

EXAMPLE-3.

$$\begin{array}{r} MN \\ \underline{\times\ \ 3} \\ LMN \end{array}$$

(Where LMN is a three digit number)

SOLUTION : We get N in the units place by multiplying N by 3. This is only possible when $N = 0$ or 5. The expanded form of MN will be $(10M + N)$.

Multiplying $(10M + N)$ by 3 is $(10M + N) \times 3$

$30M + 3N = 100\,L + 10M + N$ .....(i)

If $N = 5$ then

$20M + 15 = 100L + 5$

$20M = 100L - 10$

$$2M = 10L - 1$$

$$M = \frac{10L - 1}{2}$$

We know that L, M and N digits can be any whole number from 0 to 9.

A valid whole number M is not possible for any value of L from 0 to 9. Therefore N = 5 is not possible which implies it will be N = 0.

Come, let us now find out the value of M:-

Now similarly multiplying 3 by digit M in the tens place, we should get M in the tens place of product LMN which means either M = 5 or M = 0.

If we put N = 0 and M = 0 in equation (i) then we will get L = 0 which is not possible because LMN is a three digit number.

This implies M = 5.

If we put N = 0 and M = 5 in equation (i) then we will get L = 1.

Therefore L = 1, M = 5 and N = 0.

Try This

1. Find out the value of B if 1B × B = 96
2. Find out the value of M and N in 73 M ÷ 8 = 9N

Exercise - 6.1

Find out value of letters A, B, X, Y, Z, L, M, N as used in following questions.

(i) $\begin{array}{r} BA \\ +33 \\ \hline 12B \end{array}$ (ii) $\begin{array}{r} 3XY \\ +YY2 \\ \hline 1018 \\ \hline \end{array}$ (iii) $\begin{array}{r} MN \\ \times 6 \\ \hline MLN \\ \hline \end{array}$ (iv) $\begin{array}{r} 1Z \\ \times Z \\ \hline 7Z \\ \hline \end{array}$

(v) $\begin{array}{r} XX \\ 6 \\ +YYY \\ \hline 461 \\ \hline \end{array}$ (vi) $\begin{array}{r} 2PQ \\ +PQ1 \\ \hline Q18 \\ \hline \end{array}$ (vii) $\begin{array}{r} ML \\ \times 6 \\ \hline LLL \\ \hline \end{array}$

Number Riddle (Puzzle)

चित्र-२

Alok and Neha are making number riddles and asking each other:

Neha : Think of any 3 digits and write them.
Don't show me and don't take any zeros.

Alok : Jotted down (Writes on copy 3, 2 and 7)

Neha : Now write all two digit number which can be made by these 3 digits.

Alok : Alright. He writes, 32, 27, 23, 72, 37 and 73.

Neha : Now add all these two digit numbers.

Alok : 32 + 27 + 23 + 73 + 37 + 73 = ...
(He writes in the copy and adds them)

Neha : If you divide this sum by 22 then it will be completely divisible and the quoitient will be sum of digits that you selected.

Alok : 264 ÷ 22 = 12, Yes it is completely divisible.
Now the sum of 3, 2 and 7 (3 + 2 + 7) is 12.
Oh yes! How did you do it? I hope you didn't see what I wrote.

Neha : No, this will work for any three digits you take. I don't need to know your digits.

Try This

Select any 3 number of 3 digits and do Neha's activity.

Did you find sum of digit as Alok did?

Let us Understand Why this is So

If we select a, b and c as three digits then the two digit number which can be made from these numbers are ab, bc, ac, cb, ca and ba.

Expanded form of these are:-

$ab = 10a + b$ $\quad$ $bc = 10b + c$ $\quad$ $ac = 10a + c$

$cb = 10c + b$ $\quad$ $ca = 10c + a$ $\quad$ $ba = 10b + a$

By adding all these we get-

$$= \quad 10a + b + 10b + c + 10a + c + 10c + b + 10c + a + 10b + a$$

$$= \quad 22a + 22b + 22c = 22\,(a + b + c)$$

This means that the sum is a multiple of 22 so on dividing this sum by 22, the answer is sum of the digits ($a + b + c$).

Think and Discuss

1. Take any two digit number. Now get a new number by inter changing the place of digits. Add both the numbers. Now their sum will be completely divisible by 11. Can you say how is happens?

2. Think of a three digit number. Now get a new number by placing the digits in reverse order. Subtract the smaller number that we thus obtained, from the bigger number. Is this a multiple of 99? Why?

Which is Divisible by What

Checking the Divisibility

Do you know that how to check which number can be divided by divisors like 10, 5, 2, 3, 9 etc. How does it work? Let us see:-

Divisibility Rule for 10

Checking the divisibility by 10 is easy in comparison to other numbers. See some multiples of 10-

10, 20, 30, 40, 50, ...

For comparing, see some non multiples of 10- 12, 25, 33, 46, 57, 64, 77, 89, ... we can see that numbers that have a zero in the units place are multiples of 10. Whereas the numbers in which units place is not zero are not the multiples of 10. We get the rule of checking the divisibility by 10 through this analysis.

Now we will see how does this rule work? For this we have to use the rules of place value.

Take any number..... *cba*. In expanded form this can be written as

$$.......... + 100c + 10b + a$$

Here a is a digit in the units place, b is a digit in the tens place and c is a digit in the hundreds place. The..... dots shows that there can be more digits on left side of c.

Here 10, 100, 1000 etc. are divisible by 10 therefore $10b$, $100c$, ... will also be divisible by 10. About a we can say that if the given number is divisible by 10 then a also has to be divisible by 10, which is only possible if $a = 0$.

Obviously, any number will be divisible by 10 when the digit in the units place is 0.

Now give examples of some numbers which are divisible by 10.

Divisibility Rule for 5

See some multiples of 5:-

5, 10, 15, 20, 25, 30, 35, 40, 45, 50.......

we see that the digit of unit place are alternatively either 5 or 0.

This gives us the rule of divisibility for 5, that is if the digit in the units place of any number is either 5 or 0 then that number is divisible by 5. Now, for understanding this rule better, take any number .... *cba*.

Write it in expanded form .... $100c + 10b + a$

Here a is a digit in the units place, b is a digit in the tens place and c in a digit in the hundreds place and there can be more digits on left side of c. Since 10, 100, ... are divisible by 10. Therefore, $10b$, $100c$, ... will also be divisible by 10. These numbers are therefore also divisible by 5. $(10 = 5 \times 2)$.

Now, about a we can say that if a number is divisible by 5 then a has to be divisible by 5. It is clear that a must be either 0 or 5.

Try This

Are all numbers that are divisible by 5, also divisible by 10? Explain why.

Divisibility Rule for 2

Following are multiples of 2:-

2, 4, 6, 8, 10, 12, 14, 16, 18, 20, ... which are all even numbers.

Where the digits in the units place are 2, 4, 6, 8, 0. Therefore one can say that if the units digit of any number is 0, 2, 4, 6, 8 then that number will be divisible by 2.

Now we will check this rule. For this take any number..... *cba*

Its expanded form will be .... + 100*c* + 10*b* + *a*

Here *a* is the digit in the units place, *b* is a digit in the tens place and *c* is a digit in the hundreds place. 10, 100, ... are divisible by 10 and so are also divisible by 2. Therefore 10*b*, 100*c*, ... will be divisible by 2. Let us talk of *a* now. If a given number is divisible by 2 then *a* has to be divisible by 2. But it is only possible when *a* = 0, 2, 4, 6, 8.

Divisibility Rule for 9 and 3

You have seen the rules to check for the divisibility of 10, 5 and 2. Have you seen anything special about these rules? In all these rules divisibility is decided by the digit of units place in the given number.

The divisibility rule of 9 and 3 are different. Take any number 3429. Its expanded form will be:-

$3 \times 1000 + 4 \times 100 + 2 \times 10 + 9 \times 1$

This can be written in the following way:-

$3 \times (999 + 1) + 4 \times (99 + 1) + 2 \times (9+1) + 9 \times 1$

$= 3 \times 999 + 4 \times 99 + 2 \times 9 + (3 + 4 + 2 + 9)$

$= 9\,(3 \times 111 + 4 \times 11 + 2 \times 1) + (3 + 4 + 2 + 9)$

We see that the given number will be divisible by 9 or 3 only when the number (3 + 4 + 2 + 9) is divisible by 9 and 3.

Here 3 + 4 + 2 + 9 = 18 which is divisible by both 9 and 3.

Now take one more example:-

Say the number 3579. Its expanded form will be-

$3 \times 1000 + 5 \times 100 + 7 \times 10 + 9 \times 1$

This can be written in following way:-

$3 \times (999 + 1) + 5 \times (99 + 1) + 7 \times (9 + 1) + 9 \times 1$

$= 3 \times 999 + 5 \times 99 + 7 \times 9 + (3 + 5 + 7 + 9)$

Here 3 + 5 + 7 + 9 = 24 which is not divisible by 9 but is divisible by 3.

If we take any number is *abcd*, then its expanded form will be:-

$$1000a + 100b + 10\,c + d = 999a + 99b + 9c + (a + b + c + d)$$
$$= 9\,(111a + 11b + c\,) + (a + b + c + d)$$

Therefore the divisibility of 9 and 3 is only possible when the sum of all digits of that four digit number is divisible by 9 or 3.

(i) A number is divisible by 9 if sum of its digits is divisible by 9.

(ii) A number is divisible by 3 if sum of its digits is divisible by 3.

Checking the Divisibility Rule for 6

We know that any number that is divisible by 6, will have to be divisible by the prime factors of 6 i.e. 2 and 3.

Hence in order to check the divisibility of the number by 6, we will have to check for its divisibility by both 2 and 3.

Let us recall the rule of divisibility by 2 and 3-

The digit in the units place has to be an even number to make the number divisible by 2.

The sum of all digits of a number has to be divisible by 3 to make the number divisible by 3.

For example- take the number 1248

1248 has 8 in the units place, therefore, 1248 is divisible by 2 and the sum of all four digits of 1248 is 1 + 2 + 4 + 8 = 15 which is divisible by 3.

Hence, 1248 is divisible by 6.

Divisibility Rule for 7 and 11

Rule for checking divisibility rule by 7

Divide 91 by 7. Is 91 completely divisible by 7? On dividing, is the remainder zero? If yes, then is 91 divisible by 7?

Yes 91 is divisible by 7. How did we come to this conclusion? If we have to test the divisibility by 7 then we divide the number by 7. If remainder in zero then that number is divisible by 7. Check of divisibility for 7 of two digit number is done directly by division. But do we check the divisibility of a three digit number also by the same process? Or is there any other method?

There are many methods to check the divisibility by 7. We will check the divisbility by some of these methods:-

Take a three digit number *abc*.

Its expanded form will be- $100a + 10b + c$

$$100a + 10b + c = 98a + 7b + (2a + 3b + c)$$

Let's write it such that it shows 7 as common factor:-

$$7\,(14a + b) + (2a + 3b + c)$$

Here $7\,(14a + b)$ divisible by 7. Now if $(2a + 3b + c)$ is divisible by 7 then number *abc* will also be divisible by 7.

Try This

Check the divisibility of following numbers by 7 using the above methods-

373, 644, 343, 861

Now think of some more three digit numbers and check the divisibility by 7.

There are some more methods for checking the divisibility of numbers with more than three digits by 7. Come, let us see one more method-

$$\begin{array}{l} \underline{36484} \mid 7 \\ \downarrow \\ 36484 \\ \underline{-\quad 14} \leftarrow (7\times2) \\ 36470 \end{array}$$

$$\begin{array}{l} \underline{3647} \mid 0 \\ \downarrow \\ 3647 \\ \underline{-\quad 0} \leftarrow (0\times2) \\ 3647 \end{array}$$

$$\begin{array}{l} \underline{364} \mid 7 \\ \downarrow \\ 364 \\ \underline{-14} \leftarrow (7\times2) \\ 350 \end{array}$$

We can see from this whole process that we double the units digit of the number and subtract it from the number leaving out the units digit.

35 | 0

35

-0 ← (0×2)

35

$\because$ 35 is divisible by 7, therefore 364847 is also divisible by 7.

Check the Divisibility for 11

Take any number *abcd*. Its expanded form is $(1000a + 100b + 10c + d)$

This can also be written as:-

$$(1001 - 1)\,a + (99 + 1)\,b + (11 - 1)\,c + d$$

Let's write it such that it shows 11 as a factor-

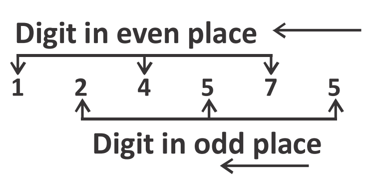
$$11\,(91a + 9b + c) - (a - b + c - d)$$

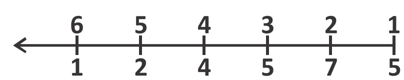
$11\,(91a + 9b + c)$ is obviously divisible by 11.

If $a - b + c - d$ is 0 or divisible by 11 then the whole number *abcd* is also divisible by 11.

Therefore, the rule is that if the difference between sum of digits in even places and sum of digits in odd places of the number is either 0 or divisible by 11 then the number is divisible by 11.

Check for number 124575-





Difference = (Sum of digits in odd places) – (Sum of digits in even places)

$= (5 + 5 + 2) - (7 + 4 + 1)$

$= 12 - 12$

$= 0$

Here the meaning of taking difference is to substract the smaller sum from bigger sum.

Difference is zero, therefore number is divisible by 11.

Try This

To check the divisibility of number 19151 by 11.

Similarly there are interesting rules of checking the divisibility by 13. Think and try to find by discussing with friends.

Think and Discuss

1. Are all numbers that are divisible by 3, also divisible by 9? Why or why not?
2. Choose any one number which are completely divisible by 6. Divide this number by 2 and 3 and see. Is this number is divisible by 2 and 3 both? What can you say about the divisibility rule of 6?
3. Using digits 2, 5, 4 and 7, make all the numbers that are completely divisible by 15. Without actually dividing by 15 how will you find out which of these numbers will be divisible by 15? (Use the divisibility rules)
4. Find the smallest number that will be divisible by 7 and 11 both.

Exercise - 6.2

1. Which of the following numbers are multiples of 5 and 10.

   316, 9560, 205, 311, 800, 7936
2. If a number that is divisible by 3, has 8 in its unit place, then what can be possible digits in tens place?
3. If number 35P is a multiple of 5 then find the value of P?
4. If 6A 3B is a number that is divisible by 9, find the value of A and B?
5. Check the divisibility by 7 for following numbers.

   (i) 672 (ii) 905 (iii) 2205 (iv) 9751
6. Check the divisibility by 11.

   (i) 913 (ii) 987 (iii) 3729 (iv) 198
7. Check the divisibility by 13.

   (i) 169 (ii) 2197 (iii) 3146 (iv) 5280

What Have We Learnt

1. We can write numbers in expanded form like a two digit number ab can be written as $10a + b$ and a three digit number abc can be written as $100a + 10b + c$.
2. We take help of expanded form of number in solving puzzle or number games.
3. We learnt about the divisibility rules of 2, 3, 5, 6, 7, 9 and 11.

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ung BH

Let us know the history of mathematics.....

The content that has been included in this chapter under the name of commercial mathematics has necessarily not always been called by that name. When we turn the pages of world history we see that mathematics has been presented in some form or the other at all times. Whether it is construction of a house for living, preparing the land for agriculture, exchange of commodities useful for living or for other things, mathematics has always been part of public life.

No human being can produce or make all the things that they need themselves. Everyone has to be dependent on other people for their needs. In earlier times also commodities and labour were exchanged for other commodities and labour. Examples of this can be seen in small villages where farmers help plowing each other land by turns. Payment for labour through grains is still prevalent. In Arabian stories the mention of traders occurs repeatedly. The arrival of the English and Portuguese for trade in India is only just about 4 -5 centuries old.

If we look at the history of mathematics, we can see many examples of its use in trade. We are giving two examples here.

In 1881, handwritten manuscripts were found during excavations in the village Bhakshali of Peshawar district. These were in the ancient Sharada script. Three parts of these manuscripts were published in 1927 and 1933 by the archaeological department of Calcutta. Doctor Hornel has written three articles on these manuscripts.

The contents of these manuscripts include elite mathematics, income expenditure and profit and loss. Experts believe these manuscripts to be belong to the third or fourth century. The articles of Doctor Hornel also contain mention of questions involving interest. The questions mentioned in the articles are diverse and involve the use of many algebraic ideas. One of these questions is-

"There are three traders, one of them has 7 horses, the second one has 9 mules and the third one has 10 camels. Each of them contributes three animals to a kitty to be distributed among them so that the total property of each of them is the same. What was the original value of the holding of each of the trader and what is the value of each of the animals?"

Bhaskar is identified as a special person of mathematics from the first half of 12^{th} century. He was born in the year 1114 CE and was the director of Ujjain Observatory. His book 'Leelawati" is world famous. The book has been translated in Persian by Abdul Faizi in 1587 and into English by Taylor in 1816. This English translation was published in Calcutta in the year 1827. This volume on mathematics, 'Leelawati", contains units on different aspects of mathematics. Some of the important ones in these are integers and fractions, interest, series and their progression as well as commercial mathematics.

This information is collected from different sources. Students and teachers can find more information on commercial mathematics from other sources.

a q

Whenever we think of comparing quantities of the same type we first think of ratio and proportion. What is a ratio? Is proportion different from ratio? Let us look at some examples to understand this.

Children of a class were measuring the distance from their ankle to the knee using their handspan.

Neelam : the distance of my knee from my ankle is 2 hand spans and 4 fingers

Kiran : this distance for me was also 2 hand spans and 4 fingers

Then each child one by one said that the distance for me is also 2 hand spans and 4 fingers.

Measure the distance from your knee.

Can we say that ratio of the length of the hand span to the distance from the knee to the ankle remains the same for all?

One More Measurement

Mahi said let's measure the length of the arm (shoulder to the wrist) using hand spans.

Saurabh : This length is also 2 hand spans and 4 fingers

Mahi : It is the same for me

The measure of the distance from the shoulder to the wrist was 2 hand spans and 4 fingers for all children.

You also measure this distance. We can say that the ratio of the length of the hand span to the distance from the shoulder to the wrist remains the same for all.

We therefore see that the ratio of the length of the hand span to the distance from the shoulder to the wrist and its ratio to the distance from the knee to the ankle is the same for all. We say that these have the same proportion.

Look at these also

If there are three black and nine white balls in a bag then we can say;

1. The number of black balls is one third $\left(\frac{1}{3}\right)$ the number of white balls

2. White balls are three times the number of black balls

Black balls	**:**	**White balls**
3	:	9
1	:	3

Similarly if the height of a banana plant is 20 cm and that of a mango plant is 60 cm then the ratio of the heights is-

Height of the banana plant	**:**	**Height of the mango plant**
20	:	60
1	:	3

This is the same ratio as we saw between black and white balls. These are therefore proportional.

Think and Discuss

Give five examples from your daily experience where you find definite ratio or proportion between different quantities.

Rehana studies in class 8 and Farida is in the class 9. Both of them got their results for the maths exams. They went and told their results to their mother.

Rehana : I got 80 marks out of 100

Mother : Ok! And Farida you?

Farida : I got 110 marks out of 150.

Rehana : Wow! Didi has better result compared to me.

Mother : How?

Rehana : Didi has got more marks than me, therefore her result is better.

Farida : Oh yes! I've got 110 marks and Farida has got only 80

Mother : But Farida, Rehana has got 80 marks out of hundred and you have got 110 out of 150, so how do you know your result is better than Rehana's?

Farida : I did the following-

Rehana got- 80 marks

And I got- 110 marks

and then I subtracted 80 marks from 110.

$110 - 80 = 30$

So I got 30 more marks than Rehana, hence, my result is better.

Mother : But the total marks for your cases are different, therefore we cannot say who has a better result. If the look at the ratio of the marks obtained against total marks, then we can say-

Rehana got- $\frac{80}{100} = \frac{8}{10} = \frac{4}{5}$ and

Farida got- $\frac{110}{150} = \frac{11}{15}$

To compare the two ratios we make the denominators the same, we get then

Rehana's marks- $\frac{4}{5} \times \frac{3}{3} = \frac{12}{15}$ that is 12 out of 15

Farida's marks- $\frac{11}{15}$ that is 11 out of 15

And because $\frac{12}{15}$ is more than $\frac{11}{15}$, Rehana's result is better.

Farida : Can we do this some other way?

Mother : Yes! We can also use percentages for this.

What are Percentages?

Percentage is another way of comparing quantities. When we compare on the basis of percentages (%) we assumed the total to be 100.

For example if out of 12 balls in a bag 3 are black then the percentage of black balls is $\frac{3}{12} \times 100 = 25\%$

The percentage of white balls is $\frac{9}{12} \times 100 = 75\%$

We can then say that 25% of the balls are black and 75% are white.

Percentage means per 100 and is shown by % sign.

Think and Discuss

Consider the following examples carefully and discuss which way of comparison, ratio or percentages would be better?

1. The mixture for Idli contains

$\frac{1}{3}$ urad lentil $\frac{2}{3}$ rice	33.3% urad lentil 66.6% rice

2. The distribution of children in the class is

Girls : Boys 3 : 2	60% Girls 40% Boys

Application of Percentage

Discount

Deepak wants to buy some material for the school.

He went to the market and saw a sign board outside the stationery shop saying discount of 6% on all you buy.

Deepak bought a packet of pencils and asked the shopkeeper the price?

Shopkeeper : The price is Rs.50 but due to 6% discount you pay 47.

Deepak : Ok! How did you calculate that?

Shopkeeper : The discount is calculated on the marked price of the item, therefore when we sell the item we deduct the discount amount and therefore the selling price is less than the marked price.

Discount = Marked Price – Selling price

The discount is expressed in percentage and calculated on the marked price.

EXAMPLE-1. The price of a shirt is Rs 300 and the shopkeeper is selling it at a discount of 20%. find out the discount amount and the selling price.

SOLUTION : Market Price = 300, Discount percentage = 20%

For a marked price of Rs 100 the discount is Rs. 20

Therefore, for a marked price of Rs. 1 the discount will be Rs. $\frac{20}{100}$

Discount on Rs. 300 = $\frac{20}{100} \times 300$ = Rs. 60

Selling price = 300 – 60 = 240

EXAMPLE-2. A book was sold for Rs. 45 after a discount of 10%. What was the marked price of the book?

SOLUTION : On a marked price of Rs. 100 the discount is Rs. 10.

Its selling price 100 – 10 = Rs. 90

Comparing-	**Selling price**	**Purchase Price**
	90	100
	45	?

When a book is sold for Rs.90 then it's marked price is Rs 100

If the book is sold for Rs.45 then the marked price is $\frac{100}{90} \times 45$ = Rs. 50

Therefore the marked price would be Rs. 50

Try This

1. A sari with the marked price of Rs. 800 is sold at a discount of 15%. Find out the selling price of the sari?
2. During sale a shop gives a discount of 15% on the marked price of all the goods being sold. What would a customer have to pay for a pair of jeans and a shirt with marked prices Rs.17 50 and Rs 650 respectively?
3. Mahi bought a pair of shoes from a sale on a discounted price of 20%. If he paid Rs.1200 for the shoes, what was the marked price of the shoes?

Sale Tax

Manish and Sahil bought some things from a shop. The shopkeeper gave them the bill. They looked at the bill carefully

BILL

Bill No. : 27 Date : 13.02.15

Name : Manish

S.No.	Name of the Item	Qty.	Rate	Amount
1.	Soap	10	19	190
2.	Toothpaste	1	40	40
3.	Coconut Oil (200 *ml.*)	2	85	170
	Total Amount	-	-	400
	Sales Tax 5%	-	-	20
	Total			420

Sahil : What is this sales tax?

Manish : This is the tax charged by the government on anything sold.

EXAMPLE-3. Rajan bought a cooler for Rs.2700 including sales tax at 8%. Find out the price of the cooler before the tax was added.

SOLUTION : 8% sales tax means that if the price without tax is Rs. 100 then price with tax is Rs. 108.

Now, if the price with taxes is Rs. 108 then the actual price is Rs. 100

Hence when the price with tax is Rs. 2700 then actual price is

$$= \frac{100}{108} \times 2,700$$

= 2,500 Rs.

Try This

1. Whenever you go shopping with your family look at the bill and see which are the products on which the shopkeeper charges sales tax. Make a list of these. Discuss with your friends in your class.

2. If you go to the market to buy something and have a 10% discount but have to pay 5% sales tax on materials purchased, then which of these two will be better for you-

   (i) Take a discount of 10% and add 5% sales tax

   (ii) Add 5% sales tax and then take 10% discount

Exercise - 7.1

1. Mohan saves Rs. 3,950 after spending 75% of his salary. What is his monthly salary?

2. In class 10, 60% are girls. Their number is 18, so what is the ratio of numbers of boys to girls in the class?

3. An item marked at Rs. 950 was sold for Rs. 760. Find the discount and discount percent given?

4. Lalita bought a motorcycle for Rs. 9,016 including 12% sale Tax. What was the cost before adding the sale Tax?

5. A flour mill costs Rs. 1,30,000. The sales tax is taken at 7% on this. If Shweta buys this mill, find the price she paid for it?

6. A machine is sold with 8% discounted price at Rs. 1,748. What is the marked price of the machine?

7. We have two baskets. One has 8 mangoes and 4 watermelons and the other has 14 mangoes and 7 watermelons. find out if the ratio of fruits in both the baskets is the same or different?

8. If 5% sales tax is added to the purchase of the following items, then find the selling price of each:-

   (i) Shirt with price of Rs. 200/-

   (ii) 5 *kg* sugar at the rate of Rs. 30/-*per kg*

9. Sarita bought a cooler for Rs. 5,750/- including a tax of 15%. What was the price of cooler before tax.

10. Suraj brought a television for Rs. 10260/- including a sales tax of 8%. What was the price before tax.

Interest

The extra amount charged for the use of some borrowed amount for a certain period is called interest. It is calculated in two ways:-

1. Simple Interest
2. Compound Interest

Simple Interest

You are familiar with simple interest and how to calculate it. Let us recall through an example:

EXAMPLE-4. Find out the simple interest on Rs. 10,000 for 3 years at 10% annual rate of interest.

SOLUTION. Principal = Rs. 10,000 Rate = 10% Time = 3 years

We know, simple interest $= \dfrac{\text{Principal} \times \text{Rate} \times \text{Time}}{100}$

$= \dfrac{10,000 \times 10 \times 3}{100}$

= Rs. 3,000

Interest and Compound Interest

Sudhir borrowed Rs. 2,00,000 for 4 years at 8% annual rate of interest. The condition was that he would have to pay the interest at the end of each year. He was to deposit Rs. 16,000/- as the interest for the first year. He was not able to deposit that for some reason next year he went deposit the earlier due amount and the interest for the second year. Ajay told Sudhir- you have to give Rs. 33,280 as the interest for the two years.

Sudhir said the interest for the two years is (Rs. 16000 + Rs. 16000) Rs. 32,000.

Ajay said- We are not taking any extra amount. The additional amount is the interest on the interest amount that was due but not paid last year. This is called compound interest.

EXAMPLE-5. Sadhna borows Rs. 10,000/- for 3 years at 10% annual rate of interest. She has to return the amount with the extra amount (interest). Find out the compound interest and the total amount to be paid after 3 yeras.

1. Principal amount for the first year, P_1 = Rs. 10,000/-

Interest for the first year $10,000 \times \frac{10}{100} = 1,000$

SI_1 = Rs. 1,000/-

2. The principal for second year, $P_2 = P_1 + SI_1$

$= 10,000 + 1,000$ = Rs. 11,000

Hence interest for second year

$SI_2 = 11,000 \times \frac{10}{100}$ = Rs. 1,100/-

3. The principal for third year, $P_3 = P_2 + SI_2$

$= 11,000 + 1,100$

= Rs. 12,100

The interest for third year $SI_3 = 12,100 \times \frac{10}{100}$

= Rs. 1,210

4. Hence amount to be paid $P_3 + SI_3$

$= 12,100 + 1,210$

= Rs. 13,310

Total interest $= SI_1 + SI_2 + SI_3$

$= 1,000 + 1,000 + 1,210$

= Rs. 3,310

Will the amount of compound interest be different from simple interest?

Lets find out:-

Simple interest for 3 years $= \frac{\text{Principle} \times \text{Time} \times \text{Rate}}{100}$

$= \frac{10,000 \times 3 \times 10}{100}$

= Rs. 3,000/-

We see that because of compound interest an extra amount Rs. 310 has to be paid. For simple interest the principle amount remains the same while for compound interest it changes each year.

Let us try to find a formula to calculate the compound interest.

Suppose Principal amount = P Rate = R%

Time = t Total amount = A

The interest for the first year $I_1 = \dfrac{P_1 \times R \times 1}{100}$

The total amount after first year $\therefore A_1 = P_1 + \dfrac{P_1 \times R \times 1}{100}$

$$= P_1\left(1 + \frac{R}{100}\right)$$

$= P_2$ (Principal amount for the second year)

Interest for the second year $I_2 = \dfrac{P_2 \times R \times 1}{100}$ $= P_2\left(\dfrac{R}{100}\right)$

$$= P_1 \times \left(1 + \frac{R}{100}\right) \times \frac{R}{100}$$

$\therefore$ Total amount after 2 years

$$A_2 = P_2 + I_2$$

$$A_2 = P_1\left(1 + \frac{R}{100}\right) + P_1\left(1 + \frac{R}{100}\right) \times \frac{R}{100}$$

$$= P_1\left(1 + \frac{R}{100}\right) \times \left(1 + \frac{R}{100}\right)$$

$$= P_1\left(1 + \frac{R}{100}\right)^2$$

$= P_3$ (Principle for third year)

Interest for the third year $I_3 = \dfrac{P_3 \times R \times 1}{100}$ $= P_3\left(\dfrac{R}{100}\right)$

$$I_3 = P_1\left(1 + \frac{R}{100}\right)^2 \times \frac{R}{100}$$

$\therefore$ Total amount after three years $A_3 = P_3 + I_3$

$$A_3 = P_1\left(1+\frac{R}{100}\right)^2 + P_1\left(1+\frac{R}{100}\right)^2 \times \frac{R}{100}$$

$$A_3 = P_1\left(1+\frac{R}{100}\right)^2 \times \left(1+\frac{R}{100}\right)$$

$$A_3 = P_1\left(1+\frac{R}{100}\right)^3$$

Hence total amount after t years

$$A_n = P_1\left(1+\frac{R}{100}\right)^t$$

In standard form $\quad A = P\left(1+\frac{R}{100}\right)^t$

$$\text{Total amount} = \text{Principal amount} \times \left(1+\frac{Rate}{100}\right)^{Time}$$

Compound interest = Total amount – Principal amount

$$C.I. = A - P$$

$$C.I. = P\left(1+\frac{R}{100}\right)^t - P$$

$$C.I. = P\left[\left(1+\frac{R}{100}\right)^t - 1\right]$$

We use this formula to calculate the compound interest.

EXAMPLE-6. Find out the compound interest for 2 years on Rs. 5600/- at 5% rate of annual interest.

SOLUTION : We know $\mathrm{A} = \mathrm{P}\left(1+\frac{\mathrm{R}}{100}\right)^t$

Here, Principal (P) = Rs. 5,600; Rate (R) = 5% annual; Time (t) = 2 years

Hence, $A = 5,600 \times \left(1 + \frac{5}{100}\right)^2$

$A = 5,600 \times \left(1 + \frac{1}{20}\right)^2$

$A = 5,600 \times \left(\frac{21}{20}\right)^2 \quad = 5,600 \times \left(\frac{21}{20}\right) \times \left(\frac{21}{20}\right)$

A = Rs. 6,174/-

$\therefore$ Compound interest = Total amount – Principal amount

= 6,175 – 5,600 = Rs. 574/-

EXAMPLE-7. Shyam deposited Rs. 64,000/- in a nationalised bank. If the annual rate of compounded interest is $2\frac{1}{2}$% then what is the total amount he would get after 3 years? What is the interest he would get?

SOLUTION : We know that $A = P\left(1 + \frac{R}{100}\right)^t$

Here, Principal (P) = Rs. 64,000/-

Rate (R) $2\frac{1}{2}\% = \frac{5}{2}\%$ annual; Time (t) = 3 years

Hence, $A = 64,000 \times \left(1 + \frac{5/2}{100}\right)^3$

$A = 64,000 \times \left(1 + \frac{5}{2 \times 100}\right)^3$

$A = 64,000 \times \left(1 + \frac{1}{40}\right)^3$

$A = 64,000 \times \left(\frac{41}{40}\right)^3$

$A = 64,000 \times \left(\frac{41}{40}\right) \times \left(\frac{41}{40}\right) \times \left(\frac{41}{40}\right)$

$A = 41 \times 41 \times 41$ = Rs. 68,921/-

Amount of interest from bank = Total amount – Principal amount

= 68,921 – 64,000 = Rs. 4,921/-

Try This

1. Find out the compounded interest after 2 years on Rs. 15,000/- at an annual rate of interest of 6%.
2. Find out the compounded interest after 3 years on Rs. 8,000/- at an annual rate of interest of $5\frac{1}{2}$%.
3. Find out the total amount and compound interest:-
   (i) On Rs. 10,700/- after 3 years at an annual rate of interest of $14\frac{1}{2}$%.
   (ii) On Rs. 18,000/- after 2 years at an annual rate of interest of 10%.

Calculation when Compounded Quarterly or Half-Yearly

In many situations including in banks the interest is now not calculated annualy but after each quarter or after 6 months. This means the principal charges every 6 months or 3 months.

When interest is compounded half yearly then we have 2 periods of 6 months in each year and the rate of interest is halved. Similarly if the interest is compounded quarterly then each year has 4 quarters and the rate of interest be comes one fourth.

The calculation are done as follows:-

1. For compounding half yearly

$$\text{Total amount} = \text{Principal} \times \left(1 + \frac{\text{half-yearly rate of interest}}{100}\right)^{\text{No. of half years}}$$

For compounding quarterly

$$\text{Total amount} = \text{Principal} \times \left(1 + \frac{\text{quarter rate of interest}}{100}\right)^{\text{No. of quarters}}$$

Try This

1. Calculate the compound interest for one and a half-year Rs. 4,000/- at 5% rate of interest?

2. If interest is compounded half-yearly then find the compound interest for $1\frac{1}{2}$ years on Rs. 1,500/- at 10% annual rate of interest.

3. Deepak borrowed Rs. 80,000/- from a bank for 2 years. If the annual rate of interest is 10% then find the total amount due when:-

   (i) Interest is compounded annually (ii) Interest is compounded half-yearly.

EXAMPLE-8. Find out the compound interest for $1\frac{1}{2}$ years on Rs. 15,625 at annual rate of interest of 8%, if the interest is compounded half yearly.

SOLUTION : The interest is compounded every half-year.

Hence, time $(t) = 1\frac{1}{2}$ year $= 3$ half-year

Rate (R) = 8% annual = $\frac{8}{2}$% half-yearly = 4% half-yearly

Principal (P) = Rs. 15,625/-

$$\therefore \text{ Total amount} = \text{Principal} \times \left(1+\frac{\text{half-yearly rate of interest}}{100}\right)^{\text{No. of half years}}$$

$$\text{Total amount } = 15,625 \times \left(1+\frac{4}{100}\right)^3 = 15,625 \times \left(1+\frac{1}{25}\right)^3$$

$$\text{Total amount } = 15,625 \times \left(\frac{26}{25}\right)^3 = 15,625 \times \left(\frac{26}{25}\right) \times \left(\frac{26}{25}\right) \times \left(\frac{26}{25}\right)$$

$$\text{Total amount } = 26 \times 26 \times 26 = \text{Rs. } 17,576/-$$

$\therefore$ Compound interest = Total amount – Principal

= 17,576 – 15,625 = Rs. 1,951

EXAMPLE-9. Calculate the interest for a quarterly compounded interest over a peirod of 9 months on Rs. 1,000/- for 8% annual rate of interest.

SOLUTION : Here interest is compounded quarterly

Hence, ... t) = 9 months = 3 quarters

$$\text{Rate (R)} = 8\% \text{ annual} = \frac{8}{4}\% \text{ quarterly} = 2\% \text{ quarterly}$$

Principal (P) = Rs. 1,000/-

$$\therefore \text{ Total amount} = \text{Principal} \times \left(1 + \frac{\text{quarterly rate of interest}}{100}\right)^{\text{No. of quarters}}$$

$$\text{Total amount} = 1,000 \times \left(1 + \frac{2}{100}\right)^3 = 1,000 \times \left(1 + \frac{1}{50}\right)^3$$

$$\text{Total amount} = 1,000 \times \left(\frac{51}{50}\right)^3 = 1,000 \times \left(\frac{51}{50}\right) \times \left(\frac{51}{50}\right) \times \left(\frac{51}{50}\right)$$

$$\text{Total amount} = \frac{1,32,651}{125} = \text{Rs. } 1,061.20$$

$$\therefore \text{ Compound interest} = \text{Total amount} - \text{Principal}$$

$$= 1,061.20 - 1,000 = \text{Rs. } 61.20$$

EXAMPEL-10. What principal amount would become Rs. 2,809/- after 2 years of compound interest at annual rate of 6%.

SOLUTION : Here, Total amount (A) = Rs. 2,809/- Principal (P) = ?

Rate (R) = 6% annual Time (t) = 2 years

$$\therefore A = P\left(1 + \frac{R}{100}\right)^t$$

$$2,809 = P\left(1 + \frac{6}{100}\right)^2 = P\left(1 + \frac{3}{50}\right)^2$$

$$2,809 = P\left(\frac{53}{50}\right)^2 = P \times \frac{53 \times 53}{50 \times 50}$$

$$2,809 = P \times \frac{2,809}{2,500}$$

$$\therefore P = \frac{2,809}{1} \times \frac{2,500}{2,809} = \text{Rs. } 2,500/-$$

EXAMPLE-11. At what rate of interest will the principal amount of Rs. 1,000/- become Rs. 1,331/- after $1\frac{1}{2}$ year of half yearly compounded interest?

SOLUTION : Here, Total amount (A) = Rs. 1,331/- Principal (P) = Rs.1,000

Rate (R) = ? Time (t) = $1\frac{1}{2}$ years = 3 half-yearly

$$\therefore\ A = P\left(1+\frac{R}{100}\right)^{t}$$

$$1,331 = 1,000\left(1+\frac{R}{100}\right)^{3}$$

$$\frac{1,331}{1,000} = \left(1+\frac{R}{100}\right)^{3}$$

$$\left(\frac{11}{10}\right)^{3} = \left(1+\frac{R}{100}\right)^{3}$$

If the terms on both sides of an equation have the same exponent then the bases are equal.

Hence, $\frac{11}{10} = 1+\frac{R}{100}$

$$\frac{11}{10} - 1 = \frac{R}{100}$$

$$\frac{1}{10} = \frac{R}{100}$$

$$R = \frac{100}{10} = 10\%\ \text{(half-yearly)}$$

Since rate of interest calcualted is half yearly,

Hence, R = 10 × 2 = 20% (annual)

EXAMPLE-12. Mohan borrowed Rs. 31,250/- at 8% annual interest. After how much time would be have to return Rs. 39,336/-

SOLUTION : Here, Total amount (A) = Rs. 39,366/- Principal (P) = Rs. 31,250

Rate (R) = 8% Time (t) = ?

$$\therefore\ A = P\left(1+\frac{R}{100}\right)^t$$

$$39{,}366 = 31{,}250\left(1+\frac{8}{100}\right)^t$$

$$\frac{39{,}366}{31{,}250} = \left(1+\frac{2}{25}\right)^t$$

$$\frac{19{,}683}{15{,}625} = \left(\frac{27}{25}\right)^t$$

$$\frac{27\times27\times27}{25\times25\times25} = \left(\frac{27}{25}\right)^t$$

$$\left(\frac{27}{25}\right)^3 = \left(\frac{27}{25}\right)^t$$

The bases on either side of the equation being equal the exponents would also be equal.

Hence $t = 3$ years.

Exercise - 7.2

1. Find the compound interest and the total amount for the following:-

(i)	Principal = Rs. 7,000	Rate = 10% yearly	Time = 4 years
(ii)	Principal = Rs. 6,250	Rate = 12% yearly	Time = 2 years
(iii)	Principal = Rs. 16,000	Rate = 5% yearly	Time = 3 years

2. Rahul borrowed Rs. 1,25,000/- at 12% annual compound interest from a bank to buy car. What would be the total amount and the amount of interest after 3 years?

3. Find out the compound interest on Rs. 50,000/- deposited in a bank at 15% annual rate after 2 years.

4. Calculate the simple and compound interest at 5% annual rate on Rs. 1,260/- after 2 years?

5. Find the difference between the amount of compound interest and simple interest on a principal amount of Rs. 8,000/- at an annual interest rate of 10% after 2 years.

6. Find out the compound interest after $1\frac{1}{2}$ years on an amount of 3,000/- at 8% annual rate of interest when the compounding of interest is half yearly.

7. How much interest would we get from the bank on Rs. 9,000/- in a year with quarterly compounded interest at 8% annual rate of interest.

8. Find the compound interest on Rs. 3,500/- after one year if compounded half yearly .

9. A man borrowed Rs. 25,000/- what total amount would he pay after one year, if the rate of interest is 20% annual and is compounded half yearly.

10. Find the compounded interest for six months on Rs.10,000 at an annual rate of interest of 12% if the interest is compounded every 3 months.

11. How much money should Surjeet deposit in the Punjab National bank so that he gets Rs.6 615 after two years with the rate of interest 5% annually?

12. What amount would become Rs.18,522 at an annual rate of interest of 10% after one and a half years if the interest is calculated every six months?

13. Kabir borrowed Rs. 15,625 from the Indian bank. He returned Rs.17,576 after 3 years. What was the rate of interest charged by the bank?

14. At what annual rate of interest Rs 6,000 would become Rs.6 ,615 in two years?

15. In how much time would the compounded amount become Rs.9,261 from a principal of Rs.8,000 if the interest is compounded annually at a rate of 5% interest?

16. Ahmed got an interest amount of Rs.5853 on his deposit of Rs.46,875 at an annual rate of interest of 8%. if the interest was compounded every six months then find the time of deposit?

17. What is the principal amount on which the difference between the compound interest and simple interest is Rs. 40, after two years at the rate of annual interest 5%?

The Use of the Formula for Compound Interest

The formula for finding the compound interest is used in many situations.

(i) For finding the increase and decrease of population

(ii) For finding the increase and decrease in price of an item

(iii) For finding the total amount when the rate of interest is different in different years.

(a) If the population of a town increases, then

$$\text{expected population} = \text{current population}\left(1+\frac{\text{Rate}}{100}\right)^{time}$$

(b) If the population of a town decreases, then

$$\text{expected population} = \text{current population}\left(1-\frac{\text{Rate}}{100}\right)^{time}$$

(c) If the price of an object increases, then

$$\text{expected price} = \text{current price}\left(1+\frac{\text{Rate of Increase}}{100}\right)^{time}$$

(d) If the price of an object decreases, then

$$\text{expected price} = \text{current price}\left(1-\frac{\text{Rate of Decrease}}{100}\right)^{time}$$

(e) If the rate of interest in successive years is $R_1\%$, $R_2\%$, $R_3\%$.... then,

$$A = P\left(1+\frac{R_1}{100}\right)\left(1+\frac{R_2}{100}\right)\left(1+\frac{R_3}{100}\right)\ldots\ldots\ldots\ldots$$

EXAMPLE-13. The present population of the town is 128000, if the population increases at 5% annually than what would be the population after three years?

SOLUTION : The present population is 128000 Rate of increase = 5% annually

Time = 3 Years Expected population = ?

$$\because \quad \text{Expected population} = \text{Current population}\left(1+\frac{\text{Rate of Increase}}{100}\right)^{time}$$

$$\text{Expected population} = 1{,}28{,}000\left(1+\frac{5}{100}\right)^3 = 1{,}28{,}000\left(1+\frac{1}{20}\right)^3$$

$$\text{Expected population} = 1{,}28{,}000\left(\frac{21}{20}\right)^3 = 1{,}28{,}000\times\frac{21}{20}\times\frac{21}{20}\times\frac{21}{20}$$

$$\text{Expected population} = 16 \times 21 \times 21 \times 21 = 1{,}48{,}176$$

EXAMPLE-14. Kishore bought a bike at Rs. 55,000. If the bike depreciates at 8% annually than what would be the value of the bike after two years?

SOLUTION : Here the price of the bike = Rs.55,000; Rate of depreciation = 8% annually

Time = 2 Years; Expected Price = ?

$$\because \text{ Expected price} = \text{Current price} \times \left(1-\frac{\text{Rate of Depreciation}}{100}\right)^{time}$$

$$= 55,000\left(1-\frac{8}{100}\right)^2 = 55,000\left(1-\frac{2}{25}\right)^2$$

$$= 55,000\times\left(\frac{23}{25}\right)^2 = 55,000\times\frac{23}{25}\times\frac{23}{25}$$

$$= 88\times 23\times 23 = \text{Rs. } 46,552/-$$

EXAMPLE-15. A car is priced at Rs.450,000. The car depreciates at 4% annually for the first two years and 10% annually for the next two years. What is the price of the car after four years?

SOLUTION : Here the price of the car = 4,50,000/-

Rate of depreciation = 4% annually

Time (t_1) = 2 years

And then Rate of depreciation = 10% annually

Time (t_2) = 2 years

Desired Value = ?

$$\because \text{ Desired Value} = \text{Original price} \times \left(1-\frac{R_1}{100}\right)^{t_1} \times \left(1-\frac{R_2}{100}\right)^{t_2}$$

$$= 4,50,000 \times \left(1-\frac{4}{100}\right)^2 \times \left(1-\frac{10}{100}\right)^2$$

$$= 4,50,000\times\left(\frac{24}{25}\right)^2\times\left(\frac{9}{10}\right)^2$$

$$= 4,50,000\times\frac{24}{25}\times\frac{24}{25}\times\frac{9}{10}\times\frac{9}{10}$$

$$= 7.2\times 24\times 24\times 9\times 9$$

= Rs. 3,35,923

Try This

1. The present population of the village is 8000. If the population is decreasing at the annual rate of 5%. What would be the population after 3 years?
2. The population of the town is 9600. If the rate of increase of the population is 15% annually, then what would be the population of the town after 2 years?

Instalment Scheme (Purchasing in Instalments)

You would have come across many advertisements of this kind-

"Buy the house of your dreams at minimum interest rate with easy instalments" or "Pay only Rs. 30,000 and take the car of your choice to your house, the rest in easy instalments or buy TV or fridge in Rs 1000 and the rest of the payment in easy monthly/ half yearly/ annual instalments"

Under such schemes people can buy costly items without full payment therefore these schemes enable people to buy costly items comfortably.

In instalment schemes, the customer while purchasing the item pays a part amount on the spot. This is called initial down payment. They subsequently sign a contract takes the item in use. They pay the remaining amount in instalments. This is called the instalment amount. The monthly, quarterly, half yearly or annual instalments are fixed with the consent of the customer and the shopkeeper.

Let us understand the instalment scheme through an example-

Finding the rate of interest of an instalment scheme:-

In the instalment scheme only a part of the total cost is paid by the customer at the time of purchase. The rest is paid in instalments and for this the seller takes some extra amount from the customer. This extra amount is the interest.

EXAMPLE-16. The cash down price of a table is Rs.1000/-. Ramesh buys it for a down payment of Rs.400 and monthly instalments of Rs.310 each. Find the rate of interest in the scheme.

SOLUTION : The cash price of table = Rs. 1000/-

The cash down payment = Rs. 400/-

The due amount for instalments= Rs.1000 - Rs.400 = Rs.600

Suppose that the rate of interest for the instalment scheme is r% annually, then the total amount of Rs.600= Principal+ Interest

$$= 600 + \frac{600 \times r \times 2}{100 \times 12}$$

$$= 600 + r \qquad .....(1)$$

The total amount after two months of the monthly instalment of Rs.310 paid after the first month

$$= 310 + \frac{310 \times r \times 1}{100 \times 12}$$

$$= 310 + \frac{31r}{120} \qquad .....(2)$$

The total amount of Rs.310 given after two months=Rs 310

Therefore combining the two monthly instalments at the end of two months

$$\text{Total amount } = 310 + \frac{31r}{120} + 310 = 620 + \frac{31r}{120}$$

Using both (1) and (2)

$$600 + r = 620 + \frac{31r}{120}$$

$$r - \frac{31r}{120} = 620 - 600$$

$$\frac{120r - 31r}{120} = 20$$

$$89r = 20 \times 120$$

$$r = \frac{2,400}{89} = 26.97\% \text{ (approximately)}$$

Therefore, the annual rate of interest in the instalment scheme=26.97%

Using another method:

The cash price of table = Rs.1000

Down cash payment = Rs.400

The amount due in instalments = Rs. 1000 – Rs. 400

= Rs.600

Number of equal instalments = 2

The total amount paid in instalments = 2 × 310 = Rs. 620

The total interest paid = 620 – 600 = Rs. 20

The principal amount for the first month = Rs. 1000 – Rs. 400 = Rs. 600

The principal amount for the second month = 600 – 310 = Rs. 290

Total principal amount (for one month) = 600 + 290 = Rs. 890

Suppose the annual rate of interest is r%, then

$$\text{Total interest} = \frac{890 \times r \times 1}{100 \times 12} = 20$$

$$\text{Or } r = \frac{20 \times 100 \times 12}{890}$$

$$r = \frac{2,400}{89} = 26.97\% \text{ (approximately)}$$

Finding the Instalment Amount

Let us now try to understand how the seller calculates the instalment amount. The shopkeeper buys the product at some price. He knows that using the instalment scheme more items can be sold. Therefore, to get an appropriate interest rate he wants to decide the cash down amount, the instalment amount and the number of instalments.

EXAMPLE-17. The price of a television set is Rs.12,000. This is sold at Rs.3,000 down payment and two equal monthly instalments. If the rate of annual interest is 18% then find the amount of each instalment.

SOLUTION :

The cash price of television	= Rs. 12,000
Cash down paid	= Rs. 3,000
Payment to be made in instalments	= Rs. (12000 – 3000) = 9000
Rate of Interest	= 18% Annual
Let each instalment be	= Rs. x

$$\text{Interest} = \frac{\text{Principal} \times \text{Rate} \times \text{Time}}{100}$$

The amount of one month interest on Rs. 9000 = $\dfrac{9000 \times 18 \times \frac{1}{12}}{100}$ = Rs. 135

The principal amount after one month = 9000 + 135 = Rs. 9135

The amount to be paid after paying the first instalment = Rs. $(9135 - x)$

The total amount of this amount would be the due instalment after one month.

Hence the interest on Rs. $(9135 - x)$ for one month $= \dfrac{(9135 - x) \times 18 \times \frac{1}{12}}{100}$

$$= \frac{3(9135 - x)}{200} \text{ Rs.}$$

The total amount after one month = Principal + Interest

$$= (9135 - x) + \frac{3(9135 - x)}{200}$$

$$= \frac{200(9135 - x) + 3(9135 - x)}{200}$$

$$= \frac{1827000 - 200x + 27405 - 3x}{200}$$

Hence $$= \frac{1827000 - 200x + 27405 - 3x}{200} = x$$

$1854405 - 203x = 200x$

Or $184405 = 200x + 203x$

$403x = 1854405$

Hence, $x = \dfrac{1854405}{403}$ = Rs. 4601.50

Hence, the amount of each instalment is Rs. 4,601.50

Finding the Cash Price

If for an instalment scheme the amount of each equal instalment, number of instalments, rate of interest and cash down payment amount is given then we can find the cash price of the item. Let us understand this through an example.

EXAMPLE-18. A bicycle is available for Rs.500 cash down and two annual equal instalments of Rs.1210 each. If the rate of interest is 10 % annually then find the cash price of the bicycle.

SOLUTION : Suppose the cash price of the bicycle is Rs. x.

Cash down payhment = Rs. 500

First Instalments = Rs. 1210, Rate of Interest = 10%

The amount due after cash down payment = Rs. $(x - 500)$

The interest for one year on $(x - 500) = \dfrac{\text{Principal} \times \text{Rate} \times \text{Time}}{100}$

$$= \frac{(x-500)\times 10 \times 1}{100} = \frac{(x-500)}{10} \text{ Rs.}$$

The total amount after one year = Principal + Interest

$$= (x - 500) + \frac{x-500}{10}$$

$$= (x - 500)\left(1 + \frac{1}{10}\right) = (x - 500)\frac{11}{10}$$

The amount due after the first instalment (the principal amount for the second year)

$$= [(x - 500)\frac{11}{10} - 1210]$$

The total amount at the end of second year = Principal $\left\{1+\frac{\text{Rate}}{100}\right\}^{\text{Time}}$

$$= [(x - 500)\frac{11}{10} - 1210]\,[1 + \frac{10}{100}]^1$$

$$= [(x - 500)\frac{11}{10} - 1210]\frac{11}{10}$$

The total amount at the end of second year is equal to the due instalment for the second year.

$$[(x - 500)\frac{11}{10} - 1210]\frac{11}{10} = 1210$$

$$[(x - 500)\frac{11}{10} - 1210]\,11 = 12100$$

Or $\quad [(x - 500)\frac{11}{10} - 1210] = 1100$

Or $\quad (x - 500)\frac{11}{10} = 1100 + 1210$

Or $\quad (x - 500) = 2310 \times \frac{10}{11}$

Or $\quad x - 500 = 2100$

$x = \text{Rs. } 2600$

Hence, the cash price of the bicycle is Rs. 2,600/-

Problems Involving Compound Interest

In a monthly instalment scheme when the total time is less than one year simple interest is used. However sometimes the seller charges compound interest even for less than one year. In such cases interest is compounded half yearly or quarterly.

Sometimes instalments are given for more than one year, in such cases also compound interest is used.

EXAMPLE-19. A fridge with a cash price of Rs.15,000, is available for Rs 2,250 cash down and two equal half yearly instalments at 8% rate of annual interest. If interest is compounded every six months then find the amount of each instalment.

SOLUTION : The cash price of the Fridge = Rs. 15,000

Cash down payment = Rs. 2,250

Remaining payable amount = Rs. (15,000 – 2,250)

= Rs. 12,750

Rate of Interest = 8% annually = 4% half yearly

Let the amount of each half yearly instalment be Rs. x, and P_1 and P_2 the principal amounts for the first and second term.

$$\therefore \quad x = P_1\left(1+\frac{4}{100}\right)^1 \text{ and } x = P_2\left(1+\frac{4}{100}\right)^2$$

$$x = P_1\left(\frac{26}{25}\right) \text{ and } x = P_2\left(\frac{26}{25}\right)^2$$

$$P_1 = \left(\frac{25}{26}\right)x \text{ and } P_2 = \left(\frac{25}{26}\right)^2 x$$

Hence, $$12,750 = \left(\frac{25}{26}\right)x + \left(\frac{25}{26}\right)^2 x$$

$$12,750 = \left(\frac{25}{26}\right)x\left[1+\frac{25}{26}\right]$$

$$12,750 = \left(\frac{25}{26}\right)x\left(\frac{51}{26}\right)$$

$$x = \frac{12,750 \times 26 \times 26}{25 \times 51} = \text{Rs. } 6760/-$$

The amount of each instalment = Rs. 6,760/-

Exercise - 7.3

1. A chair is sold for cash price of Rs. 450 or a cash down payment of Rs.210 and two equal monthly instalments of Rs.125 each. Find out the rate of interest charged in the scheme.
2. A TV stand is sold for Rs.3,000 cash or cash down payment of Rs.600 with two equal monthly instalments of Rs.1,250. Find out the rate of interest charged in the scheme.
3. The cash price of a fan is Rs.1,940. In the instalment scheme it is available for a cash down payment of Rs 620 and two equal monthly instalments. If the rate of annual rate of interest charged in under the scheme is 16% then find the amount of each instalment.

4. The cash price of a microwave oven is Rs. 20,100. if this is available for Rs.3700 cash down payment with two equal monthly instalments at 10% annual rate of interest then find the amount of each instalment.

5. An Iron is bought for cash down payment of Rs. 210 and two equal monthly instalments of Rs.220 each. If the rate of interest is 20% annually then find the cash price of the Iron.

6. The cash price of a scooter in a showroom is Rs.20,000. Under an instalment scheme this is available for cash down payment of Rs.11,000 with two equal annual instalments at 25% annual rate of interest. If the interest is compounded every year then find the amount of each instalment.

7. Washing machine is available for Rs 12,000 cash payment or after a cash down payment of Rs.3600, two equal half yearly instalments. If the annual rate of interest is 20% and the interest is compounded every six months, then find the amount of instalment.

8. A sewing machine is available for Rs 3,000 cash or a cash down payment of Rs 450 and two equal half yearly instalments. If the compound interest is 4% annually then what would be the amount of each instalment.

What Have We Learnt

1. We compare quantities on the basis of ratios and percentages etc.
2. Percentages are used to calculate the amount of discount, sales tax and interest.
3. The reduction in the marked price is called discount.
4. The government charges sales tax on any item sold. The seller adds the sales tax amount to the bill amount.
5. The interest charged is of two types, simple interest and compound interest.
6. (i) If the interest is annually compounded then $A = P\left(1+\frac{R}{100}\right)^{t}$.

   (ii) When the interest is compounded half yearly then time periods are number of half years and the rate of interest is halved then total amount $A = P\left(1+\frac{R}{200}\right)^{2t}$.

   (iii) When interest is compounded quarterly, the time period are number of quarters and rate is the fourth the annual rate. So $A = P\left(1+\frac{R}{400}\right)^{4t}$.

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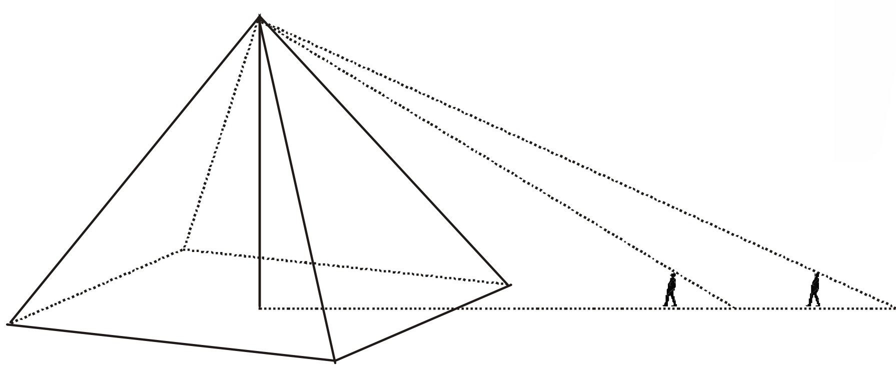
Come, let us know the history of Trigonometry.....

Trigonometry was developed in order to satisfy the different day to day needs of humans- For example, in India in order to study the speed and position of the celestial bodies in the fields of Astronomy and Astrology; in Greece to study the relation between circles and chords in the field of Astronomy and in Egypt to know the height of Pyramids. When these ideas got together, trigonometry was developed. Although Trigonometry means measurement related to triangles, the knowledge is used extensively in relation to angles. In all of these, triangles and the imaginary right angled triangles are identified and its ratio of sides is used to find height, distance, speed, state etc.

It is believed that the first list of trigonometric ratios was given by Heparcus in second centry BC in India. In the field of astronomy, by regular use, five principles were assigned indicating the relation between angles and chord. The most important amongst these was the sun principle which defines 'Sine' used today. In the fifth century Aryabhatt took this further and used Jya (Sine) and Kojya (Cosine). In the 7^{th} Century, Bhaskaracharaya I, gave the formula for calculating Sin x, so for each x you can obtain Sin x, with an error of less than two percent. Later again in the seventh centry, Brahmagupta used angles like $(\circledR - x)$, $\left(\frac{\circledR}{2} - x\right)$ and established a relation between Sine x and Cosine $\left(\frac{\circledR}{2} - x\right)$. He also established a relation between Sine and Cosine of the sum of two angles with the Sine and Cosine of the individual angles like Sin (A+B) = SinAcosB + cosAsinB etc. Along with these two, in the twelfth century Bhaskaracharaya has mentioned the use of tangets in his book 'Goladhyay'.

In the fourteenth century Mahadev analysed several trigonometric functions and their infinite series and came up with some important expansion formulae, which were discovered much later in the western world. These are used even today.



This information has been collected from different books and been presented here. Teachers and children are free to obtain information on Trigonometry from other sources as well.

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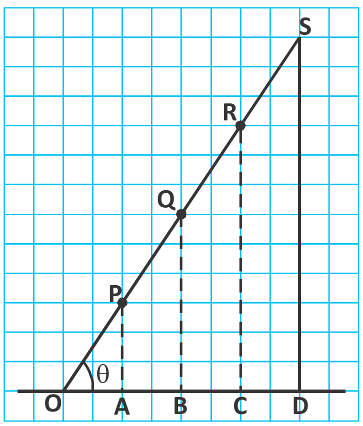
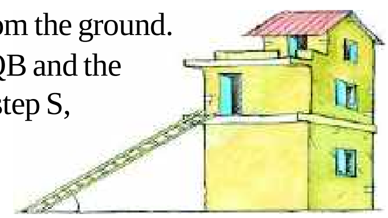


Fig. 1

With every step that we climb on the stair case, we go higher from the ground (*Fig*.1)

The height of first step P is PA from the ground. Similary the second step Q is at height QB and the third step R is at a height RC, the fourth step S, is at SD.



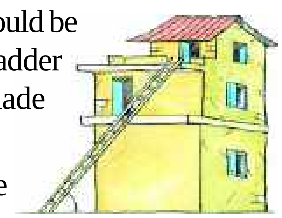
On each step we not only go higher from the ground but we also go closer to the building.

Do we go the same distance closer as we go higher? Is there a relation between these two?

Here $\dfrac{PA}{OA} = \dfrac{QB}{OB} = \dfrac{RC}{OC} = \dfrac{SD}{OD}$

Thus the ratio of the height we climb, to the distance we move ahead, is the same.

If the height to which we need to climb is a bit more what would be required, if the ladder was the same? We would have to move the ladder closer to the base of the house (*Fig*.2), thus increasing the angle made by the ladder with the ground.



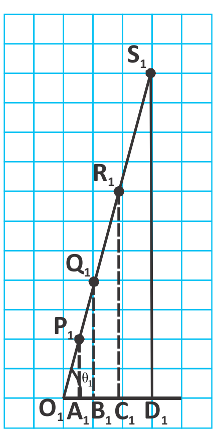


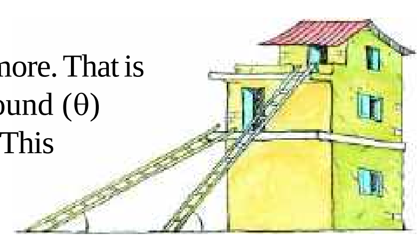
Fig. 2

Now count the squares and say whether the ratio of the distance to height is the same?

$$\frac{P_1A_1}{O_1A_1} = \frac{Q_1B_1}{O_1B_1} = \frac{R_1C_1}{O_1C_1} = \frac{S_1D_1}{O_1D_1}$$

We see that it is indeed the same.

Though the value of the ratio in second case is more. That is when the angle made by the ladder with the ground (θ) increased, the ratio of height to distance increased. This ratio is known as the tangent of this angle (θ).



It means tangent $\theta = \dfrac{PA}{OA} = \dfrac{QB}{OB}$

and tangent $\theta_1 = \dfrac{P_1A_1}{O_1A_1} = \dfrac{Q_1B_1}{O_1B_1}$

Other Ratios:- If we look at this movement of going higher and towards the wall in form of a line diagram, then it is as if we are making a right angled triangles from every point representings a step as a vertex.

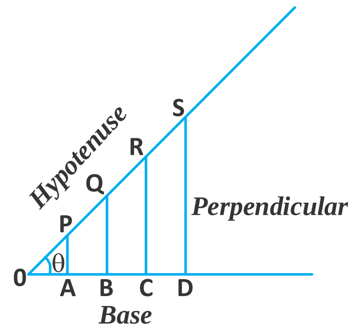


Fig. 3

If we denote the angle made by the ladder with the ground as θ, the height to which we climb is the perpendicular and distance from the wall is the base and the ladder would represent the hypotenuse.

We have said above tangent θ is $\dfrac{PA}{OA}$, which in the form of perpendicular and base

tanθ would be = $\dfrac{\text{Perpendicular}}{\text{Base}}$. The value of which is the same in all the triangles of *Fig*.3. It remains the same for the same value of θ.

In short this is called tan θ.

$$\tan\theta = \frac{PA}{OA} = \frac{QB}{OB} = \frac{RC}{OC}$$

Will there be any other ratios we get from these Line Drawings

Can we get some other fixed ratios from these values of sides in the given right angled triangle? Let us see the ratio of perpendicular to hypotensuse:-

$$\text{Ratio} = \frac{PA}{OP}, \frac{QB}{OQ}, \frac{RC}{OR}$$

Also let us see the ratio of base to hypotenuse.

$$\frac{OA}{OP}, \frac{OB}{OQ}, \frac{OC}{OR}$$

Check if these ratios are the same.

The ratio of the perpendicular to hypotenuse for a fixed angle is known as the sineθ (in short sinθ).

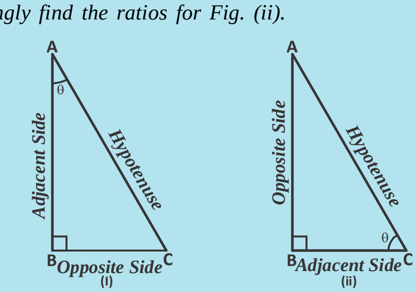
Hence $\sin\theta = \dfrac{PA}{OP} = \dfrac{QB}{OQ} = \dfrac{RC}{OR}$

Here in right angled △ABC, ∠B = 90° and ∠A = θ (in Fig. (i)) then side opposite to θ is BC and side AB is the adjacent side. Also AC is the hypotenuse.

Similarly in right angled △ABC (in Fig. (ii)) ∠B = 90°, ∠C = θ then the opposite side to θ is AB and the adjacent side is BC, AC is the hypotenuse.

in Fig. (i) $\sin\theta = \dfrac{\text{Opposite side}}{\text{Hypotenuse}} = \dfrac{BC}{AC}, \cos\theta = \dfrac{AB}{AC}, \tan\theta = \dfrac{BC}{AB}$

Accordingly find the ratios for Fig. (ii).



Similarly the ratio of base to hypotenuse is known as cosineθ (in short cosθ).

$$\cos\theta = \frac{OA}{OP} = \frac{OB}{OQ} = \frac{OC}{OR}$$

(sinθ, cosθ, tanθ etc. are known as trigonometric ratios)

Exercise - 8.1

If in a right angled triangle ABC, ∠B is a right angle; then find the value of sin A, cos C and tan A.

Given:-

(i)	AC = 5	AB = 3	BC = 4
(ii)	AB = 12	BC = 5	AC = 13
(iii)	AB = 5	AC = 13	BC = 12
(iv)	BC = 12	AB = 9	AC = 15

Relationship between the Ratios

Relation between sin¯ , cos¯ and tan¯ : If in right angled triangle ABC, ∠B is right angle and if ∠C = θ, then:-

$$\tan\theta = \frac{AB}{BC}$$

$$= \frac{AB}{AC} \times \frac{AC}{BC}$$

$$= \frac{AB}{AC} \div \frac{BC}{AC}$$

$$= \sin\theta \div \cos\theta$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

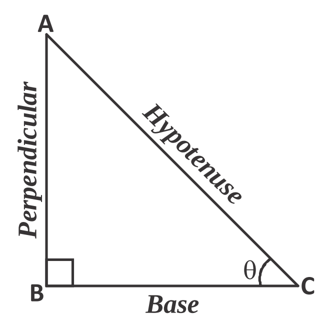


Fig. 4

Some other Trigonometric Ratios

We saw that in a right angled triangle ABC, right angled at B and ∠C = θ:-

$$\frac{\text{Perpendicular}}{\text{Hypotenuse}} = \sin\theta, \qquad \frac{\text{Base}}{\text{Hypotenuse}} = \cos\theta, \qquad \frac{\text{Perpendicular}}{\text{Base}} = \tan\theta$$

The reciprocals of these three give three more ratios whose names are:-

$$\frac{\text{Hypotenuse}}{\text{Perpendicular}} = \text{cosecant}\theta \ (\text{or cosec}\theta) = \frac{1}{\sin\theta}$$

$$\frac{\text{Hypotenuse}}{\text{Base}} = \text{secant}\theta \ (\text{or sec}\theta) = \frac{1}{\cos\theta}$$

$$\frac{\text{Base}}{\text{Perpendicular}} = \text{cotanget}\theta \ (\text{or cot}\theta) = \frac{1}{\tan\theta}$$

Try This

If $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, then can you write the ratio $\cot\theta$ in terms of $\sin\theta$ and $\cos\theta$?

Trigonometric Ratio and Pythagoras Theorem

The concept of all trigonometric ratios can be understood by using a right angled triangle. Pythagoras theorem also gives a relation between the sides of a right angled triangle. Let us use this to find a relation between the trigonometric ratios.

$a^2 = a \times a$
$b^2 = b \times b$

Let the lengths of the two sides of a right angled traingle be a and b and the length of hypotenuse be c, then by Pythogoras theorem, the relation between a, b and c is:-

$$a^2 + b^2 = c^2 \qquad (\text{Perpendicular}^2 + \text{Base}^2 = \text{Hypotenuse}^2) \qquad \ldots(1)$$

$\sin^2\theta = \sin\theta \times \sin\theta$

Now if the hypotenuse with length c makes an angle θ with the base of length b then,

$$\sin\theta = \frac{a}{c} \text{ and } \cos\theta = \frac{b}{c}$$

Squaring and adding the two gives us:-

$$\sin^2\theta + \cos^2\theta = \frac{a^2}{c^2} + \frac{b^2}{c^2}$$

$$\sin^2\theta + \cos^2\theta = \frac{a^2 + b^2}{c^2}$$

$$\sin^2\theta + \cos^2\theta = \frac{c^2}{c^2} \qquad [\because a^2+b^2=c^2]$$

$$\sin^2\theta + \cos^2\theta = 1$$

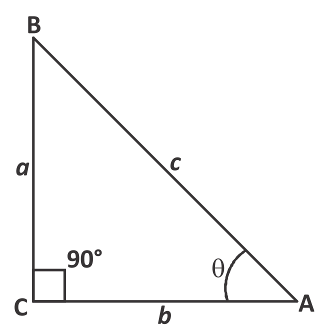


Fig. 5

or we can write it as:-

$$\sin^2\theta = 1 - \cos^2\theta \qquad \text{or} \qquad \cos^2\theta = 1 - \sin^2\theta$$

The above three statements of the relation between $\sin^2\theta$ and $\cos^2\theta$ are in the form of equations. These hold ture for all values of θ from 0° to 90°. In right angled triangles, these are known as trigonometric identities for an angle θ $(0 \le \theta \le 90°)$.

Some other identities exist which give a relation between $\tan^2\theta$ and $\sec^2\theta$ as also between $\cot^2\theta$ and $\text{cosec}^2\theta$. These can be obtained as follows, observe and understand:-

Identity-1 $\quad \sin^2\theta + \cos^2\theta = 1$

Dividing throughout by $\sin^2\theta$ we get,

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \cot^2\theta = \text{cosec}^2\theta \ \textbf{(Identity-2)} \ \left(\because \frac{\cos\theta}{\sin\theta} = \cot\theta\right)$$

Again dividing identity-1 throughout by $\cos^2\theta$ we get:-

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta \qquad \text{or} \qquad 1 + \tan^2\theta = \sec^2\theta \ \textbf{(Identity-3)}$$

Try This

Like Identity-1, write Identity-2 and Identity-3 in different ways.

To find Trigonometric Ratios

We have seen that all six trigonometric ratios are related to each other. We have also seen that if we know the value of one trigonometric ratio, then we can obtain the ratio of the sides of any right angled triangle with the same angle θ.

We can do this using the Pythogras Theorem. Using one trigonometric ratio, we can find all the remaining ratios.

EXAMPLE-1. ΔPQR is a right angled triangle in which $\angle Q$ is the right angle and $\angle R = \theta°$.

Given $\sin\theta = \frac{3}{5}$. Can we find the remaining five ratios?

SOLUTION : $\because \sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{PQ}{PR} = \frac{3}{5}$

We can write this as $\sin\theta = \dfrac{3x}{5x}$ (As the ratio of $3x$ to $5x$ is 3 : 5)

Then we can say $PQ = 3x$, $PR = 5x$ ......(1)

In right angled PQR, $\text{Hypotenuse}^2 = \text{Perpendicular}^2 + \text{Base}^2$

$(5x)^2 = (3x)^2 + \text{Base}^2$

$25x^2 - 9x^2 + \text{Base}^2$

$16x^2 = \text{Base}^2$

$(4x)^2 = \text{Base}^2$

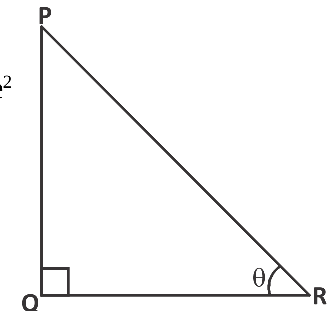


Fig. 6

(Finding a square root on both sides we get)

$\therefore$ Base (QR) $= 4x$

Now $\cos\theta = \dfrac{\text{Base}}{\text{Hypotenuse}} = \dfrac{4x}{5x} = \dfrac{4}{5}$

Similarly we can find the other trigonometric ratios.

EXAMPLE-2. If $\sin\theta = \dfrac{5}{13}$, find the remaining trigonometric ratios.

SOLUTION : Given $\sin\theta = \dfrac{5}{13}$ ..... (1)

Using this how do we find value of $\cos\theta$

We know-

$$\sin^2\theta + \cos^2\theta = 1$$

To find $\cos\theta$ we rewrite this as

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos^2\theta = 1 - \left(\frac{5}{13}\right)^2 \qquad \left[\text{Given } \sin\theta = \frac{5}{13}\right]$$

$$\cos^2\theta = 1 - \frac{25}{169} = \frac{169 - 25}{169} = \frac{144}{169}$$

$$\cos^2\theta = \left(\frac{12}{13}\right)^2$$

$$\therefore \quad \cos\theta = \frac{12}{13} \qquad .....(2)$$

Now we know sinθ and cosθ, so lets find value of tanθ.

We know that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ or $\sin\theta \div \cos\theta$

$$\therefore \tan\theta = \frac{5}{13} \div \frac{12}{13}$$

$$= \frac{5}{13} \times \frac{13}{12}$$

$$\tan\theta = \frac{5}{12}$$

Now we can find the other ratios secθ, cosecθ and cotθ

Since we know $\sec\theta = \dfrac{1}{\cos\theta}$, $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$, $\cot\theta = \dfrac{1}{\tan\theta}$

So $\quad \sec\theta = \dfrac{1}{\cos\theta} = \dfrac{1}{12/13} = \dfrac{13}{12}$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{5/13} = \frac{13}{5}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{5/12} = \frac{12}{5}$$

EXAMPLE-3. If $\sec A = \dfrac{5}{3}$, then find the other trigonometric ratios for ∠A.

SOLUTION : We are given $\sec A = \dfrac{5}{3}$ ..... (1)

(i) But as $\sec A = \dfrac{1}{\cos A}$ (secA is the reciprocal of cosA)

$$\therefore \cos A = \frac{1}{5/3} = \frac{3}{5}.$$

(ii) Using identity, we will find value of sinA

$$\sin^2 A = 1 - \cos^2 A$$

$$= 1 - \left(\frac{3}{5}\right)^2 \quad = 1 - \frac{9}{25}$$

$$= \frac{25-9}{25} = \frac{16}{25}$$

$$\sin^2 A = \left(\frac{4}{5}\right)^2$$

$$\sin A = \frac{4}{5}$$

(iii) As $\tan A = \frac{\sin A}{\cos A}$ or $\sin A \div \cos A$

So, $\tan A = \frac{4}{5} \div \frac{3}{5}$

$$= \frac{4}{5} \times \frac{5}{3} = \frac{4}{3}$$

$$\therefore \quad \tan A = \frac{4}{3}$$

(iv) The reciprocal of tanA is cotA

Hence $\cot A = \frac{1}{\tan A} = \frac{1}{4/3} = \frac{3}{4}$

(v) $\because \ \operatorname{cosec} A = \frac{1}{\sin A} = \frac{1}{4/5} = \frac{5}{4}$

Hence $\operatorname{cosec} A = \frac{5}{4}$

EXAMPLE-4. If $5\tan\theta = 4$, then find value of $\dfrac{5\sin\theta - 3\cos\theta}{\sin\theta + 2\cos\theta}$

SOLUTION : $5\tan\theta = 4$

So, $\tan\theta = \dfrac{4}{5}$

Now, $\dfrac{5\sin\theta - 3\cos\theta}{\sin\theta + 2\cos\theta}$

$$= \frac{5\dfrac{\sin\theta}{\cos\theta} - \dfrac{3\cos\theta}{\cos\theta}}{\dfrac{\sin\theta}{\cos\theta} + 2\dfrac{\cos\theta}{\cos\theta}}$$ (Dividing both numerator and denominator by $\cos\theta$)

$$= \frac{5\tan\theta - 3}{\tan\theta + 2}$$ $\left(\because \dfrac{\sin\theta}{\cos\theta} = \tan\theta\right)$

$$= \frac{5\left(\dfrac{4}{5}\right) - 3}{\left(\dfrac{4}{5}\right) + 2}$$ $\left(\because \tan\theta = \dfrac{4}{5}\right)$

$$= \frac{4-3}{(4+10)/5} = \frac{1}{14/5}$$

$$= \frac{5}{14}$$

EXAMPLE-5. If $\tan\theta = 1$ in a right angled ΔABC right angled at B, prove that $2\sin\theta\cos\theta = 1$

SOLUTION : In ΔABC, $\tan\theta = \dfrac{BC}{AB} = 1$

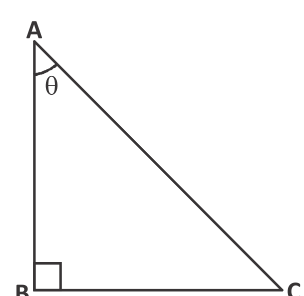


Fig. 7

Or $BC = AB$

Say $AB = BC = k$ (k is some positive number)

Now $AC = \sqrt{(AB)^2 + (BC)^2} = \sqrt{k^2 + k^2} = k\sqrt{2}$

Hence, $\sin\theta = \dfrac{BC}{AC} = \dfrac{1}{\sqrt{2}}$ and $\cos\theta = \dfrac{AB}{AC} = \dfrac{1}{\sqrt{2}}$

So $2\sin\theta\cos\theta = 2\left(\dfrac{1}{\sqrt{2}}\right)\left(\dfrac{1}{\sqrt{2}}\right) = 1 \quad \left(\because \sqrt{2}\times\sqrt{2} = 2\right)$

Or $2\sin\theta\cos\theta = 1$

Exercise - 8.2

1. One of the trigonometric ratios is given below. Find the remaining trigonometric ratios:-

(i) $\tan\theta = \dfrac{3}{4}$ (ii) $\sin\theta = \dfrac{5}{13}$ (iii) $\cos\alpha = \dfrac{1}{3}$

(iv) $\cot\theta = 1$ (v) $\text{cosec}A = \dfrac{5}{4}$ (vi) $\sec\beta = 2$

(vii) $\text{cosec}A = \sqrt{10}$

2. If $\cot\theta = \dfrac{21}{20}$, then find the value of $\sin\theta \times \cos\theta$.

3. If $\cos A = \dfrac{4}{5}$, then find the value of $\dfrac{\cot A - \sin A}{2\tan A}$.

4. If $\sec\theta = \dfrac{5}{3}$, then find the value of $\dfrac{\tan\theta - \sin\theta}{1 + \tan\theta.\sin\theta}$.

5. If $\sin A = \dfrac{1}{3}$, then find the value of $\cos A$, $\text{cosec}A + \tan A$ and $\sec A$.

6. In a right angled ΔABC, right angle is at $\angle C$ and $\tan A = \dfrac{1}{\sqrt{3}}$, then find the value of $\sin A\cos B + \cos A\sin B$.

7. If $\cot A = \dfrac{3}{4}$, then find the value of $\dfrac{\sin A + \cos A}{\sin A - \cos A}$.

8. If $\sin\theta = \dfrac{4}{5}$, then find the value of $\dfrac{4\tan\theta - 5\cos\theta}{\sec\theta + 4\cot\theta}$.

Trigonometric Ratios for Some Special Angles

We can find the trigonometric ratios for $\theta = 0°, 30°, 45°, 60°$ or $90°$ in a right angled triangle using geometry. Let us see how:-

Trigonometric Ratios of 45°

Triangle ABC is right angled triangle with right angle at $\angle B$ and $\angle C = 45°$

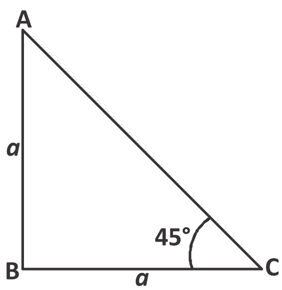


Fig. 8

Clearly $\angle A$ will also be 45°

So if BC = a then

AB = a (Why?)

(In a triangle, sides opposite equal angles are equal)

Now $AC^2 = AB^2 + BC^2$ (By Pythagoras Theorem)

$= a^2 + a^2 = 2a^2$

$AC = a\sqrt{2}$

$\because$ For $\angle C$ (45°), BC is base, AB is perpendicular and AC is the hypotenuse.

$$\therefore \quad \sin C = \sin 45° = \frac{AB}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\cos 45° = \frac{BC}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\tan 45° = \frac{AB}{BC} = \frac{a}{a} = 1$$

$$\cot 45° = \frac{1}{\tan 45°} = \frac{1}{1} = 1$$

$$\sec 45° = \frac{1}{\cos 45°} = \frac{1}{1/\sqrt{2}} = \sqrt{2}$$

$$\operatorname{cosec} 45° = \frac{1}{\sin 45°} = \frac{1}{1/\sqrt{2}} = \sqrt{2}$$

Trigonometric Ratios of 30°

Consider an equilateral triangle ABD whose each side is of length $2a$ and each angles is 60°.

Draw a perpendicular from B to AD, which meets AD in C

$\therefore$ AC = CD = a (Why?)

$\angle ABC = \angle DBC = 30°$ (Why?)

(A perpendicular drawn from the vertex of an equilateral triangle bisects the opposite side and also the angle at this vertex)

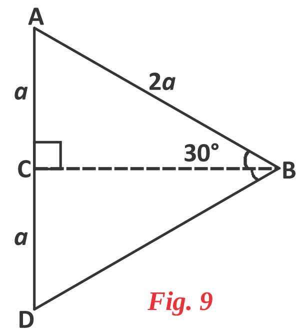


Fig. 9

Now ΔACB is right angled with a right angle at C.

$\angle ABC = 30°$ and the base for this angle is BC, AC is the perpendicular and AB is the hypotenuse.

$$BC^2 = AB^2 - AC^2 \qquad \left(\text{from } BC^2 + AC^2 = AB^2\right)$$
$$= (2a)^2 - (a)^2 = 4a^2 - a^2$$
$$= 3a^2 = a^2.3$$
$$BC = a.\sqrt{3}$$

Now we have the lengths of AB, BC and AC.

You can write the trigonometric ratios for 30° in your copies. Compare your answers with your friends.

Trigonometric Ratios for 60°

ΔABC has $\angle A = 60°$

For this angle, the perpendicular BC $\left(= a\sqrt{3}\right)$ and base AC (= a), whereas hypotenuse AB (= $2a$).

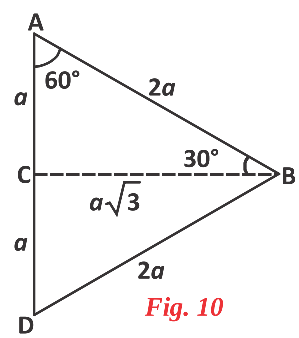


Fig. 10

$$\sin 60° = \frac{BC}{AB} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{AC}{AB} = \frac{a}{2a} = \frac{1}{2}$$

$$\tan 60° = \frac{BC}{AC} = \frac{a\sqrt{3}}{a} = \sqrt{3}$$

Find the remaining ratios with the help of your friends.

Trigonometric Ratios for 0°

To find trigonometric ratios for 0° angle, we will have to think about a right angled triangle with one angle 0°. Do you think that such a triangle is possible? (Discuss this with your friends)

We shall think about how in a right angled triangle, if one of its acute angles is continuously reduced, will affect the lengths of the sides.

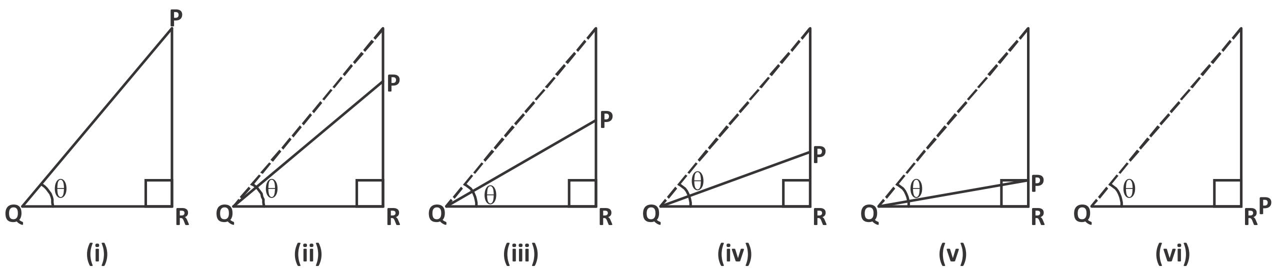


Fig. 11

ΔPQR is right angled triangle. $\angle$PQR is the angle which has to be reduced till it become 0°. The figures (i) to (vi) shows the continuous reduction of the angle θ.

As the angle reduces, what changes can be observed in the perpendicular PR?

Is the hypotenuse QP also changing?

We can see that as θ is reduced, PR is also getting smaller. So when θ is going towards 0º, PR is also going towards 0

Hence when θ = 0, perpendicular PR = 0.

Along with this QP is also getting smaller and almost equal to the base QR.

Hence for θ = 0, base QR = hypotenuse QP

$$\therefore \sin 0° = \frac{PR}{QP} = \frac{0}{QP} = 0$$

$$\cos 0° = \frac{QR}{QP} = 1 \quad (\because QR = QP \text{ given})$$

$$\tan 0° = \frac{PR}{QR} = \frac{0}{QR} = 0$$

$$\cot 0° = \frac{QR}{PR} = \frac{QR}{0} = \text{Not defined}$$ (in a rational number, if denominator is zero, the number is undefined)

$$\sec 0° = \frac{QP}{QR} = 1$$

$$\text{cosec } 0° = \frac{QP}{PR} = \frac{QP}{0} = \text{Not defined}$$

Trigonometric Ratios for 90°

To get trigonometric ratios for 90°, we have to take a right angled triangle and see what happens as we increase θ to 90°.

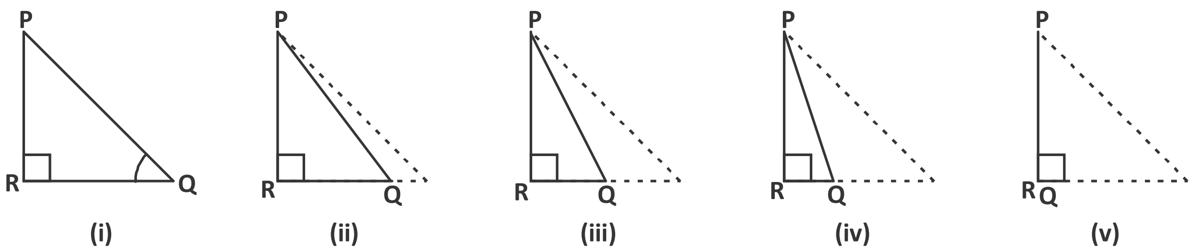


Fig. 12

PQR is a right angled triangle in which ∠PQR is increased till it becomes 90°. in *Fig.*12, you can see waht happens as we continuously increase the ∠Q till it becomes 90°.

As we increase ∠Q, can you see a change is the base QR?

Is there any change in hypotenuse PQ?

As you can see, as we increase the value of Q, the base QR gets reduced and when ∠Q = 90°, then QR = 0. Also the length of the hypotenuse reduces and becomes almost equal to perpendicular PR.

Hence when ∠Q = 90°, hypotenuse PQ = perpendicular PR and base QR = 0

So $$\text{Sin } 90° = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{PR}{PQ} = 1$$

$$\text{Cos } 90° = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{QR}{PQ} = \frac{0}{1} = 0$$

$$\tan 90° = \frac{\text{Perpendicular}}{\text{Base}} = \frac{PR}{QR} = \frac{PR}{0} = \text{Not defined}$$

Similarly we can find the other ratios.

The Trigonometric Ratios of the Special Angles

TABLE - 1

Angles/ Ratio	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
$\cot\theta$	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
$\operatorname{cosec}\theta$	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

EXAMPLE-6. Find the value of

$\cos 60° \ \cos 30° + \sin 60° \ \sin 30°$

SOLUTION : $\cos 60° \ \cos 30° + \sin 60° \ \sin 30°$

$$= \frac{1}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \times \frac{1}{2} \quad \text{(On replacing values)}$$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = 2 \times \frac{\sqrt{3}}{4}$$

$$= \frac{\sqrt{3}}{2}$$

EXAMPLE-7. Determine the value of:-

$$\frac{5\sin^2 30° + \cos^2 45° - 4\tan^2 30°}{2\sin 30° \ \cos 30° + \tan 45°}$$

$\tan 90°$, $\sec 90°$, $\cot 0°$ and $\operatorname{cosec} 0°$ are not defined. If we take an angle slightly less than 90°, then $\tan\theta$ and $\sec\theta$ will have a very large value. So when it reaches 90°, this value will be infinite.

Similarly $\cot\theta$ and $\operatorname{cosec}\theta$ become infinitely large as θ approaches 0, so we cannot determine their value.

SOLUTION : $\dfrac{5\sin^2 30° + \cos^2 45° - 4\tan^2 30°}{2\sin 30° \cos 30° + \tan 45°}$

$$= \frac{5\times\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 - 4\left(\frac{1}{\sqrt{3}}\right)^2}{\left(2 \times \dfrac{1}{2} \times \dfrac{\sqrt{3}}{2}\right) + 1} \qquad \text{(On replacing values)}$$

$$= \frac{\dfrac{5}{4} + \dfrac{1}{2} - \dfrac{4}{3}}{\dfrac{\sqrt{3}}{2} + 1}$$

$$= \frac{\dfrac{15 + 6 - 16}{12}}{\dfrac{\sqrt{3} + 2}{2}} = \frac{\dfrac{21 - 16}{12}}{\dfrac{\sqrt{3} + 2}{2}}$$

$$= \frac{5}{12} \times \frac{2}{\sqrt{3}+2} = \frac{5}{6\left(\sqrt{3}+2\right)}$$

$$= \frac{5}{6\left(2+\sqrt{3}\right)} \times \frac{\left(2-\sqrt{3}\right)}{\left(2-\sqrt{3}\right)} \qquad \text{(On rationalising the denominator)}$$

$$= \frac{5\left(2-\sqrt{3}\right)}{6\left(4-3\right)} = \frac{5\left(2-\sqrt{3}\right)}{6} \qquad \left((2+\sqrt{3})(2-\sqrt{3}) = (2)^2 - \left(\sqrt{3}\right)^2\right)$$

EXAMPLE-8. Prove that $\cos^2 30° - \sin^2 30° = \cos 60°$

SOLUTION : $\cos^2 30° - \sin^2 30°$

$$= (\cos 30°)^2 - (\sin 30°)^2$$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

$$= \frac{3}{4} - \frac{1}{4} = \frac{3-1}{4} = \frac{2}{4}$$

$$= \frac{1}{2} = \cos 60°$$

Exercise - 8.3

1. Choose the correct option from the following:-

   (i) $\dfrac{1 - \tan^2 45^\circ}{1 + \tan^2 45^\circ} =$

   (a) 1 (b) tan90° (c) 0 (d) sin45°

   (ii) $\dfrac{2\tan 30^\circ}{1 - \tan^2 30^\circ} =$

   (a) sin 60° (b) sin 30° (c) tan 60° (d) cos 60°

2. Evaluate the following:-

   (i) $\cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ$ (ii) $\tan 30^\circ \sec 45^\circ + \tan 60^\circ \sec 30^\circ$

   (iii) $\operatorname{cosec} 30^\circ + \cot 45^\circ$ (iv) $\dfrac{\cot 60^\circ}{\sec 30^\circ - \tan 45^\circ}$

   (v) $\tan^2 60^\circ + \tan^2 45^\circ$ (vi) $\dfrac{\cos 30^\circ + \sin 60^\circ}{1 + \cos 60^\circ + \sin 30^\circ}$

   (vii) $\dfrac{\sin^2 45^\circ + \cos^2 45^\circ}{\tan^2 60^\circ}$ (viii) $\dfrac{\sin 30^\circ - \sin 90^\circ + 2\cos 0^\circ}{\tan 30^\circ \tan 60^\circ}$

3. Check whether true or false:-

   (i) $\sin 30^\circ \cos 60^\circ + \cos 30^\circ \sin 60^\circ = \tan 90^\circ$

   (ii) $1 - 2\sin^2 30^\circ = \cos^2 60^\circ$ (iii) $2\cos^2 45^\circ - 1 = \cos 90^\circ$

   (iv) $\sin^2 45^\circ = 1 - \cos^2 45^\circ$ (v) $\sin^2 60^\circ + \cos^2 60^\circ = 1$

Trigonometric Equations

Just as on solving an algebraic equation, we evaluate the value of unknowns like x, y, z, ..., similarly in trigonometric by solving equations we can evaluate the values of angle θ.

In this section we shall try to study equations which give the value of unknown θ lying between 0° to 90°

Example-8. Solve for θ the equation $2\sin\theta - 1 = 0$ if $0^\circ \le \theta^\circ \le 90^\circ$

Solution : $2\sin\theta - 1 = 0$

$$2\sin\theta = 1 \text{ or } \sin\theta = \frac{1}{2}$$

$\sin\theta = \sin 30°$ $\quad \left(\because \sin 30° = \dfrac{1}{2}\right)$

$\therefore \theta = 30°$

Example-9. Solve for θ the equation $\sqrt{3}\tan\theta = 1$ when $0° \leq \theta° \leq 90°$

Solution : $\sqrt{3}\tan\theta = 1$ or $\tan\theta = \dfrac{1}{\sqrt{3}}$

$\tan\theta = \tan 30°$ $\quad \left(\because \tan 30° = \dfrac{1}{\sqrt{3}}\right)$

$\therefore \theta = 30°$

Exercise - 8.4

Solve the following equations for the value of θ, when $0° \leq \theta° \leq 90°$

1. $\sin\theta = \cos\theta$
2. $2\cos\theta = 1$
3. $2\sin^2\theta = \dfrac{1}{2}$
4. $3\tan^2\theta - 1 = 0$
5. $2\sin\theta = \sqrt{3}$
6. $\tan\theta = 0$
7. $3\operatorname{cosec}^2\theta = 4$
8. $2\cos^2\theta = \dfrac{1}{2}$
9. $4\sin^2\theta - 3 = 0$
10. $4\sec^2\theta - 1 = 3$
11. $\cot^2\theta = 3$

Other Applications of Trigonometric Ratios

Till now the trigonometric ratios that we have studied have been for angles of a right angled triangle.

In fact, besides right angled triangles, the trigonometric ratios also exist and are defined for other triangles, quadrilaterals, pentagons and polygons. These are special characteristics of the angles. So if we know the value of the angles in a figure, we can use trigonometric ratios to determine the length of the sides. This can be understood by the following example:-

Example-10. Consider a triangle ABC in which $\angle B = 45°$ and $\angle C = 30°$, AB = 5 *cm*. Thus the triangle is not a right angled traingle.

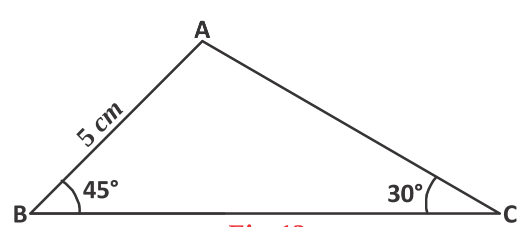


Fig. 13

Solution : Can we find AC and BC using the information given?

We can construct a perpendicular from the vertex A to the side BC, intersecting the side in D.

Now consider ΔABD,

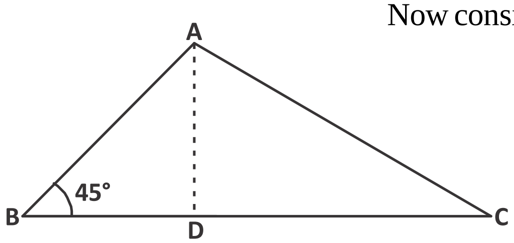


Fig. 14

$$\sin 45° = \frac{AD}{AB} = \frac{AD}{5}$$

$$\text{Or } AD = 5 \sin 45° = \frac{5}{\sqrt{2}}$$

Now we can find BD. Rehana said that in the ΔABD, $AD = BD$

Do you think that is correct?

Are they equal?

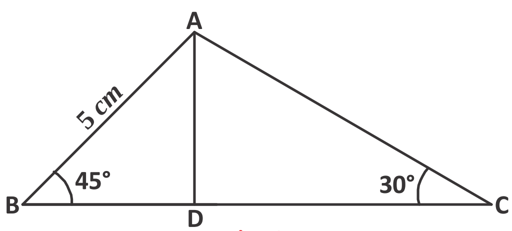


Fig. 15

Now take ΔADC

$$\sin 30° = \frac{AD}{AC}$$

$$AC = \frac{AD}{\sin 30°}$$

$$\text{Or} \quad AC = \frac{5}{\sqrt{2}} \times \frac{1}{\sin 30°}$$

$$= \frac{5}{\sqrt{2}} \times 2 = 5\sqrt{2}$$

$$\text{And } DC = AC \cos 30° = 5\sqrt{2} \times \frac{\sqrt{3}}{2}$$

$$= \frac{5 \times \sqrt{3}}{\sqrt{2}}$$

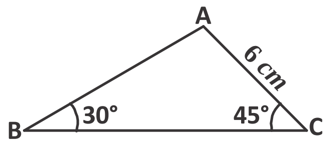
BD and DC, both add to give BC

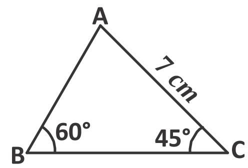
$$BC = \frac{5}{\sqrt{2}} + \frac{5\sqrt{3}}{\sqrt{2}} = \frac{5\left(1+\sqrt{3}\right)}{\sqrt{2}}$$

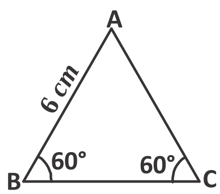
$$\text{Hence } AB = 5,\ AC = 5\sqrt{2} \text{ and } BC = \frac{5\left(1+\sqrt{3}\right)}{\sqrt{2}}$$

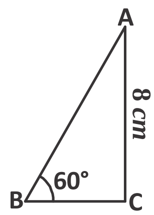
Try This

Find all the sides in the following triangles:

1. 

2. 

3. 

4. 

What Have We Learnt

1. The trigonometric ratios can be found using the following:-

$$\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} \qquad \cos\theta = \frac{\text{Base}}{\text{Hypotenuse}} \qquad \tan\theta = \frac{\text{Perpendicular}}{\text{Base}}$$

$$\text{cosec}\,\theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}} \qquad \sec\theta = \frac{\text{Hypotenuse}}{\text{Base}} \qquad \cot\theta = \frac{\text{Base}}{\text{Perpendicular}}$$

2. The relation between the various trigonometric ratios is:-

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \text{cosec}\,\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

3. If we know one trigonometric ratio of one of the acute angles of the triangle, we can find the remaining trigonometric ratios.

4. We can find the trigonometric ratios for particular angles like 0º, 30º, 45º, 60º and 90º.

5. The value of sinA or cosA can never exceed 1 whereas the value of secA or cosecA is always 1 or more than 1.

6. The three identities are:-

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \cot^2\theta = \text{cosec}^2\theta \qquad \text{where } \theta \neq 0^\circ$$

$$1 + \tan^2\theta = \sec^2\theta \qquad \text{where } \theta \neq 90^\circ$$

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Let us know something about the history of geometry.....

The moment human started distinguishing between shapes around him, it led to the creation of geometry. Since then there are many objects whose name is based on their geometrical shapes. To understand more about these shapes human started to draw these shapes in different ways and in this effort he created different lines and shapes.

In this process he also studied spatial relationship. This developed the understanding of angles and construction of shapes. In India geometry was predominantly used in construction of monuments and to locate and forecast the position of celestial bodies. There were many formulae to do this. Initial geometry was an endeavour based more on experience and extracting rules from these examples. Those rules were created to find and calculate length, breadth, height, angle, area, volume etc. easily. The aim of this was to find the use in daily needs like survey of land, construction of building ,bridges and for geological and other technical uses. But as often happens the scope of geometry was also to make further discoveries and it slowly got more extensive.

Geometry which means land measurement shows the reason for its creation. Many formulaes of those times were as complicated and deep as todays contemporary mathematics, 'which is not easy to find even today. This was the begininng of formal geometry. In the era of Harappan Civilization people were experts in measurement as well as creating geometrical shapes. Likewsie in Shulva-Sutra there is a description of how to construct and find area of triangles, squares, rectangles and other complicated geometrical shapes. These formulae could also be used more comprehensively. Thus, the formulae related to triangles and quadrilaterals can be used for all types of triangles and quadrilaterals.

Few examples of Shulva-Sutra:-

1. To construct a square equal to the sum of the area of two given squares.
2. To construct a square with double the area of the given square.

To construct a square with double the area of the given square, we need to find the side for that square, Katyayan and Apstambh had given the following Shulva-Sutras for this:-

$$\text{New side} = \text{old side}\left(1+\frac{1}{3}+\frac{1}{3\times 4}-\frac{1}{3\times 4\times 34}\right) = 1.4142156 \times \text{old side}$$

This value is the same as value of $\sqrt{2}$ correct up to five decimal places.

We find several examples of this kind of geometrical rules and formulae in Egyptian, Indian (Indus valley and Harappan), Babylonian, Arabian civilization.

This information has been collected from different books and presented. Teachers and students can collect more information about geometry from other sources.

S I 2 W

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There are lot of shapes hidden around us. In the picture given below we can see a door, frame of a window, top surface of a book, roof-top etc., all these shapes are rectangular.

Fig. 1

Some other objects may have surfaces that are triangular, pentagonal or any other shape. Every angel rectangular object has all angles equal to 90° and has opposite side that are equal. Other shapes also have some line segments that are equal and their angles could be equal.

See the window grill in the picture. There are lots of line segments in the picture which intersect each other. In this grill and the other grid type we find several line segments which meet at different intersecting points. Is there any relation between the angles made at the intersecting points? In this chapter we will study the angles made by lines at their points of intersection.

Line Segment and End Points

Draw a line on your copy. What symbols are used to express it?

Now draw a ray. Which symbols are use for this?

Does a line have any end point? And in a ray? Discuss with your classmate.

See the picture given below:-

How many end points are there? This picture does not show a line or a ray. This is a line segment.

Fig. 2

This can also be marked on a line.

How many line segment can be there on a line?

Discuss with your friends.

Identifying a Line Segment

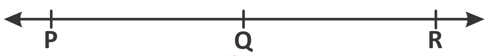


Fig. 3

1. How many points are marked on this line?
2. How many line segment are there on this line? Which are they?

Try to identify all these three things in figures 4, 5, 6 and 7.

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Number of points on the line	Name of the points	Name of line segment	Numbers of line segments
3	P, Q, R	PQ, PR, QR	3
4	P, Q, R, S		
5			
6			
7			

Here PQ and QP are the name for the same line segment.

If only two points are marked on a line then how many line segments would there be?

According to this table what relationship is there between the number of points and numbers of line segments.

Likewise if there are 8 points on the line then the number of line segments are

$$1 + 2 + 3 + 4 + 5 + 6 + 7.$$

If there are n points on the line then the number of line segment would be

$$1 + 2 + 3 + 4 + 5 + \ldots\ldots\ldots\ldots\ldots + (n - 1)$$

Do you agree with this? Discuss with your friends.

Collinear Points

In the above table P, Q, R, S etc. on a single line. These are collinear points. This means all the points which lie on the same line are called collinear points.

Here points A, B and C are collinear. (*Fig.*8)

Fig. 8

In *Fig.*9, are the points A, B, C and D collinear? Can we also say that B, C and D not collinear?

And are C and D also not collinear?

Can we draw a line on which both points C and D lie? Yes, line segment CD lies on this line.

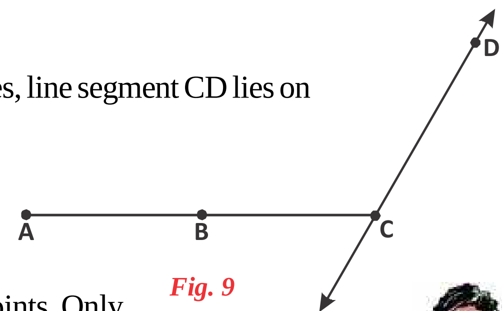


Fig. 9

Clearly any two points must have lie on any one line, and on this line they will be collinear.

So, whether the points are colinear or not, is a question which is relavant only when we are taking about at least three points. Only then would you ask the question whether the points are collinear or not?

Think and Discuss

Can three collinear points make a triangle?

Line and Angle

In *figure*-9 angle DCA at point 'C' is more than 90°, so this is an obtuse angle.

We know different kinds of angle like acute angle, right angle, obtuse angle, straight angle and reflex angle.

Try This

Draw each type of angle we have mentioned and write their names.

Adjacent, Complementary & Supplementary Angles

Here we will see which pair of angles comes into the category of adjacent angles, complementary angles and supplementary angles.

Adjacant Angles

See the given *Fig.*10(i) and (ii).

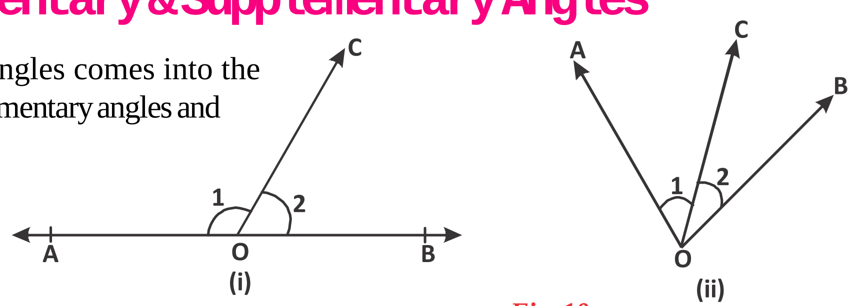


Fig. 10

There are two angles 1 and 2 both in *Fig.*10(i) and (ii). In which O is a vertex and side OC is in middle and is a common side.

Therefore in *Fig.*10 angles 1 and 2 are adjacent angles.

Now see *Fig.*11(i), (ii) and (iii):-

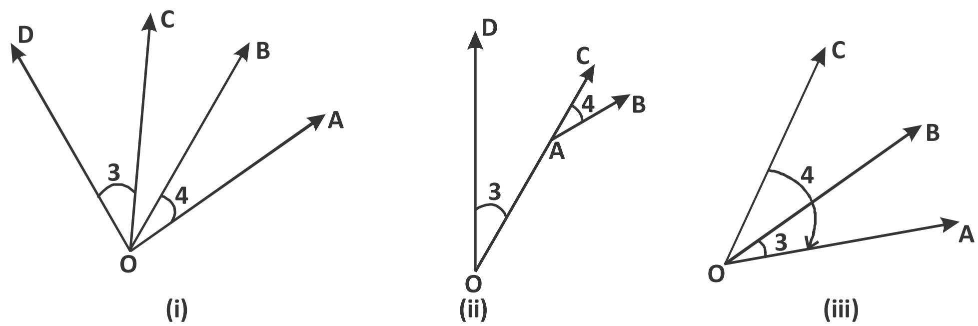


Fig. 11

In *Fig.*11(i), 3 and 4 have same vertex but there is no common side in these angles.

In *Fig.*(ii) angle 3 and 4 have different vertices, for angle 3 vertex is O, while for angle 4 it is A.

In *Fig.*11(iii) angle 3 and 4 have some common side OA, but angle 3 is a part of angle 4.

Therefore in all the *Fig.*11, angles 3 and 4, are not adjacent.

Two angles are adjacent when they have a common side, a common vertex and both angles do not overlap each other.

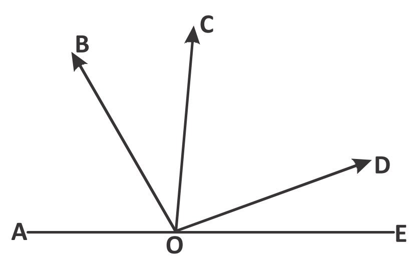
Try This

1. Draw two angle which are not adjacent.
2. See the figure and say whether the following are adjacent or not?

   (i) $\angle AOB$ and $\angle BOC$

   (ii) $\angle BOC$ and $\angle DOE$

   (iii) $\angle EOD$ and $\angle DOC$



Think and Discuss

Where are two angles adjacent ?

(i) When both the angles are obtuse (ii) Both are acute

(iii) One angle is obtuse and the other is acute.

Complementary Angles

Look at the figures below, each figure has two angles. What is the sum of each of these angle pairs?

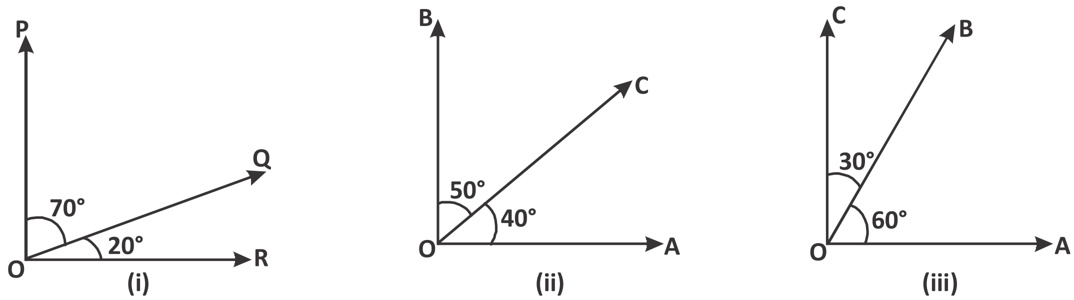


Fig. 12

When sum of a pair of angles is 90° , then each angle is complementary to the other.

See these complementary angles are also adjacent.

You too draw some more adjacent complementary angle like the ones given in *Fig*.12.

Supplementary Angle

What is the sum of $\angle 1$ and $\angle 2$ given in figure below?

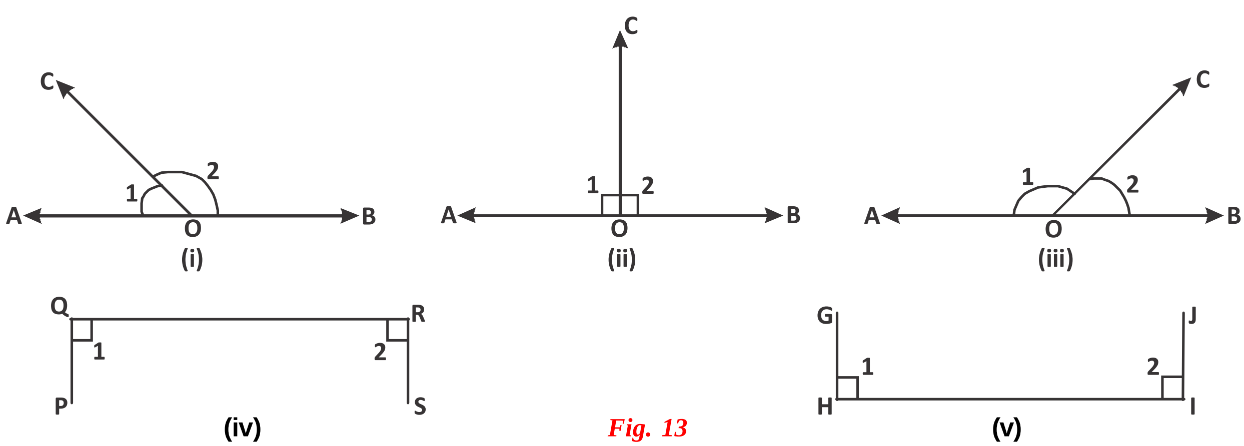


Fig. 13

Sum of all these pair angles is 180°, that means each angle is supplement to other.

Are the angles in *Fig.*13(i), (ii) and (iii) adjacent angles? Are the angles (iv) and (v) also adjacent?

Here in *Fig.*13(i), (ii) and (iii) the sum of pair of adjacent angles makes a straight line, this is also called straight angle. This kind of pair of angles is also called linear pair of angles.

Can we say each pair of supplementary angles is also a linear pair of angles?

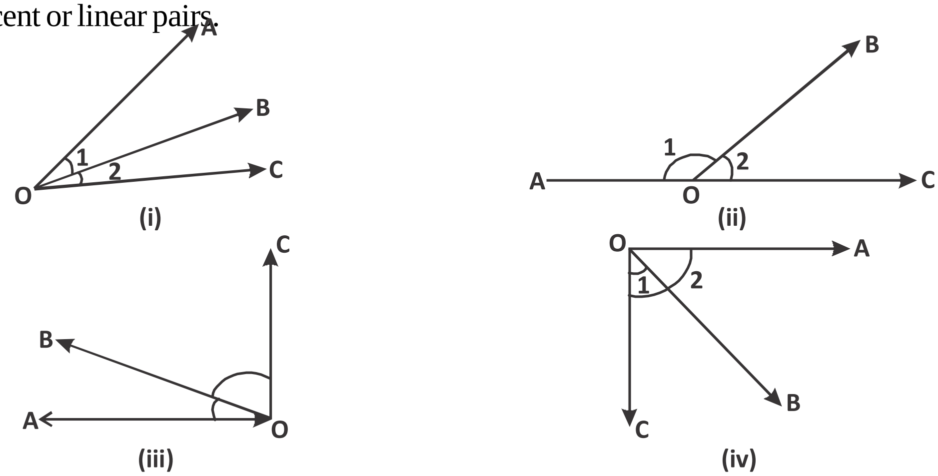
Are angles in *Fig.*13(iv) and (v) also a linear pair?

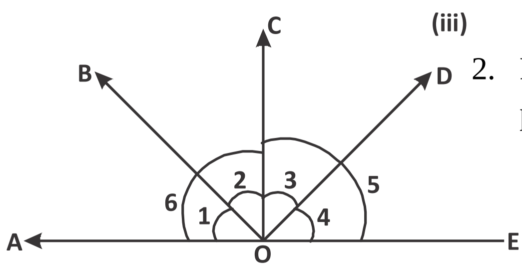
Think and Discuss

1. Can two right angles be a complementary angles?
2. Is each linear pair also supplementary angles?

Try This

1. Which of the following angles are complementary or supplementary? Which ones are adjacent or linear pairs.





2. From the following figure, which angles represent the following pair

(i) $\angle 1$ and $\angle 2$ (ii) $\angle 5$ and $\angle 6$ (iii) $\angle 6$ and $\angle 3$

(iv) $\angle 5$ and $\angle 2$ (v) $\angle 3$ and $\angle 4$

Intersecting and Non-intersecting Lines

In *Fig.*14(i) and (ii) if we increase the lines, then which pair of lines intersect each other?

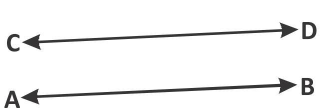


Fig. 14 (i)

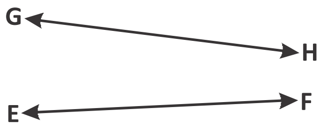


Fig. 14 (ii)

Here line AB and CD do not intersect each other. Where as lines GH and EF when extended to H and F meet at point O (*Fig.*15)

Fig. 15

Therefore AB and CD are non-intersecting lines. And EF or GH are intersecting lines.

Angle Made by Two Intersecting Lines

When tow lines intersect each other on any point, some angles are formed at the point of intersection .

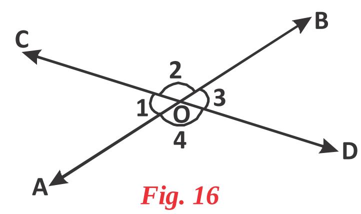


Fig. 16

Look at the *Fig.*16. Lines AB and CD intersect each other at point O. This forms $\angle 1$, $\angle 2$, $\angle 3$ and $\angle 4$. Is there any similarity between $\angle 1$ and $\angle 3$ or $\angle 2$ and $\angle 4$.

Here we can see that $\angle 1$ and $\angle 3$ meet at point O and are opposite to each other. Likewise $\angle 2$ and $\angle 4$. These angles are called vertrically opposite angles.

Propery of Vertically Opposite Angles

Here $\angle 1$, $\angle 3$ and $\angle 2$, $\angle 4$ are vertically opposite angles.

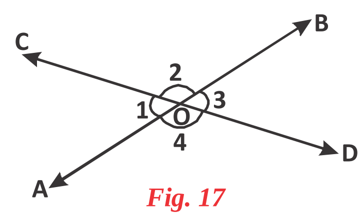


Fig. 17

Above the line CD

$\angle COB + \angle BOD = \angle COD$

What is $\angle COD$, here?

This is straight angle.

So, $\angle COB + \angle BOD = 180°$

Or $\angle 2 + \angle 3 = 180°$ ......(i)

Can you find the same type of relation above line AB.

$\angle AOC + \angle COB = \angle AOB$

Or $\angle 1 + \angle 2 = 180°$ .....(ii)

Here $\angle AOB$ is a straight line angle.

Now in (i) and (ii)

$\angle 2 + \angle 3 = \angle 1 + \angle 2$

Or $\angle 3 = \angle 1$

So $\angle 1 = \angle 3$ .....(iii)

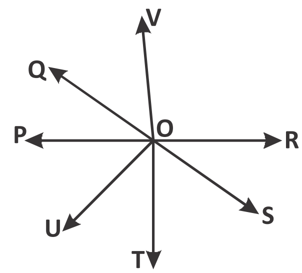
Similary, you can add the angles above line CD and add the angles below line AB and find the following relationship :

$$\angle 2 = \angle 4 \qquad .....(iv)$$

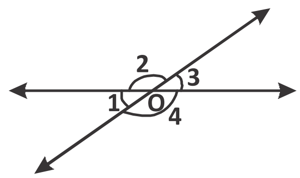
Here $\angle 1$ and $\angle 3$, $\angle 2$ and $\angle 4$ are vertically opposite angles. From the equation (iii) and (iv) we can say that these angles are equal. Therefore vertically opposite angles are equal.

Try This

1. Show all vertically opposite angles in the figure.



2. In figure, $\angle 2 = 110°$ then find the measurement of $\angle 1$ and $\angle 4$.



Example-1. In *Fig.*18, $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are opposite rays. What is the measurement of $\angle AOC$ and $\angle BOC$?

Solution : $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are opposite rays.

which make $\angle AOC$ and $\angle BOC$ a linear pair.

Hence $\angle AOC + \angle BOC = 180°$

$$(2x + 20^o) + 2x = 180^o$$

$$4x + 20^o = 180^o$$

$$x = 40^o$$

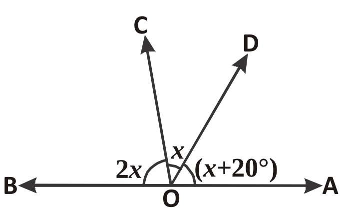


Fig. 18

Now $\angle AOC = 2x + 20^o$

$$= 2(40^o) + 20^o$$

$$= 100^o$$

$$\left\{ \begin{aligned} &\angle AOC = \angle COD + \angle DOA \\ &= x + (x + 20^o) \\ &= 2x + 20^o \end{aligned} \right\}$$

And $\angle BOC = 2x$

$$= 2(40^o)$$

$$= 80^o$$

Hence $\angle AOC = 100^o$ and $\angle BOC = 80^o$

EXAMPLE-2. In the given *Fig.*19, lines AB and CD intersect at a point O. If $\angle AOC : \angle COB = 7:8$, find the measures of all angles.

SOLUTION : According to the questions $\angle AOC : \angle COB = 7:8$

So if $\angle AOC = 7x$ .....(i)

then $\angle COB = 8x$ .....(ii)

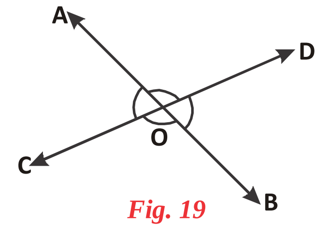


Fig. 19

The ray OC stand on the line AB. (linear pair)

$\angle AOC + \angle COB = 180^o$

$7x + 8x = 180^o$

$15x = 180^o$

$x = 12^o$ .....(iii)

Now $\angle AOC = 7x$

$= 7(12^o)$

$= 84^o$

And $\angle COB = 8x$

$= 8(12^o)$

$= 96^o$

Also $\angle BOD = \angle AOC = 84^o$ (Vertically opposite angles)

पुनः $\angle AOD = \angle COB = 96^o$ (Vertically opposite angles)

EXAMPLE-3. In the given *Fig.*20, $\angle COD = 90^o$, $\angle BOE = 72^o$ and if AOB is a straight line, find the measures of $\angle AOC$, $\angle BOD$ and $\angle AOE$.

SOLUTION : AOB is a straight line,

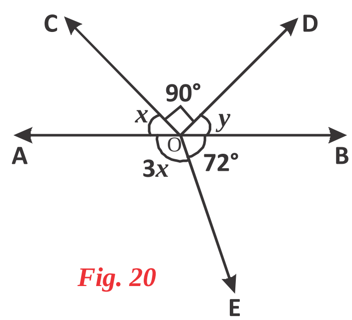


Fig. 20

$\angle AOE + \angle BOE = 180^o$ (linear pair)

$\Rightarrow 3x + 72^o = 180^o$

$\Rightarrow 3x = 108^o$

$\Rightarrow x = 36^o$ .....(i)

Similarly, $\angle AOC + \angle COD + \angle DOB = 180^o$ (straight angle)

$\Rightarrow x + 90° + y = 180^o$

$\Rightarrow 36^o + 90^o + y = 180^o$

$\Rightarrow 126^o + y = 180^o$

$\therefore\ y = 54^o$ .....(ii)

Hence $\angle AOC = x = 36^o$

$\angle BOD = y = 54^o$

And $\angle AOE = 3x = 3 \times 36^o = 108^o$

EXAMPLE-4- In the given *Fig.*21, ray OS starts from the point O on line PQ. Other rays OR and OT bisect angles $\angle$POS and $\angle$SOQ respectively. Find measures of $\angle$ROT.

SOLUTION : Ray OS, is on the line PQ, hence by the a linear pair axiom,

$\angle POS + \angle SOQ = 180^o$

If $\angle POS = x$

then $x + \angle SOQ = 180^o$

$\angle SOQ = 180^o - x$ .....(i)

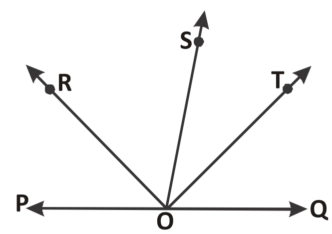


Fig. 21

But ray OR, besects $\angle$POS,

hence $\angle ROS = \frac{1}{2} \times \angle POS$

$= \frac{1}{2} \times x = \frac{x}{2}$ .....(ii)

Similarly Ray OT bisects $\angle$SOQ,

Hence $\angle SOT = \frac{1}{2} \angle SOQ$

$= \frac{1}{2}(180^o - x)$ (from (1))

$= 90^0 - \frac{x}{2}$ .....(iii)

It is clear from the figure,

$\angle ROT = \angle ROS + \angle SOT$

$= \frac{x}{2} + \left(90^o - \frac{x}{2}\right) \quad = \frac{x}{2} + 90^o - \frac{x}{2}$

Hence $\angle ROT = 90^o$

EXAMPLE-5. In the adjjacent figure, there are four rays OP, OQ, OR and OS, Prove that $\angle POQ + \angle QOR + \angle SOR + \angle POS = 360^o$

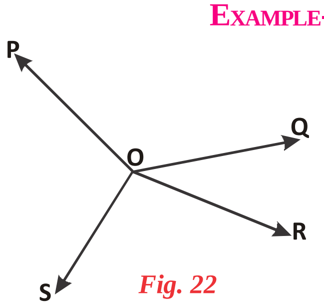


Fig. 22

SOLUTION : In the given *Fig.*22, We need to extend one of the rays OP, OQ, OR or OS to a point beyond O. (Why ?)

Extend ray OQ to a point T. (*Fig.*23) so that TOQ is a straight line.

Now it is clear from the figure that ray OP on TQ and so the linear pair axiom applies

$\angle TOP + \angle POQ = 180^o$ .....(i)

Similarly ray OS is on TQ and so by linear pair axiom,

$\angle TOS + \angle SOQ = 180^o$ .....(ii)

By adding equations (i) and (ii),

$\angle TOP + \angle POQ + \angle TOS + \angle SOQ = 360^o$ .....(iii)

It is clear from the figure that-

$\angle TOP + \angle TOS = \angle POS$

$\angle TOP = \angle POS - \angle TOS$ .....(iv)

And $\angle SOQ = \angle SOR + \angle QOR$ .....(v)

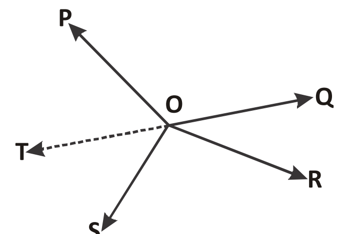


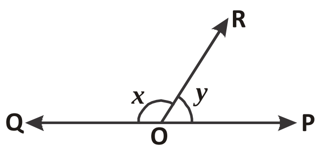
Fig. 23

Keeping values of equations (iv) and (v)in equation (iii)

$\angle POS - \angle TOS + \angle POQ + \angle TOS + \angle SOR + \angle QOR = 360^o$

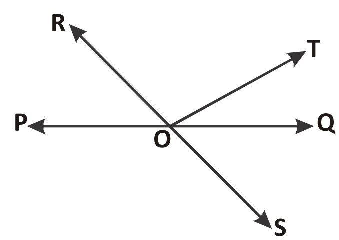
Hence, proved that $\angle POQ + \angle QOR + \angle SOR + \angle POS = 360^o$

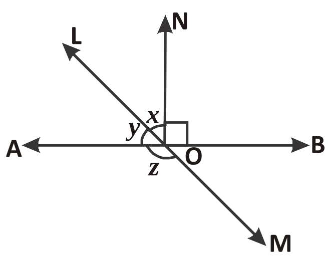
Exercise 9.1



1. In the figure alongside, $\angle POR$ and $\angle QOR$ are a linear pair. If $x - y = 80^o$ find the values of x and y.

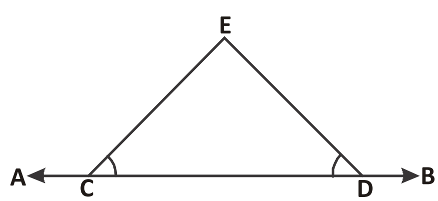
2. In the given figure lines PQ and RS intersect in point O. If $\angle POR + \angle QOT = 70^o$ and $\angle QOS = 40^o$, then find the measures of $\angle QOT$ and reflex $\angle ROT$.

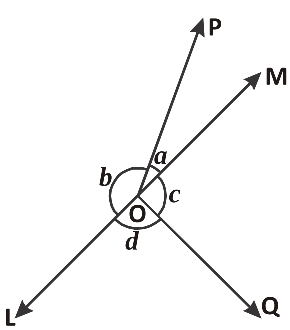




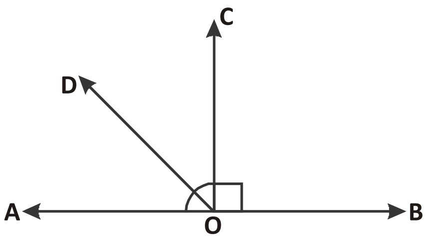
3. In the adjoining figure, lines AB and LM intersect in a point O. If $\angle NOB = 90^o$ and $x : y = 2 : 3$ then find the value of z.

4. In the given figure $\angle ECD = \angle EDC$ then prove that $\angle ECA = \angle EDB$





5. In the given figure $a + b = c + d$ then prove that LOM is a straight line.



6. In the given figure, AOB is a line Ray OC is perpendicular to AB. There is another ray OD between rays OA and OC. Prove that $\angle COD = \frac{1}{2}(\angle BOD - \angle AOD)$

7. If $\angle ABC = 64^o$ and AB is extended to a point X, Draw a figure to show this information. If ray BY biscects $\angle CBX$, then find measures of $\angle ABY$ and reflex $\angle YBX$.

Parallel Lines and Transversal Lines

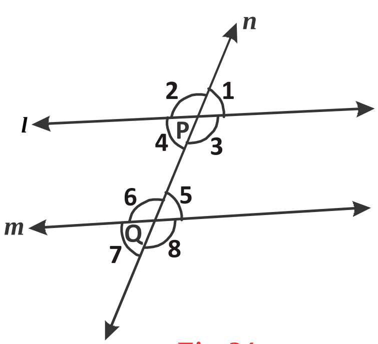


Fig. 24

What sort of lines ℓ, m can you see in *Fig*.24 and 25?

The lines ℓ, m in *Fig*.24 are non intersecting lines and in *Fig*.25 are interesecting lines.

In both the figures line *n* is intersecting both lines ℓ and *m* in points P and Q respectively. This line *n* is called a transversal.

Observe the *Fig*.24 and 25. In which of the two, is the distance same at all points of lines ℓ and *m*.

Here the lines ℓ, m in *Fig*.25 are intersecting and the distance between them is unequal and in *Fig*.24 they are non intersecting and the distance between them is the same. These are known as parallel lines.

When a transversal cuts two other lines, the following angles are formed in both *Fig*.24 and 25.

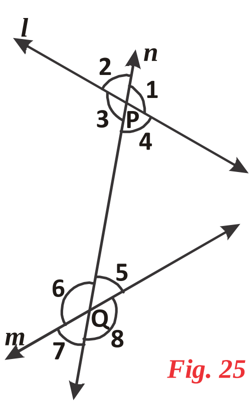


Fig. 25

(i) Corresponding angles

(a) $\angle 1$ and $\angle 5$ (b) $\angle 2$ and $\angle 6$

(c) $\angle 3$ and $\angle 7$ (d) $\angle 4$ and $\angle 8$

(ii) Alternate interior angles

(a) $\angle 4$ and $\angle 6$ (b) $\angle 3$ and $\angle 5$

(iii) Alternate exterior angle

(a) $\angle 1$ and $\angle 7$ (b) $\angle 2$ and $\angle 8$

(iv) The interior angles on the same side of the transversal

(a) $\angle 4$ and $\angle 5$ (b) $\angle 3$ and $\angle 6$

Interior angles on the same side of the transversal are also referred as consecutive interior angles or allied angles or co-interior angles. Quite often the alternate interior angles are simply refered as alternate angles.

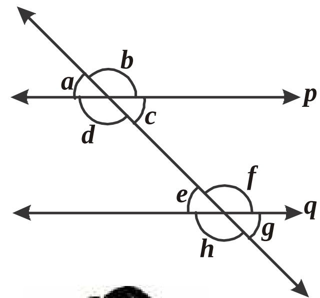
(v) The angles which are on the same side of transversal but on the exterior side of the two lines, are known as, consecutive exterior angles or allied exterior angles or coexterior angle.

(a) $\angle 1$ and $\angle 8$ (b) $\angle 2$ and $\angle 7$

Try This

Complete the table by observing the given figure-

S. No.	Pair of angles	Angles	Number of pairs of angles
1	Alternate Interior angle		2
2	Interior angles on same side of transversal		
3		$\angle a$ and $\angle g$ $\angle b$ and $\angle h$	
4		$\angle a$ and $\angle h$ $\angle b$ and $\angle g$	
5	Corresponding angles		4



Properties of Corresponding angle & Alternate Angle

Corresponding and alternate angles are formed when a transversal intersects two other lines. Is there a relation between the pairs so formed?

When a transversal intersects two intersecting lines, the pairs of corresponding and alternate angles are not equal, but if these two lines are parallel, then both pairs of corresponding angles and alternate angles are equal.

Think and Discuss

If a transversal intersect two other lines such that the corresponding pair of angles are equal, then can we say that the two lines are parallel?

Now the question is whether based on this property of corresponding angles, can we say something of the proportions of other angles pairs formed by the transversal with the parallel lines like the alternate interior angles or alternate exterior angles? Yes, in order to do this we take two parallel lines ℓ and m, which are cut by a transversal n at points P and Q. Look at *Fig.*26.

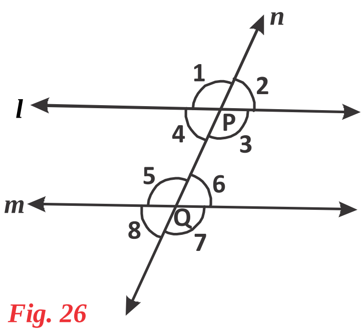


Fig. 26

Here $\angle 1 = \angle 5$ (Corresponding angles) – (i)

$\angle 1 = \angle 3$ (Vertically opposite angles) – (ii)

from (i) and (ii), $\angle 5 = \angle 3$

What angles are these? These are alternate interior angles.

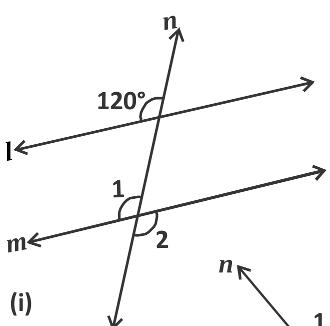
So we can say that when a transversal cuts two parallel lines, the alternate interior angles are equal.

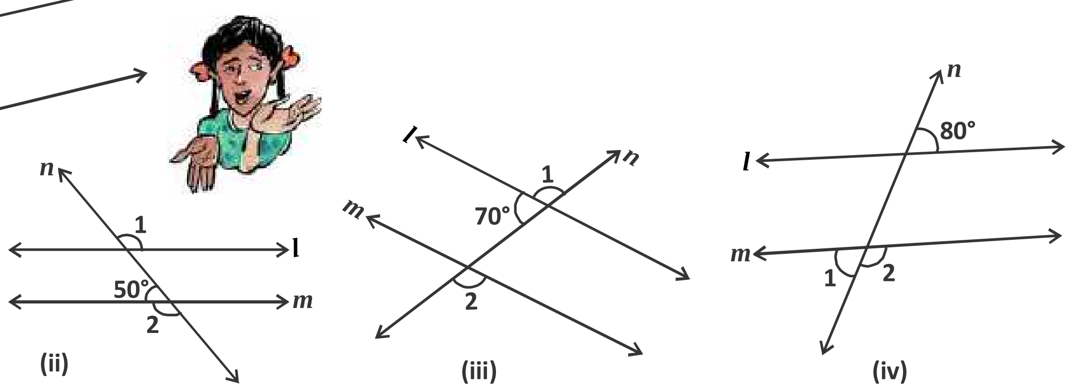
Think and Discuss

If a transversal intersects two lines such that the alternate interior angles are equal, can we say that the lines are parallel?

Try This

In the given figures, find the values of $\angle 1$ and $\angle 2$, if transversal n intersects two parellel lines l and m. Give reasons for your answers.





Lines Parallel to the Same Line

If two lines are parallel to the same line, are they parallel to each other?

For inspecting this, draw three lines l, m, n as shown in adjoining figure, such that line m is parallel to line l and line n is parallel to line l.

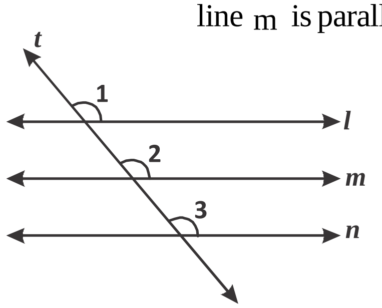


Fig. 27

Draw a transversal t intersecting lines l, m and n.

By the corresponding angles postulate,

$\angle 1 = \angle 2$ ....(1)

$\angle 1 = \angle 3$ ....(2)

Hence, from (1) and (2), We deduce

$\angle 2 = \angle 3$

But $\angle 2$ and $\angle 3$ are the corresponding angles formed by transversal *t* with lines m and n, hence we can say that line m and n are parallel.

This result can be written as a theorem as follows :

Theorem

It is that statement which can be proved logically using known facts and logic.

Theorem-1 । **The lines which are parallel to the same line, are parallel to each other.**

EXAMPLE-6. Find values of x, y, z and a, b, c from the given *Fig.*28.

SOLUTION : From the figure,

$y = 110^\circ$ ($\because$ Corresponding angles)

and $x + y = 180^\circ$ (Linear pair)

$\Rightarrow$ $x + 110^\circ = 180^\circ$

$\Rightarrow$ $x = 70^\circ$

$z = x = 70^\circ$ ($\because$ Corresponding angles)

Again $c = 65^\circ$ ($\because$ Alternate angles)

And $a + c = 180^\circ$ ($\because$ Linear pair)

$\Rightarrow$ $a + 65^\circ = 180^\circ$

$\Rightarrow$ $a = 115^\circ$

Also $b = c = 65^\circ$ ($\because$ Vertically opposite angles)

Hence $a = 115^\circ$, $b = 65^\circ$, $c = 65^\circ$

$x = 70^\circ$, $y = 110^\circ$, $z = 70^\circ$

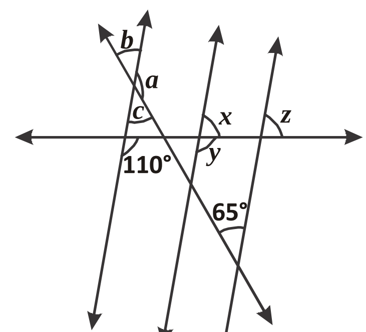


Fig. 28

EXAMPLE-7. In the given figure, if $PQ \parallel RS, \angle MXQ = 135^\circ$ and $\angle MYR = 40^\circ$ then find measure of $\angle XMY$.

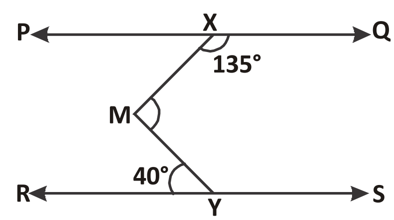


Fig. 29

SOLUTION : Construct a line AB parellel to PQ and passing through M.

Now, $AB \parallel RS$ and $PQ \parallel RS$.

Hence, $\angle QXM + \angle XMB = 180^\circ$ .....(1)

($\because$ $AB \parallel PQ$, and these are cointerior angles on same side of transversal) fig–30

According to the question, $\angle QXM = 135^\circ$, so by equation (1)

$135^\circ + \angle XMB = 180^\circ$

$\angle XMB = 180^\circ - 135^\circ$

$\therefore$ $\angle XMB = 45^\circ$ .....(2)

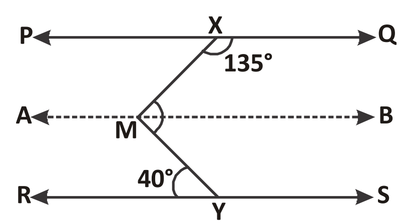


Fig. 30

Again $\angle BMY = \angle MYR$ .....(3) $(\because AB \parallel RS,$ Alternate angles$)$

According to the questions, $\angle MYR = 40^\circ$, and by equaction (3)

$$\angle BMY = 40^\circ \quad .....(4)$$

Adding equation (2) and (4) we get,

$$\angle XMB + \angle BMY = 45^\circ + 40^\circ$$

That is $\angle XMY = 85^\circ$

EXAMPLE-8. In the given figure $AB \parallel CD$ find the value of x.

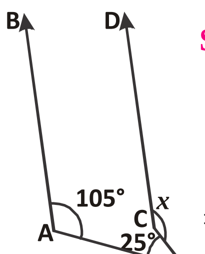


Fig. 31

SOLUTION : Draw $EF \parallel AB$ passing through point E. So obviously $EF \parallel CD$

Now $EF \parallel CD$ and CE is a transversal, So

$$\angle DCE + \angle CEF = 180^\circ$$

$\Rightarrow$ $x + \angle CEF = 180^\circ$ $(\because \angle DCE = x)$

$\angle CEF = 180^\circ - x$ .....(1)

$EF \parallel AB$ and AE is a transversal

So $\angle BAE + \angle AEF = 180^\circ$ $(\because$ Cointerior angles$)$

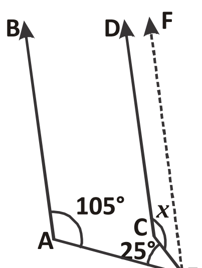


Fig. 32

$\Rightarrow$ $105^\circ + (\angle AEC + \angle CEF) = 180^\circ$

$(\because \angle BAE = 105^\circ)$

$\Rightarrow$ $105^\circ + 25^\circ + (180^\circ - x) = 180^\circ$

$(\because \angle AEC = 25^\circ$ and from equation (1))

$\Rightarrow$ $310^\circ - x = 180^\circ$

Hence $x = 130^\circ$

EXAMPLE-9. If a transversal intersects two lines such that the bisectors of the pair of corresponding angles are parallel, then prove that the lines are parallel to each other.

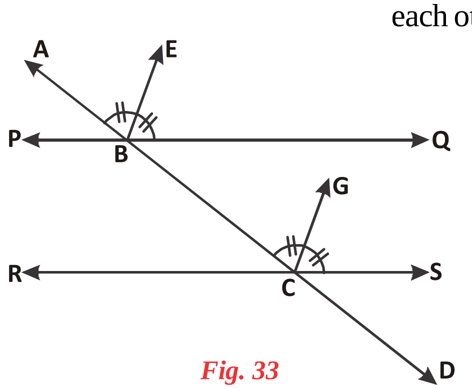


Fig. 33

SOLUTION : According to the *Fig.*33, AD is a transversal intersecting lines PQ and RS in points B and C respectively. $\angle ABQ$ and CG bisects $\angle BCS$. Also $BE \parallel CG$

To Prove that : $PQ \parallel RS$

We are given that BE bisects $\angle ABQ$,

Hence $\angle ABE = \frac{1}{2}\angle ABQ$ .....(1)

Similarly CG bisects $\angle BCS$,

Hence $\angle BCG = \frac{1}{2}\angle BCS$ .....(2)

But $BE \parallel CG$ and AD is a transversal, hence

$\angle ABE = \angle BCG$ .....(3)

From equctions (1), (2) and (3)

$$\frac{1}{2}\angle ABQ = \frac{1}{2}\angle BCS$$

Or $\angle ABQ = \angle BCS$

But these are cooespoding angles formed by the transveral AD on the lines PQ and RS .

Hence $PQ \parallel RS$

EXAMPLE-10. In the given figure, $AB \parallel CD$ and $CD \parallel EF$, Also $EA \perp AB$. If $\angle BEF = 55^o$, find the value of x, y and z.

SOLUTION : Given $AB \parallel CD$ and $CD \parallel EF$ Hence $AB \parallel EF$. In the *Fig.*34 BE is extended to a point G.

Now, $\angle DEF + \angle FEG = 180^o$ (Linear pair)

$55^o + \angle FEG = 180^o$ $(\because \angle DEF = 55^o)$

$\angle FEG = 125^o$

Thus $\angle FEG = y = x = 125^o$

(Corresponding angles)

Again $\angle CED + \angle DEF = 90^o$

$(\because EA \perp AB$ and $AB \parallel EF)$

Thus $z + 55^o = 90^o$

$\therefore$ $z = 35^o$

Hence $x = 125^o$, $y = 125^o$, $z = 35^o$

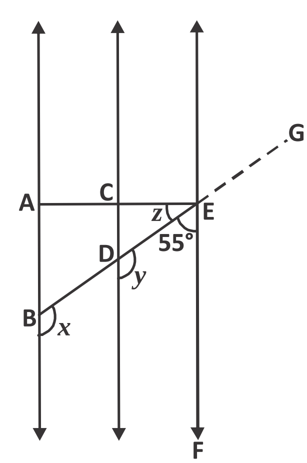
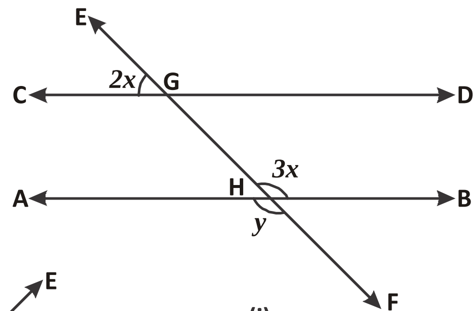


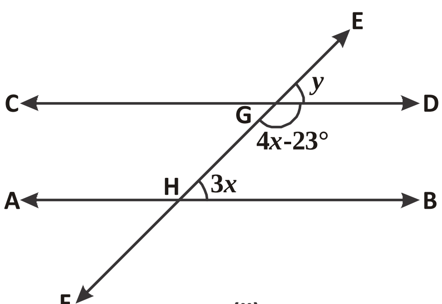
Fig. 34

Exercise-9.2

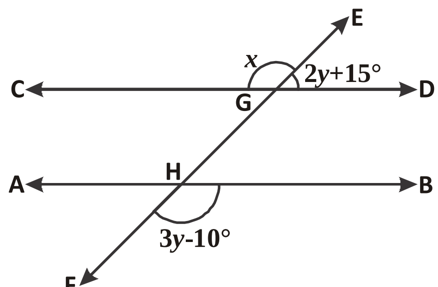


(i)

1. In the given figure $AB \parallel CD$ and EF is a transversal which intersects these line at H and G. Find the value of x and y.

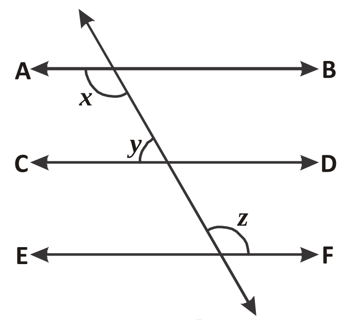


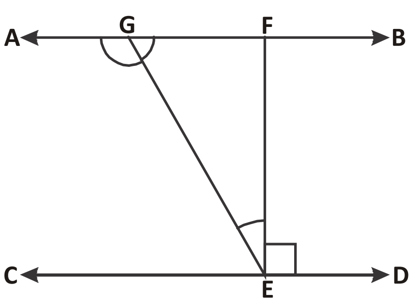
(ii)



(iii)

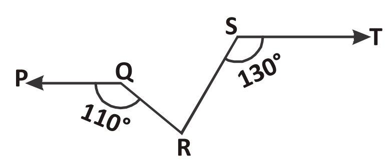
2. In the given figure, if $AB \parallel CD$, and $CD \parallel EF$. Further $y : z = 3 : 7$ then find the value of x.



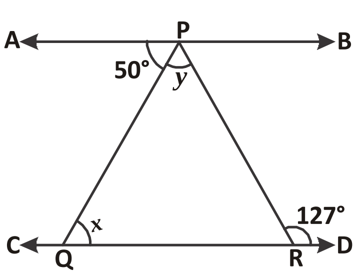


3. In the adjoining figure if $AB \parallel CD$, $EF \perp CD$ and $\angle GED = 126°$ then find measures of $\angle AGE, \angle GEF$, and $\angle FGE$.

4. In the given figure, if $PQ \parallel ST$, $\angle PQR = 110°$ and $\angle RST = 130°$ then the measure of $\angle QRS$.

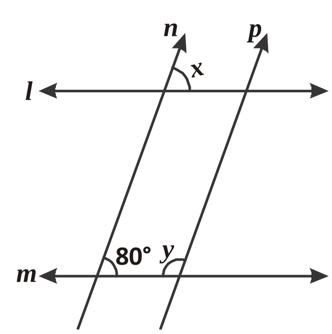


(**Hint**– Construct a line $\parallel$ to ST passing through R)

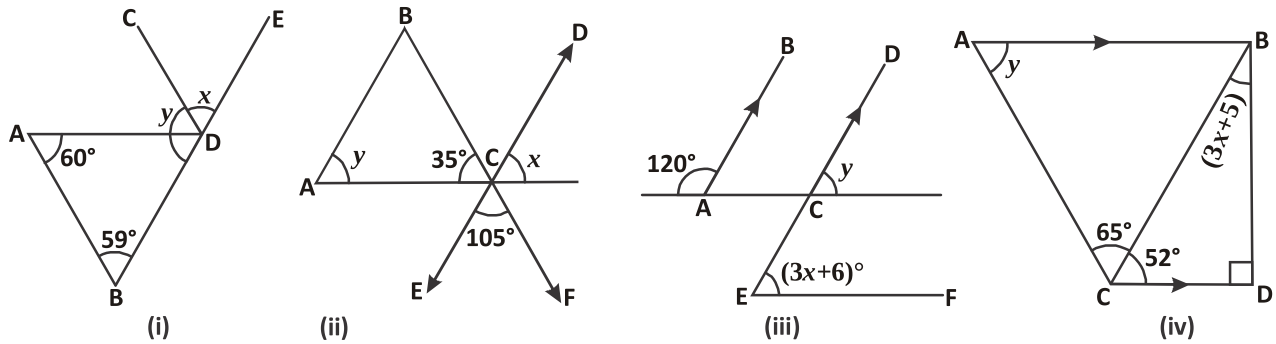


5. In the adjoining figure, if $AB \parallel CD$, $\angle APQ = 50°$ and $\angle PRD = 127°$ then find value of x and y.

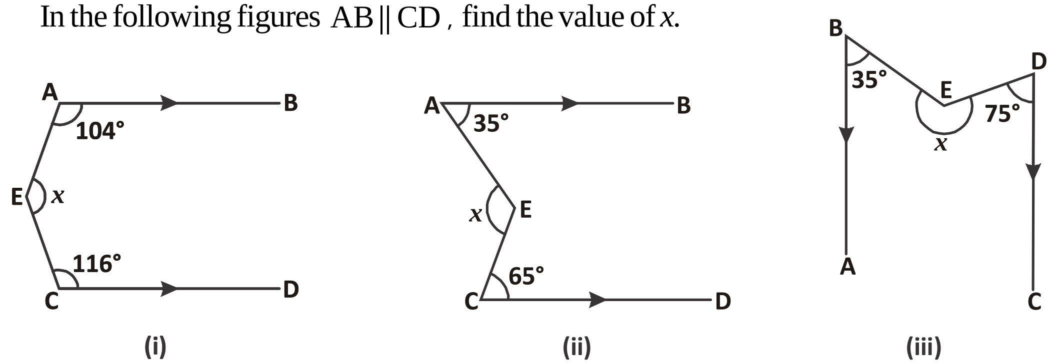
6. Find value of x and y. (Given $\ell \parallel m, n \parallel p$)



7. Find the values of x and y in the given figures here $AB \parallel CD$ है।



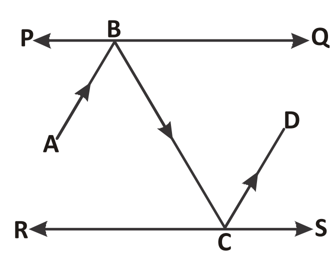
8. In the following figures $AB \parallel CD$, find the value of x.



9. Complete the following table :

S.N.	Name of triangle	Measure of angles	Speciality and other properties
1.	Acute angled triangle		
2.		One angle 90°	
3.	Obtuse angled triangle		
4.		Each angle 60°	
5.			Two sides are equal
6.	Scalene triangle		

10. In the given figure PQ and RS are two mirrors kept parallel to each other. Incidental ray AB, strikes mirror PQ at B and reflects back along BC which strikes mirror RS at C and is reflected along CD. Show that $AB \parallel CD$
(**Hind :** Perpendicular to parallel lines are parallel to each other)



To Prove Mathematical Statement

We have proved using a protractor and paper cutting activity that sum of the internal angles of a triangle is 180°. Now we will prove this statement using parallel lines and related postulates and theorems.

Theorem–2 : The sum of the internal angles of a triangle is 180^0 .

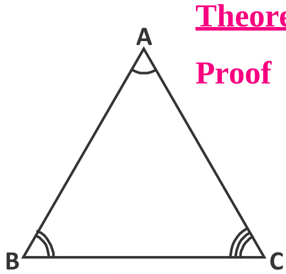


Fig. 35 (i)

Proof : Given Δ ABC with angle $\angle 1$, $\angle 2$ and $\angle 3$.

We have to prove that $\angle 1 + \angle 2 + \angle 3 = 180^o$ ।

To prove this draw a line PQ parallel to BC passing through A. (*Fig.*35(ii))

Now lines BC and PQ are parallel to each other. AB and AC are transversals. It is clear from the figure that $\angle 4$, $\angle 2$, and $\angle 5$, $\angle 3$ are alternate pairs of angles. Hence

hence $\angle 4 = \angle 2$ .....(1)

$\angle 5 = \angle 3$ .....(2)

But PAQ is a straight line, hence

$$\angle 4 + \angle 1 + \angle 5 = 180^o \quad .....(3)$$

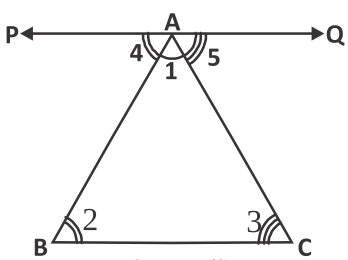


Fig. 35 (ii)

Replacing values of $\angle 4$ and $\angle 5$ from equations (1) and (2) in (3), We get

$$\angle 2 + \angle 1 + \angle 3 = 180^o$$

Or $$\angle 1 + \angle 2 + \angle 3 = 180^o$$

Hence, we can say that sum of interior angle of a triangle is 180^0.

Think and Discuss

Are the following statements true or false. Give reasons.

S.No.	Statement	True/False	Reason
1.	A triangle can have two right angles		
2.	A triangle can have two obtuse angles		
3.	Two angle of a triangle can be acute angles		
4.	A triangle can have all angles measuring less than 60^0		
5.	A triangle can be have all angles measuring more than 60^0		
6.	A triangle can have all angles that are exactly 60^0 .		

The Exterior Angles of a Triangle

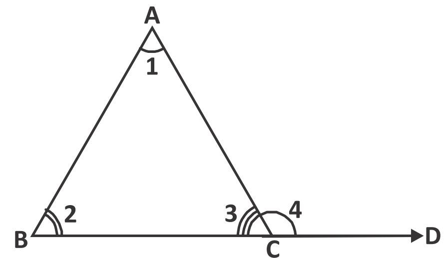


Fig. 36

In the given figure is a ΔABC whose side BC when extended to C forms an exterior angle $\angle ACD$

By the linear pair axiom we can say

$\angle 3 + \angle 4 = 180^o$ .....(1)

In Δ ABC, the sum of three interior angles is 180^0

Hence $\angle 1 + \angle 2 + \angle 3 = 180^o$ .....(2)

From equaction (1) and (2) we get,

$\angle 1 + \angle 2 + \angle 3 = \angle 3 + \angle 4$

or $\angle 1 + \angle 2 = \angle 4$

This result can be written as the following theorem :

Theorem–3 : If the side of a triangle is extended, the exterior angle so formed is equal to the sum of the interior opposite angles.

This is known as the exterior angle theorem. It is also clear from this theorem that the exterior angle is greater than each of the interior opposite angles.

Statement-1 : Prove that the sum of the four interior angle of a quadrilateral ABC is 360^0

SOLUTION : Given that quadrilaterals ABCD has four interior angles $\angle A, \angle B, \angle C$ and $\angle D$ -

We have to prove

$\angle A + \angle B + \angle C + \angle D = 360^o$

As shown in the *Fig.*37 join A to C dividing the quadrilateral into two triangle ΔADC and ΔABC .

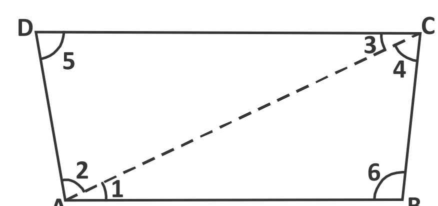


Fig. 37

By the angle sum property in ΔABC,

$\angle 1 + \angle 6 + \angle 4 = 180^o$ .....(1)

Similarly by angle sum property in ΔADC,

$\angle 2 + \angle 5 + \angle 3 = 180^o$ .....(2)

Adding equations (1) and (2) we get,

$\angle 1 + \angle 6 + \angle 4 + \angle 2 + \angle 5 + \angle 3 = 360^o$

or $(\angle 1 + \angle 2) + (\angle 3 + \angle 4) + \angle 5 + \angle 6 = 360^o$

or $\angle A + \angle C + \angle D + \angle B = 360^o$

That is $\angle A + \angle B + \angle C + \angle D = 360^o$

From the above example it is clear that you can find the sum of interior angle of any polygon by dividing it into triangles. For example :

Name of polygon	Number of triangles	Sum of interior angle
Quadrilateral	2	$2 \times 180^0 = 360^0$
Pentagon	3	$3 \times 180^0 = 540^0$
Hexagon	4	....
Octagon	....	....

We can now say that an *n* sides polygon can be divided into *n-2* triangles with common vertices, so the sum of the interior angles of a polygon with *n* sides

$= (n - 2) \times 180^o$

EXAMPLE-11. If the three angles of a triangle are $(2x+1)^\circ$, $(3x+6)^\circ$ and $(4x-16)^\circ$ respectively, find the measure of each angle.

SOLUTION : The sum of interior angles of a triangle is 180^0, hence

$(2x+1)^\circ+(3x+6)^\circ+(4x-16)^\circ=180^0$

$\Rightarrow 9x-9=180^o$

$\Rightarrow 9x=189^o$

$\Rightarrow x=21^o$

Hence $(2x+1)=(2\times 21^o+1)=43^o$

$(3x+6)=(3\times 21^o+6)=69^o$

$(4x-16)=(4\times 21^o-16)=68^o$

The angles are therefore $43^o, 69^o$ and 68^o respectively.

EXAMPLE-12. In the given figure $AB \parallel QR$, $\angle BAQ=142^o$ and $\angle ABP=100^o$ Find value of the following–

(i) $\angle APB$ (ii) $\angle AQR$ (iii) $\angle QRP$

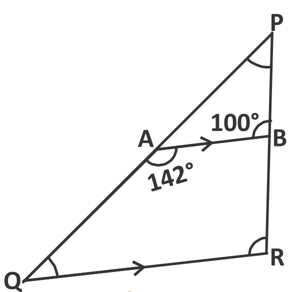


Fig. 38

SOLUTION : (i) The side PA of ΔAPB is extended till Q, hence by exterior angle theorem,

$\angle BAQ=\angle ABP+\angle APB$

$\Rightarrow 142^o=100^o+\angle APB$

$\Rightarrow \angle APB=142^o-100^o$

$\Rightarrow \angle APB=42^o$

(ii) $\angle BAQ + \angle AQR = 180^\circ$

(Sum of cointerior angles is 180°)

$\Rightarrow 142^\circ + \angle AQR = 180^\circ$

$\Rightarrow \angle AQR = 180^\circ - 142^\circ$

$\Rightarrow \angle AQR = 38^\circ$

(iii) As $AB \parallel QR$ and PR is a transversal, hence

$\angle QRP = \angle ABP$ (corresponding angles)

$\therefore \angle QRP = 100^\circ$

EXAMPLE-13. In the given figure if $BE \perp EC$, $\angle EBC = 40^\circ$, $\angle DAC = 30^\circ$ find the values of x and y.

SOLUTION : In ΔEBC

$90^\circ + 40^\circ + x = 180^\circ$ (By the property sum of interior angles of a triangle)

$\Rightarrow 130^\circ + x = 180^\circ$

$\Rightarrow x = 50^\circ$ .....(1)

Now in ΔADC

$\angle ADE = \angle DAC + \angle ACD$ (By exterior angle theorem)

$\Rightarrow y = 30^\circ + x$

$\Rightarrow y = 30^\circ + 50^\circ$ (from equation (1))

$\therefore y = 80^\circ$

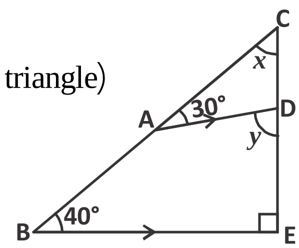


Fig. 39

EXAMPLE-14. Find the value of x from the given *Fig*.40

SOLUTION : ABCD in the figure is a quadrilateral. Join AC and extend it to E as shown in *Fig*.41.

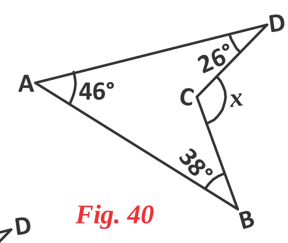


Fig. 40

Assume $\angle DAE = p^o$

$\angle BAE = q^o, \angle DCE = z^o$ and

$\angle ECB = t^o$

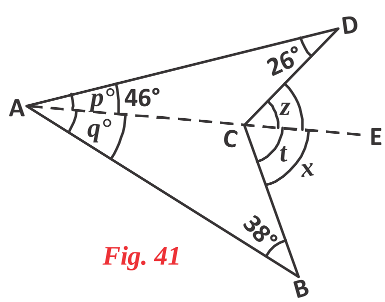


Fig. 41

By the exterior angle theorem in ΔACD

$$\angle DCE = \angle CAD + \angle ADC$$

$$z^o = p^o + 26^o \qquad .....(1)$$

Again in ΔABC

$$\angle BCE = \angle BAC + \angle ABC$$

$$t^o = q^o + 38^o \qquad .....(2)$$

Adding equations (1) and (2) we get

$$z^o + t^o = p^o + 26^o + q^o + 38^o$$

$$x = p + q + 64° \qquad \left[\begin{array}{l} \because z° + t° = x \\ p + q = 46° \end{array}\right]$$

$$x = 46° + 64°$$

$$x = 110°$$

EXAMPLE-15. In the given figure $\angle A = 40^o$. If BO and CO are the respective bisectors of $\angle B$ and $\angle C$ then find the measure of $\angle BOC$

SOLUTION : Say $\angle CBO = \angle ABO = x$ and $\angle BCO = \angle ACO = y$

($\because BO$ is bisector of $\angle B$ and CO is bisector of $\angle C$)

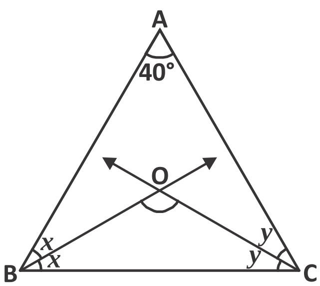


Fig. 42

Then , $\angle B = 2x, \angle C = 2y$

By angle sum property we can say,

$$\angle A + \angle B + \angle C = 180^o$$

$$\Rightarrow 40^o + 2x + 2y = 180^o \quad [\because \angle A = 40^o]$$

$$\Rightarrow 2(x + y) = 180^o - 40^o$$

$$\Rightarrow x + y = 70^o \qquad .....(1)$$

Again by angle sum property of ΔBOC,

$$x + \angle BOC + y = 180^o$$

$$\Rightarrow \angle BOC = 180^o - (x + y)$$

$$\Rightarrow \angle BOC = 180^o - 70^o \qquad \text{(From equation (1))}$$

$$\therefore \angle BOC = 110^o$$

EXAMPLE-16. In the given figure, the sides AB and AC of ΔABC are extended to E and D respectively. If the bisectores of $\angle CBE$ and $\angle BCD$, that is BO and CO respectively meet in point O, then prove that $\angle BOC = 90° - \frac{1}{2}\angle BAC$

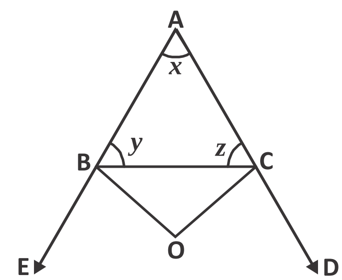


Fig. 43

SOLUTION : Ray BO bisects $\angle CBE$, hence

$$\angle CBO = \frac{1}{2}\angle CBE$$

$$= \frac{1}{2}(180^o - y)$$

$$= 90^o - \frac{y}{2} \qquad .....(1)$$

Similarly CO bisects $\angle BCD$,

hence $\angle BCO = \frac{1}{2}\angle BCD$

$$= \frac{1}{2}(180^o - z) \quad = 90^o - \frac{z}{2} \qquad .....(2)$$

Now in ΔBOC, by angle sum property,

$$\angle BOC + \angle BCO + \angle CBO = 180^o \qquad .....(3)$$

Substituting (1) and (2) in (3), we get

$$\angle BOC + 90^o - \frac{z}{2} + 90^o - \frac{y}{2} = 180^o$$

$$\Rightarrow \angle BOC + 180^o = 180^o + \frac{z}{2} + \frac{y}{2}$$

$$\therefore \angle BOC = \frac{1}{2}(z + y) \qquad .....(4)$$

Now in ΔABC by angle sum property,

$$x + y + z = 180^o$$

$$y + z = 180^o - x \qquad .....(5)$$

Substituting (5) in (4) we get,

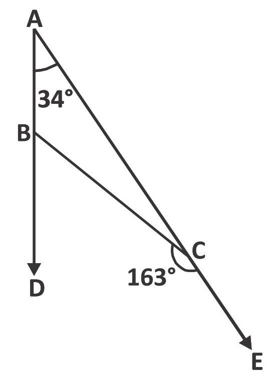
$$\angle BOC = \frac{1}{2}(180^o - x)$$

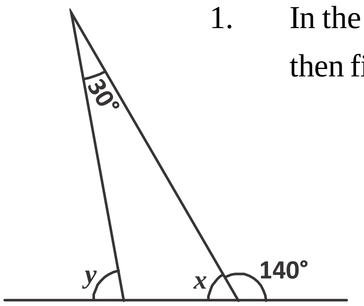
$$= 90^{\circ} - \frac{x}{2}$$

$$\angle BOC = 90^{\circ} - \frac{1}{2}\angle BAC$$

Exercise-9.3

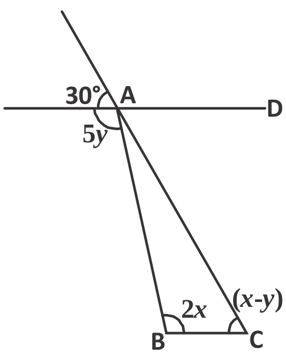
1. In the adjoining figure if $\angle BAC = 34^{\circ}$, $\angle BCE = 163^{\circ}$ then find measure of $\angle ACB$, $\angle ABC$ and $\angle DBC$.





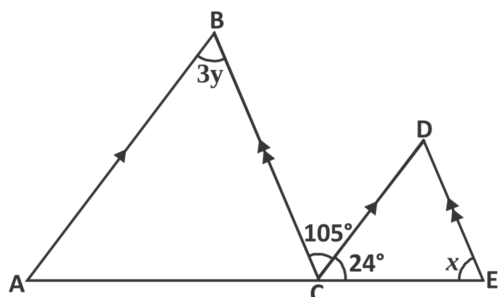
(i)

2. Find the value of x and y from the given figures :



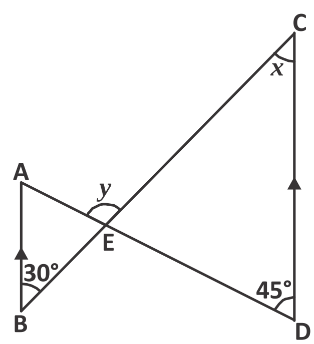
(ii)

(Hint AD || BC)



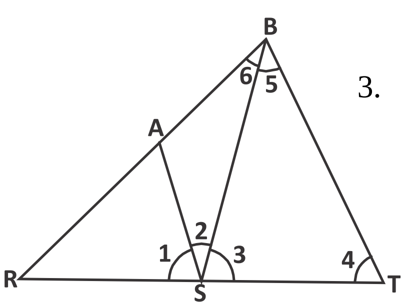
(iii)

(Hint AB || CD, BC || DE)



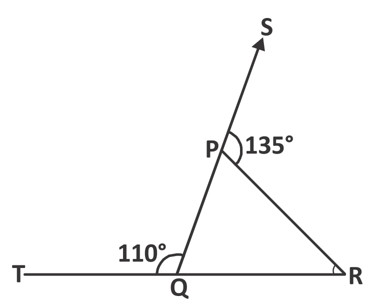
(iv)

(Hint AB || CD)

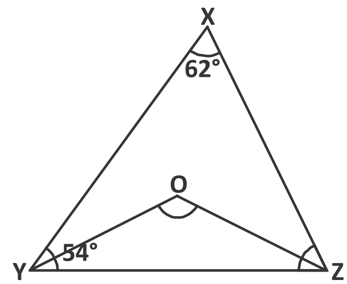


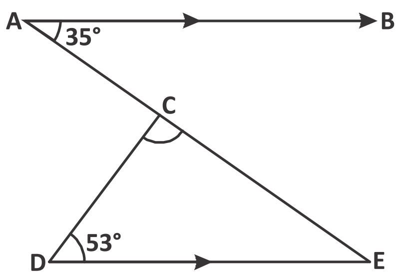
3. In the given figure $AS \parallel BT$, $\angle 4 = \angle 5$ and SB is an angle bisector of $\angle AST$, Find the value of $\angle 1$.

4. In the given figure, the side QP and RQ of the ΔPQR are extended to points S and T respectively. If $\angle SPR = 135^{\circ}$ and $\angle PQT = 110^{\circ}$, find the measure of $\angle PRQ$.



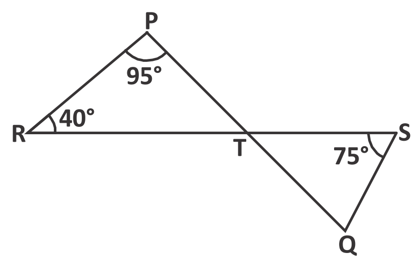
5. In the given figure $\angle ZXY = 62^{o}$ and $\angle XYZ = 54^{o}$. If YO of ZO are the bisectors of $\angle XYZ$ and $\angle XZY$ respectively in ΔXYZ, find value of $\angle OZY$ and $\angle YOZ$.

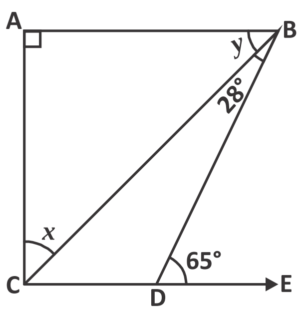




6. In the given figure $AB \parallel DE$, $\angle BAC = 35^{o}$ and $\angle CDE = 53^{o}$ find the measure of $\angle DCE$.

7. In the given figure lines PQ and RS intersect at a point T. If $\angle PRT = 40^{o}$, $\angle RPT = 95^{o}$ and $\angle TSQ = 75^{o}$, find value of $\angle SQT$

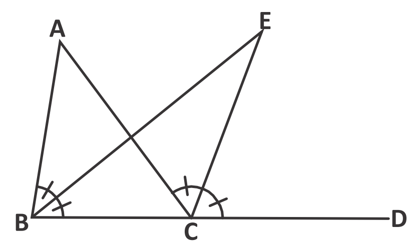




8. In the figure, $AB \perp AC$, and $AB \parallel CD$, if $\angle CBD = 28^{o}$, $\angle BDE = 65^{o}$, find the values of x and y

9. In the adjoining figure the side BC of ΔABC is extended upto D. If the bisectors of $\angle ABC$ and $\angle ACD$ meet in a point E then prove that

$$\angle BEC = \frac{1}{2}\angle BAC$$



(**Hint–** Angle sum of $\Delta ABC =$ Angles sum of ΔBEC and $\angle ACD = \angle BAC + \angle ABC$)

What Have We Learnt

1. If a ray stands on a straight line, the sum of the two adjacent angles so formed is 180^{o} and these are known as a linear pair.

2. If the sum of two adjacent angles is 180^{o}, then the sides which are not common, form a straight line.

(The above two axioms together are known as the linear pair axiom)

3. The vertically opposite angles formed by two intersecting lines are equal.

4. If a transversal cuts two parallel lines, then –
   (i) Each pair of corresponding angles are equal
   (ii) Each pair of alternate interior angles are equal.
   (iii) Each pair of cointerior angles on one side of the transversal are supplementary.

5. Two lines are parallel if the transversal which intersects is such that-
   (i) One pair of corresponding angles is equal or
   (ii) One pair of alternate interior angles is equal or
   (iii) One pair of cointerior angles on one side of the transversal is supplementary.

6. Those lines which are parallel to the same line, are parallel to each other.

7. The sum of the interior angles of a triangle is 180°.

8. If any one side of a triangle is extended, the exterior angle so formed is equal to the sum of the interior opposite angles.

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What is Congruency?

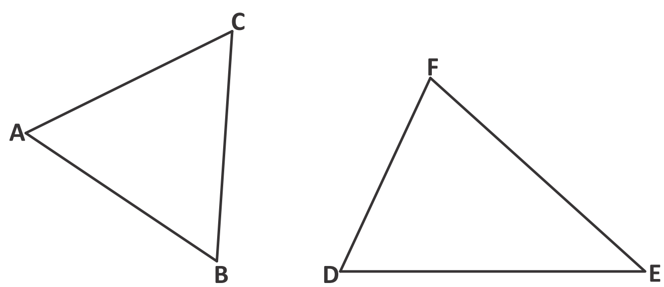


Fig. 1

See the triangles in *Fig.*1. Are they of same measure? If you put one triangle on the other, do they cover each other? In this figure we can see that they are not same, as the sides of the triangle are not same.

How can we know whether two shapes cover each other completely or not? In the given triangles it will only happen when point A falls on point D, point B on point E and point C falls on point F and that is only possible if all the sides and angles of one triangle are equal to the other. That means the two triangles are congruent.

Congruence means that all parts are equal. Those shapes, in which all their parts are equal will cover each other completely.

In reference to triangles it means all sides and angles of one triangle are equal to the corresponding side and angle of the other triangle. Similarly we can also check congruency in quadrilaterals and pentagons. But is it necessary to check the equality of all parts when two figures are congruent ? Or is there some special curcumstances when we can see some parts of those figures and comment on the congruency?

Circle, Square and Rectangle

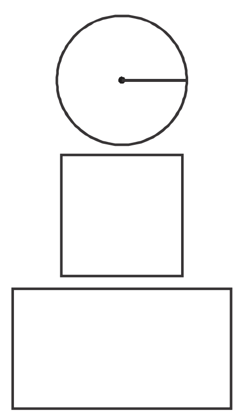


Fig. 2

A Square has four sides and four angles and each of its side is equal and every angle is 90°. If the side of two squares are equal, we can say that both are congruent, and they would cover each other completely.

But if one side of a rectangle is equal to the corresponding side of an other rectangle, will they be congruent? Obviously it won't be so. When the two adjacent sides of one rectangle is equal to the corresponding sides of the other rectangle only then will they congruent. Now think when we can say the two circles are congruent? Just by equality of the radii of those circles, we can say that they are congruent.

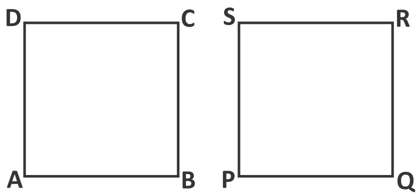


Fig. 3

Precisely we can say-

1. Two squares ABCD and PQRS are congruent if AB = PQ.

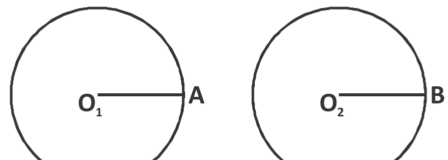


Fig. 4

2. Two circles are congruent if their radii are equal that is $OA_1 = O_2B$.

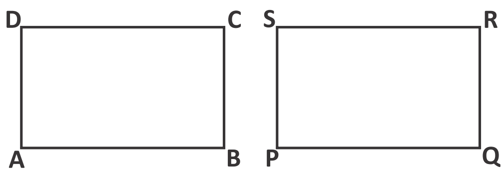


Fig. 5

3. In this way two rectangles are congruent. If there corresponding adjucent sides are equal that is AB = PQ, AD = PS. Like wise in triangles, can we finds some conditions for the congruency? In this chapter we will investigate about this in detail.

Congruency of triangle

In geometry, triangle is a closed figure which is made by the least number of line segments.

All the polygons are made up of triangles, that is why triangle congruency helps in checking the congruency of polygons.

We know that the triangles are congruent if their corresponding sides and angles are equal .

Corresponding Part of a Triangle

Look the triangles ABC and PQR, if we consider sides AB and PQ to be corresponding sides then the remaining corresponding parts of the triangles are as given in the table below.

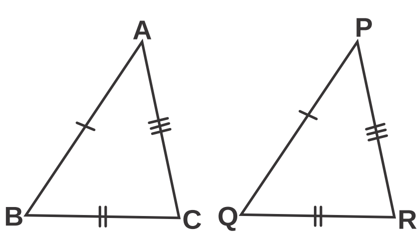


Fig. 6

Corresponding sides	Corresponding Angle	Vertex
$AB \leftrightarrow PQ$	$\angle A \leftrightarrow \angle P$	$A \leftrightarrow P$
$BC \leftrightarrow QR$	$\angle B \leftrightarrow \angle Q$	$B \leftrightarrow Q$
$AC \leftrightarrow PR$	$\angle C \leftrightarrow \angle R$	$C \leftrightarrow R$

é sign indicates correspondancy and] sign relates to congruency.

If two triangles ΔABC and ΔPQR are congruent. That is $\Delta ABC \cong \Delta PQR$, than $AB = PQ$, $BC = QR$, $AC = PR$, $\angle A = \angle P$, $\angle B = \angle Q$, $\angle C = \angle R$

We can also write $\Delta BCA \cong \Delta QRP$ or $\Delta CAB \cong \Delta RPQ$.

We know that ΔABC and ΔBCA or ΔCAB all are same.

We can write ΔBCA congruent to ΔQRP. But we shouldn't write-

$$\Delta ABC \cong \Delta RQP \quad \text{or} \quad \Delta BCA \cong \Delta RPQ \text{ (why?)}$$

In triangles congruency it is necessary to write angle and vertices in the correct sequence. An abbreviation for corresponding parts of congruent triangles is CPCT.

How to check triangle congruency

Is it necessary to show the equality of all parts of a triangle when looking for congruency in triangles? Let us try to find conditions for triangles congruency mathematically.

(i) **Side-angle-side congruency (SAS congruency)-** "Two triangles are congruent if two sides and the included angle of one triangle are equal to the two sides and the included angle of the other triangle".

(ii) **Angle-side-angle congruency (ASA congruency)-** "Two triangles are congruent if two angles and the included side of the one are equal to the two angles and the included side of the other.

(iii) **Side-side-side congruency (SSS congruency)-** Two triangles are said to be congruent if three sides of one triangles are equal to the three sides of the other triangles.

These three self evident postulate , can be used to identify the congruence of triangles, and on the basis of these tests we can find new tests to check for congruency of triangles. But identification is only possible if we can prove these like the theorems.

AXIOM, POSTULATE AND THEOREM

While learning mathematics we come across words like Axioms, postulates, theorem, corollary. Let us understand these words in brief-

Axiom and Postulate : Both are self evident, based on this we make new statements and prove them. In general, logical self evident statements are used in all subjects. We call them Axioms. Like in euclidian geometry some axioms are (i) if a is equal to b, and a also equals c, then b is equal to c. (2) Whole is bigger than its part. Like wise we can take some others axioms.

Postulate : Those self evident truths which are related to some specific subjects are know as postulates. Often axioms and postulates are treated as synonyms. Some geometrical postulates are- A straight line can be drawn using two given points and the points will be on this line. Or any line segment can be extended on either side to form a line.

Theorem and corollary : All those mathematical statements which are logically proved by using axioms, postulate and definitions are called theorems. Like the sum of interior angles of a triangle is 180°

On the basis of theorems, axioms and postulate some more theorems can be proved, these are called corollaries. In geometry there is no clear distinction between corollary and theorems. They often used one instead of the other.

(iv) Angle-angle-side congruence (AAS congruence theorem):

Theorem-10.1 : Two triangles are congruent if any two pairs of angles and one pair of corresponding sides are equal.

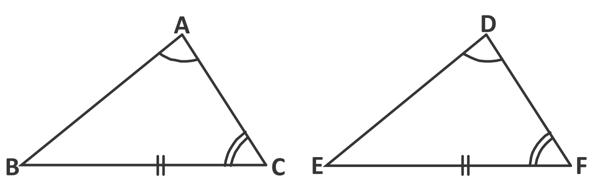


Fig. 7

Given $\angle A = \angle D$, $\angle C = \angle F$

and $\overline{BC} = \overline{EF}$

Two angles is in triangle ΔABC and ΔDEF are equal so the third angle is also equal.

$\because \angle A = \angle D$ and $\angle C = \angle F$ (given)

$\therefore \angle B = \angle E$ (sum of interior angles of a triangle is 180°)

Because side $\overline{BC}$ lies in between $\angle B$ and $\angle C$,

We can use ASA congruence rule to prove that ΔABC and ΔDEF are congruent.

Therefore $\Delta ABC \cong \Delta DEF$ (ASA congruency)

(v) Right angle hypotenuse side theorem:

Theorem-10.2 : Two triangles are congruent if the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and corresponding side of the other.

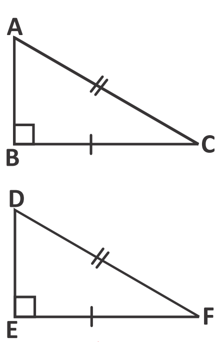


Fig. 8

Given in triangle ΔABC and ΔDEF

$\angle B = \angle E = 90°$, $AC = DF$ and $BC = EF$

We have to prove that $\Delta ABC \cong \Delta DEF$,

Produce DE to P so that $EP = AB$, join PF.

$\therefore$ $\Delta ABC \cong \Delta PEF$ (by S.A.S.)

$\therefore$ $\angle A = \angle P$ ... (1) C.P.C.T.

$AC = PF$ ...(2) C.P.C.T.

But $AC = DF$ given.

$\therefore$ $DF = PF$ and $\angle D = \angle P$ ... (3) (Angles opposite to equal sides of ΔDPE are equal)

From (1) and (3)

$\angle A = \angle D$

Now again in ΔABC and ΔDEF

$BC = EF$, $AC = DF$ (known)

$\angle ACB = \angle DFE$

$\therefore\ \Delta ABC \cong \Delta DEF$ (SAS Congruency)

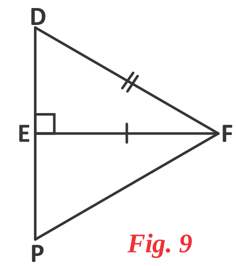


Fig. 9

THEOREMS AND SELF PROOFS

There are few statements in geometry, where it is not sure, whether they be considered as theorem or postulates. To prove theorem in a simple way and understand logical relationships in Geometry, we can select postulates in different ways. In some books AAS and ASA are taken as postulate to prove the theorem SSS. While in some others, AAS is taken as a postulate to prove the theorem ASA .

In this textbook we will consider ASA, SAS and SSS as postulates (self proved) and AAS, RHS as theorems.

Solution of the problems can be demonstrated in different ways. Some are shown below:-

EXAMPLE-1. In this example we use SAS condition to know more about this given figure.

In figure OA = OD and OB = OC.

Prove that

1. $\Delta AOB \cong \Delta DOC$
2. $AB \parallel CD$

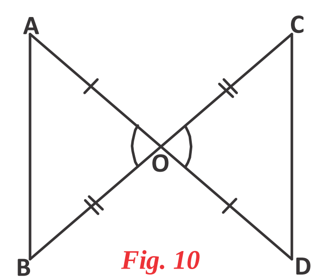


Fig. 10

SOLUTION : 1. In triangle ΔAOB and ΔDOC-

$OA = OD$ given .....(1)

$\angle AOB = \angle DOC$ (vertically opposite angles are equal) ....(1)

$OB = OC$ given .....(3)

Equation (1), (2) and (3) fulfill all the three conditions for congruence.

Therefore by SAS rule of congruency we have proved

$\Delta AOB \cong \Delta DOC$

2. The corresponding parts of the congruent triangles ΔAOB and ΔDOC are also equal.

Therefore $\angle OBA = \angle OCD$. Because they are the alternate angles between AB and CD line segment. Hence in this example $AB \parallel CD$.

EXAMPLE-2. If in triangle ΔABC, $AP = PB$ and $CP \perp AB$. Then prove that-

1. $\Delta CPA \cong \Delta CPB$
2. $AC = BC$

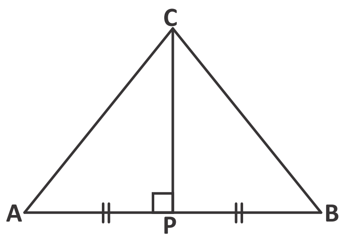


Fig. 11

SOLUTION : 1. In ΔCPA and ΔCPB

$AP = PB$	(given) .....(1)
$\angle APC = \angle BPC = 90°$	(given) .....(2)
$CP = CP$	(common side) ...(3)

Therefore by SAS congruence $\Delta CPA \cong \Delta CPB$

2. $\Delta CPA \cong \Delta CPB$ so $AC = BC$ (Corresponding parts of Congruent Triangle)

EXAMPLE-3. Angles opposite to equal sides of a triangle are equal.

SOLUTION : If we have a triangle ΔABC in which $AB = AC$

Construct a bisector of $\angle A$, which meets BC in point D.

In ΔABD and ΔACD

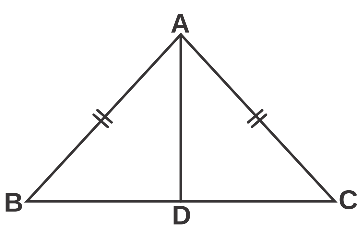


Fig. 12

$AB = AC$	(given)
$\angle BAD = \angle CAD$	(by construction)
$AD = AD$	(common)
$\therefore \Delta ABD \cong \Delta ACD$	(SAS congruence)
$\angle B = \angle C$	(C.P.C.T.)

EXAMPLE-4. If $\overrightarrow{BD}$, is a bisector of $\angle ABC$, and $\overline{AB} = \overline{BC}$ then with the help of SAS congruency prove that $\Delta ABD \cong \Delta CBD$

SOLUTION :

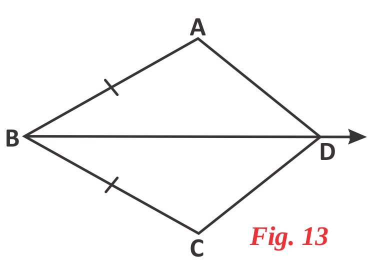


Fig. 13

Statement	Reason
$AB = BC$	given
$\overrightarrow{BD}$ is bisector of $\angle ABC$	given
$\angle ABD = \angle CBD$	By definition of angle bisector
$BD = BD$	common side
$\Delta ABD \cong \Delta CBD$	by SAS congruence

EXAMPLE-5. In figure $AC = BC$, and $\angle DCA = \angle ECB$ and $\angle DBC = \angle EAC$

Prove that- $\Delta DBC \cong \Delta EAC$ in which $DC = EC$

SOLUTION : $\because$ $AC = BC$ (given)

$\therefore$ C is the mid point of AB

$\angle DCA = \angle ECB$ (given)

add $\angle DCE$ on both side

$\angle DCA + \angle DCE = \angle ECB + \angle DCE$

$\Rightarrow$ $\angle ACE = \angle BCD$

$\angle DBC = \angle EAC$ (given)

$\therefore$ $\Delta DBC \cong \Delta EAC$ (by ASA congruence)

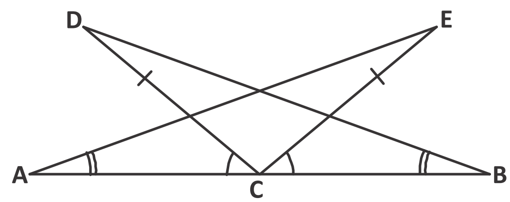


Fig. 14

EXAMPLE-6. Ray $\overrightarrow{AZ}$ bisects angle A, and B is any point on ray $\overrightarrow{AZ}$. BP and BQ are perpendiculars from B to the arms of angle A. Show that-

1. $\Delta APB \cong \Delta AQB$

2. $BP = BQ$ that means point B is equidistant from the sides forming angle A.

SOLUTION : Given $\overrightarrow{AZ}$ is bisector of $\angle QAP$

$\therefore$ $\angle QAB = \angle PAB$

$\angle Q = \angle P = 90°$

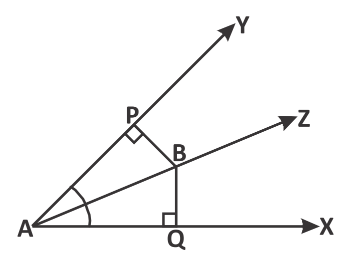


Fig. 15

1. Now in $\Delta APB = \Delta AQB$

$AB = AB$ (common)

$\angle APB = \angle AQB = 90°$ (given)

$\angle PAB = \angle QAB$

$\therefore$ $\Delta APB \cong \Delta AQB$ (by AAS congruence)

2. $\because$ $\Delta APB \cong \Delta AQB$

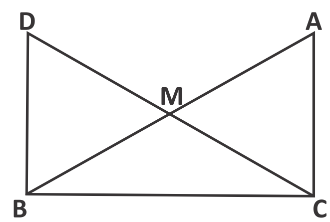
$\therefore$ $BP = BQ$ ($\because$ corresponding parts are equal)

Hence, perpendicular distance of B from AP = perpendicular distance of B from AQ. Therefore, point B is equidistant from $\angle A$.

Try This

In a right angle triangle ΔABC. $\angle C$ is right angle, M mid point of hypotenuse AB. Join C to M and extend it to D such that DM = CM. Jion Point D to B. Show that

1. $\Delta AMC \cong \Delta BMD$
2. $CM = \frac{1}{2} AB$
3. $\Delta DBC \cong \Delta ACB$
4. $\angle DBC = 90°$



EXAMPLE-7. Given GE is a bisector of $\angle DGF$ and $\angle DEF$.

Prove that- $\Delta GDE \cong \Delta GFE$

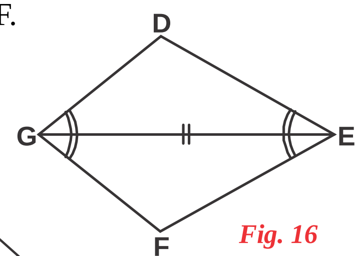


Fig. 16

SOLUTION :

GE is bisector of $\angle DGF$ (Given) → $\angle DGE = \angle FGE$ (By definition of Bisector) →

GE bisector of $\angle DEF$ (given) → $\angle DEG = \angle FEG$ (By definition of Bisector) →

GE = GE (Common side) →

$\Delta GDE \cong \Delta GFE$ (ASA congruence)

EXAMPLE-8. In a triangle XYZ of $\angle Y = \angle Z$ and XP is a bisector of $\angle X$, than prove that P is the midpoint of YZ and $XP \perp YZ$

SOLUTION : In ΔXYP and ΔXZP

$\angle Y = \angle Z$ (given)

$\angle YXP = \angle ZXP$ (XP is angle bisector)

$XP = XP$ (Common side)

$\Delta XPY \cong \Delta XZP$ (AAS congruence)

$\therefore YP = PZ$ (by CPCT)

Hence P is the mid point of YZ

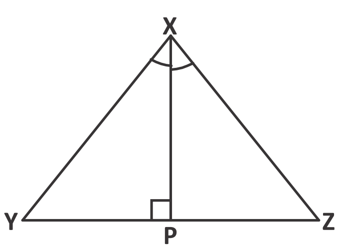


Fig. 17

$\angle YPX = \angle ZPX$ (by CPCT)

$\therefore \quad \angle YPX + \angle ZPX = 180°$ (linear pair)

$\angle YPX + \angle YPX = 180°$ $(\because \angle YPX = \angle ZPX)$

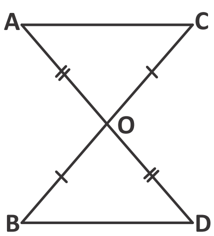
$\angle YPX = 90° = \angle ZPX$

$\therefore \quad XP \perp YZ$

Try This

Prove that the two triangles made by the diagonal of parallelogram are always congruent to each other.

Exercise - 10.1



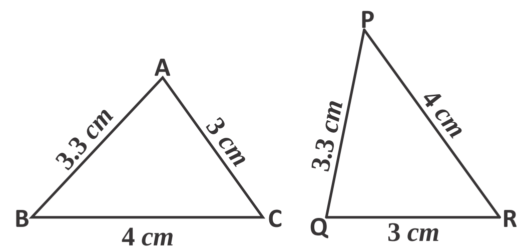
1. In a figure if OA = OD and OB = OC then which of the given statements is true-

   a) $\Delta AOC \cong \Delta BDO$

   b) $\Delta AOC \cong \Delta DOB$

   c) $\Delta CAO \cong \Delta BOD$

2. See the given figure of ΔABC and ΔPQR and tell which statement is true-



   a) $\Delta ABC \cong \Delta PQR$

   b) $\Delta ABC \cong \Delta QPR$

   c) $\Delta ABC \cong \Delta PRQ$

3. From the following which is not the condition for congruence.

   a) SSS b) SAS c) AAA

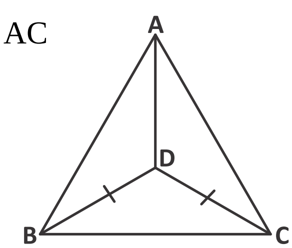
4. Besides equivalance of two corresponding angle, which least element is necessary to state that the given two triangles are congruent.

   (a) No corresponding side is equal

   (b) At least one corresponding side is equal.

   (c) Third corresponding angle is equal

5. In the figure $\angle B = \angle C$, BD and CD are the bisector of them. Then AB : AC will be-

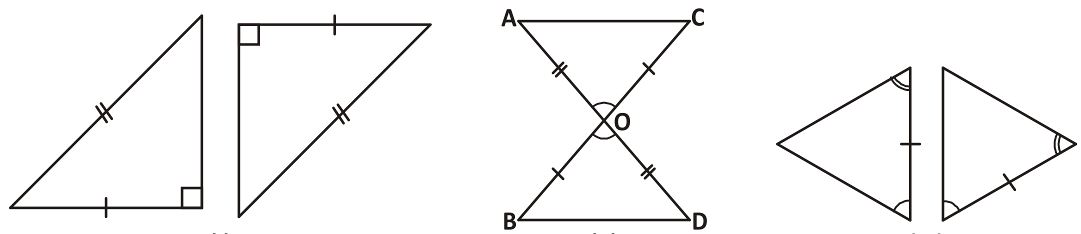


   a) 2 : 1

   b) 3 : 2

   c) 1 : 1

6. See the triangle pairs and state which condition of congruence applies to which pair.

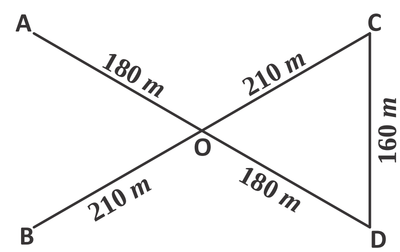


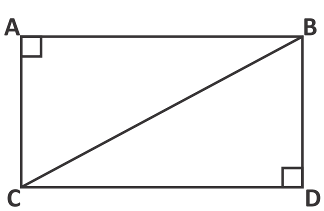
(i) (ii) (iii)

(iv) (v)

7. If $\Delta ABC \cong \Delta PQR$, AC = $3x + 2$, PR = $6x - 13$ and BC = $5x$ then find the value of QR.

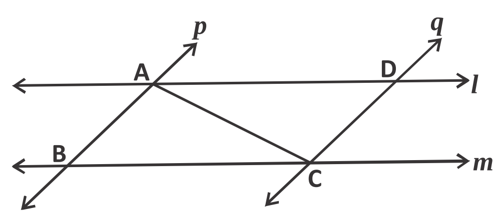
8. With the help of the given points, find the distance between A and B, also give reason to your statement.

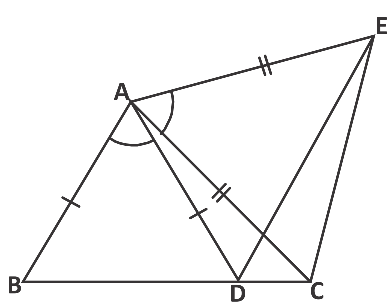




9. For a running competition there is a special arrangment made for two teams. One team runs from A to B and then from B to C and returns to the starting point A. While the other team starts from C and via D to B and then B to the point C again. If $\overline{AB} \parallel \overline{CD}$ and $\angle A = \angle D = 90°$, then the length of the travel done by the teams are equal. Also explain your answer.

10. l and m are parallel lines, which have been intersected by the parallel lines p and q. Show $\Delta ABC \cong \Delta CDA$ (write answer in flow-chart way).





11. In a figure AC = AE, AB = AD and $\angle BAD = \angle EAC$. Then show that BC = DE.

Property of Isosceles Triangle

So far we have studied about the rules of congruency in triangles. Let us apply these rules to know some more characteristics of triangles.

A triangles which has two equal sides is called an isosceles triangle. Let us understand some characteristics of isosceles triangles.

Try This

Construct a triangle which has two sides of measure 4.5 cm and another side is of 6 cm.

Measure the angles opposite each side, are they equal. You will find, angle opposite to equal sides are equal.

Make some more isosceles triangles with different sides, which shows this important characteristic of isosceles triangle.

Theorem-10.3 : Any triangle which has equal sides will have equal angles opposite to these.

Let us prove this mathematical statement.

We have taken isosceles triangle ABC

Which has sides AB = AC

We have to prove that $\angle B = \angle C$

For this we draw angle bisector of $\angle A$ which meet to side BC at point D.

After angle bisector we can see two triangles

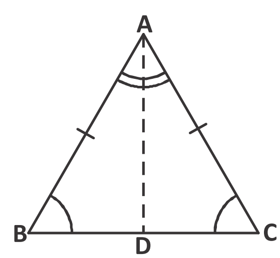


Fig. 18

In- ΔBAD and ΔCAD

$AB = AC$	(given)
$\angle BAD = \angle CAD$	(by construction)
$AD = AD$	(common side)
Therefore, $\Delta BAD \cong \Delta CAD$	(side-angle-side congruency law)

$\therefore\ \angle B = \angle C$ (by CPCT)

Therefore this statement is true for every isosceles triangle. Now let us consider its converse and think about it.

Theorem-10.4 : (Converse of theorem-10.3) If two angles of a triange are equal, then the sides opposite to these must be equal.

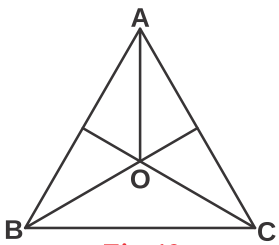


Fig. 19

EXAMPLE-9. In a isosceles triangle ABC, AB = AC. The angle bisectors of $\angle B$ and $\angle C$ and intersect at point O. Show that-

1. OB = OC   2. AO bisects angle A

SOLUTION : ΔABC

Statement	Reason
AB = AC	given
$\therefore \angle C = \angle B$	angles opposite to equal sides are equal
$\therefore \frac{1}{2}\angle B = \frac{1}{2}\angle C$	
$\angle OCB = \angle OBC$	bisector
OB = OC	sides opposite to equal angles are equal

2. In ΔABO and ΔACO

Statement	Reason
AB = AC	given
OB = OC	already proved
$\angle OBA = \angle OCA$	$\because \frac{1}{2}\angle C = \frac{1}{2}\angle B$
$\therefore \Delta ABO \cong \Delta ACO$	by SAS congruency
$\Rightarrow \angle OAB = \angle OAC$	C.P.C.T

Therefore AO bisect angle A.

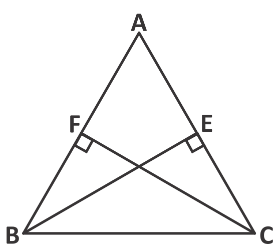


Fig. 20

EXAMPLE-10. ABC is an isosceles triangle in which altitudes BE and CF are drawn to equal sides AC and AB respectively. Show that these attitudes are equal.

SOLUTION : Given, In Δ ABC, AB and AC are equal.

We have to prove that altitude BE = CF

1. From the given reasons select the appropriate reason for each statement.

Angle opposite to equal sides are equal

Each angle is of 90°

Common side

Given

By ASA congruence

Corresponding part of congruent triangles are equal

Statement	Reason
$AB = AC$	
$\angle ACB = \angle ABC$	
$\angle BFC = \angle BEC$	
$BC = BC$	
$\angle BEC = \angle CFB$	
$\therefore \Delta BEC \cong \Delta CFB$	
$BF = CF$	

EXAMPLE-11. In a given figure M is a mid point of AB. $\overline{CA} = \overline{CB}$ then prove that $\Delta ACM \cong \Delta BCM$.

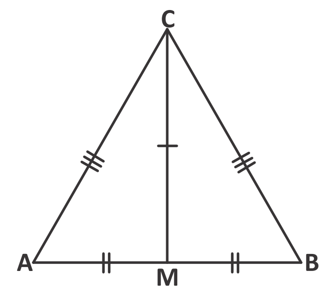
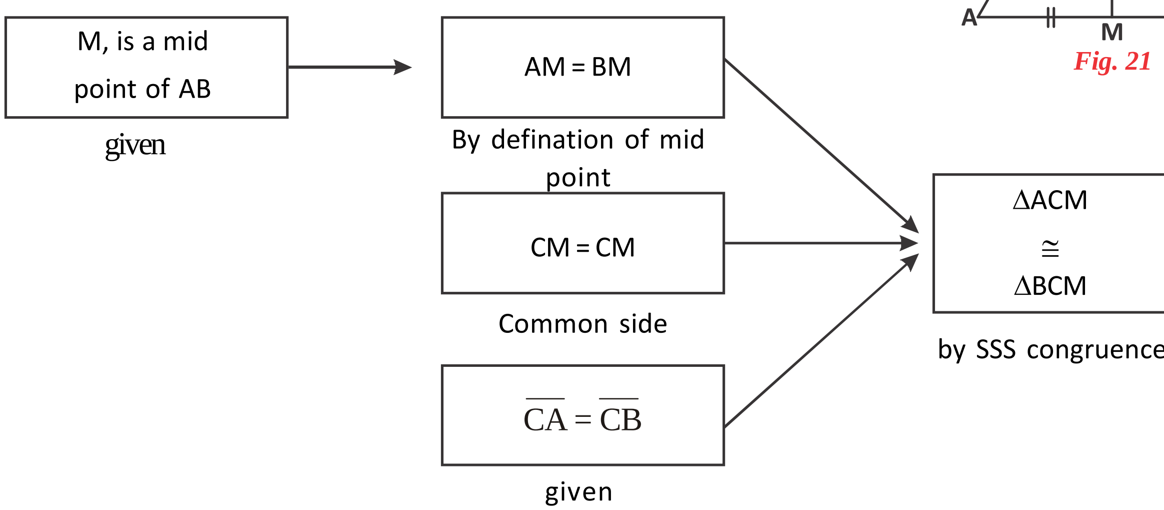


Fig. 21

SOLUTION : Given- M, is a mid point of AB and $\overline{CA} = \overline{CB}$

We have to prove that $\Delta ACM \cong \Delta BCM$.



EXAMPLE-12. Given $\angle O = \angle P = 90°$, $\overline{MN} = \overline{QR}$, $\overline{OM} = \overline{PQ}$

Prove that $\Delta MOR \cong \Delta QPN$

SOLUTION :

Statement	Reason
$\angle O = \angle P = 90°$	given
$OM = PQ$	$\overline{OM} = \overline{PQ}$ (given)
$MN = QR$	$\overline{MN} = \overline{QR}$ (given)
$MN+NR = QR+NR$	adding NR in both the sides
$MR = NQ$	by figure
$\Delta MOR \cong \Delta QPN$	by RHS congruence theorem

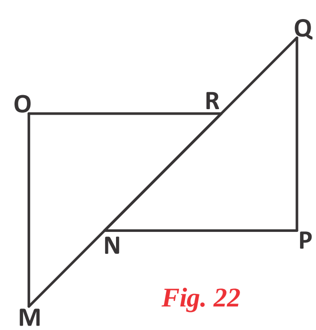
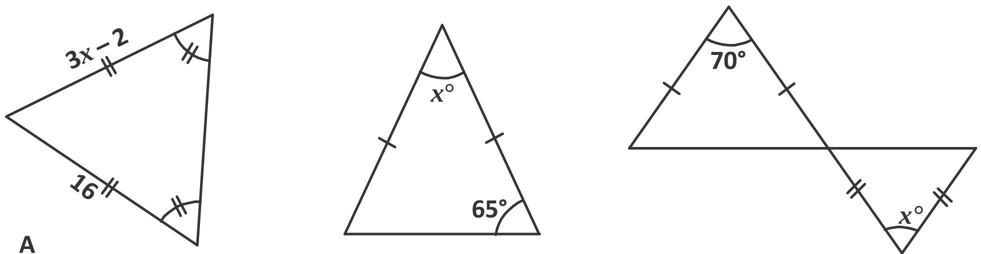
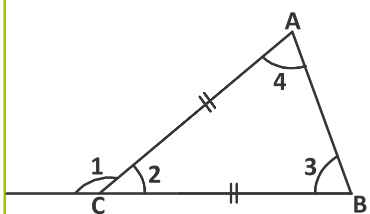


Fig. 22

Try This

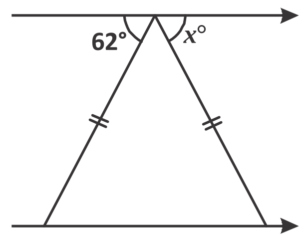
1. In a given isosceles triangle find the value of x.





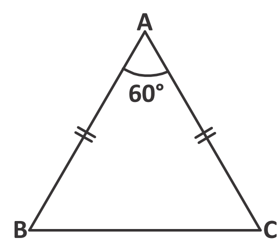
2. $BC \cong AC$ (given) and $\angle 1 = 140°$ then find the measurement of angle $\angle 2$, $\angle 3$ and $\angle 4$.

3. In the given figure find the value of x.



Exercise - 10.2

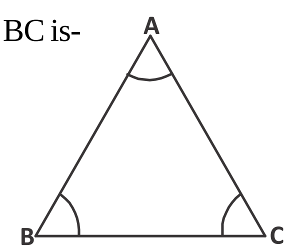
1. In the given figure $AB = AC$ and $\angle A = 60°$ then the measurement of $\angle C$ is -



(i) 35° (ii) 45°
(iii) 60° (iv) 180°

2. In the given figure if $\angle A = \angle B$ then $AC : BC$ is-

(i) 1:1 (ii) 1:2
(iii) 2:1
(iv) None of these



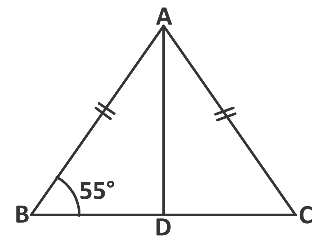
3. In ΔABC if $AB = AC$ and $\angle B = 50°$ then find the measurement of $\angle A$.

(i) 50° (ii) 180° (iii) 100° (iv) 80°

4. In triangle ΔABC if $\angle C = \angle A$ and $AB = 4$ cm, $AC = 5$ cm then BC would be-

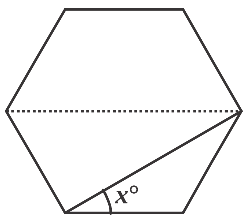
(i) 2 cm. (ii) 3 cm. (iii) 4 cm. (iv) 9 cm.

5. In the given figure $\angle B = 55°$. If D is a mid point of BC and AB = AC. Then the measurement of $\angle BAD$ is-



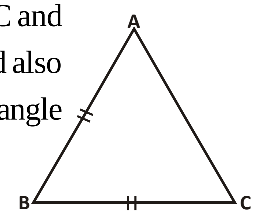
(i) 70° (ii) 55°

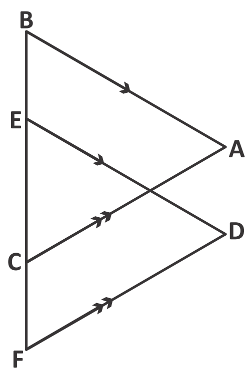
(iii) 35° (iv) 180°



6. For the given hexagon find the value of x.

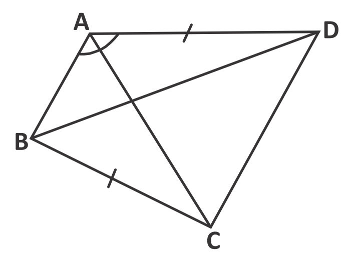
7. In an isosceles triangle ABC, AB = BC with base AC and $\angle A = 2x + 8$, $\angle B = 4x - 20$ then find the value of x and also verify whether this triangle is acute, obtuse or a right angle triangle.



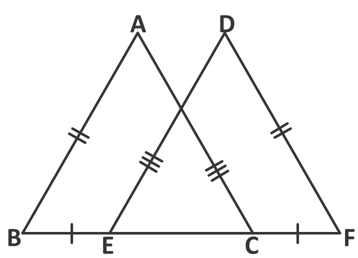


8. In the given figure AB || ED, CA || FD and BC = EF prove that $\Delta ABC \cong \Delta DEF$.

9. ABCD is a quadrilateral in which AD = BC and $\angle DAB = \angle CBA$. Prove that-



1. $\Delta ABD \cong \Delta BAC$
2. BD = AC
3. $\angle ABD = \angle BAC$



10. If AB = DF, AC = DE, BE = FC then prove that $\Delta ABC \cong \Delta DFE$.

Application of Congruency

Two figures of same shape and size are congruent. There are some conditions under which the given two triangles can be called congruent. Like side-side-side equality, angle-side-angle equality etc. Here we shall see the relationship of congruency of a figure and its area.

Will the Area of the Congruent Figures be Equal?

Look at the triangles ABC and PQR drawn on the graph paper. Are they congruent? Which congruence conditions do they satisfy?

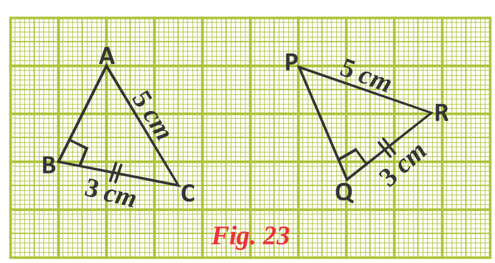


Fig. 23

In Δ ABC and ΔPQR-

$$\angle B = \angle Q = 90°, AC = PR \text{ and } BC = QR$$

That means by RHS congruency theorem ΔABC and ΔPQR are congruent.

Now we will find the area of these triangles.

In ΔABC, BC = 3 *cm* and AC = 5 *cm*.

Then, $AB = \sqrt{(AC)^2 - (BC)^2} = \sqrt{(5)^2 - (3)^2} = \sqrt{16} = 4$ (why?)

$\therefore$ AB = 4 cm.

Therefore, the area of $\Delta ABC = \frac{1}{2} \times 3 \times 4 = 6$ cm^2

Likewise in ΔPQR, PQ = 4 *cm*

Then, the area of ΔPQR is 6 cm^2

$\therefore$ Area of ΔABC = Area of ΔPQR

Make some more triangles congruent to ΔABC and ΔPQR. Are they all are equal in area?

You will find that all the congruent triangles have the same area.

Now, Look at *Fig.*23.

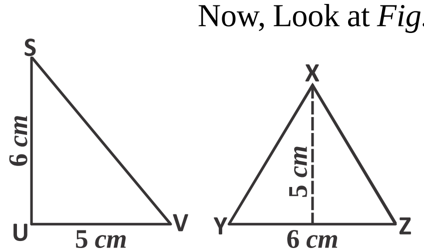


Fig. 24

Area of ΔSUV and ΔXYZ is 15 cm^2 (How?)

Are ΔSUV and ΔXYZ congruent? These two triangles are not congruent because they are not of same shape and size.

Make triangles with area of 15 cm^2 and test their congruency.

We can say that if two figures are congruent then their area must be equal. But if the area of two figures is the same they may and may not be congruent.

This characteristic is not limited to triangles only but we can also see this in other geometrical shapes like circle, quadrilateral, pentagon etc.

This charateristic of congruent figures can be used in finding the area of different figures in different contexts. Now we will consider some situations where we use this characteristic to find some new information or some new relationships.

Figures on the Same Base & between Same Parallels

See the figures given below :

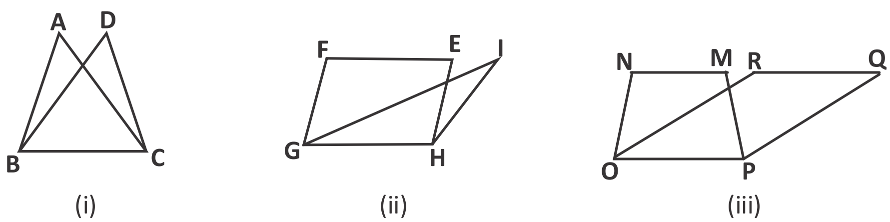


Fig. 25

You can see that in figure (i), ΔABC and ΔDBC have the same common base BC. In figure (ii) quadrilateral EFGH and ΔIGH have the common base GH. Likewise in figure (iii) trapezium MNOP and parallelogram QROP have a common base OP. Now if we construct in figure (i), (ii) and (iii) we find some new situations.

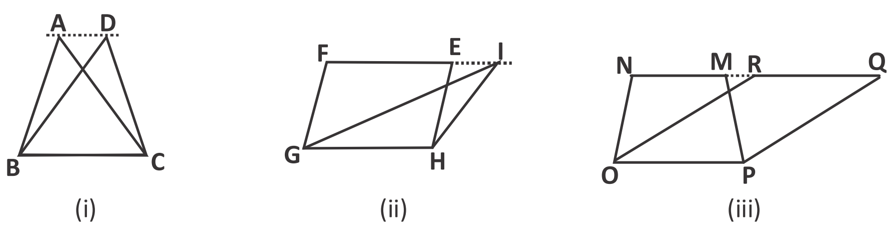
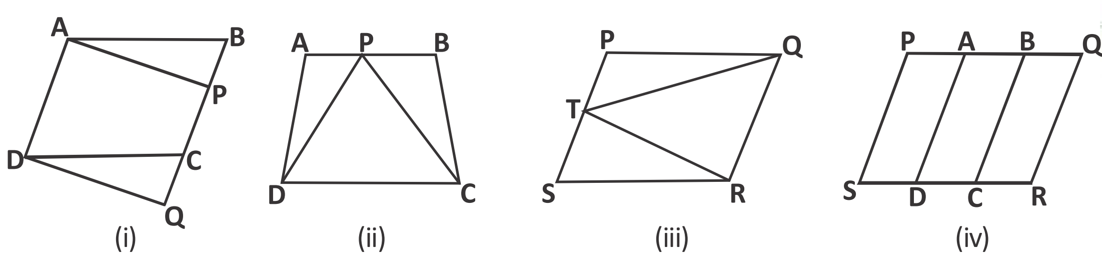


Fig. 26

After the construction we see that in figure (i) AD || BC ΔABC and ΔDBC are on the same base and in between same parallel lines AD and BC. Likewise EFGH and GHI are on the same base GH and in between same parallel lines EI and GH. In figure (iii) Trapezium MNOP and parallelogram OPQR are on the same base OP and in between same parallel lines NQ and OP.

Try This

Find the figures which are situated on the same base and in between the same parallel lines?



Area of the Figures which are on the Same Base and between Same Parallel Lines.

Now we will see the relationship between the area of those figures which are on the same base and between same parallel lines.

Suppose two parallelograms ABRS and PQRS are on the base SR and in between same parallel line AQ and SR.

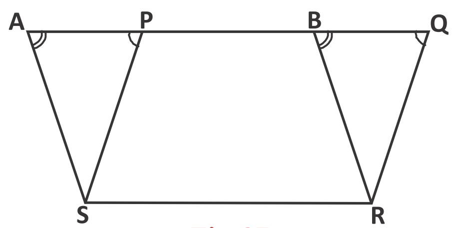


Fig. 27

In ΔAPS and ΔBQR, $AS \| BR$ and AQ is transversal.

$\angle SAP = \angle RBQ$ (corresponding angle)

and $PQ \| SR$ and AQ is transversal

$\angle SPA = \angle RQB$ (corresponding angle)

and $AS = BR$ ($\because$ ABRS is parallelogram)

$\therefore \Delta SPA \cong \Delta RQB$

therefore, area of ΔSPA = area of ΔRQB

So, area of ABRS = area of ΔAPS + area of trapezium PBRS (why?)

Area of parallelogram PQRS = area of ΔBQR + area of trapezium PBRS

Area of parallelogram ABRS = area of parallelogram PQRS

Clearly the area of parallelograms which are on the same base and in between the same parallel lines is equal.

So, parallelograms which are drawn on same base and lie between same parallel lines have an equal area. This is clearly a theorem which can be written as follows:

Theorem-10.5 : Prallelograms which are on the same base and between the same parallels have an equal area.

Example-13. PQRS is a parallelogram and PQTV is a rectangle. SU is a perpendicular on PQ.

Prove that (i) area of PQRS = area of PQTV

(ii) area of PQRS = PQ × SU

Solution : (i) Rectangle is also a parallelogram, and we have to prove that area of parallelogram PQRS = area of rectangle PQTV.

Can we prove this through the help of *Fig.*27?

Yes, we can see in the figure that parallelogram PQRS and rectangle PQTV are on the same base PQ and these two figures lie between parallel lines PQ and VR.

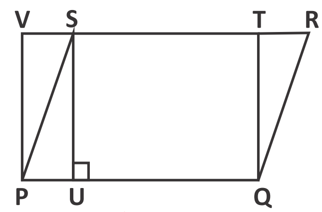


Fig. 28

We already know that the area of the parallelograms which are on the same base and between the same parallels are equal to one another, therefore,

area of parallelogram PQRS = area of rectangle PQTV

(ii) area of PQRS = area of PQTV

= PQ × TQ

= PQ × SU (SU is a perpendicular on PQ , so SU = TQ why?)

Therefore, area of PQRS = PQ × SU

So, the area of a parallelogram is the product of any parallel side to the height with reference to these parallel lines.

EXAMPLE-14. If triangle ABC and parallelogram ABEF are on the same base AB and in between parallel lines AB and EF, then prove that-

area of $\Delta ABC = \frac{1}{2} \times$ area of parallelogram ABEF

SOLUTION : According to the question construct ΔABC and parallelogram ABEF on the same base AB and in between parallel lines AB and EF.

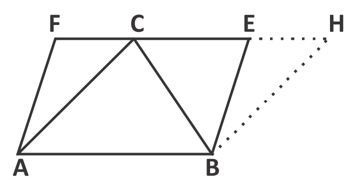


Fig. 29

Area of $\Delta ABC = \frac{1}{2} \times$ area of parallelogram ABEF to prove this

Construct BH parallel to AC which intersect FE extended in H.

Through construction we get parallelogram ABHC. BC is a diagonal which divides it into two triangles ΔABC and ΔBCH.

area of $\therefore$ ΔABC = area of ΔBCH (why?)

You know that diagonal of parallelogram divides it into two congruent triangles.

therefore, area of parallelogram ABHC = area of ΔABC + area of ΔBCH

area of parallelogram ABHC = area of ΔABC + area of ΔABC

area of parallelogram ABHC = 2 area of ΔABC

or $\frac{1}{2} \times$ area of parallelogram ABHC = area of ΔABC

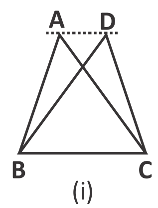
Here area of ABHC = area of ABEF (why?)

(Because ABHC and ABEF are on the same base and lie between same parallel lines)

therefore, area of triangle $\Delta ABC = \frac{1}{2} \times$ area of ABEF

Triangles on the Same Base and lying between Same parallel Lines

Let there be two triangle ABC and DBC on the same base BC and between parallel lines AD and BC.



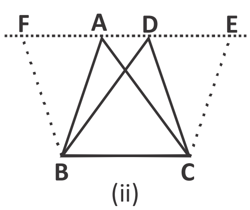


Fig. 30

Now we construct CE‖BA and BF‖CD so we get parallelograms AECB and FDCB on the same base BC and between parallel lines BC and EF.

Where the area of AECB = area of FDCB (why?)

area of $\Delta ABC = \frac{1}{2} \times$ area of AECB ....(1)

(diagonal of parallelogram divides it into two congruent triangle)

and area of $\Delta DBC = \frac{1}{2} \times$ area of FDCB

($\because$ area of AECB = area of FDCB)

area of $\Delta DBC = \frac{1}{2} \times$ area of AECB ....(2)

therefore, from eq. (1) and (2) we can say that-

area of ΔABC = area of ΔDBC

It is clear now that triangles which are on the same base and between the same parallel lines are equal in area. This is a theorem and can be written as-

Theorem-10.6 : Triangles which are on the same base and in between same parallel lines are equal in area.

Now we will find the relation of the area of a triangle to its base and the corresponding height (altitude)

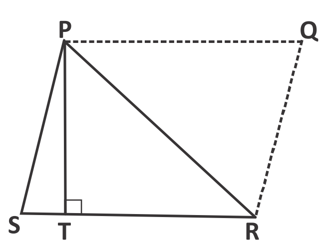


Fig. 31

Assume PSR is a triangle, where SR is base and PT its height.

PT $\perp$ SR now construct PQ and RQ such that PQ‖RS and RQ‖SP so that we get PQRS as parallelogram in which area of ΔPSR = area of ΔPQR (why?)

The diagonal divides a parallelogram into two congruent triangles.

Area of parallelogram PQRS = area of ΔPSR+ area of ΔPQR

Area of parallelogram PQRS = area of ΔPSR + area of ΔPSR

$$\text{Area of } \Delta PSR = \frac{1}{2} \times \text{areaof } PQRS$$

$$= \frac{1}{2} \times SR \times PT$$

Clearly the area of a triangle is half of the products of base and its corresponding height.

Theorem-10.7 : The area of a triangle is half of the product of its base and its corresponding height.

We know that triangles on the same base and between same parallels are equal in area. Now can we say that two triangles with same base and equal areas lie between the same parallel?

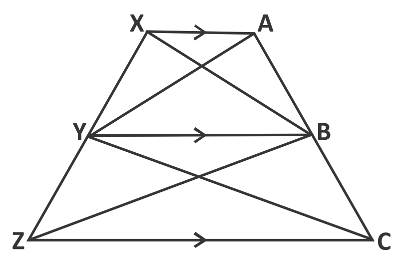


Fig. 32

EXAMPLE-15. According to the figure XA || YB || ZC. Prove that the area of (ΔXBZ) = area of (ΔAYC).

SOLUTION : ΔXYB and ΔABY, on the same base YB and between same parallel lines XA and YB.

$\therefore$ area (ΔXYB) = area (ΔABY) .....(i)

Similarly,

ΔYBZ and ΔBYC on the same base YB and between same parallel lines YB and ZC.

$\therefore$ area of (ΔYBZ) = area of (ΔBYC) .....(ii)

Here area of (ΔXBZ) = area of (ΔXYB) + area of (ΔYBZ)

and area of (ΔAYC) = area of (ΔAYB) + area of (ΔBYC)

by adding equation (i) and (ii),

area of (ΔXYB) + area of (ΔYBZ) = area of (ΔABY) + area of (ΔBYC)

therefore, area of (ΔXBZ) = area of (ΔAYC)

Inequalities in Triangles

We have learnt the relation between triangles based on the equality of sides and angles, but there are a lot of geometrical shapes which are not same but we still compare them for example, the length of segment AB is more than segment CD or $\angle A$ is bigger than $\angle B$.

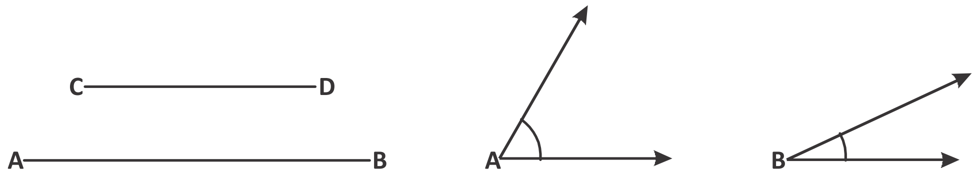


Fig. 33

We will learn about the relation between unequal sides and the angles of a triangle through this activity.

Activity- On a drawing board take two points B and C, put a thread using pin on B and C, now we have a side BC of a triangle.

Put one end of a different thread on C and fix its second end to a pencil. Draw ray $\overrightarrow{CQ}$. Mark a point A with a pencil. Join A to B. Now mark another point P on the same ray. Join P to B, and also Q to B. Now compare the length of PC and AC.

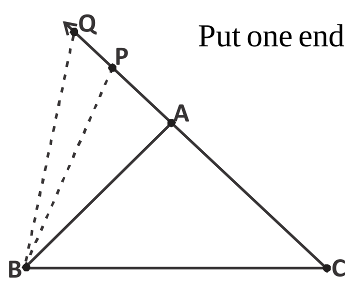


Fig. 34

Is PC > AC? Yes ...... (comparing the length)

Comparing triangle ΔABC and ΔPBC $\angle PBC > \angle ABC$

Likewise if we mark points on CA and keep joining them to B, we will see that as we increase the length of AC, the measurement of $\angle B$ also increases.

Do this with different triangles. We can see other important and interesting inequilities of triangles, some of them are given below in the form of theorems.

Theorem-10.8 : **If in a triangle two sides are unequal then the angle opposite to the longer side is greater.**

Theorem-10.9 : **Side opposite to greater angle is longer in a triangle.**

Theorem-10.10 : **Sum of lengths of any two sides of a triangle is always greater then the third side.**

We will understand theorem 10.10 through an activity.

Fix nails (A, B and C) on a drawing board such that they make a triangular shape.

Now join these three points by a thread and compare the length of the thread of any one side of a triangle to the other two side threads together, you will always find the length of two threads together is always greater then the third thread.

Measure the three sides AB, BC and CA and compare the sum of any two sides in different group to the third side. You will see that-

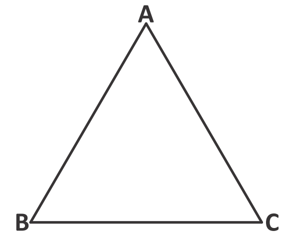


Fig. 35

(i) AB + BC > CA

(ii) BC + CA > AB

(iii) CA + AB > BC

Similarly we can find more results and prove them in the form of theorems.

Let us some examples based on these theorems.

EXAMPLE-16. Prove that hypotenuse is the longest side of any right angles triangle.

SOLUTION : Given in ΔABC

$\angle B = 90°$

We have to prove that AC > AB

and AC > BC

In ΔABC

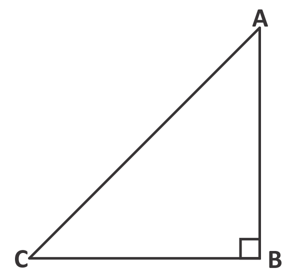


Fig. 36

$\angle B = 90°$ (given)

So $\angle A + \angle C = 90°$ (sum of internal angles of triangle is 180°)

$\therefore \angle A + \angle C = \angle B$

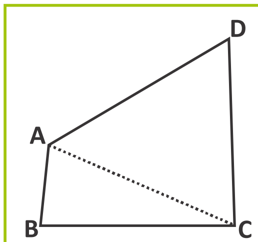
that means $\angle A < \angle B$ and $\angle C < \angle B$

So we can say that $\angle B$ is the biggest angle of ΔABC

We know that the side opposite to the biggest angle is the largest side.

Therefore AC > AB and AC > BC

Think and discuss



AB and CD are respectively the smallest and largest side of a quadrilateral ABCD. Show that $\angle A > \angle C$ and $\angle B > \angle D$.

EXAMPLE-17. In triangle ABC prove that- $\angle ABC > \angle ACB$

Construction- Take point D on AC. such that AB = AD, join B to D.

Corollary : In ΔABD

$AB = AD$ (by construction)

$\angle ABD = \angle ADB$ ....(i) (angle opposite to equal sides)

But $\angle ADB$, is a exterior angle of $\angle BCD$

$\angle ADB > \angle BCD$ ....(ii) (by exterior angle theorem)

by equation (i) and (ii)

$\angle ABD > \angle BCD$

$\angle ABC > \angle ABD$ (by construction)

$\angle ABC > \angle ACB$

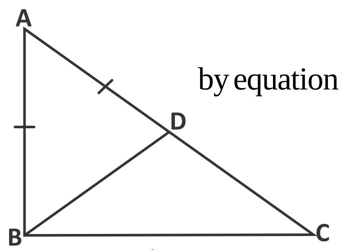


Fig. 37

Try This

Choose right option.

1. From the following given measurement, which make a triangle-

   (i) $10cm, 5cm, 4cm$ (ii) $8cm, 6cm, 3cm$

   (iii) $5cm, 8cm, 3cm$ (iv) $14cm, 6cm, 7cm$

2. In triangle ABC if $\angle C > \angle B$ then which of the following is right-

   (i) $EF > DF$ (ii) $AB > AC$

   (iii) $AB < AC$ (iv) $BC > CA$

3. From the following given measurement, by which can you make a triangle-

   (i) $35°, 45°, 95°$ (ii) $40°, 50°, 100°$

   (iii) $21°, 39°, 120°$ (iv) $110°, 80°, 20°$

4. If in a triangle ABC, AD is a median, then which of the following statement is false-

   (i) $AB + BC > AD$ (ii) $AC + BC > AD$

   (iii) $AB + BC < AD$ (iv) $AB + BD > DC$

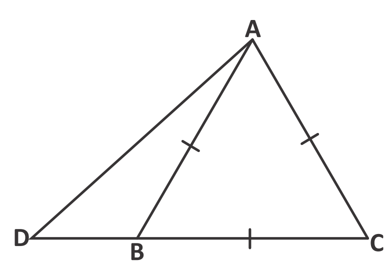
5. In a given figure if $AB = AC = BC$ then which of the following statement is true.

   (i) $AD = AC$

   (ii) $AD < AB$

   (iii) $BC = BD$

   (iv) $AD > AB$



EXAMPLE-18. In a given figure PR > PQ and PS is a angle bisector of ∠QPR, then prove that ∠PSR > ∠PSQ.

SOLUTION : Because PR > PQ

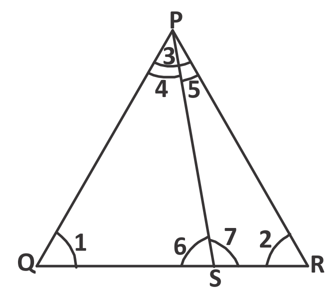


Fig. 38

∴ ∠1 > ∠2

in ΔPQS ∠1 + ∠4 + ∠6 = 180° .....(i)

in ΔPRS ∠2 + ∠5 + ∠7 = 180° .....(ii)

therefore in these two triangles

∠4 = ∠5 .....(iii) (angle bisector of ∠3)

∠1 > ∠2 .....(iv)

by adding (iii) and (iv)

⇒ ∠4 + ∠1 > ∠5 + ∠2 .....(v)

by equation (i) ∠1 + ∠4 = 180° – ∠6

by equation (ii) ∠5 + ∠2 = 180° – ∠7

by putting value in equation (v)

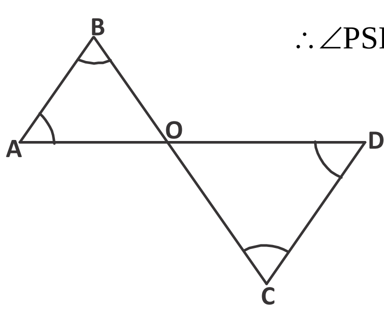
180° – ∠6 > 180° – ∠7

– ∠6 > – ∠7 (by changing side)

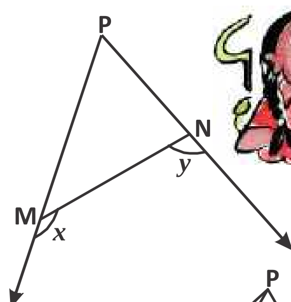
or ∠7 > ∠6

∴ ∠PSR > ∠PSQ

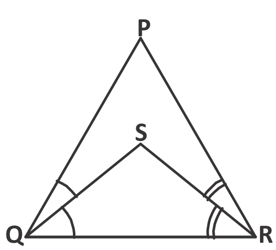
Exercise-10.3



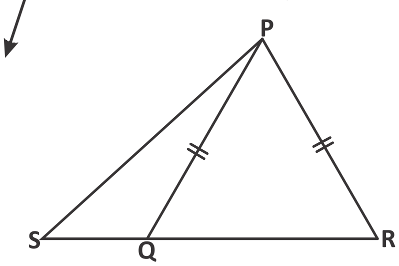
1. In figure ∠B < ∠A and ∠C < ∠D. Show that AD < BC.



2. In a given figure if $x > y$ then prove that MP > NP.



3. In a given figure PQ > PR and QS and RS are angle bisectors of ∠Q and ∠R respectively. Prove that SQ>SR



4. In a given figure PQ = PR then prove that PS > PQ.

Uses of Congruency

Congruency and congruent figures are not only useful in our daily life but this is also seen in the field of engineering when constructing bridges, buildings and towers.

What Have We Learnt

1. Geometrical shapes are congruent if they have the same shape and size.
2. Circle with the same length of radii are congruent.
3. Two squares are congruent if they have sides of equal length.
4. Two triangles are congruent when their corresponding sides and corresponding angles are equal.
5. If two sides and the included angle of one triangle are equal to the corresponding two sides and included angle of the triangle, then the triangles are congruent. (SAS)
6. If any two angles and their included side of triangle are equal to the corresponding two angles and included side of the other triangle, then the triangles are congruent. (ASA)
7. If two angles and a non-included side of one triangle are equal to the corresponding parts of another triangle, then the triangles are congruent.
8. Angles opposite to equal sides are equal.
9. Sides opposite to equal angles are equal.
10. If in two triangles, all three sides in one triangle are equal to the corresponding sides in the other, then the triangles are congruent.
11. If in two right triangles the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and corresponding side of the other triangle, the triangles are congruent.
12. Angle opposite to greater side of a triangle is greater.
13. Side opposite to greater angle of a triangle is greater.
14. In a triangle, the sum of any two sides of the triangle is greater than the third side.
15. Each angle of an equilateral triangle is 60°.
16. If ΔABC and ΔPQR are congruent then we write it like $\Delta ABC \cong \Delta PQR$.
17. We represent the corresponding parts of congruent triangles as CPCT.

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Look at the figures given below. Which of them are triangles?

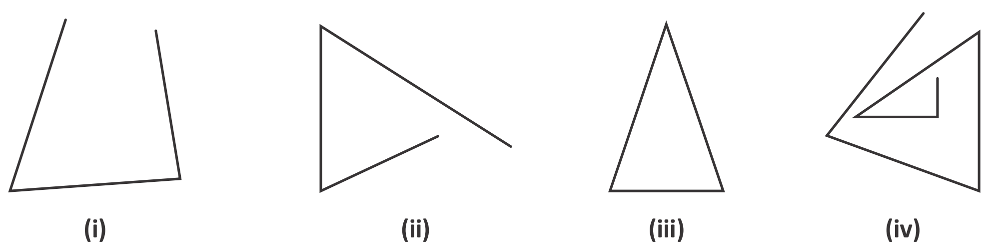


Fig. 1

A figure formed by joining three non-collinear points is called a triangle. A triangle is a figure enclosed by three line segments. It has three sides, three angles and three vertices. What are the other features of a triangle? Discuss.

Look at *Fig*.2(i-iv). In each of them, four points are joined together.

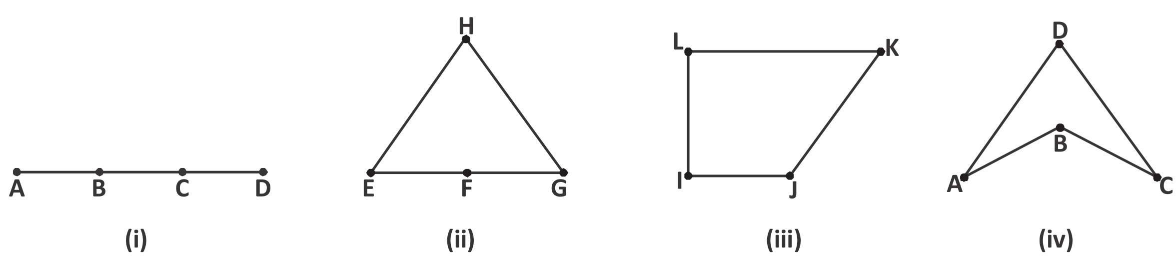


Fig. 2

In *Fig*.2(i), all four points lie on the same line (are collinear) so we get a line segment when we join them. In *Fig*.2(ii), three of the points are collinear but not the fourth. Here, we get a triangle when we join the points.

Is a quadrilateral formed in *Fig*.2(iii) or *Fig*.2(iv)? It is clear that to form a quadrilateral, three points out of four have to be non-collinear.

If three out of four points on the same plane are not lying on the same line (i.e. are non-collinear) then they will form a quadrilateral when we join them together in an order.

Try This

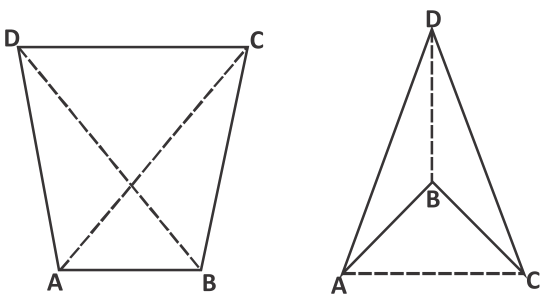
Define a quadrilateral based on the properties described so far. Discuss your definition with your friends.

Find different objects in your school or classroom that have surfaces in the shape of quadrilaterals. For example, the blackboard, window panes, each page of a book etc.

Think and Discuss

Many of the objects around us are rectangular. A rectangle is also a quadrilateral. Why?

Types of Quadrilateral



(i) (ii)

Fig. 3

We saw that *Fig.*2(iii) and *Fig.*2(iv) are quadrilaterals. Let us draw the diagonals of these quadrilaterals (*Fig.*3).

We find that both the diagonals in *Fig.*3(i) lie inside but in *Fig.*3(ii), one diagonal lies inside and the other is outside the quadrilteral. What is different in the two quadrilaterals?

All angles of the quadrilteral in which both diagonals are inside are less than 180°. Such quadrilaterals are called convex quadrilaterals. For example, quadrilateral PQRS (*Fig.*4).

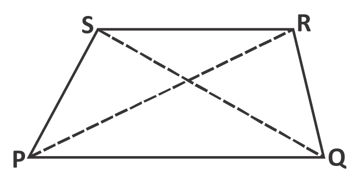


Fig. 4

A quadrilateral in which one of the angles is more than 180º will have one diagonal inside and the other outside of it. Such a quadrilateral is called a concave quadrilateral.

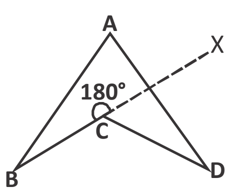


Fig. 5

In *Fig.*5 $\angle BCX = 180^\circ$. Therefore, the interior angle $\angle BCD$ of quadrilateral ABCD is more than 180°.

In this chapter, we will only study convex quadrilaterals like those shown in *Fig.*4.

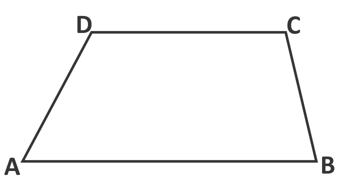
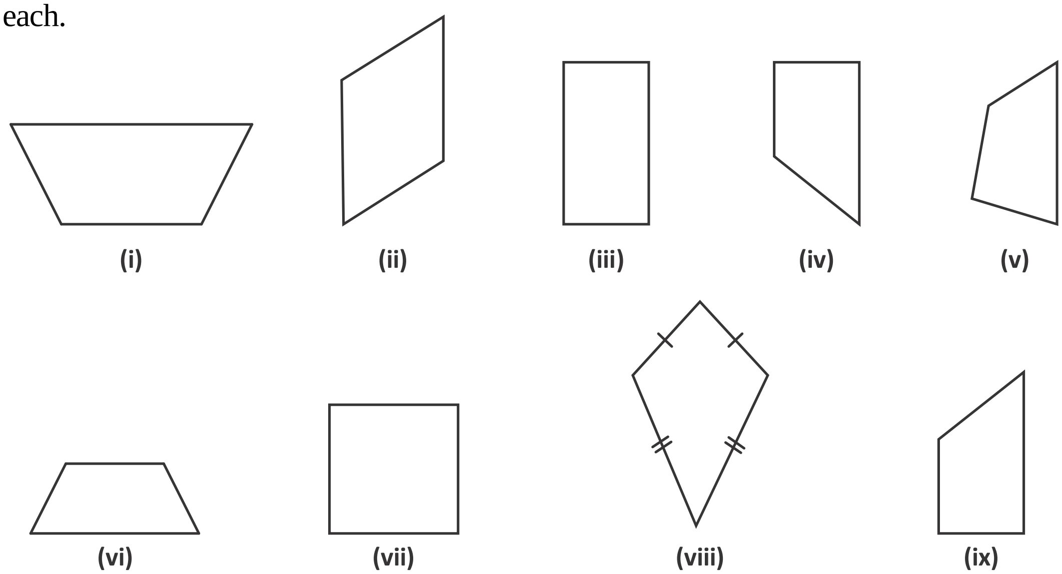


Fig. 6

In quadrilateral ABCD, the sides AB and DC are parallel to each other. It is a trapezium. We can say that a quadrilateral in which only one pair of sides is parallel, is a trapezium.

Try This

Look at the figures given below. Which of them are **not** trapeziums? Give reasons for each.



Parallelogram


Fig. 7

We know that if one pair of opposite sides of a quadrilateral are parallel then it is called a trapezium. If both pairs of opposite sides of a quadrilateral are parallel then it is called a parallelogram.

Rhombus

When all sides of a parallelogram are equal it is called a rhombus.

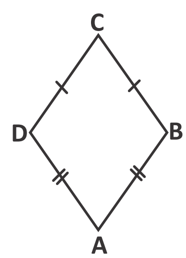
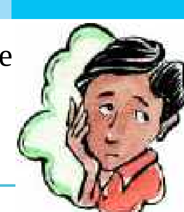


Fig. 8

Think and Discuss



1. You have read about many types of quadrilaterals (trapezium, rhombus, square etc.). Identify which of them are parallelograms.
2. Are all parallelograms also trapeziums? Discuss.

Now draw a square and a rectangle. Are they examples of parallelograms? Yes, a square is a special type of parallelogram and each of its interior angle measures 90^{o}.

Try This

1. Is a rectangle also a square?
2. Try to draw a parallelogram where three of the angles are right angles but which is not a rectangle. Is such a parallelogram possible? Discuss.

Suppose all sides of a rectangle are equal. Then, what will be? Such a rectangle is a square (*Fig*.9).

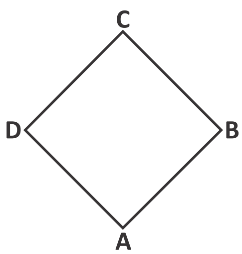


Fig. 9

(i)	Is a square a rectangle?	(ii)	Are squares parallelograms?
(iii)	Is a square a rhombus?	(iv)	Is a rhombus also a parallelogram?

Now, we will learn how to prove some properties and theorems related to quadrilaterals.

We know that each diagonal of a quadrilateral divides it into two triangles.

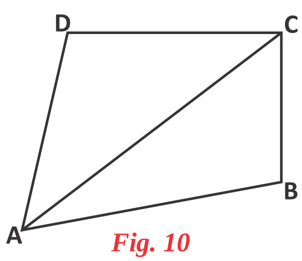


Fig. 10

Assume that ABCD is a quadrilateral and AC is its diagonal. Then, diagonal AC divides the quadrilateral into two triangles ΔABC and ΔADC (*Fig*.10).

From the angle sum property of triangles we know that the sum of the three interior angles of a triangle is 180°.

In ΔADC, $\angle ADC + \angle DCA + \angle CAD = 180°$ .....(1)

Similary in ΔABC, $\angle ABC + \angle BCA + \angle CAB = 180°$ .....(2)

By adding equations (1) and (2)

$\angle ADC + \angle DCA + \angle CAD + \angle ABC + \angle BCA + \angle CAB = 180° + 180°$

$\angle ADC + (\angle DCA + \angle BCA) + (\angle CAD + \angle CAB) + \angle ABC = 360°$

$\angle ADC + \angle BCD + \angle BAD + \angle ABC = 360°$

Therefore, the sum of the four interior angles of quadrilateral ABCD is equal to 360°.

Similarly, the sum of the interior angles of any quadrilateral is equal to 360°.

Try This

1. The interior angles of a quadrilateral are in the ratio 3:5:7:9. Find the measure of each interior angle.
2. If all the angles of the quadrilateral are equal then what is the measure of each angle?

Take a piece of paper and draw a parallelogram on it. Draw any one of its diagonals. Take a pair of scissors and first cut out the parallelogram and then cut along the diagonal as shown in *Fig*.11. Place the cut parts on each other. Do they overlap? Turn the cut parts around if needed.

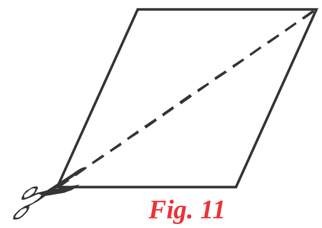


Fig. 11

Are the parts overlapping because of some special property of the parallelogram? Which property is it?

We will now study the properties of a parallelogram and logically verify them.

THEOREM-11.1 : A diagonal of a parallelogram divides it into two congruent triangles.

PROOF : Let ABCD be a parallelogram and AC be one of its diagonals (*Fig*.12).

In parallelogram ABCD,

AB||DC and AC is a transveral

$\angle DCA = \angle CAB$ (pair of alternate angles)

Similarly, DA||CB where AC is a transversal

$\angle DAC = \angle BCA$

Now, in ΔACD and ΔCAB

$\angle DCA = \angle CAB$

AC = CA (common side)

$\angle DAC = \angle BCA$

$\therefore\ \Delta ABC \cong \Delta CDA$ (by ASA congruency)

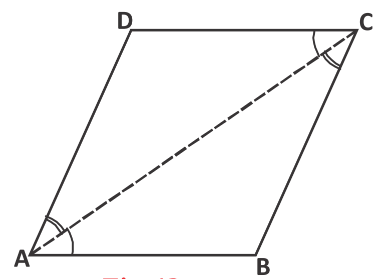


Fig. 12

That is, diagonal AC divides the parallelogram ABCD into two congruent triangles.

Clearly, a diagonal of a parallelogram divides it into two congruent triangles.

THEOREM-11.2 : In a parallelogram, opposite sides are equal.

PROOF : Let ABCD be a parallelogram. Join the vertex A to C. This is diagonal AC. Diagonal AC divides the quadrilateral ABCD into two triangles ABC and ACD (*Fig*.13).

Now, in ΔABC and ΔACD

$\angle DAC = \angle BCA$ (alternate interior angles)

(AD || BC)

Similarly, by alternate interior angles

$\angle DCA = \angle BAC$ (AB || DC)

Also AC = CA is a common side

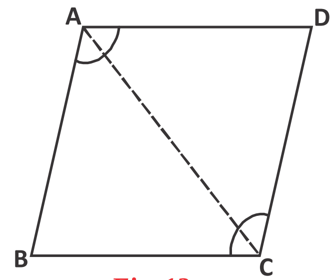


Fig. 13

$\therefore\ \Delta ABC \cong \Delta CDA$ (by A-S-A congruency)

Therefore, AD = BC and AB = CD (by C.P.C.T.)

That is, opposite sides of a parallelogram are equal (hence proved).

THEOREM-11.3 (CONVERSE) : If each pair of opposite sides of a quadrilateral is equal then it is a parallelogram.

PROOF : Given, quadrilateral ABCD in which AB = CD and BC = AD. Now, draw diagonal AC (*Fig.*14).

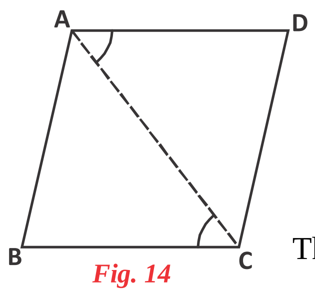


Fig. 14

In ΔABC and ΔCDA,

BC = AD (given)

AB = DC (given)

AC = CA (common side)

$\therefore\ \Delta ABC \cong \Delta CDA$ (by S-S-S congruency)

Therefore, $\angle BCA = \angle DAC$ (by C.P.C.T.)

and $AD \parallel BC$ .....(1)

where AC is a transversal.

Since $\angle ACD = \angle CAB$,

because CA is a transversal

therefore, $AB \parallel CD$ .....(2)

Therefore, by (1) and (2), ABCD is a parallelogram.

We have seen that each pair of opposite sides of a parallelogram is equal and conversely, if each pair of opposite sides of a quadrilateral is equal then it will be a parallelogram.

Now, we will prove such a result for those quadrilaterals in which pairs of opposite angles are equal.

THEOREM-11.4 : Opposite angles of a parallelogram are equal.

PROOF : Quadrilateral ABCD is a parallelogram (*Fig.*15)

in which $AB \parallel DC$

$\because$ Line segment AD intersects the parallel lines AB and DC.

$\angle A + \angle D = 180°$ (interior angles on the same side of the transversal)

and DC intersects the lines AD and BC.

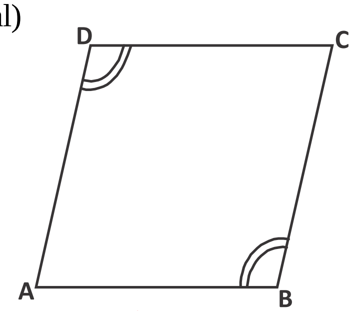


Fig. 15

$\angle D + \angle C = 180°$ (interior angles on the same side of the transversal)

therefore, $\angle A + \angle D = \angle D + \angle C$

that is, $\angle A = \angle C$

similarly $\angle B = \angle D$ can also be proven.

It is clear that opposite angles in a parallelogram are equal.

Now, what will happen if opposite angles of a quadrilateral are equal. We will find the logical possibility of such a quadrilateral being a parallelogram.

THEOREM-11.5 (Converse of Theorem-11.4) : If in a quadrilateral each pair of opposite angles are equal then it is a parallelogram.

PROOF : In quadrilateral ABCD, $\angle A = \angle C$ and $\angle B = \angle D$ (*Fig.*16) .....(1)

We know that sum of all interior angles of a quadrilateral is 360°.

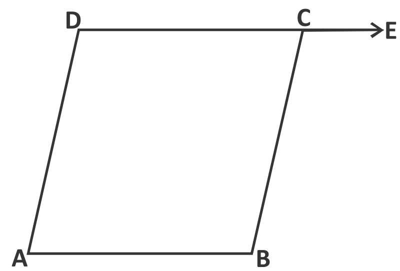


Fig. 16

$$\angle A + \angle B + \angle C + \angle D = 360^o$$

$$\angle A + \angle B + \angle A + \angle B = 360^o \text{ (by -1)}$$

$$2\angle A + 2\angle B = 360^o$$

$$\angle A + \angle B = \frac{360^o}{2}$$

$$\angle A + \angle B = \angle C + \angle D = \frac{360^o}{2}$$

$$\therefore \angle C + \angle D = 180^o \qquad .....(2)$$

Now, extend DC upto E-

We see that $\angle C + \angle BCE = 180^o$ .....(3)

Therefore, $\angle BCE = \angle ADC$ by equation (2) and (3)

Since, $\angle BCE = \angle D$ and DC is a transversal

therefore, AD || BC

Similarly, AB || DC and so ABCD is a parallelogram

Properties of Diagonals of a Parallelogram

Draw a parallelogram on paper and draw both its diagonals. Cut the parallelogram into four parts along the diagonals as shown in *Fig.*17.

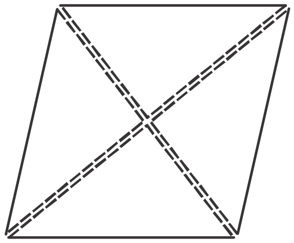


Fig. 17

Do the parts appear similar to each other or different from each other?

We are getting four triangles and they are actually two pairs of congruent traingles. Do the diagonals bisect each other?

Let us check the validity of the statement given in theorem 11.6.

THEOREM-11.6 : Diagonals of a parallelogram bisect each other.

PROOF : ABCD is a parallelogram in which AB = DC and AB || DC

Also AD = BC and AD || BC (*Fig.*18)

When we join A to C and B to D then AC and BD intersect each other at point O.

In ΔAOB and ΔCOD

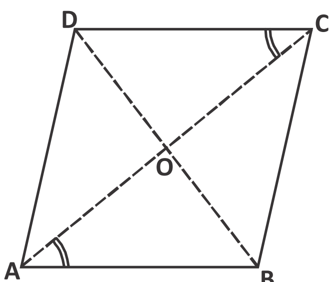


Fig. 18

$\angle OAB = \angle OCD$ .....(1)

(AB || DC and are cut by the transversal AC)

$\angle ABO = \angle ODC$ .....(2)

(AB || DC and are cut by the transversal BD)

AB = DC

$\therefore$ $\Delta AOB \cong \Delta COD$ (A-S-A congruency)

$\therefore$ Side AO = OC and BO = OD (by C.P.C.T.)

So, we can say that diagonals of a parallelogram bisect each other.

EXAMPLE-1. Prove that if diagonals of a parallelogram are equal then it is a rectangle.

PROOF : Let ABCD be a parallelogram in which AC and BD are diagonals

and AC = BD (*Fig.*19)

Now in ΔABC and ΔDCB

AB = DC (opposite sides of a parallelogram)

BC = CB (common side)

AC = BD (given)

$\Delta ABC \cong \Delta DCB$ (S-S-S congruency)

therefore, $\angle ABC = \angle DCB$ .....(1)

Since $\angle ABC$ and $\angle DCB$ are situated on the same side of the transversal BC of parallel lines AB and CD, therefore,

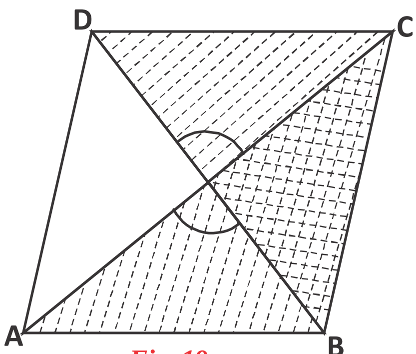


Fig. 19

$\angle ABC + \angle DCB = 180°$ .....(2)

by (1) and (2)

$\angle ABC + \angle ABC = 180^o$

$2\,\angle ABC = 180^o$

$\angle ABC = 90^o$

that is, $\angle DCB = 90^o$

Similarly, we can prove that $\angle A = \angle D$.

$\angle A = \angle D = 90°$

therefore, $\angle A = \angle B = \angle C = \angle D = 90°$

Hence, parallelogram ABCD is a rectangle.

Clearly, if diagonals of a parallelogram are equal then it is a rectangle.

Hence proved.

Try This

Similarly, try to prove that if diagonals of a rhombus are equal then it is a square.

EXAMPLE-2. If diagonals of a parallelogram are perpendicular to each other then it is a rhombus.

PROOF : Let ABCD be a parallelogram in which diagonals AC and BD are perpendicular to each other. We need to prove that ABCD is a rhombus (*Fig*.20).

Now, in ΔAOD and ΔCOD

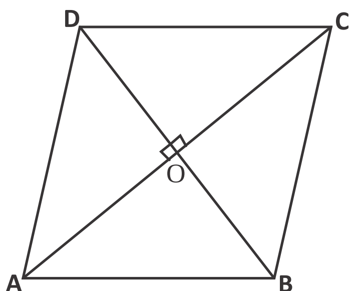


Fig. 20

AO = CO (diagonals of a parallelogram bisect each other)

$\angle AOD = \angle COD$ (each angle is right angle)

OD = OD (common side)

$\Delta AOD \cong \Delta COD$ (SAS congruency)

Therefore, AD = CD (by C.P.C.T.)

Also, AB = CD and AD = BC ($\because$ opposite sides of a parallelogram are equal)

$\therefore$ AB = BC = CD = AD

Clearly, parallelogram ABCD is a rhombus. Therefore, it can be said that if the diagonals of a parallelogram are perpendicular to each other then it is a rhombus.

Try This

Can you show that the diagonals of a rhombus are perpendicular to each other?

Example-3. Prove that the diagonals of a rhombus are perpendicular to each other.

Proof : A rhombus is a parallelogram in which all sides are equal. Consider the rhombus ABCD (*Fig.*21). We see that in rhombus ABCD, the diagonals AC and BD intersect each other at O. We need to prove that AC is perpendicular to BD.

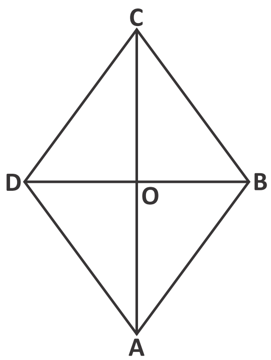


Fig. 21

In ΔAOB and ΔBOC

AO = OC (diagonals of a parallelogram bisect each other)

OB = OB (common side)

AB = BC (sides of a rhombus)

$\therefore \quad \Delta AOB \cong \Delta BOC$ (SSS congruency)

Therefore, $\angle AOB = \angle BOC$

Now, because $\angle AOB + \angle BOC = 180°$ (Linear pair)

$\therefore \quad \angle AOB + \angle AOB = 180°$

or $\quad 2\angle AOB = 180°$

or $\quad \angle AOB = \dfrac{180°}{2}$

or $\quad \angle AOB = 90°$

Similarly, we can prove that $\angle BOC = \angle COD = \angle AOD = 90°$. That is, the diagonals of a rhombus are perpendicular to each other. Hence proved.

Example-4. Prove that the angle bisectors of a rhombus make a rectangle.

Proof : ABCD is a parallelogram as shown in *Fig.*22. Bisectors of $\angle A$, $\angle B$, $\angle C$ and $\angle D$ intersect at P, Q, R and S forming quadrilateral PQRS (*Fig.*22).

In ΔASD,

Since DS bisects $\angle D$ and AS bisects $\angle A$, therefore

$$\angle DAS + \angle ADS = \frac{1}{2}\angle BAD + \frac{1}{2}\angle ADC$$

$$= \frac{1}{2}(\angle A + \angle D)$$

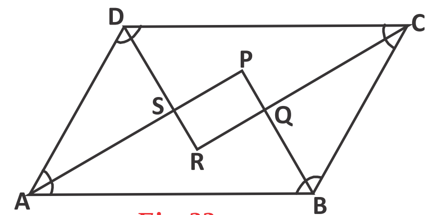


Fig. 22

$= \frac{1}{2} \times 180° = 90°$ ($\angle A$ and $\angle D$ are interior angles on the same side of the transversal) .....(1)

In Δ ASD,

$\angle DAS + \angle ADS + \angle DSA = 180°$ (Why?) .....(2)

From, equation (1) and (2)

$90^o + \angle DSA = 180°$

$\angle DSA = 90°$

Therefore, $\angle PSR = 90°$ (Being vertically opposite to $\angle DSA$)

Similarly, $\angle BQC = \angle PQR$

In ΔAPB,

$\angle PAB + \angle APB + \angle PBA = 180°$ (Sum of angles of a triangle)

but $\angle PAB + \angle PBA = 90°$ ($\angle A$ and $\angle B$ are interior angles on the same side of the transversal)

$\therefore \angle APB = 90°$

Similarly, $\angle SRQ = 90°$. Thus, PQRS is a quadrilateral in which all the angles are right angles. Therefore, quadrilateral PQRS is a rectangle.

Think and Discuss

1. Diagonals of a rectangle are of equal length (***Hint***- a rectangle is a parallelogram)
2. Diagonals of a square are equal and they bisect each other at right angles.

EXAMPLE-5. If the diagonals of a parallelogram ABCD intersect at point O and if OA = 3 *cm* and OB = 4 *cm*, then find the lengths of the line segments OC, OD, AC and BD.

SOLUTION : ABCD is a parallelogram where AC and BD intersect at O (*Fig*.23).

OA = 3 *cm* OB = 4 *cm*

because diagonal of the parallelogram AC and BD bisects each other.

OC = OA

$\therefore$ OC = 3 *cm*

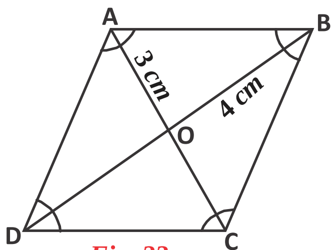


Fig. 23

and $OD = OB$

$\therefore$ $OD = 4\ cm$

Now, $AC = AO + OC = 3 + 3 = 6\ cm$

$BD = OB + OD = 4 + 4 = 8\ cm$

Therefore, it is clear that $AC = 6\ cm$ and $BD = 8\ cm$.

THEOREM-11.7 : If in a quadrilateral, a pair of opposite sides is equal and parallel then it is a parallelogram. (Prove with the help of teacher)

EXAMPLE-6. In triangle ABC, median AD was drawn on side BC and extended to E such that AD = ED. Prove that ABEC is a parallelogram.

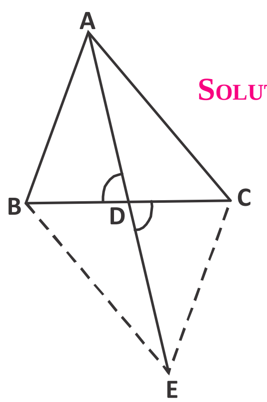


Fig. 24

SOLUTION : Let ABC be the triangle and AD be its median on BC (*Fig.*24).

Extend AD to E such that AD = ED

Now join BE and CE

In ΔABD and ΔECD

$BD = DC$ (since D is the mid point of BC)

$\angle ADB = \angle EDC$ (vertically opposite angles)

$AD = ED$ (given)

$\therefore$ $\Delta ABD \cong \Delta ECD$ (by SAS congruency)

Now, $AB = CE$ (sides of congruent triangles)

and $\angle ABD = \angle ECD$

Both are a pair of alternate interior angles made between the lines AB and CE by the transversal line BC.

$\therefore$ $AB \parallel CE$

So, in quadrilateral ABEC

$AB \parallel CE$ and $AB = CE$

Therefore, ABEC is a parallelogram.

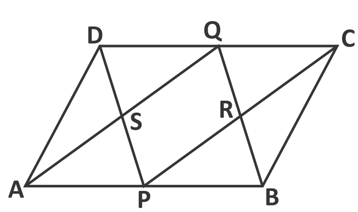


Fig. 25

EXAMPLE-7. ABCD is parallelogram in which P and Q are the mid points of opposite sides AB and CD respectively (*Fig.*25). If AQ intersects DP at point S and BQ intersects CP at point R then, show that:-

(i) APCQ is a parallelogram.

(ii) DPBQ is a parallelogram.

(iii) PSQR is a parallelogram

SOLUTION : (i) In quadrilateral APCQ

$AP \parallel QC$ (because $AB \parallel CD$) .....(1)

$AP = \frac{1}{2} AB$

$CQ = \frac{1}{2} CD$ (given)

Since $AB = CD$

therefore, $AP = QC$ .....(2)

by equation (1) and (2) APCQ is a parallelogram.

(ii) Similarly, quadrilateral DPBQ is a parallelogram because $DQ \parallel PB$ and $DQ = PB$

(iii) In quadrilateral PSQR

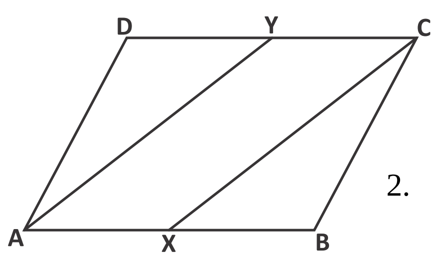
$SP \parallel QR$

(where SP is a part of DP and QR is a part of QB)

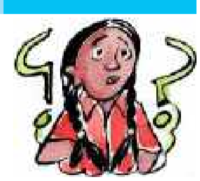
Similarly $SQ \parallel PR$

Therefore, PSQR is a parallelogram

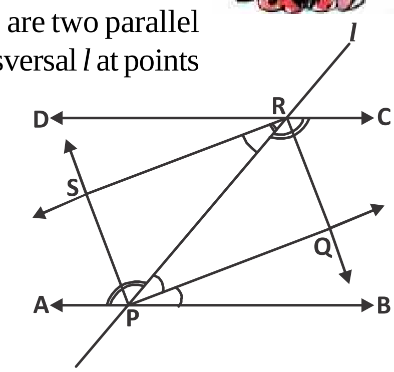
Exercise - 11.1

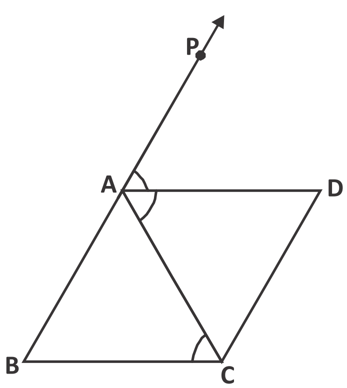


1. X and Y are the mid points of opposite sides AB and CD of parallelogram ABCD. Prove that AXCY is a parallelogram.



2. In the adjacent figure, AB and DC are two parallel lines which are intersected by transversal l at points P and R respectively. Prove that the bisectors of the interior angles make a rectangle.



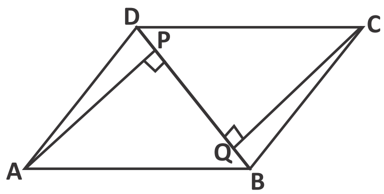


3. ABC is an isosceles triangle in which AB = AC, AD bisects the exterior angle PAC and $CD \parallel BA$.

Prove that:-

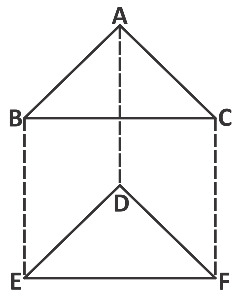
(i) $\angle DAC = \angle BCA$

(ii) Quadrilateral ABCD is a parallelogram.



4. ABCD is a parallelogram and BD is one of its diagonals. AP and CQ are perpendiculars on BD from the vertices A and C respectively. Prove that:-

   (i) $\Delta APB \cong \Delta CQD$ (ii) $AP = CQ$



5. ABCD is a rectangle in which diagonal AC bisects both the angles A and C. Then prove that:-

   (i) ABCD is a square.

   (ii) Diagonal BD bisects both angles B and D

6. ΔABC and ΔDEF are such that AB and BC are equal and parallel to DE and EF respectively. Prove that AC and DF are equal and parallel.

The Mid-Point Theorem

You have studied many properties of triangles and quadrilaterals. Let us study a property of triangles which is related to the mid-point of its sides. Let us look at the theorem:-

THEOREM-11.8 : A line segment joining the mid-points of two sides of a triangle is always parallel to and half of the third side.

PROOF : Let us take ΔABC to prove this statement. In ΔABC, D and E are the mid points of AB and AC respectively. Draw a line segment DF by joining mid points D and E such that E is the mid-point of DF. Join C and F (*Fig.*26).

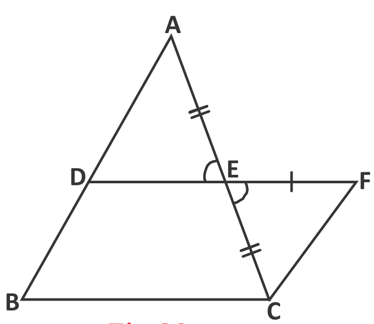


Fig. 26

Now, you can see that in ΔADE and ΔCFE

AE= CE (E is the mid-point of AC)

$\angle AED = \angle CEF$ (vertically opposite angles)

DE = EF (by construction)

$\therefore$ $\Delta ADE \cong \Delta CFE$ (by SAS congruency)

AD = CF and $\angle ADE = \angle CFE$ (by C.P.C.T.)

now BD = AD and AD = CF

$\therefore$ BD = CF .....(1)

Also $\angle ADE = \angle CFE$ are equal (proved above). But these are alternate interior angles for AD and CF intersected by DF.

$\therefore$ AD||CF

or BD||CF .....(2)

By (1) and (2), BD and CF in quadrilateral BDFC are both equal and parallel. You know that if a pair of opposite sides is both equal and parallel in a quadrilateral, then it is a parallelogram. Therefore, DBCF is a parallelogram.

Since opposite sides of a parallelogram are equal so DF = BC. Also, DE + EF = DF, DE = EF, so DF=2DE

$$\therefore \ BC = 2DE \quad \text{and} \quad DE = \frac{1}{2}\ BC$$

Try This

Now write the converse of theorem 11.8 and verify it.

THEOREM-11.9 : ***l*, *m* and *n* are the three parallel lines intersected by transversals *p* and *q* such that *l*, *m* and *n* cut off equal intercepts DE and EF on *p*. Show that *l*, *m* and *n* cut off equal intercepts AB and BC on *q* also.**

PROOF : Parallel lines *l*, *m* and *n* are intersected by transversal lines *p* at point D, E and F such that DE = EF

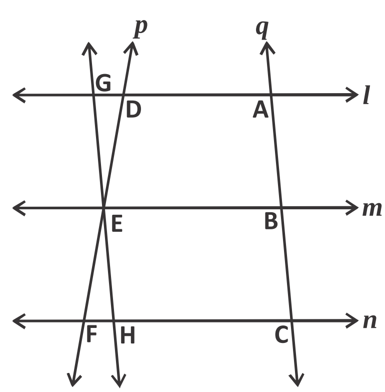


Fig. 27

If transversal line *q* intersects parallel lines *l*, *m* and *n* at points A, B and C respectively, then we need to prove that AB = BC.

Now, to prove this we will draw a line which is parallel to *q*, passes through the point E and intersects *l* and *n* at G and H respectively.

Clearly AG || BE (because *l* || *m* and A, G and B, E lie on *l* and *m* respectively)

GE || AB (by construction)

Then, AGEB is a parallelogram

$\therefore$ AG = BE and GE = AB .....(1)

Similarly, BE || CH (because m || n and B, E and C, H lie on m and n respectively)

EH || BC (by construction)

Then, BEHC is a parallelogram

$\therefore$ BE=CH and EH = BC .....(2)

Now, in ΔGED and ΔHEF

$\angle DGE = \angle EHF$ (alternate angles)

$DE = EF$ (given)

$\angle DEG = \angle HEF$ (vertically opposite angles)

$\therefore \Delta GED \cong \Delta HEF$ (by ASA congruency)

therefore, $GE = EH$

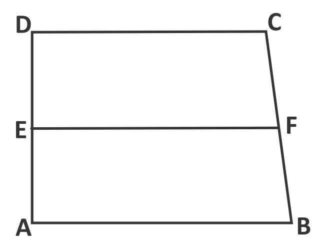
$\because GE = AB, EH = BC$ by (1) and (2)

$\therefore AB = BC$

Hence, proved.

Exercise - 11.2

1. ABCD is a trapezium where $AB \parallel DC$. E is the midpoint of AD. A line drawn from E, parallel to AB, meets BC at point F. Prove that F is the mid point of BC.

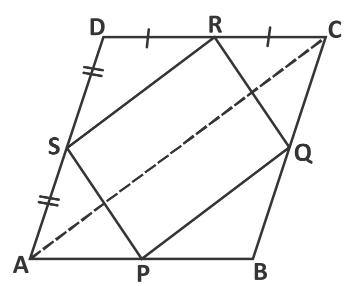


2. ABCD is rhombus and P, Q, R, S are the mid points of the sides AB, BC, CD and DA respectively. Show that quadrilateral PQRS is a rectangle.
3. ABCD is a rectangle in which P, Q, R, S are the mid points of sides AB, BC, CD and DA respectively. Show that quadrilateral PQRS is a rhombus.
4. ABCD is a quadrilateral in which P, Q, R and S are mid-points of the sides AB, BC, CD and DA. AC is a diagonal. Show that :

   (1) $SR \parallel AC$ and $SR = \frac{1}{2} AC$

   (2) $PQ = SR$

   (3) PQRS is a parallelogram



What have We Learnt

1. Sum of the interior angles of a quadrilateral is 360°.
2. A diagonal of a parallelogram divides it into two congruent triangles.

3. There are many types of quadrilaterals. Some types of quadrilaterals are:-

(i)	Parallelograms	(ii)	Rhombus	(iii)	Trapezium
(iv)	Rectangle	(v)	Square		

4. A quadrilateral is a parallelogram, if
   (i) Both pairs of opposite sides are equal;
   (ii) Both pairs of opposite angles are equal;
   (iii) diagonals bisect each other;
   (iv) a pair of opposite sides is both equal and parallel.
5. Diagonals of a rectangle bisect each other and are equal and vice-versa.
6. Diagonals of a square bisect each other at right angles and are equal and vice-versa.
7. Diagonals of a rhombus bisect each other at right angles and vice-versa.
8. The line-segment joining the mid-points of any two sides of a triangle is parallel to the third side and is half of it.

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At the house of Salma's friend a large table was needed for the birthday party.

Salma said, "I have a large table at home. We can bring that here."

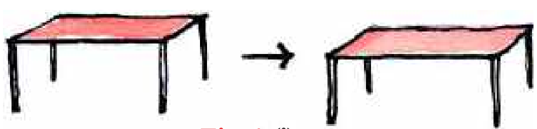


Fig. 1 (i)

At Salma's house, they first brought the table from the corner of the room near the door.

Then they began thinking about how to take the large table out of the door?

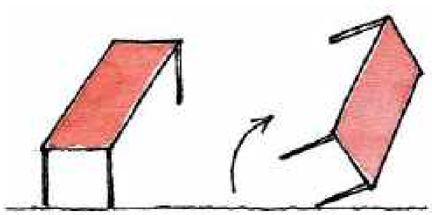


Fig. 1 (ii)

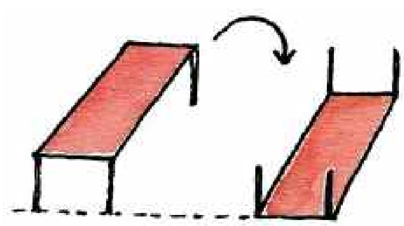


Fig. 1 (iii)

For this, they first tilted the table so that it could come out of the door and then inverted to place it on the van that would transport it. In this process, the orientation of the table changed many times.

In the first step table was displaced from one place to another. In the next step the table was rotated and then it was turned over.

We note that in this entire transportation the orientation of the table changed many times. There was however, no change in its shape and size. Thus, by moving, turning or tilting does not affect the shape or size of a thing.

Let us now suppose that Salma wants to make pictures of this process of moving the table in her notebook.

Would the size of the picture of the table be different from the actual size of the table? *Fig.*1(iv)

Discuss with your friends.

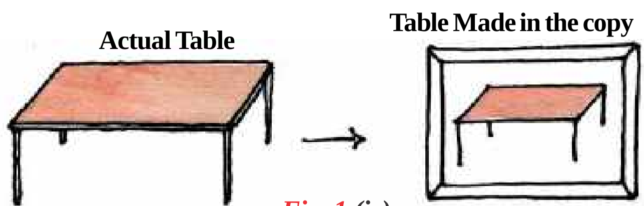


Fig. 1 (iv)

Around us we see shapes that are triangular, circular, spherical, and rectangular in our daily life. In actual life all things are three dimensional but if we look at the face of the three dimensional objects from the front, or from the top or from either the right or the left side then, we see only the face as a two dimensional shape.

Do This

We have given examples of some concrete materials here. Observe these objects from the top, front, right or the left and complete the following table accordingly:

S. No.	Name of the Object	Three dimen-sional shape	Seeing it from various perspective		
			Top	In front	Left/Right
1.	Dice	Cube	Square	....	....
2.	Toothpaste box	Cuboid	Rectangle	Rectangle	....
3.	Battery of a torch	Cylinder	....	....	....
4.	Ball	Sphere	....	....	....

Transformation

We have just seen that on inverting, rotating or inclining objects and observing them gives different appearances. In many situations, the shape of the objects changes, while for many others it stays the same. Such observations are usually made in our daily life activities. Deepti gave this example; "When I organize the furniture in my room then I rotate, shift, change positions of the furniture items like, sofas, tables, chairs and beds in many different ways". In order to change the place of a picture on the wall we shift the picture from one place to the other.

Ashwin said, “We keep the used utensils facing up and after washing they are placed facing down”. The position of the utensils as they were originally and after turning over is not the same. The utensil seems different in these two positions.

Akanksha says, "When I make a picture of my school building, the picture has the structure and shape of the building but the size is smaller".

When you get a passport size picture enlarged into a bigger picture, is this also a transformation?

Dipti thought for a while and said, “I have found that in some situations the transformed shape appears to be the same as the original and is also congruent but in other situations the size changes after transformation, i.e. They are not congruent. That means any action on a shape that changes its position, shape or size can be called a transformation.”

Consider the operations performed in *Fig.*2(i) and (ii)

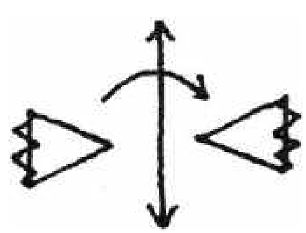


Fig. 2 (i)

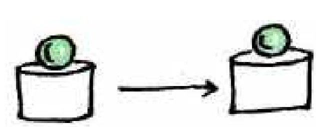
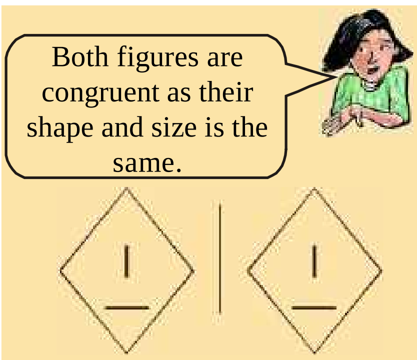


Fig. 2 (ii)

Look at the *Fig*.2(i), if we rotate the initial shape about the line *l*, then we will get the second shape. In *Fig*.2 (ii), the shape has been moved from one position to another.



In these situations if we place the original shape on the transformed shape, then would both cover each other?

We know that similar shapes that are of the same size are also congruent. This means the transformed and the original shapes in *Fig*.2(i) and (ii) are congruent.

(These figures are congruent because they are of same shape and size)

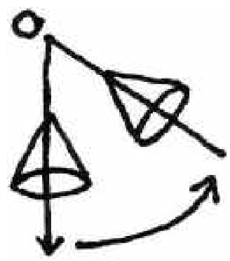


Fig. 2 (iii)

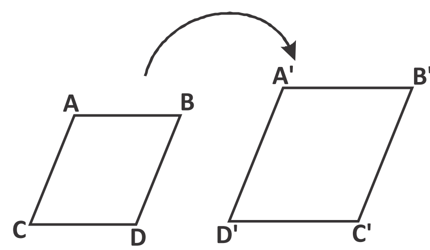


Fig. 2 (iv)

Now say, are the figures obtained after rotation and scaling up respectively in *Fig*.2(iii) and 2(iv) congruent to the original ?

Think and Discuss

Write two examples from your daily life in which you change the size, position or the shape of objects.

Think about these transformations, discuss with your friends and complete the table below:

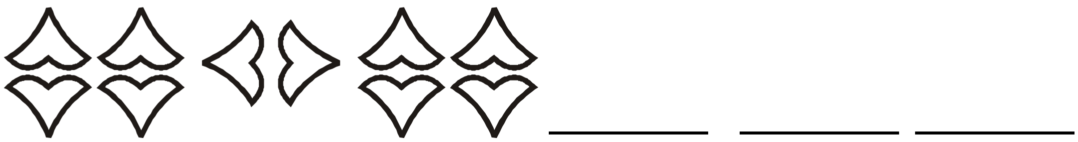
S.No.	What is happening	Are the figures same in size	Are the figures same in shape	Are the figures Congruent
(i)	Overturning	Yes		
(ii)	Shifting		Yes	
(iii)	Rotating			Yes
(iv)	Scaling up	No	Yes	No

What conclusions can you draw from the above table?

Maria immediately said, "In *Fig*.2(i) over turning and 2(ii) shifting and in 2(iii) rotating the initial shape is congruent to the shape after the transformation. However, in *Fig*.2(iv) there is no congruence due to the scaling up."

Playing with Geometrical Shapes

This is a design of a border on the wall, please extend it:-



The motif of this design is , we can get the entire border by rotating, inverting or sliding the above motif. Let us see how this can be done?

We get the first figure of the border  by inverting the motif. The second figure namely  of the border is got by sliding the motif and then inverting it. Finally rotating the motif  in an anticlockwise direction by 90°, we get the shape . By inverting this shape we get . Extend the border in this manner.

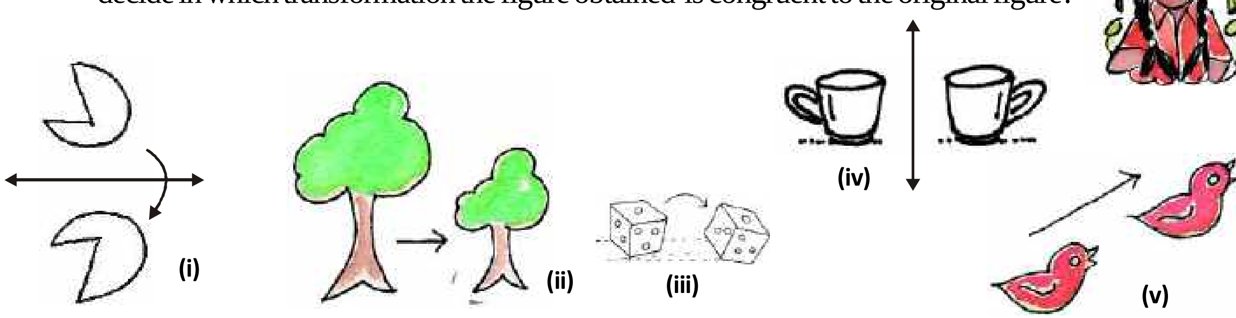
Can you make some more borders using the same motif? Think about it and make such borders.

Try This

Choose motif and using transformations make new designs and borders.

Exercise 12.1

1- What particular operation is happening in each of these figures? Observe, think and decide in which transformation the figure obtained is congruent to the original figure?



Types of Transformation

We can see two types of transformation:

1. **Rigid transformation**: Operations under which the transformed figure is congruent with the original figure are called rigid transformation.

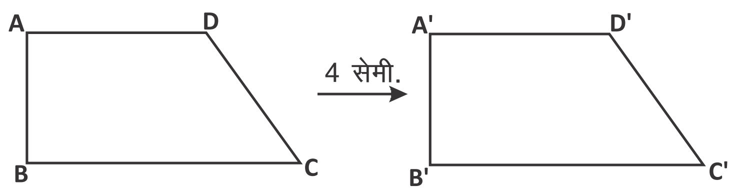


Fig. 3 *(i)*

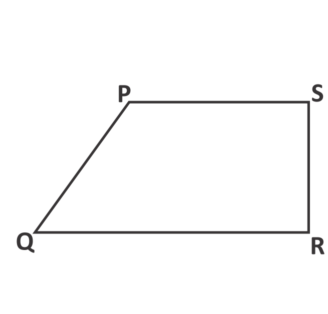


Fig. 3 *(ii)*

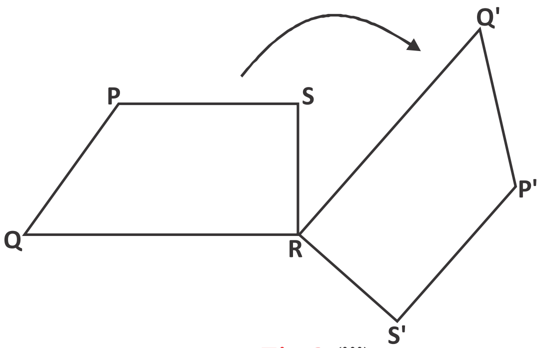


Fig. 3 *(iii)*

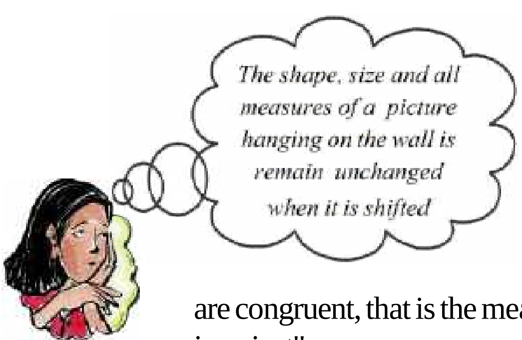
In the above set of diagrams quadrilateral ABCD is slid in *Fig.*3(i), inverted in *Fig.*3(ii) and then rotated in *Fig.*3(iii). Are all of these diagrams of the same quadrilateral?

In this we have performed three different operations on the same diagram. We can see that in these the position of the transformed diagram appears to be different from the original.

We know that in rigid transformations the shape and size of the figure does not change.

(i) Translation

Let us consider the first operation. In *Fig.*3(i) quadrilateral ABCD is moved horizontally by 4 *cm*. Will the size of all the sides remain invariant? Will the measure of all the angles remain invariant? Yes they will. (Why?)



Remember the properties of congruent shapes.

Dipti said: "Yes, in transformation (i) both the quadrilaterals are congruent, that is the measures of corresponding sides and corresponding angles remains invariant".

$\therefore$ AB = A'B', BC = B'C', CD= C'D' and DA= D'A'

Similarly the congruent angles would be equal

$\angle$ABC = $\angle$A'B'C', $\angle$BCD = $\angle$B'C'D', $\angle$CDA = C'D'A'

and DAB = $\angle$D'A'B'

Think and discuss

Would the corresponding sides and angles of a quadrilateral remain invariant under rotation and inversion as well? Think about the reasons, discuss and write.

The shape and size of the quadrilateral remains invariant under translation, therefore this is a rigid transformation. Any operation in which a shape or an object is shifted by a particular distance in a particular direction to a different location is called a translation.

(ii) Reflection

Look at the operation in *Fig.*4 -

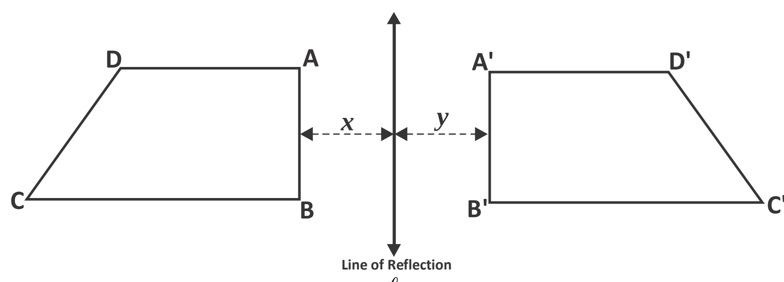


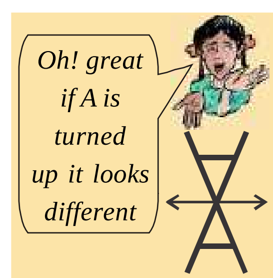
Fig. 4

If you place a mirror on line ℓ, then how will the image of quadrilateral ABCD appear?

Would it look like quadrilateral A'B'C'D'?

Ravi says if I invert (rotate by 180) quadrilateral ABCD about the line ℓ, then I get quadrilateral A'B'C'D'.

Do you agree with Ravi's method? Take another shape and rotate it by 180, around a particular line and see what kind of shape you get? Is this picture the same as what you would get when you reflect the shape on a plane mirror placed on the same line?



This operation is called reflection and the line around which the shape is reflected is called the line of reflection. In this shape is inverted or rotated around a specific line by 180 to obtain the transform shape.

If we consider the distance of line 'ℓ' from the quadrilateral ABCD to be x and distance of the quadrilateral A' B' C' D' from 'ℓ' to be y, then would x be equal to y i.e is $x = y$? Yes, the original shape and the transform shape would be equidistant from the line 'ℓ'.

Notice that this is a particular property of reflections. Let us try to understand this with an example-

In *Fig.*5 here, square MNOP is transformed to M'N'O'P'. Join the corresponding vertices of the two squares with line segments. The line segments PP', OO' and NN', intersect with the line 'ℓ' at points A, B and C respectively.

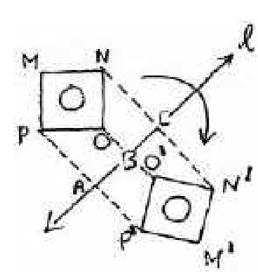


Fig. 5

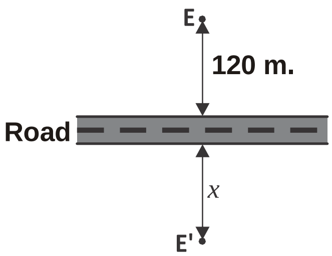
Can we say PA= P'A

Sakshi says: "The distance of point A from the vertex P is the same as a distance of point A from the vertex P'. This means PA= P'A.

Would the statements OB = OB' and NC=NC' be similarly true? (why)

From the above discussion we can conclude that the distance of the two squares from the line 'ℓ' is the same.

Try This



1. Reflection is a rigid transformation. Why? Discuss in your group and write with reason.
2. On one side of the road an electric pole E, is fixed. The distance of the road from the pole is 120 m. Taking the road as the line of reflection, reflect the pole. After reflection what would the perpendicular distance (x) of the image of the electric pole (E'), formed on the other side of the road be from the road?

(iii) Rotation

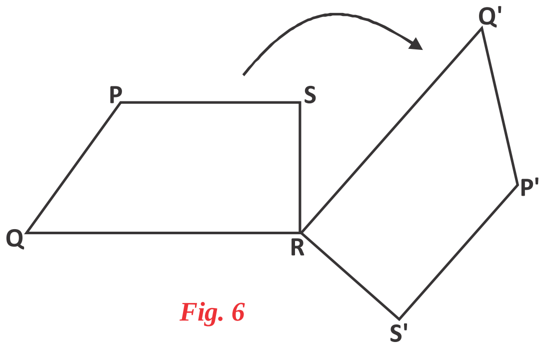


Fig. 6

Now we discuss the third kind of transformation.

Here the quadrilateral PQRS has been rotated about a point 'R' at the centre in a clockwise direction. This transformation is called rotation.

Look at the *Fig.*6 and say

Is quadrilateral PQRS ≅ quadrilateral P' Q' R S' ?

(≅ is the sign of congruence)

This rotation is a rigid transformation. Why?

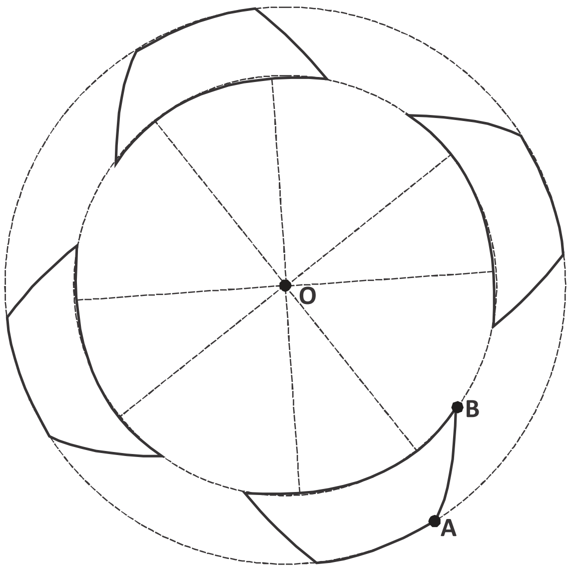


Fig. 7

You would have seen many rides in fairs that go round in big circles. One such ride is shown in *Fig.*7. The ride is revolving around the point 'O'. That is the point 'O' is the point of rotation. This point of rotation is located outside the body. If we see a moving ride going around in a circle we will notice that it revolves in a circular path around the point 'O'. (*Fig.*7) If we look carefully we realise any point A or B on the ride would move in a circular trajectory.

Let us discuss this process using a different example-

Look at *Fig.*8: in this the flagpole PQ is rotated clockwise by 30° with 'O' as the point of rotation. This point of rotation 'O' is also located outside the body.

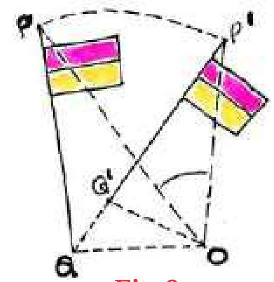


Fig. 8

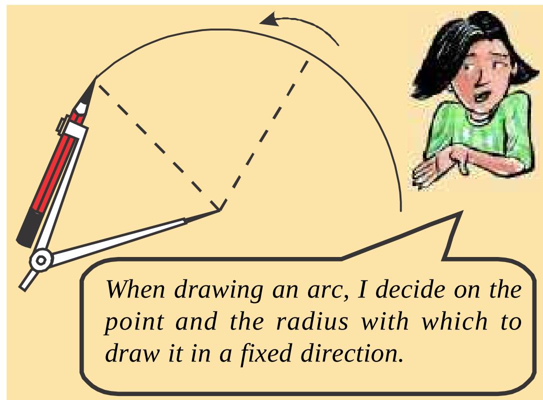
In this case, to rotate the flafpole by an angle of 30° we take any two points P and Q on it. We draw lines joining point P and Q respectively to the point 'O'. Keeping the lengths of the lines invariant, rotate each of them clockwise by 30°. The flagpole has rotated by 30° and the points P and Q have also rotated by 30°.

For the process of rotation we need to pay attention to the following three things-

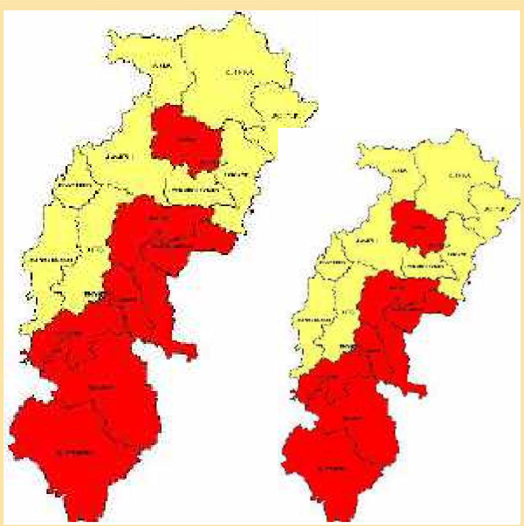
First- **Centre of rotation**: The fixed point around which the body will be rotated. This point can be on the body or outside it.

Second- **the direction of rotation**, which can be clockwise or anticlockwise.

Third- **the measure of angle of rotation**, apart from deciding the centre of rotation and the direction of rotation we need to know the angle of rotation.

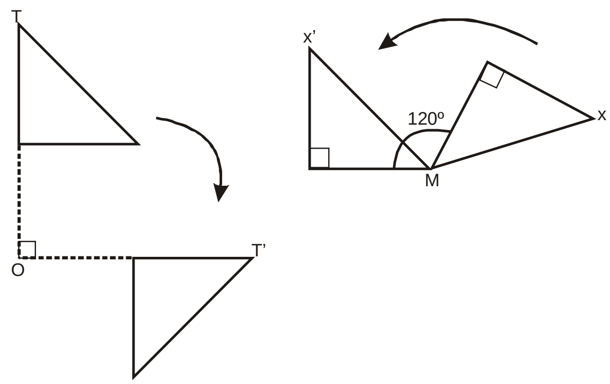


2. **Non-rigid Transformation** : Transformations under which the transformed object and original object are not congruent, for example scaling (magnifying or reducing) of the original object, are called non rigid transformations.



Look at the two maps of Chhattisgarh drawn above. The shape of both the maps is the same but the sizes are different.

Try This



Look at the rotations in the two figures and answer the questions below separately for each-

1. What is the point of rotation?
2. Does the point lie inside or outside?
3. Direction of rotation
4. The angle of rotation?

Exercise 12.2

1. You have to make a border for the wall of a room.

   Complete the following border.

   A B C D E F

   Look at the pattern and answer the following questions.

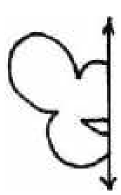
   (i) If you only had the figure 'A' as above, would you be able to make this complete pattern by translating, rotating or reflecting A?

   (ii) Which figures can be obtained by transforming A? Which transformation will you use for this purpose?

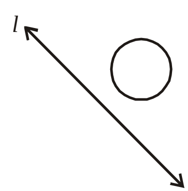
   (iii) Which transformation will you use to get D from B?

2. Taking 'l' to be the line of reflection complete the pictures-

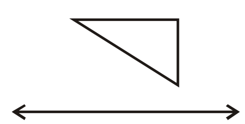
(i)



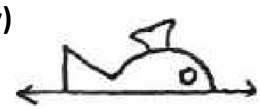
(ii)



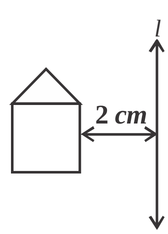
(iii)



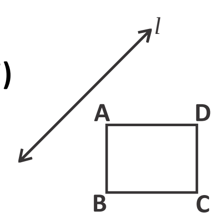
(iv)



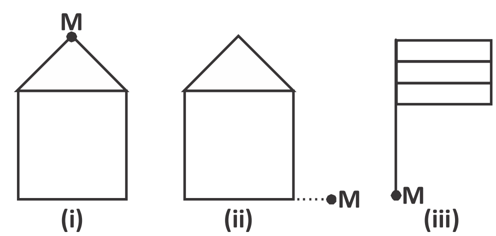
(v)



(vi)

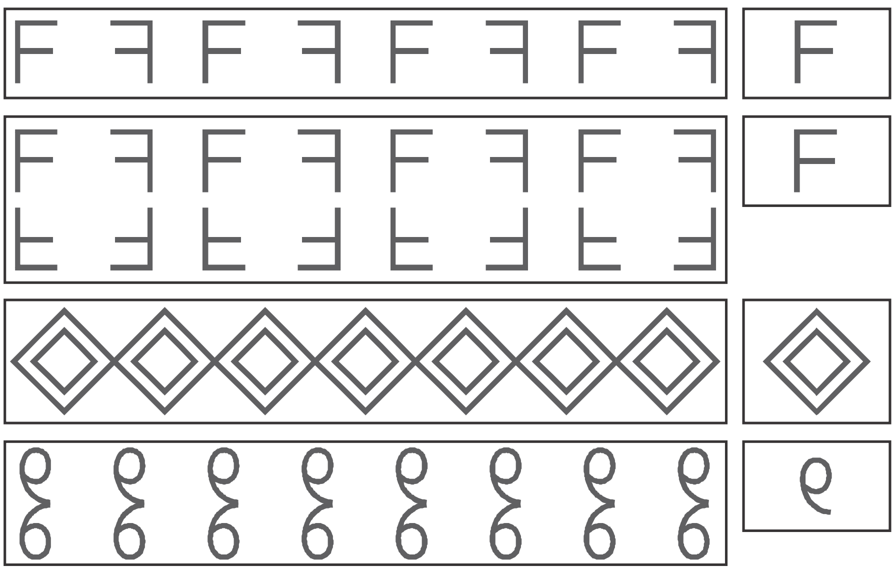


3. Taking point 'M' to be the point of rotation, rotate the following figures as directed-
   (i) Clockwise by 90°
   (ii) Anticlockwise by 30°
   (iii) Clockwise by 60°



4. Choose any shape of your choice. Using translation, rotation, reflection design a border of the table cover.

5. Look carefully at each of the basic pattern (motif) given on the right. Draw the next figure for each pattern. Which transformation did you used?



Symmetry

Look at that figures drawn here. If we fold them right in their middle we will get exactly the same shapes on the two sides of the fold.

What do we call such figures? What do we call the line that divides the figure into two identical parts?

Fig. 9

These figures are called symmetrical shapes and the line that divides the figure into two identical parts is called the line of symmetry.

(i) Linear Symmetry

In many natural objects, buildings, geometrical shapes and other things, we can see symmetry.

Fig. 10

Look at *Fig.*10. Draw a line on this that will divide it in two identical parts. The shapes on both sides of the line must be identical.

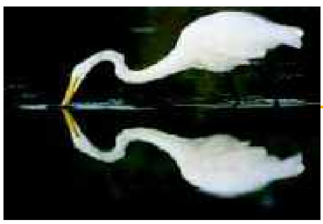


Fig. 11

You can think of some more figures like this.

Look at the *Fig.*11. The line of symmetry is horizontal here and the picture of the bird on both sides is identical. This is called as linear symmetry.

Can any other line be drawn in these which divides them in identical parts?

(ii) Rotational Symmetry

Fig. 12 (i)

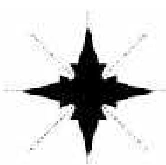
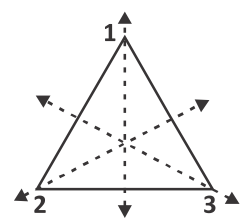
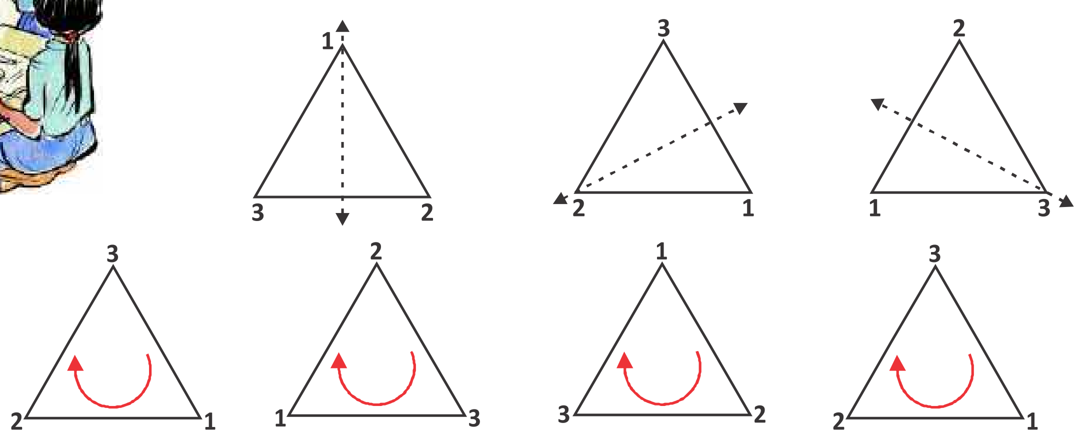


Fig. 12 (ii)

Look at *Fig.*12 (i) and 12 (ii). How many lines of symmetry do these have? You would find that on in one rotation they will look like their initial state at least once.

Let us consider the rotational symmetry of an equilateral triangle. There are three lines of symmetry in a equilateral triangle. In a complete rotation an equilateral triangle is identical to the initial state at three positions. This number is the order of rotation. In the same manner find out the order of rotation for the other two shapes in *Fig.*12.





Initial state | After rotating by 120° | After rotating by 240° | After rotating by 360° (the initial state)

Fig. 13

On the basis of the above discussion complete the following table-

Figure	How many lines of symmetry?	In one complete rotation how many times the same as original	Order of rotation
Regular Pentagon			
Equilateral Triangle	3	3	3
Rectangle			
Square			
The letter U			
The letter M			

Think and Discuss

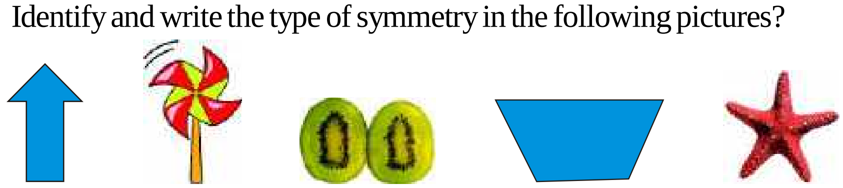
How many lines of symmetry are there in regular polygon? Is there a relation in the number of sides and order of rotations in such a polygon? What is it?

Try This

1. What symmetry can be seen in the following letters? Identify the point of symmetry and write it.

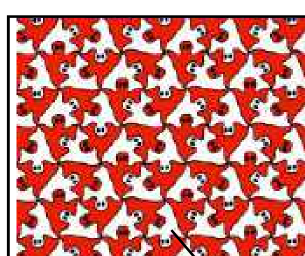
   I F N H G A O

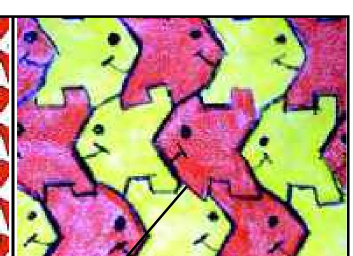
2. Identify and write the type of symmetry in the following pictures?



Applications of Symmetry and Transformations

Many designs and patterns can be seen on floors, walls, wallpapers, saries, clothes etc. These have many symmetries and often are made with transformations on one motif. For example in *Fig.*14(i) the motif can be recognised at many places. It is the same in *Fig.*14(ii).





(Motif)

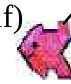


Fig. 14 (i) *Fig. 14 (ii)*

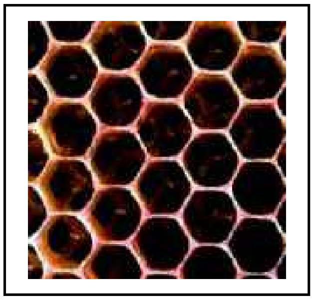
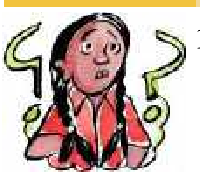


Fig. 15 (i)

Fig. 15 (ii)

Look at *Fig.*15(i) and *Fig.*15(ii) carefully. Identify the motif and draw.

Exercise - 12.3



1. Draw pictures of some objects that show-
   (i) linear symmetry
   (ii) rotational symmetry

2. Take a shape of your choice, using it as the motif, create a pattern.

3. Identify English alphabet, that have-
   (i) two lines of symmetry   (ii) No lines of symmetry
   (iii) rotational symmetry

What Have We Learnt

1. If congruent shapes are inverted, rotated or translated then they remain congruent.
2. Transformations under which the transformed shape is congruent to the original shape are called rigid transformations.
3. Rigid transformations include the three processes translation, reflection and rotation.
4. For translation the distance and direction need to be specified.
5. For rotation, the point of rotation, the angle of rotation and direction of rotation needs to be known.
6. The original shape and the transformed shape are equidistant from the line of reflection.
7. We have learned about two symmetries- linear symmetry and rotational symmetry.
8. Repeating a motif and placing it on a plane in an organised manner without any gaps or overlaps can produce patterns.

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We have learnt to draw line segment with a scale and angles with compass and protector. Now we shall learn to draw some closed shapes.

Let us Construct

If the length of three line segments are given, then is it always possible to construct a triangle with the given measurement? Discuss among yourselves.

If the three sides of a triangle are of 5 *cm*, 6 *cm* and 7 *cm*, then can we construct a triangle? Let us try:-

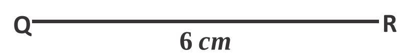


Fig. 1 (i)

1. Draw a line segment QR of 6 *cm* (*Fig*.1(i)).
2. Spread the arms of the compass upto 5 *cm*, place one arm on point Q and make an arc. (*Fig*.1(ii)).
3. Spread the arms of compass upto 7 *cm* and place it on point R and then make an arc which intersects the first arc at point P. (*Fig*.1(iii))
4. P is the intersection point.
5. Join P with R and Q.

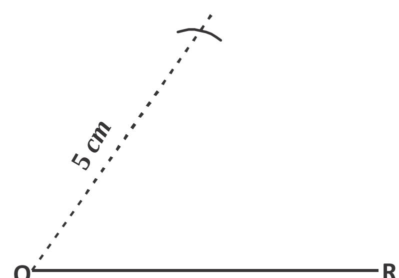


Fig. 1 (ii)

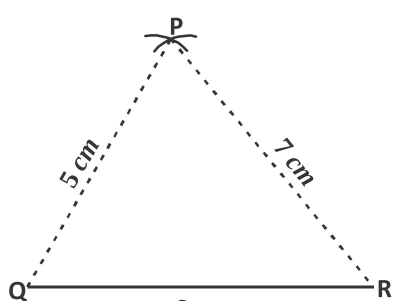


Fig. 1 (iii)

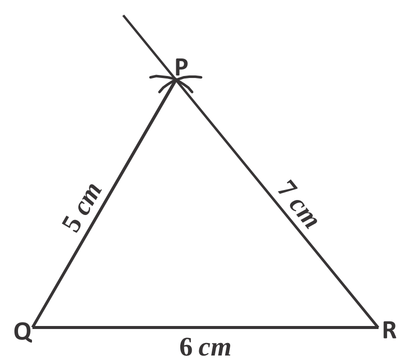
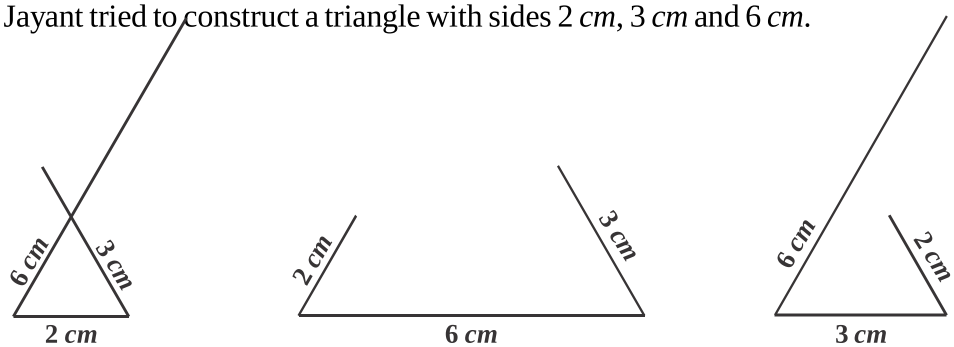


Fig. 1 (iv)

6. Thus the triangle PQR is constructed. (*Fig*.1(iv)).

Jayant tried to construct a triangle with sides 2 *cm*, 3 *cm* and 6 *cm*.



Can a triangle be constructed with these measurements? Why?

Try This

Is it possible to construct triangles with the following measurements?

(i) (2 *cm*, 3*cm*, 4 *cm*) (ii) (3 *cm*, 4 *cm*, 5 *cm*)

(iii) (2*cm*, 4 *cm*, 8 *cm*) (iv) (4 *cm*, 5 *cm*, 6 *cm*)

Of the given measurements, triangles can be formed only if the sum of two small sides is bigger than the measurement of the longest side.

Some More Constructions

Construction-1 : Construct a triangle when the measurement of two sides and the angle formed by them is given.

EXAMPLE-1. Construct a triangle ABC where AB = 5 *cm*, AC = 4 *cm* and $\angle A = 45°$.

Steps of Constructions

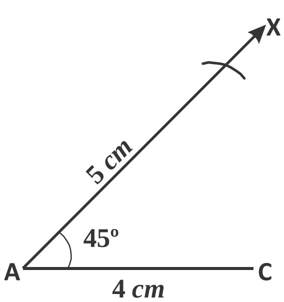


Fig. 2 (i)

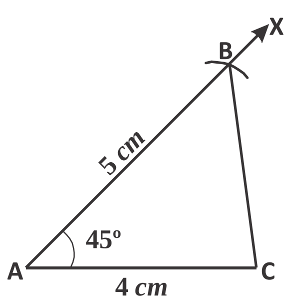


Fig. 2 (ii)

1. Draw a line segment AC of 4 *cm*.
2. Draw a ray AX on A which forms an angle of 45° with AC.
3. Draw an arc of 5 *cm* from point A, which cuts AX at point B. (*Fig*.2(i))
4. Draw a line segment joining points B and C. This way ΔABC is constructed. (*Fig*.2(ii)).

Try This

In this triangle AB = 5 *cm*, AC = 4 *cm* and $\angle$A = 45°. If we want we can draw a line segment AB of 5 *cm* and then a ray AY making on angle of 45° on AB.

Now from A make an arc of 4 *cm* on AC.

Is triangle ACB like the first triangle?

Do This Also

Construct a triangle ABC where

AB = 7 *cm*, AC = 6 *cm* and $\angle$B = 40°

Steps of construction:-

1. First of all draw a line segment AB = 7 *cm*
2. At point B draw a ray BX such that $\angle$ABX = 40º (*Fig*.3 (i))
3. From point A draw an arc of radius 6 *cm* intersecting ray BX at points C and D. (*Fig*.3(ii)).

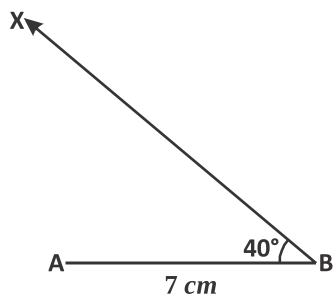


Fig. 3 (i)

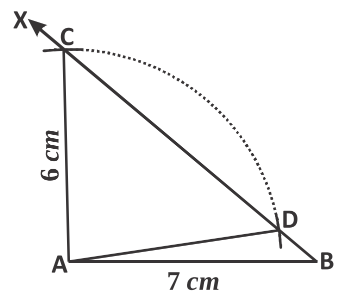


Fig. 3 (ii)

You can see that with the given conditions we get two points C and D on the ray BX. Therefore, it can be said that with the given measurements of the triangle two points A and B can be definitely determined; but the third point can be either C or D. As the third point can be either C or D, therefore the measurements given are not sufficient to construct a unique triangle.

Try This

Discuss with friends the measurements given to construct triangle:-

(i) AB = 7 *cm*, AC = 6 *cm*, $\angle$C = 40°

(ii) AB = 3 *cm*, BC = 4 *cm*, $\angle$A = 60°

(iii) PR = 6 *cm*, PQ = 5 *cm*, $\angle$Q = 75°

(iv) AB = 4.5 *cm*, AC = 6.3 *cm*, $\angle$A = 55°

You have seen that a unique triangle can be constructed only when the measurements of two sides and the angle formed by them is given.

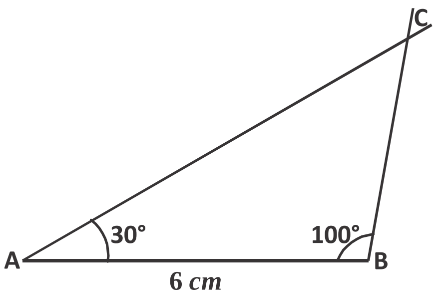


Fig. 4 (ii)

Construction-2 : Construct a triangle when the measurement of one side and the two angles on it's two end points are given.

EXAMPLE-2. To construct a triangle ABC where AB = 6 *cm*; $\angle BAC = 30°$, $\angle ABC = 100°$.

Steps of construction:-

1. Draw a line segement AB = 6 *cm*.
2. On line segment AB draw an angle of 30° at point A with the help of the protractor. (*Fig*.4 (i))
3. Similarly at point B draw an angle of 100°.
4. Extent the arms of both the angles. Let the point of intersection be 'C'.
5. Then ABC is the required triangle (*Fig*.4 (ii)).

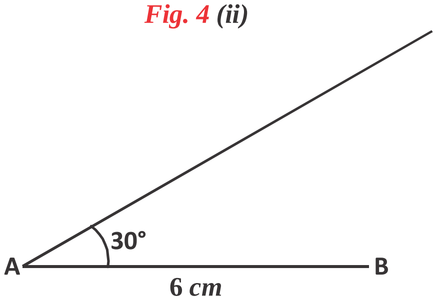


Fig. 4 (i)

Try This

1. Construct a triangle with the given measurements and discuss which type of triangle are these:-
   (i) In ΔPQR, PQ = 5 *cm*, $\angle P = 90°$, $\angle Q = 30°$
   (ii) In ΔMNP, MN = 6 *cm*, $\angle M = 90°$, $\angle N = 30°$
2. Draw and see if it is possible to construct triangle of given measurement:-
   (i) PQ = 3.5 *cm*, $\angle Q = 45°$, $\angle R = 50°$
   (ii) XY = 7.5 *cm*, $\angle Z = 70°$, $\angle Y = 40°$

Special Type of Triangles

Construction-3 : To construct such a triangle where the base, angle formed on the base and the sum of remaining two sides are given.

EXAMPLE-3. Construct a triangle PQR where QR = 4 *cm*, PQ + PR = 7.5 *cm* and $\angle PQR = 60°$.

Steps of Construction:-

1. Draw line segment QR = 4 *cm* and at point Q draw an $\angle XQR = 60°$.
2. With Q as centre, draw an arc of radius 7.5 cm intersecting QX at point S. Join RS. (*Fig*.5(i)).

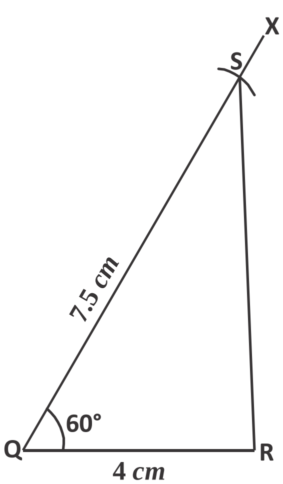


Fig. 5 (i)

3. With the help of compass draw a perpendicular bisector l of RS which cuts QS at point P and SR at point T. (*Fig*.5(ii)).

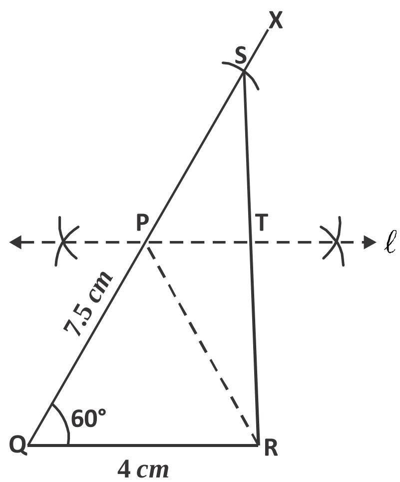
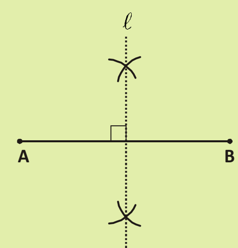


Fig. 5 (ii)

Perpendicular bisector : Perpendicular bisector is that line which divides any line segment into two equal parts by forming right angle.

Construction of Perpendicular Bisector:

1. Distance between two arms of the compass should be more than half of the line segment.
2. Now from point A cut an arc on both the sides of the line segment. Then from point B repeat the same process.
3. Join the cut points of both the arcs with a scale. This line l is the perpendicular bisector of AB.



4. Joint PR (*Fig*.5(iii)).

$\Delta PTS \cong \Delta PTR$. (Why?)

$\therefore$ PS = PR (CPCT)

QP + PS = QP + PR (=7.5 *cm*)

Therefore, ΔPQR is the required triangle.

Why is step 3 constructed like this?

We should locate point P on the side QS such that PS = PR

This could be done if both line segments could be seen as corresponding sides of two congruent triangles.

The perpendicular bisector of SR gives two such points P and T which divide ΔPSR into two congruent triangle by line segment PT.

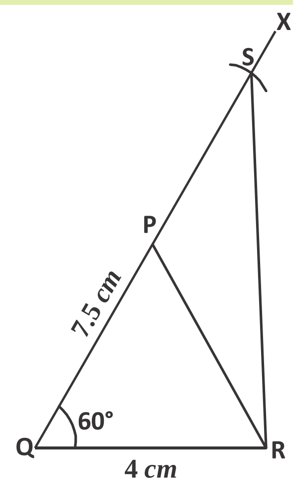


Fig. 5 (iii)

Alternate Method

Now we shall construct the same triangle in a different way.

Steps of Construction:-

1. Repeat steps 1 and 2 like. (*Fig*.6(i)).
2. Construct an $\angle SRY$ equal to $\angle QSR$. Intersecting QX at point P. (*Fig*.6(ii))

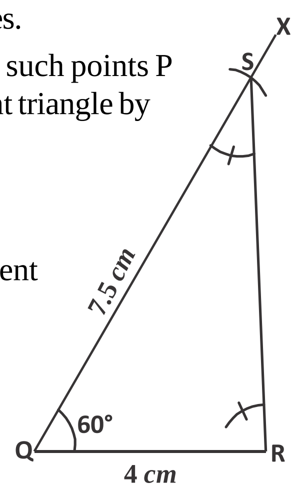


Fig. 6 (i)

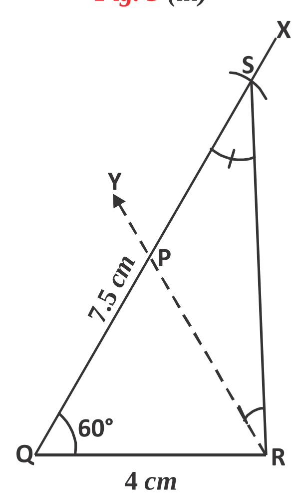
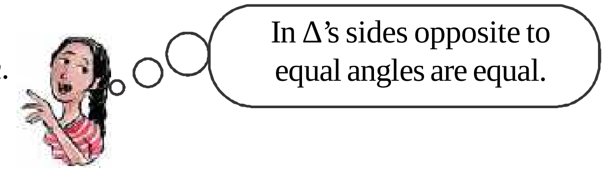


Fig. 6 (ii)

PS = PR (Why?)

QP + PS = QP + PR = 7.5 *cm*.

$\therefore$ PQR is the required triangle.



Try This

Construct a triangle ABC where BC = 6 *cm*, ∠B = 60° and AB+AC = 11 *cm*.

Construction-4 : To construct such a triangle where base, an angle on base and difference between two remaining sides are given.

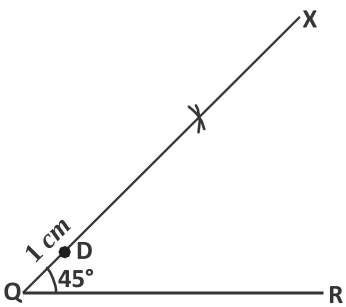
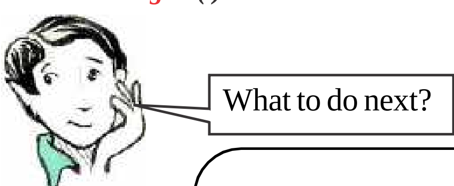


Fig. 7 (i)

EXAMPLE-4. Construct a triangle PQR where the base QR = 5.2 *cm*, ∠PQR = 45° and PQ – PR = 1 *cm*.

Steps of construction

1. Draw a line segment QR = 5.2 *cm*.
2. At point Q make an ∠XQR = 45°. On ray QX take a point D such that QD = 1 *cm* (PQ – PR = 1 *cm*) (*Fig*.7(i)).



I'll tell you!

On QX we need a point P such that PD = PR (then the difference between PQ and PR will be 1 *cm*.)

If PD and PR could be seen as corresponding sides of two congruent triangles then PD will be equal to PR. If we draw perpendicular bisector of side DR then we get two congruent triangles PDT and PRT.

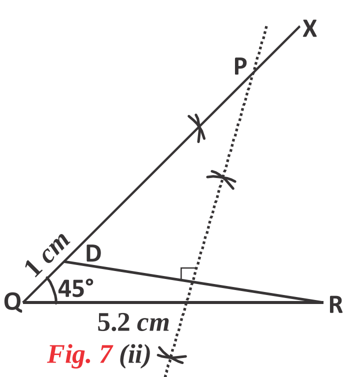


Fig. 7 (ii)

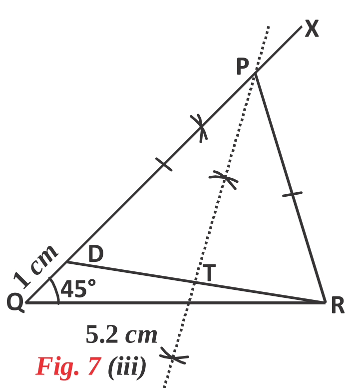


Fig. 7 (iii)

3. Join the points R and D. Draw a perpendicular bisector of the arm RD which cuts the ray QX at P. (*Fig*.7(ii)).
4. Join points P and R. Thus the required ΔPQR is constructed. (*Fig*.7(iii))

Construction-5 : To construct a triangle where the sum of three sides (perimeter) and the two base angles are given.

EXAMPLE-5. Construct a triangle ABC where $\angle B = 30°$, $\angle C = 60°$ and AB + BC + CA = 10.5 *cm*.

Steps of Construction

1. Draw a line segment PQ = 10.5 *cm* (PQ = BC + CA + AB)

2. Construct angles of measure 30° (equivalent to $\angle$B) and 60° (equivalent to $\angle$C) at point P & Q respectively. Now construct angle bisector of both the above angles & name their point of intersection as A. (*Fig*.8(i)).

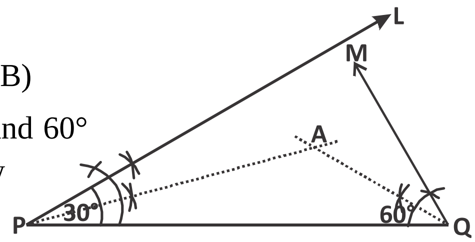


Fig. 8 (i)

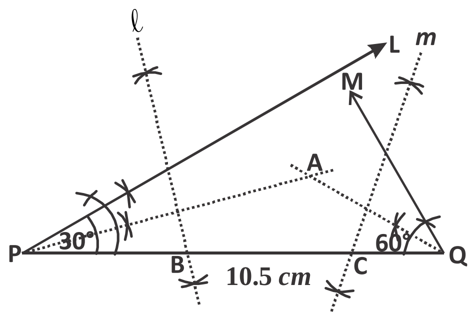


Fig. 8 (ii)

3. Now draw perpendicular bisector *l* and *m* of side PA and QA respectively which bisect the line segment PQ at the points B and C. (*Fig*.8(iii)).

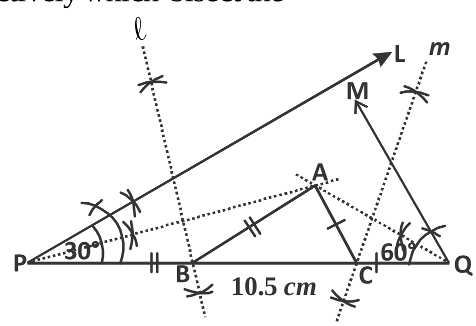


Fig. 8 (iii)

4. Join point B and point C with A. (*Fig*.8(iii))

ΔABC is the required triangle.

Try This

Measure the three sides of the triangle which you have constructed and add them. Is AB + BC + CA = 10.5 *cm?*

If one side of the triangle is extended, the external angle formed on the extended side is equal to the sum of the two internally opposite angles.

Exercise - 13.1

1. Construct triangles by the given measurements of their sides and angles.

S.No.	Traingle	Given Measurements		
(i)	ΔDEF	DE = 4.5 *cm*	EF = 5.5 *cm*	DF = 4 *cm*
(ii)	ΔPQR	$\angle$Q = 30°	$\angle$R = 30°	QR = 4.7 *cm*
(ii)	ΔABC	$\angle$B = 60°	BC = 5 *cm*	AB + AC = 8 *cm*

2. Construct a right angled triangle with a base of 4 *cm* and the sum of the other sides is 8 *cm*.
3. Construct a triangle PQR where QR = 7 *cm*, $\angle$Q = 45° and PQ – PR = 2 *cm*.
4. Construct a traingle XYZ where $\angle$XYZ = 50°, YZ = 5 *cm* and XZ – XY = 2.5 *cm*.
5. Construct a triangle ABC where AB + BC + CA = 13 *cm* and $\angle$B = 45°, $\angle$C = 70°.

Construction of Quadrilateral

So far you have constructed quadrilaterals in different situation. Now we shall construct these in some new situations.

Construction-6 : Construct a parallelogram where its two diagonals and the angle formed between them is given.

EXAMPLE-6. Construct a parallelogram ABCD where AC = 7 *cm* and BD = 6 *cm* and the angle formed between them is of 40°.

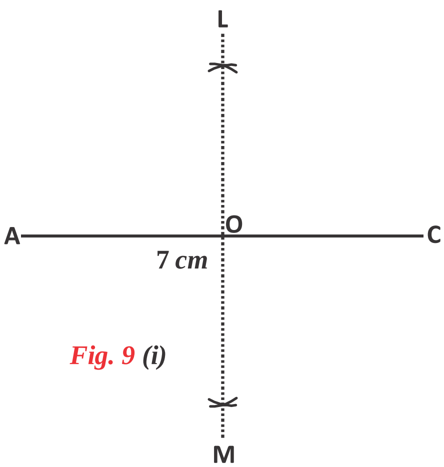


Fig. 9 (i)

Steps of construction:-

1. Draw a line segment AC = 7 *cm*.
2. Draw a perpendicular bisector LM of line segment AC which intersects AC at O. (*Fig*.9(i))
3. Draw a ray OX where $\angle$AOX = 40°. Now extend ray OX to XOX'. (*Fig*.9(ii)).
4. Taking O as the centre point, draw two arcs of 3 *cm* (half of the length of another diagonal i.e. half of 6*cm* = 3 *cm*) on XOX' namely B and D. Join both B and D with A and C. (*Fig*.9(iii)).

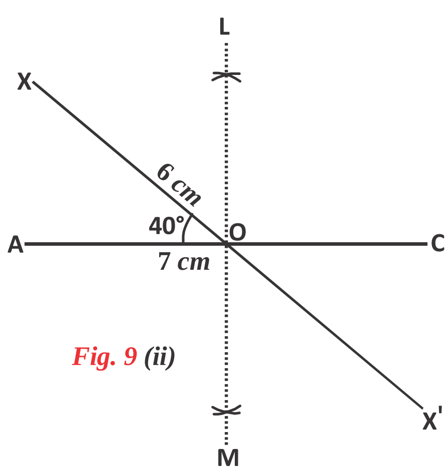


Fig. 9 (ii)

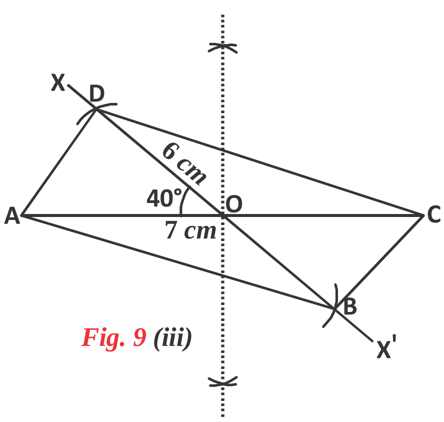


Fig. 9 (iii)

Thus, the required parallelogram ABCD is constructed.

Remember:- Diagonals in a parallelogram bisect each other. Therefore we draw perpendicular bisector of AC by which we got the central point O. On point O, we constructed an $\angle AOX = 40°$ and OB = OD = 3 *cm*.

Try This

Can you similarly construct a rectangle and a square?

Discuss with friends and form two questions based on it and construct it.

Construction-7 : Construct a trapezium when two adjacent sides, the angle formed by them and the parallel sides are known.

EXAMPLE-7. Construct a trapezium ABCD when AB = 5 *cm*, BC = 2.8 *cm*, AD = 3 *cm*, ĎA = 60° and AB || CD.

Steps of construction:-

1. Draw a line segment AB = 5 *cm*.
2. Draw a ray AX by forming an angle of 60° at point A.

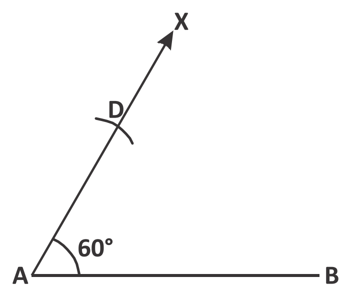


Fig. 10 (i)

3. Cut an arc AD of 3 *cm* on AX to get point D. (*Fig*.10(i)).

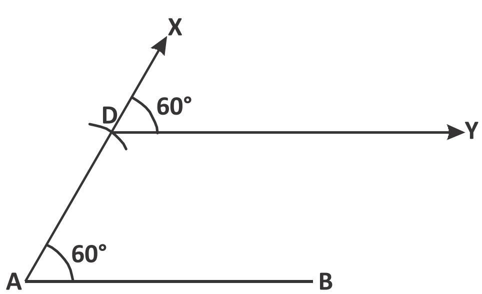


Fig. 10 (ii)

4. From point D draw a ray DY such that $\angle XDY = 60°$ (*Fig*.10(ii))

5. From point B, draw an arc of 2.8 *cm* intersecting DY at points C and C' (*Fig*.10(iii)).

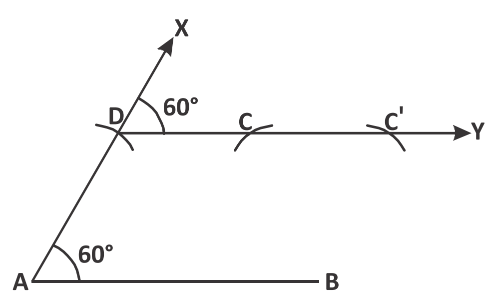


Fig. 10 (iii)

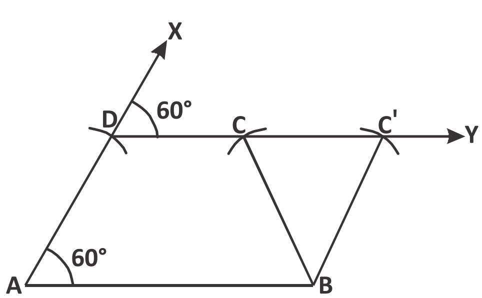


Fig. 10 (iv)

6. Join B with C and C'. (*Fig*.10 (iv)). This way the required trapezium ABCD and ABC'D is formed.

Exercise - 13.2

1. Construct a parallelogram ABCD where AD = 4 *cm*, AB = 6 *cm* and $\angle A = 65°$.
2. Construct a parallelogram where AB = 4 *cm*, AD = 3 *cm* and diagonal AC = 4.5 *cm*.
3. Construct a rectangle where one side is of 3 *cm* and the diagonal is of 5 *cm*.
4. Construct a rhombus where the two diagonals are of lengths 4.5 *cm* and 6 *cm* respectively.
5. Construct a trapezium ABCD where AB || CD, AB = 5 *cm*, BC = 3 *cm*, AD = 3.5 *cm* and the distance between the parallel lines is 2.5 *cm*.

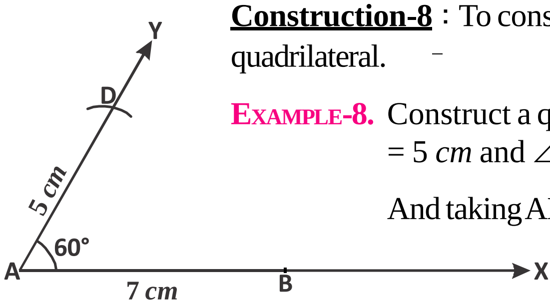


Fig. 11 (i)

Construction-8 : To construct a triangle which is equal in area to the area of a given quadrilateral.

EXAMPLE-8. Construct a quadrilateral where AB = 7 *cm*, CD = 6 *cm*, BC = 4 *cm*, AD = 5 *cm* and $\angle BAD = 60°$.

And taking AB as one side construct a triangle which is equal in area to the area of a quadrilateral.

Steps of construction:-

1. Draw a ray AX. On AX mark line segment AB of 7 *cm*.
2. On point A draw an $\angle BAY = 60°$ and cut an arc of 5 *cm* which cuts AY at D. (*Fig*.11(i))
3. Cut arc of length 4 *cm* & 6 *cm* from points B & D respectively, intersecting at point C.
4. Join BC, CD. Quadrilateral ABCD is the required quadrilateral. (*Fig*.11 (ii))

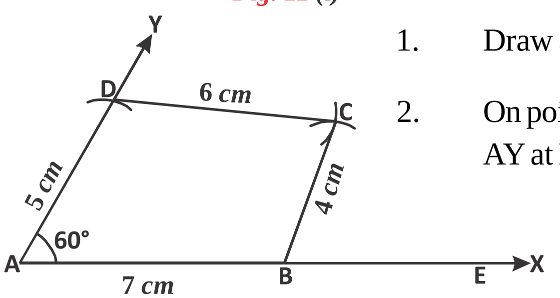


Fig. 11 (ii)

5. Now join BD. From point C draw CE || BD which cuts AX at E. (*Fig*.11(iii)). Join E and D. (*Fig*.11(iv))

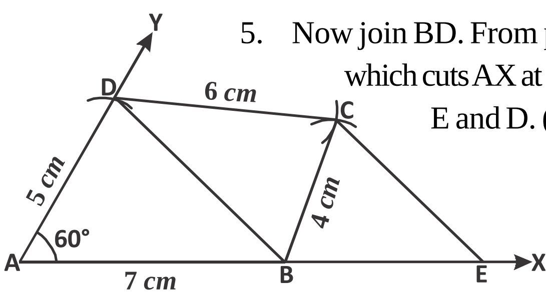


Fig. 11 (iii)

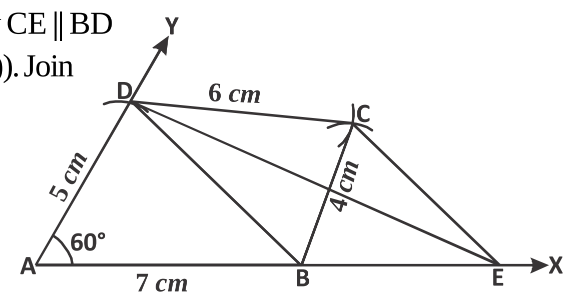


Fig. 11 (iv)

Thus, we get the required triangle which is equal in area to the area of quadrilateral ABCD.

Think and Discuss

Areas of ΔADE and quadrilateral ABCD are equal. How?

Construction-9 : Construct a parallelogram and rectangle which is equal in area to the area of a given triangle.

EXAMPLE-9. Construct a parallelogram which is equal in area to that of a triangle ABC where AB = 9 *cm*, AC = 5 *cm* and $\angle CAB = 60°$.

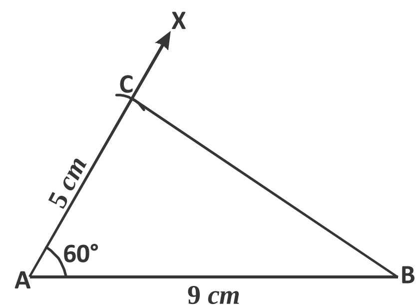


Fig. 12 (i)

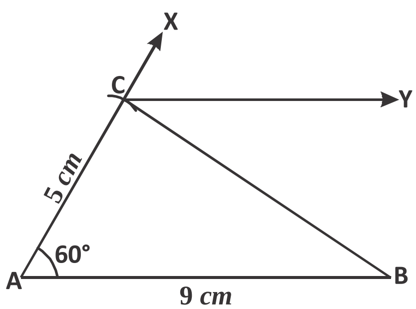


Fig. 12 (ii)

Steps of construction:-

1. Draw a line segment AB = 9 *cm* and an $\angle BAX$ of 60° at A.
2. On ray AX cut an arc of 5 *cm* at point C. Join BC. Required triangle ABC is formed (*Fig*.12(i)).
3. From point C draw a ray CY parallel to AB. (*Fig*.12(ii)).
4. Draw a perpendicular bisector PQ of side AB which bisects AB at O. (*Fig*.12(iii)).
5. From point O draw OD|| AC (*Fig*.12(iv))

Thus, we get the required parallelogram which is equal in area to the area of ΔABC. Discuss why ?

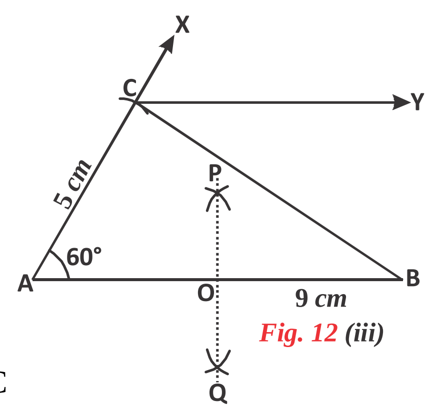


Fig. 12 (iii)

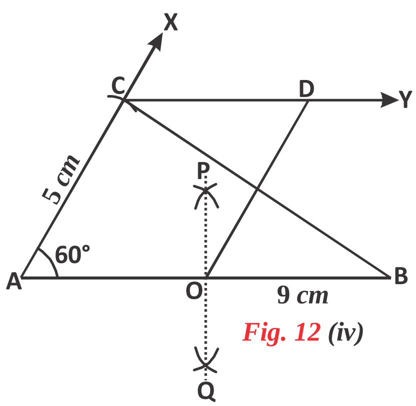


Fig. 12 (iv)

Exercise - 13.3

1. Construct a quadrilateral ABCD where AB = 5 *cm*, BC= 6 *cm*, CD = 7 *cm* and $\angle B = \angle C = 90°$. Then on AB as base construct a triangle which is equal in area to that of the quadrilateral.

2. Construct a triangle whose area is equal to the area of the rhombus whose sides are of 6 *cm* and one angle of 60°.

3. Construct an isosceles triangle with a base of 6 *cm* and base angles of 70°, construct a parallelogram and rectangle which is equal in area to that of the triangle.

4. Construct a traingle PQR where PQ = 8 *cm*, PR = 6 *cm*, ∠QPR = 65°. Construct a parallelogram whose area is equal to the area of the triangle.

Constructing a Circumscribed Regular Polygon Around a Circle and Inscribed Regular Polygon in a Circle

Construction-10 : Construct a regular pentagon inscribed in a 3 *cm* radius circle.

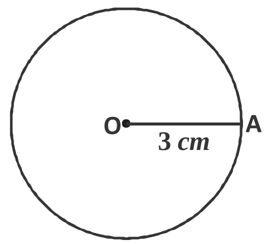


Fig. 13 (i)

Steps of construction:-

1. With centre O draw a circle of radius 3 *cm* Join O with point A on the circumference. (*Fig.*13(i)).

2. Since we have to construct a regular pentagon therefore divide the circle into 5 equal parts. The value of an angle subtended at the centre would be $\frac{360°}{5} = 72°$ (Why?).

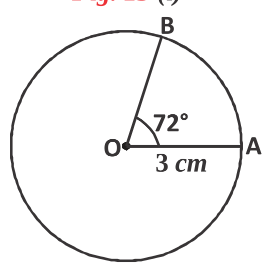


Fig. 13 (ii)

3. On OA, draw an angle of 72° at point O which cuts the circumference at B. (*Fig.*13(ii)).

4. Measure the arc AB with the compass and mark arcs on the circumference and we get points C, D and E. (*Fig.*13(iii)).

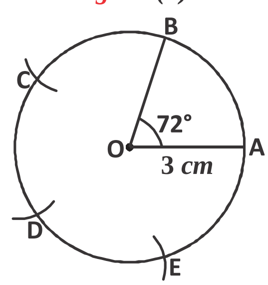


Fig. 13 (iii)

5. Joint A with B, B with C, C with D, D with E and E with A. (*Fig.*13(iv)).

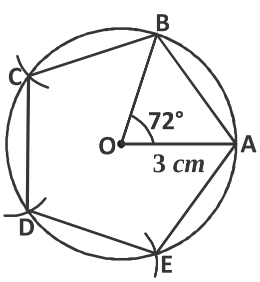


Fig. 13 (iv)

This way a required regular pentagon is obtained.

Similarly any regular polygon can be inscribed in a circle.

Think and Discuss

For pentagon the angle at the centre is $\frac{360°}{5}$, for hexagon it is $\frac{360°}{6}$, so would the angle at the centre of a polygon of *n* sides be $\frac{360°}{n}$?

Construction-11 : To construct a regular hexagon circumscribed around a circle of radius 3.5 *cm*.

Steps of construction:-

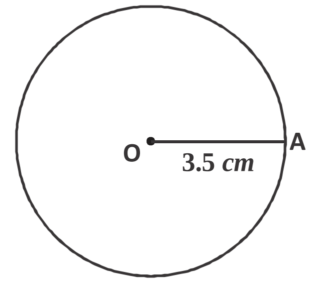


Fig. 14 (i)

1. With centre O draw a circle of radius 3.5 *cm* Take a point A on the circumference and join it with O. (*Fig*.14(i))

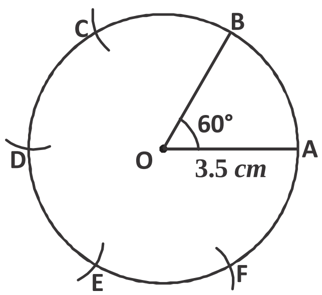


Fig. 14 (ii)

2. The value of the internal angle of regular hexagon at the circle will be = $\frac{360°}{6}$ = 60°. On OA draw an angle of 60° at point O which cuts the circumference at B.

3. As in construction-10 with arc AB mark the points C, D, E and F (*Fig*.14(ii)).

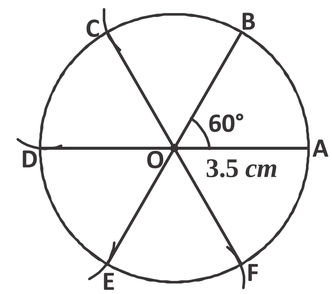


Fig. 14 (iii)

4. Joint points C, D, E and F with the centre O. (*Fig*.14(iii)).

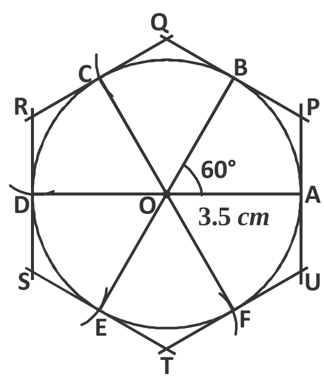


Fig. 14 (iv)

5. On OA, OB, OC, OD, OE and OF draw perpendicular lines UAP, PBQ, QCR, RDS, SET and TFU. (*Fig*.14(iv)).

Thus, we get the required hexagon PQRSTU which is the circumscribing the circle.

Similarly in any circle we can construct a regular polygon inscribing or circumscribing it.

Exercise - 13.4

1. Construct a regular quadrilateral inscribed in a circle of a radius of 2 *cm*.
2. Construct a regular octagon inscribed in a circle of a radius of 3 *cm*.
3. Construct a regular pentagon circumscribed around a circle of radius 2.5 *cm*.
4. Construct a regular octagon circumscribed around a circle of radius 3 *cm*.

What Have We Learnt

1. A triangle can be constructed only when:-
   - (i) The sum of two small sides is bigger than the measurement of the longest side.
   - (ii) Measurement of two sides and angle formed by them is given.
   - (iii) Measurement of one side and angles on both its ends are given.
   - (iv) When the base of a triangle, any one angle on the base and sum of the remaining two sides given.
   - (v) When the base of a triangle, any one angle on the base and the difference between the remaining two sides is given.
   - (vi) When perimeter of a triangle and both angles on the base are given.
2. A parallelogram can be constructed when its two diagonals and angle between them is given.
3. Trapezium can be constructed when two adjacent sides, angle formed by them and parallel sides are given.
4. Area of two triangles formed on one base and between same two parallel lines, is equal.
5. Angles formed at the centre by each sides of polygon of n sides will be $\frac{360°}{n}$.
6. A regular polygon inscribed in a circle and circumscribed around a circle can be constructed.

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Let us know the history of mensuration.....

Usually mathematical ideas originates from daily activities and experiences. In ancient times people used measure of “Bitta”, “Hath” (hand span, palm length) etc. for measuring the land, height of walls, depth of well etc. Using only these measures many big palaces, buildings, castles, ponds, roads, canals, drains etc. were constructed.

The field area was measured on the basis of the amount of seed sown in it, weight was measured by taking pebbles and other natural objects as units, volume was measured in lottas, glasses, tumblers, pots etc. Since these measures varied a lot, gradually the practice of measuring things in a standard unit evolved. In Indus valley of India fine standardized system of measuring length and weights (बाट), existed as early as 5 centuries before Christ. There were standard weights for different measures. Small weights for expensive things and big weights for things which were exchanged in large quantity. Most of the weights were cubical. Weighing balances (तराजू) with two sides were also made here. Similarly there were standard measures for measuring lengths which could even measure up to 1/16 of an inch. Many methods of measurements were adopted in Iran and Central Asia from India and vice versa. Mensuration as we know today has evolved, based on all these. Many situations are involved where these are needed. Some of them are-

1. Cost of fencing with wire around any field.
2. Cost of bricks or stones used in making parapets of wells.
3. Area of any room.
4. Volume of any tank.
5. The number and cost of tiles used to make a floor.
6. Estimate of the cost of ploughing or cutting of the crop.

For these we will need to find perimeter and area of closed two dimensional shapes and surface area and volume of solid shapes. In mensuration we will learn to find perimeter, area of triangle, quadrilateral, circle etc and surface area, volume of geometrical shapes like cube, cuboid, cylinder, cone and sphere etc.

Indian mathematicians gave many formulas for geometrical shapes and figures which are similar to formulae that we use today. For example- Aryabhatt gave following formula of area of circle:-

‘समपरिणाहस्यार्थं विष्कंभार्धहतमेव वृत्तफलम्’

i.e. Area of circle $= \frac{1}{2}$ (Circumference) $\times \frac{1}{2}$ (Diameter)

$= \pi r^2$

Where *r* is radius of circle.

The information presented here is collected from various sources. Teachers and students can get more information about mensuration from other sources also.

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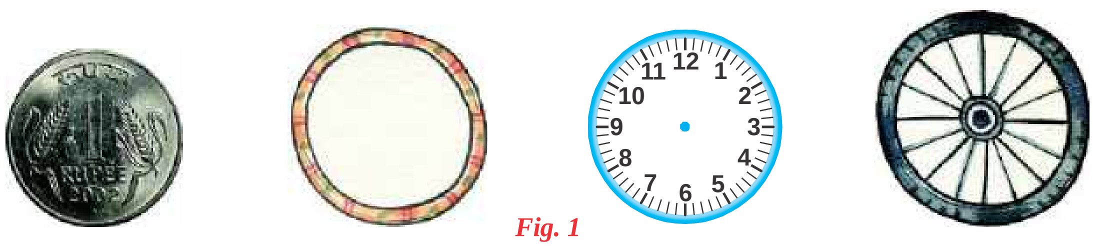

Fig. 1

We see many circular shapes like coins, bicycle wheels, dial of a clock etc. You may find many other objects around you that are circular. Can you think of few more such objects? In this chapter, we will read about circle and its properties.

Diameter of a Circle

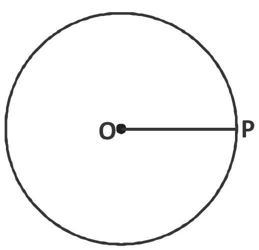

Fig. 2

You know about a circle. *Fig.*2, there is a circle with centre O and radius OP. In *Fig.*3, you can see the line segment AOB that passes through centre of the circle 'O' and has its end points at the circumference. AOB is a diameter of the circle.

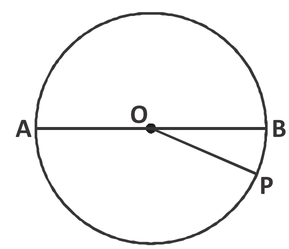

Fig. 3

Diameter of the Circle = 2 × Radius

Circumference of a Circle

In any circle drawn with any radius, the ratio of the circumference to its diameter is always fixed. This ratio is represented using a Greek letter π (pi).

$$\text{Therefore, } \frac{\text{Circumference of Circle}}{\text{Diameter}} = \pi$$

Circumference of the Circle $= \pi \times \text{diameter}$

$= \pi \times (2 \times \text{radius})$ (Since, diameter = 2 × radius)

If radius of a circle is *r*, then

Its circumference $= \pi \times (2 \times r)$

Circumference of Circle $= 2\pi r$

Try This

Take circular objects from your surroundings, find out the ratio of their circumferences to the respective diameters. Is the ratio constant? If yes, what is the value?

Area of a Circle

Draw a circle with centre 'O' and radius '*r*'. Inscribe a regular polygon with '*n*' sides in the circle (as shown in *Fig*.4). Now, make triangles by joining vertices of the regular polygon with the centre of the circle. You can see ΔOPQ as one of the triangles.

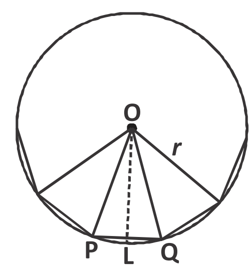

Fig. 4

$$\text{Area of Triangle OPQ} = \frac{1}{2}\ PQ \times OL$$

$$= \frac{1}{2} \times \text{side} \times \text{length of perpendicular from centre to the side.}$$

Since, the lengths of perpendicular from centre 'O' on each side of regular polygon are the same. Therefore, area of each triangle will be the same.

We know, that there are *n* such triangles.

So, area of *n* triangles

$$= n \times \frac{1}{2}\ \text{side} \times \text{perpendicular from centre to the side}$$

Now, what will happen if number of sides become infinite? In such situation, perimeter of the polygon will become same as circumference of the circle and area of the polygon will become equal to the area of circle. The length OL will become equal to 'r'.

$$\text{Therefore, Area of Circle} = \frac{1}{2} \times \text{Circumference} \times \text{Radius}$$

$$= \frac{1}{2} \times 2\pi r \times r$$

$$= \pi r^2$$

Sector of a Circle

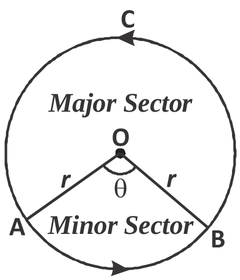

Fig. 5

Sector of circle is the portion of the circle enclosed by two radii and an arc. In *Fig.*5, you can see a circle with centre O and radius 'r'. A, B and C are any three points on the circumference of the circle. Join center 'O' with point A and B. Radii OA and OB divide the circle in two parts OAB and OBCA. These two are the sector of the circle.

Sector OAB is enclosed by radii OA, OB and an arc $\overset{\frown}{AB}$ and sector OBCA is enclosed by radii OA, OB and an arc $\overset{\frown}{BCA}$.

Let the arc $\overset{\frown}{AB}$ subtend an angle θ at the centre 'O' then the length of the arc is proportional to angle subtended by the arc at the centre.

$$\frac{\text{Length of an arc}}{\text{Circumference of circle}} = \frac{\text{Angle subtended by the arc at centre}}{\text{Angle subtended by circle at centre}}$$

$$\frac{\text{Lenth of an arc}}{\text{Circumference of circle}} = \frac{\theta}{360^{\circ}}$$

$$\text{Length of an arc} = \frac{\theta}{360^{\circ}} \times \text{Circumference of circle}$$

$$\text{Length of an arc of sector} = \frac{\theta}{360^{\circ}} \times 2\pi r$$

In the similar manner, area of the sector is proportional to the interior angle subtended at the centre by the arc enclosing it.

$$\therefore \frac{\text{Area of sector}}{\text{Area of circle}} = \frac{\text{Angle subtended at centre by the enclosing arc}}{\text{Angle subtended by circle at centre}}$$

$$\frac{\text{Area of Sector}}{\text{Area of circle}} = \frac{\theta}{360^{\circ}}$$

$$\text{Area of sector} = \frac{\theta}{360^{\circ}} \times \text{Area of circle}$$

$$\text{Area of sector} = \frac{\theta}{360^{\circ}} \times \pi r^2$$

Area of a Circular Path

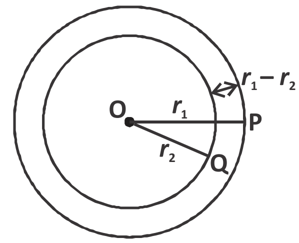

Fig. 6

Circular path is the region between two concentric circles. If the radii of the outer circle and inner circle are r_1 and r_2 respectively,

then, width of the circular path = Outer radius – Inner radius

$$= r_1 - r_2$$

Area of circular path = Area of outer circle – Area of inner circle

$$= \pi r_1^2 - \pi r_2^2$$

$$= \pi(r_1^2 - r_2^2)$$

Area of circular path = $\pi(r_1^2 - r_2^2)$

EXAMPLE-1. Diameter of a circle is 14 *cm* Find the circumference and the area of the circle.

SOLUTION : Given, diameter of circle = $2r = 14$ *cm*.

$$\therefore \text{ radius of circle, } r = \frac{14}{2} = 7 \, cm$$

We know, circumference of circle = $2\pi r$

$$= 2 \times \frac{22}{7} \times 7 = 44 \, cm$$

And area of the circle $= \pi r^2$

$$= \frac{22}{7} \times 7^2 = 154 \, sq \, cm$$

EXAMPLE-2. Find area of the circle, whose circumference is 176 *cm*.

SOLUTION : Here, circumference of the circle = 176 *cm*

$$\therefore \; 2\pi r = 176$$

$$2 \times \frac{22}{7} \times r = 176$$

$$r = \frac{176 \times 7}{2 \times 22} = 28 \, cm$$

Therefore, area of the circle = πr^2

$$= \frac{22}{7} \times (28)^2 = 2464 \, sq \, cm$$

EXAMPLE-3. Radii of two circles are 8 *cm* and 6 *cm* respectively. Find radius of the circle whose area equals the sum of the areas of the two given circles.

SOLUTION : Here, radius of the first circle $r_1 = 8\ cm$

radius of the second circle $r_2 = 6\ cm$

And radius of the required circle R = ?

We know area of required circle = Area of first circle + Area of second circle.

$$\pi R^2 = \pi r_1^2 + \pi r_2^2$$
$$\pi R^2 = \pi(r_1^2 + r_2^2)$$
$$R^2 = r_1^2 + r_2^2$$
$$R^2 = 8^2 + 6^2$$
$$R^2 = 64 + 36$$
$$R^2 = 100$$
$$R = 10\ cm$$

EXAMPLE-4. Radius of a circular ground is 35 *cm*. How long will it take for a boy to complete 10 rounds of the grounds at a speed of 5 *km* per hour.

SOLUTION : Radius of the circular ground $r = 35\ m$

Distance covered by the boy in one round (circumference) = $2\pi r$

Hence, distance covered in 10 rounds $= 10 \times 2\pi r$

$$= 10 \times 2 \times \frac{22}{7} \times 35$$

$$= 2200\ m = 2.2\ km$$

Time taken by the boy to cover 5 *km* = 60 minute

Therefore, 2.2 *km* will be covered in time $= \dfrac{60 \times 2.2}{5} = 26.4$ minute

= 26 minute and 24 second.

EXAMPLE-5. Find width of the circular path whose outer and inner circumference are 110 meters and 88 meters respectively.

SOLUTION : Let, outer radius of the circular path = r_1 meter

and, inner radius of the circular path = r_2 meter

we know that, outer circumference of circular path = 110 meter

$$2\pi r_1 = 110$$

$$2 \times \frac{22}{7} \times r_1 = 110$$

$$r_1 = \frac{110 \times 7}{2 \times 22} = 17.5 \text{ meter}$$

Inner circumference of the circular path = 88 meter

$$2\pi r_2 = 88$$

$$2 \times \frac{22}{7} \times r_2 = 88$$

$$r_2 = \frac{88 \times 7}{2 \times 22} = 14 \text{ meter}$$

Width of the circular path = $r_1 - r_2$

$$= 17.5 - 14 = 3.5 \text{ meter}$$

EXAMPLE-6. A circular garden is surrounded by a 7 meter wide road. Circumference of the garden is 352 meter. Find area of the road.

SOLUTION : Let, the outer radius of the road = r_1

And, inner radius of the road (radius of garden) = r_2

We know that, circumference of the garden = 352 meter.

$$2\pi r_2 = 352$$

$$2 \times \frac{22}{7} \times r_2 = 352$$

$$r_2 = \frac{352 \times 7}{2 \times 22} = 56 \text{ meter}$$

Therefore, outer radius of circular path (road) $r_1 = 56 + 7 = 63$ meter

$\therefore$ Area of the circular path = $\pi(r_1^2 - r_2^2)$

$$= \frac{22}{7} \times \left[(63)^2 - (56)^2\right]$$

$$= \frac{22}{7} \times (63 + 56)(63 - 56)$$

$$= \frac{22}{7} \times 119 \times 7 = 2618 \text{ } sq \text{ } m$$

EXAMPLE-7. There is a circle of radius 21 *cm*. A sector that subtends an angle 120° at the center is cut from the circle. Find length of the arc of sector cut. Also find area of the sector.

SOLUTION : Given, radius of the circle $r = 21\ cm$

Angle subtended by the sector $\theta = 120^\circ$

Therefore, length of the arc of sector $= \dfrac{\theta}{360^\circ} \times 2\pi r$

$$= \frac{120^\circ}{360^\circ} \times 2 \times \frac{22}{7} \times 21$$

$= 44\ cm$

And area of the sector $= \dfrac{\theta}{360^\circ} \times \pi r^2$

$$= \frac{120^\circ}{360^\circ} \times \frac{22}{7} \times (21)^2$$

$= 462\ sq\ cm$

EXAMPLE-8. Find the area of the shaded portion in given figure. (*Fig.*7)

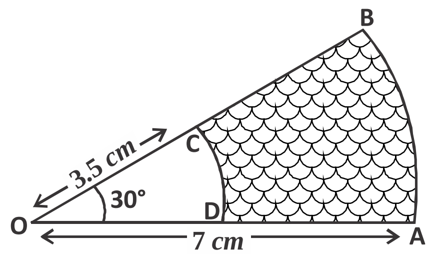

Fig. 7

SOLUTION : Area of the shaded portion ABCD

= Area of sector OAB – Area of sector OCD

$$= \frac{\theta}{360^\circ} \times \pi (OA)^2 - \frac{\theta}{360^\circ} \times \pi (OD)^2$$

$$= \frac{\theta}{360^\circ} \times \pi \left[(OA)^2 - (OD)^2\right]$$

$$= \frac{30^\circ}{360^\circ} \times \frac{22}{7} \times \left[(7)^2 - (3.5)^2\right]$$

$$= \frac{1}{12} \times \frac{22}{7} \times (7 + 3.5) \times (7 - 3.5)$$

$$= \frac{11}{6 \times 7} \times 10.5 \times 3.5$$

$= 9.625$ *sq cm* or $9.625\ cm^2$

Area of the shaded portion is 9.625 *sq cm*

Exercise-14.1

1. Find the circumference of the circle, whose radius is 17.5 *cm*.
2. Find the area of the circle, whose radius is 4.2 *cm*.
3. A horse is tied by a 14 meter long rope in a ground. What is the area of the ground, that the horse can graze, if he can move up to the full length of the rope?
4. Radius of a bicycle wheel is 35 *cm*. How much distance will it cover in 500 complete rotations?
5. Radius of a circle is 3 meter, what would be the radius of a circle whose area is 9 times the area of the first circle?
6. Inner circumference of a circular path is 440 *m*. Width of the path is 14 *m*. Find diameter of the outer circle of the circular path.
7. A circular ground of radius 50 meter is sorrounded by a 5 meter wide road. You want to cover the road with tile. Find the total cost of tiling if the rate of tiling per square meter is 30 rupees.
8. You are given a sector which subtends an angle of 70° at the centre of the circle with radius 21 *cm* Find the length of the arc and area of the given sector.
9. Area of a sector of the circle is $\frac{1}{6}$ times the area of the circle. Find the angle subtended at the centre by the given sector.
10. Area of a sector is 1540 *sq cm*, it subtends an angle of 50° at the centre of the circle. Find the radius of the circle.

What Have We Learnt

1. Diameter of the circle (2 × radius), circumference of the circle ($2\pi r$), area of the circle (πr^2). We also learnt about the sector of a circle.
2. Area of the circular path = $\pi(r_1^2 - r_2^2)$; where r_1 and r_2 are outer and inner radii respectively.

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We see a lot of things like book, pen, pencil, rubber etc. around us. These are 3D objects but we can see 2D shapes like triangle, quadrilateral, circle etc. in them. The surface, edges and vertices of these 3D objects are made up by combining these 2D shapes. We will now learn about surface area and volume of 3D objects such as cube and cuboid. To do this we will use the concept of 2D shapes and their area.

Representing 3D Shapes

We can draw 2D shapes (triangle, quadrilateral and circle etc.) on paper as per their measures.

Can we similarly depict three dimentional shapes on paper with their measurements?

Representing A Cube

While depicting 3D shapes we try to show all the faces. Those on the front as well on those of behind the faces in front. Cubes and Cuboids have 6 faces. They have 8 edges. How many vertices do they have?

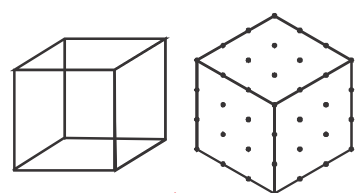


Fig. 1

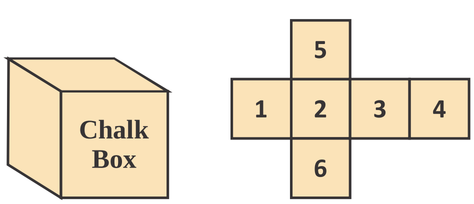
Opening A Three Dimensional Shape to make a 2D Shape

1. Surface Area of A Cube:

Take a closed chalk box and open it as shown in the figure.

Now write down the number of faces, vertices and edges. Are all the faces congruent?

We find that all the faces of chalk box are of the same size, i.e. they are all congruent. Such a shape is called a cube.



(I) *Fig. 2* (ii)

If the length of the side of a square is 'a' then it's area would be a^2.

Can you find out the area of all the surfaces?

Hamid- I can add the area of 6 faces and find the total surface area.

Total surface area of the cube $= a^2 + a^2 + a^2 + a^2 + a^2 + a^2$

$= 6\,a^2$

Try This

1. Take a card board sheet and make a cube of side 8 *cm*.
2. The length of the edge of a cube is 4 *cm*. By how many times will the total surface area increase, if the length of the edge is increased to 8*cm*.

2. Cuboid

Take a box of toothpaste and open it as shown in the picture.

Fig. 3 (*i*)

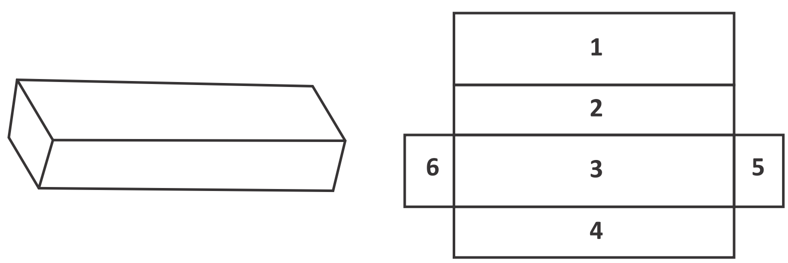


Fig. 3 (*ii*)

Now write down the number of faces, edges and vertices it has.

Are the lengths and breadths of faces 1 and 3, 2 and 4 and 5 and 6 the same?

We see that for a toothpaste box, the opposite faces have the same measure i.e. they are congruent. Such a shape is called a cuboid.

Surface Area of a Cuboid

Locate some cuboid shaped objects around you and draw their figures in your notebook. Name their vertices.

How many sides does each have (*Fig*.4)?

Which faces of the cuboid are equal to each other and rectangular?

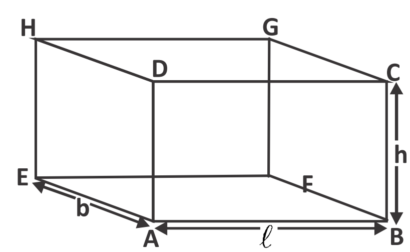


Fig. 4

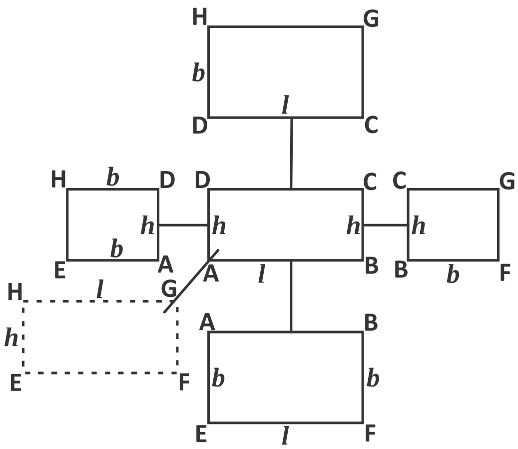


Fig. 5

How can you find the total surface area of a cuboid? Jaspal said that we can obtain the total surface area by adding area of all the rectangular surfaces.

Neha calculate the total surface area of a shoe-box.

She wrote *l* for length, *b* for breadth and *h* for height on the surfaces. Then she calculated the area of each surface seperately and added them. (*Fig*.5).

You also do the above task and see if you get the same.

So the total surface area of a cuboid $= hl + lb + hl + lb + hb + hb$

$= 2\,lb + 2\,lh + 2\,bh$

$= 2\,(lb + lh + bh)$

Try This

1. How will you find the total surface area of a cube using the formula for a cuboid?
2. What is the total surface area of a cuboid whose length is 6 *cm*, breadth is 3 *cm* and height is 2 *cm*.

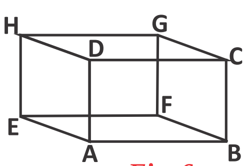


Fig. 6

Do you know which of the faces of the cuboid are the lateral faces?

The four side faces of the cuboid (excluding the top and bottom faces) are the lateral faces.

How will you find the surface area of the lateral faces of a cuboid?

Ajit- I can add the areas of faces ABCD, EFGH, BFGC and AEHD and that is the area of the lateral faces of the cuboid.

Monika- Yes, if we do this, the area of lateral faces of a cuboid $= 2hl + 2\,hb$

$= 2\,h\,(l + b)$

Can you now find the area of the lateral faces of a cube?

The area of lateral faces of a cube $= 4\,a^2$ (why?)

Think and Discuss

Keep your mathematics book as shown in the pictures and find out its total surface area (Add the third position yourself).

Volume of Solid Objects

Fig. 7

Take a glass tumbler, fill it with water till the top. Now put 2 pieces of lemon in it. What do you see? Some water will flow out of it. This tells us that lemon takes up some of the space in the tumbler. In the same way all solid objects occupy space.

If a hollow object filled with a fluid like water or air then the fluid will take the shape of that object. In that case the volume of the fluid gives us the volume of that object or capacity of the object.

The milk pot (patila) can contain more milk then in the bowl (katori). Is the capacity of the patila more than that of the katori.

Volume of Cuboid

Is the volume of the room more than the volume of the almirah kept in the room? Or which has more volume, the pencil box or the pencils and eraser kept in it. Can you find the volume of some of these objects?

We use the unit square to measure the area of any surface, similarly we use the unit cube to find the volume of a solid because the cube is the simplest solid shape. To find the area we divide the surface into unit squares. Similiarly, to find the volume of a solid we will divide it into unit cubes.

A pencil box occupies more space than a pencil or an eraser, which means the volume of the pencil box is more.

1 cubic centimeter $= 1\ cm \times 1\ cm \times 1\ cm = 1\ cm^3$

1 cubic meter $= 1\ m \times 1\ m \times 1\ m = 1\ m^3$

Volume of a cuboid - surface area of the rectangular base × height

$= \text{A} \times h = l \times b \times h$ unit cubes

Here A is the area of the surface of the base of the cuboid and h is its height.

Volume of Cube

We know that cube is a special kind of cuboid whose length, breadth and height are equal i.e. $l = b = h = a$ (suppose)

Volume of cube = Side × Side × Side

$= a \times a \times a$

$= a^3$ cubic unit

Where a is the side of the cube.

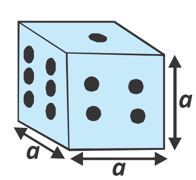


Fig. 8

To find the volume of a solid object using unit cubes is very convenient. The meaning of cubic unit of volume is number of unit length cubes included in it.

EXAMPLE-1. A cuboidal room has a length of 12 meter, breadth of 8 meter and height 4 meters. If the rate of white washing is Rs. 7 per sq m then how much money will be spent in painting its 4 walls and the ceiling?

SOLUTION : The length of the room l = 12 m, breadth b = 8 m and height h = 4 m

Since the room is a cuboid therefore,

the area of four walls of the room = perimeter of the base × height of the room

$= 2\,(l + b) \times h$

$= 2\,(12 + 8) \times 4 = 160$ sq m

Area of the ceiling = $l \times b$ $= 12 \times 8 = 96$ sq m

$\therefore$ Total area which needs to be while washed = 160 + 96 = 256 sq m

The rate of white washing is Rs. 7 per Sq. meter, therefore the cost of white washing the walls and the ceiling,

= Total area to be white washed × 7

= 256 × 7 = Rs. 1792

Try This

Measure the length, breadth and height of your classroom with your friends. Then find out:-

(i) The total surface area of the walls excluding the windows and doors.

(ii) If the room is to be white washed, then what is the total area that will need to be white washed.

(iii) What is the rate for white washing in your town/village? Using that what would be expenditure on getting the walls of the room white washed.

EXAMPLE-2. Anwar got a cubical water tank (with a lid) constructed on the terrace of his house. The length of outer edge 1.8 meter. He wants to put square tiles of side 30 *cm* on the entire surfase barring the base of the tank. If the cost of putting a dozen tiles is Rs 396, then what is the total expenditure that Anwar will incur?

SOLUTION : Since Anwar wants to get the five outer faces of the water tank tiled, therefore to find the number of tiles we need to find the surface area of these five surfaces.

Outer length of the cubical tank $a = 1.8\ m = 180\ cm$

$\therefore$ The area of the five faces of the tank $= 5\,a^2$

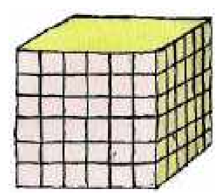
$= 5 \times 180 \times 180\ sq\ cm$ .....(1)

Area of each square tile $=$ side $\times$ side

$= 30 \times 30\ sq\ cm$ .....(2)

$$\therefore \text{ No. of tiles } = \frac{\text{Area of the five faces of the tank}}{\text{Area of each tile}}$$

$$= \frac{5 \times 180 \times 180}{30 \times 30} = 180$$



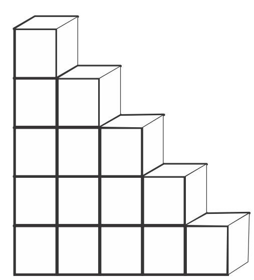
Cost of putting one dozen tiles = Rs. 396

$\therefore$ Cost of putting one tile $= \dfrac{396}{12} =$ Rs. 33

So, the cost of putting 180 tiles $= 180 \times 33 =$ Rs. 5940

EXAMPLE-3. While playing with some cubical blocks, William made the structure shown in the picture. If the side of each cube is 4 *cm*, then find the volume of the structure made by Willam.

SOLUTION : Since the length of each side of cube, $a = 4\ cm$



$\therefore$ The volume of each cube $= a \times a \times a$

$= 4 \times 4 \times 4$

$= 64\ cubic\ cm$

Cubes used in the structure = 15

$\therefore$ The volume of the structure $= 64 \times 15 = 960\ cubic\ cm$

Try This

Find the area of any rectangular sheet of your book. Now measure the height of the book and thus find its volume. In a shelf of an almirah in your school's library, how many such books can be placed? Do you read newspaper in your school every day? Now instead of the book, consider the almirah is filled with newspaper and do the same estimation and find out in how many days the almirah will be filled with newspapers?

Exercise - 15.1

1. The total surface area of a cube is 1350 m^2, find its volume.
2. The volume of a cuboid is 1200 cm^3. Its length is 15 *cm*, breadth 10 *cm* Find out its height?
3. What would the total surface area of a cubical box be if :-
   (i) each side is doubled?
   (ii) each side is tripled?
   (iii) each side is made n times?
4. The perimeter of the largest room in Priyanka's house is 250 *m*. The cost of white washing its four walls at the rate of Rs. 10 per sq m cost is Rs. 15000. What is the height of the room. (***Hint*** : Area of four walls = lateral surface area)
5. The length of the side of a cubical box is 10 *cm*. There is another cuboidal box with its length, breadth and height being 12.5 *cm*, 10 *cm* and 8 *cm* respectively.
   (i) Which box has more lateral surface area and by how much is it more?
   (ii) Which box has less total surface area and how much less?
6. The population of a village is 4000. Each person in the village requires 150 litre of water every day. There is a water tank in the village with length, breadth and height 20 *m*, 15 *m* and 6 *m* respectively. If this tank is filled to the brim with water then for how many days, the water requirement would be met?
7. A river 3 meter deep and 40 meter wide is flowing at the rate of 2 *km* per hour. Can you find out the volume of water going to the sea per minute?
8. The total surface area of a cuboid is 3328 m^2. If the length, breadth and height are in the ratio 4 : 3 : 2, then what will be its volume?
9. A cuboid is made by joining the edges of three identical cubes each having a volume of 125 cm^3. Find the total surface area of the cuboid thus formed?
10. A water body is of cuboidal shape of (Cuboid is a right angled parllelo pipped). Its length 20 *m*. When 18 *l*. water is taken out of the water body, its level goes down by 15 *cm*. Find the breadth of the water body.
    (***Hint*** : 1 Litre = 1000 *ml*, 1 *ml* = 1 cm^3)
11. A 10 meter long, 4 meter high and 24 cm thick wall is to be constructed in an open ground. If the wall is to be built using bricks of the size 24 *cm* × 12 *cm* × 8 *cm*, then how many bricks will be needed?

12. Find out the expenditure on digging a cuboidal pit of 8 *m* length, 6 *m* width and 3 *m* depth. If the cost of digging is Rs. 30 per cubic meter.

13. The dimensions of a godown are 60 $m \times 25$ $m \times 10$ m What is the maximum number of wooden crates of dimensions 1.5 $m \times 1.25$ $m \times 0.5$ m that can be stacked in this godown?

14. A solid cube of side 12 *cm* is cut into 8 identical cubes each having the same volume. What will be the side of the new cubes? Also find out the surface area of both the cubes?

15. Mary wants to keep her christmas tree in a wooden box covered with coloured paper. She wants to know how much paper she needs to buy. The above mentioned box is 80 *cm* long, 40 *cm* wide and 20 *cm* high. And if the paper is 40 *cm* square sheets then how many sheets are needed.

What Have We Learnt

1. Cube is a regular hexagonal solid object. It has 4 lateral faces and 8 vertices.
2. If the side of the cube is of length 'a' then

   The total surface area of the cube $= 6a^2$

   Lateral surface area of the cube $= 4a^2$

   Volume of the cube $= a^3$
3. If the length of a cuboid is 'l', breadth 'b' and height 'h' then

   Total surface area of the cuboid $= 2(lb+bh+lh)$

   Lateral surface area of the cuboid $= 2h\,(l+b)$

   Volume of the cuboid $= lbh$
4. Solid objects that have more volume occupy more space.

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Let us know the history of statistics.....

Statistics has being used in India since ancient days. As early as between 321 BC to 296 BC, we find in Arthshastra written by Kautilya the use of data in a variety of ways. This book describes in detail the agricultural data, figures for rural and urban population as well as economic data and the processes how they were collected during Maurya dynasty. This process of collecting data continued in the times of Mughal Emperor Akbar as well. Abul Fazal in his book ‘Aina-E-Akbari’, written around 1596 – 1597, describes this process of data collection and its use.

In the British period, East India Company needed to keep a record of its accounts as well as detailed information about the areas under its control. In 1807, the company got a survey done in its states. This survey included one crore 50 lakh people spread over 60,000 square miles area. This report included information on many significant aspects. The more important of these include the geographical description of each district, the religion as well as rites and rituals of the citizens, the natural wealth of the country, fisheries, agriculture situation and industrial situation. A government officer, A. Shakespeare, presented in 1848, the first census report. This was related to the area and the revenue of all districts of the Northwest province. The first effort to collect the detailed census data of India was made in the years 1867 to 1872. The first nationwide census took place in 1881. Since then nationwide census is being done every 10 years.

After independence the need for an appropriate statistical structure was felt for the economic and social development of the country. Prof P.C. Mahalanobis was the first statistical adviser to the Indian cabinet in 1949. His contribution to development of statistics in India is unforgettable. Prof Mahalanobis was the founder director of Indian Statistical Institute set up in Calcutta in the year 1932. This was declared as an institute of national importance in 1959. Besides this in 1949, Central Statistical Organisation was set up. From the 20th century to now, efforts to develop statistical methods, concepts and uses are continuing.

It appears that the word Statistics comes from the Latin word ‘Status’, which means political state or administration.

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Knowingly or unknowingly we keep using data all the time. We organise the information from our prior experiences, analyse it and draw conclusions. For example in the month of July if the sky is cloudy and the wind blows from the east, we say that it would rain today. Similarly people who travel know that some trains generally come on time but there are some that are often late. While buying pulses, wheat grain, rice etc., we examine a small part of the material and decide if it is worth buying or not. In the cricket match we consider the rate at which runs are being scored and at what rate they are further required etc. In the newspaper every day we look at the maximum and minimum temperatures, average humidity, time for sunrise and sunset etc. Being able to see these data with comprehension and drawing conclusions from it helps us analyse better and to make better judgements.

Individuals, families, panchayats, State and Indian Government and all other institutions and organisations use data for taking decisions and planning. The better our ways of collecting and organising data and the sharper our analysis is, the better would be our decisions and their implementation.

Data Collection and Representation

Suppose you have 30 students in a class and you are asked to collect the following data how will you go about it?

1. Information about the blood group of each student of the class.
2. The number of students of the class who come walking and those who use other means

Students of a class started collecting this data. They decided to do this in two groups. Each group went to every student and asked about their blood group and the means of transport to the school. Group 1 made the following table:-

TABLE - 1

A		B		AB		O	
Rh^+	Rh^-	Rh^+	Rh^-	Rh^+	Rh^-	Rh^+	Rh^-
卌	|||	卌 ||||	||||	|||	||	|||	|

Group 2 made the following table:-

TABLE - 2

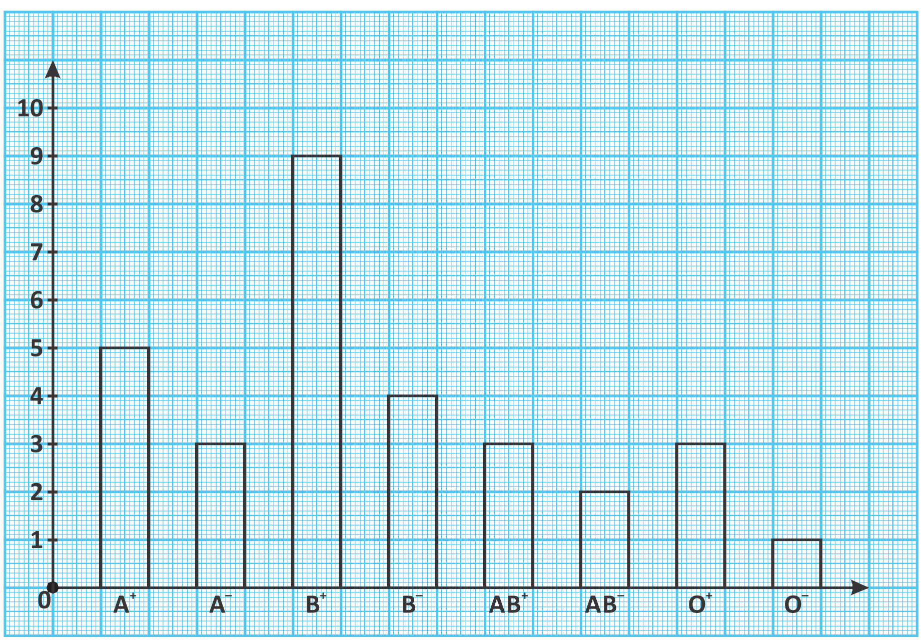
On Foot	Bicycle	Scooter	Bus	Others
卌 ||	|||	卌 卌	卌 |||	||

Frequency Table

In order to understand the collected data better Group 1 re-organised the data and made table-3:-

TABLE - 3

Blood Group	Tally Mark to Count	In Number
A^+	卌	5
A^-	|||	3
B^+	卌 ||||	9
B^-	||||	4
AB^+	|||	3
AB^-	||	2
O^+	|||	3
O^-	|	1
Total		**30**



In this table along with the tally marks the frequency count is also written as a number, for example 9 written opposite B^+ shows that there are nine people who have the blood group B^+. In the same manner the other numbers show the frequency of other blood groups. Such a table is called frequency table. The group then made a bar diagram based on the table.

Try This

1. Write any five conclusions that can be drawn from the bar diagram.
2. Similarly also represent the data of group -2 as a bar diagram.
3. Make the attendance table for the students of your class in the month of January and answer the following:-
   (i) On which day was the attendance maximum?
   (ii) When was the attendance minimum? (iii) Write some more conclusions.
4. From hockey, cricket, kabbadi, football and volleyball, which is the sport your class fellows like the most? Collect data for this and make a frequency table to find the answer to the following-
   (i) Which is the most popular sport?
   (ii) Which sport is liked by less children?

Place in Ascending and Descending Orders

Our data may have values which are repeated or may not. If the data is not large then we can draw conclusions by simply placing it in ascending-descending order. For example-

In a class the marks obtained in mathematics exam of 15 students out of 100 are as follows:

45, 35, 56, 22, 99, 71, 80, 63, 42, 36, 18, 77, 54, 82, 41

Writing these in ascending order-

18, 22, 35, 36, 41, 42, 45, 54, 56, 63, 71, 77, 80, 82, 99

Writing in descending order-

99, 82, 80, 77, 71, 63, 56, 54, 45, 42, 41, 36, 35, 22, 18

Now you can answer the following:-

1. What is the lowest and the highest marks obtained.
2. What is the difference between them?

Discuss with your friend and find out some more conclusions can be drawn from the data.

Grouped Frequency Table

1. Inclusive Class

When the number of data points is large and they have a large range between their maximum and the minimum then frequency table will be very large. In such situations instead of finding the frequency of one number and find the frequency of small groups. (We will call these groups classes)

EXAMPLE-1. In a 50 over match the number of runs made by a team in each over is given below.

7, 8, 2, 5, 7, 12, 6, 20, 18, 9, 11, 5, 19, 10, 3, 6, 12, 8, 16, 0, 12, 7, 8, 11, 15,
13, 4, 7, 1, 22, 2, 17, 1, 6, 21, 4, 9, 15, 0, 5, 1, 9, 26, 10, 14, 3, 16, 2, 6, 8

When we make a frequency table for this data we will need to find the number of overs in which no runs were scored, the number of overs where one run was scored etc. In this way we will have to go up to 26. This is because in one over 26 runs were scored. This would be a huge table.

Can we therefore reason in the following manner:-

How many overs in which runs from 1 to 6 were scored? How many overs in which runs from 7 to 12 were scored?

And in the same manner number of overs in which 13 to 18 or 19 to 24 and then 25 to 30 runs were scored. They call these groups classes. Depending upon our needs the groups can be smaller or bigger. In this example you can choose the groups to be 1 to 4, 5 to 8, 9 to 12 or from 1 to 5, 6 to 10, 11 to 15 etc. You could choose any other group size as well.

How do we find out the frequency of these groups?

From the above suggested grouping choose anyone. Look at the number of runs made in each over. Put a tally mark in the appropriate group. Do this for all the 50 overs. You will get the following frequency table:-

TABLE - 4

Number of runs scored	Tally mark	Number of overs (frequency)
0-4	𝍸 𝍸 ||	12
5-9	𝍸 𝍸 𝍸 |||	18
10-14	𝍸 ||||	09
15-19	𝍸 ||	07
20-24	|||	03
25-29	|	01
	Total	**50**

While using such tables we use some terms, for example class interval, lower class limit, upper limit, centre point, inclusive class, non-inclusive class etc. Let us try to understand these.

In the above example 0-4, 5-9, 10-14 etc., are all classes.

Look at any two classes of this frequency table. You will find that the lower limit of the next group starts where the upper limit of the first group ends. This means that the upper limit of any group is not the same as the lower limit of the next group. These groups/ classes are called inclusive because the lower and upper limits are both included in that group. The group 0 to 4 includes overs in which 0, 1, 2, 3, 4 runs were scored. There are five such situations and hence the class interval is also 5. In the same manner the group of 5 to 9, includes overs in which 5, 6, 7, 8, 9 runs were scored, the class interval here is also 5. For the first group 0 to 4 the lower limit is 0 and the upper limit 4. Similarly, the other groups the lower limits are 5, 10, 15, ... and the upper limits are 9, 14, 19… In each group the difference between the lower and upper limit is 4.

The midpoint of the class 0 to 4 is $= \frac{0+4}{2} = 2$

and of class 5 to 9 $= \frac{5+9}{2} = \frac{14}{2} = 7$

We can similarly find the midpoints of other classes which are known as class marks.

Look at the frequency table given below. The heights of a group of people is recorded in inclusive classes:-

TABLE - 5

Class Interval (Height in Centimetre)	141-150	151-160	161-170	171-180	**Total**
Frequency	9	11	15	10	**45**

Now discuss the following questions with your friends:

1. How many classes are there in this frequency table?
2. Which group has 180 as its upper limit?
3. What are the lower and upper limits of the class 151 to 160?
4. Which class has the highest frequency?
5. What is the frequency of the first class?
6. What is the meaning of the statement that the class 171 to 180 has a frequency 10?
7. Are these classes inclusive? Yes or no, give reason for your choice?

This table helps to show the data in a simple and concise form and we can see the main features of the data at a glance. Such a table is called grouped frequency distribution table.

2. Exclusive Class

In table 5 above you saw that the number of people having heights between 141 to 150 *cm* is nine. The number of people with heights between 151 to 160 *cm* is 11. If there is a person who has a height between 150 and 151 *cm* which group would you place that person in?

Similarly if the height of a person is 160.4 or 160.6 *cm* which group would you place that person in?

For this we will have to examine the way we are building the classes. Can we do something such that the upper limit of one group is the lower limit of the next group so that there is no gap in between? For example:-

EXAMPLE-2. The weights of class IX students were measured, the data is shown in the following frequency table:-

Weight (In kilograms)	30-33	33-36	36-39	39-42	42-45
Frequency (No. of children)	4	9	12	7	3

EXAMPLE-3. The monthly income of all families of a village is given below:-

Income (In rupees)	0-1000	1000-2000	2000-3000	3000-4000
Frequency (No. of families)	12	30	13	5

These are examples of exclusive classes. Such an arrangement sometimes creates a problem. For example if a family has an income of Rs.2000 per month, then in which group it will be placed, in group 2 or in 3? Similarly in example 1, if the weight of a child is exactly 39 kg then which group would she go in?

In such cases it is assumed that whenever the value is equal to the value of the upper limit of a group, it will be placed in the next group. On this basis we can say that the family with income Rs.2,000 would go in class 3 (2000 – 3000). And the 39 kg child would be counted in class 4 (39 - 42).

Changing Inclusive Classes to Exclusive Classes

When inclusive classes are changed to exclusive classes then lower limits of all class are decreased by half the class interval between classes and the upper limit is increased by the same amount.

TABLE - 6

Inclusive Classes		**Exclusive Classes**	
Class Interval	**Frequency**	**Class Interval**	**Frequency**
6 - 10	8	5.5 - 10.5	8
11 - 15	11	10.5 - 15.5	11
16 - 20	10	15.5 - 20.5	10
21 - 25	15	20.5 - 25.5	15
26 - 30	6	25.5 - 30.5	6

As can be seen in this example the difference between the lower limit of one group and upper limit of the next group is 1 (The second group has a lower limit of 11, which is one more than the upper limit of the lower group.) Thus half of this that is 0.5 is subtracted from all lower limits and 0.5 added to the upper limit values. That makes the lower limit in the first group to be 5.5 and the upper limit to be 10.5. Similarly for the last group the limit are 25.5 to 30.5. The class interval remains 5.

Try This

Change the inclusive classes of Table 5 to exclusive classes and add two people with heights 150.5 *cm* and 160.5 *cm* to the data.

EXAMPLE-4. The class marks of a distribution are 104,114,124, 134, 144, 154 and 164. Find the class size and the class limits.

SOLUTION : The class size is the difference between the adjoining class values.

Thus the size of the class is $114 - 104 = 10$

We need classes of size 10 whose mid points are respectively 104, 114,124, 134, 144, 154 and 164.

Therefore the lower limit of the first group is $= \left(104 - \dfrac{10}{2}\right) = 99$

The upper limit of the first group is $= \left(104 + \dfrac{10}{2}\right) = 109$

The other class interval values would be

99-109, 109-119, 119-129, 129-139, 139-149, 149-159, 159-169

Exercise - 16.1

1. Explain the following:-

   Class interval, size of a class, class mark, class frequency, class limits

2. Give the difference between inclusive and exclusive classes.

3. The weather department has given the following as the data of maximum temperatures in August for Delhi. Make a frequency table of this data.

   32.5, 33.3, 33.8, 31.0, 28.6, 33.9, 33.3, 32.4, 30.4, 32.6, 34.7, 34.9, 31.6, 35.2, 33.3, 33.3, 36.4, 36.6, 37.0, 34.5, 32.5, 31.4, 34.4, 33.6, 37.3, 37.5, 36.9, 37.0, 36.3, 36.9, 36.9

4. The following are the distances of the work places of 40 teachers from their homes:-

7, 9, 5, 3, 7, 8, 10, 20, 3, 5, 11, 25, 15, 12, 7, 13, 18, 12, 11, 3

12, 6, 12, 14, 7, 2, 9, 15, 6, 15, 17, 2, 16, 32, 19, 10, 12, 17, 18, 11

Make a frequency distribution table of class size 5 for this data.

5. The value of Pi up to 50 places of decimal is given below-

3.14159265358979323846264338327950288419716939937510

i. Make a frequency distribution table of the digits from 0 to 9 occuring in this expansion.

ii. Which is the number that occurs the least?

iii. Which is the number that occurs the most? What can we conclude from this?

6. The per hectare production of rice in 40 fields of a village in quintals is given below, make a frequency distribution for this data.

31, 20, 25, 18, 28, 20, 18, 26, 15, 12, 25, 16, 30, 20, 22, 24, 45, 28, 30, 16,

30, 40, 20, 30, 20, 30, 28, 47, 40, 35, 28, 45, 20, 35, 32, 18, 20, 26, 23, 16

Write the conclusions that you draw from this freqency distribution table.

7. 40 children were asked about the number of hours they watched TV in the previous week, their responses were:

1, 5, 6, 2, 7, 4, 10, 12, 5, 8, 10, 12, 36, 22, 6, 15, 3, 1, 2, 4, 21, 16, 17,
13, 14, 2, 7, 9, 23, 26, 31, 33, 5, 35, 25, 26, 29, 30, 9, 31

i. Make a frequency distribution table of class size 5.

ii. What is the lower limit of the first class?

iii. Give the limits of the fourth class

iv. What is the class mark of the 7^{th} class

v. How many children watched television for 20 or more hours in the week?

The Pictorial Depiction of Data

There is a saying that "a picture is better than thousand words" The comparis of different sets of data can be represented with the help of graph. We will discuss the following diagrams here:

(i) Histograms

(ii) Frequency polygons

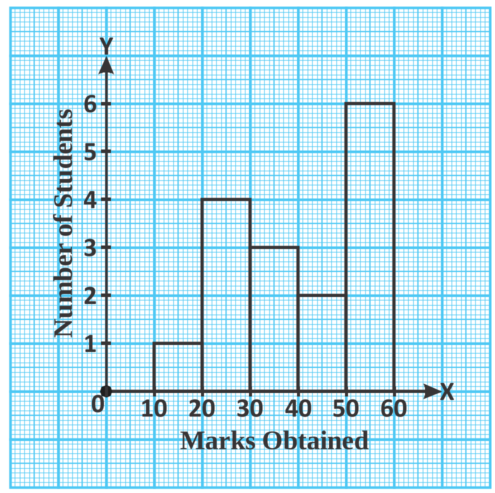
(iii) Cumulative frequency curve or ogive

Histrogram

This is a simple and elegant method of displaying frequency distribution. While constructing this class interval (the independent variable) is taken on the X axis and the frequencies (dependent variable) is shown on the Y axis. In this we make rectangles with the class interval as base with height in proportion to the frequency of that class. Thus we see a continuous series of rectangles with equal bases. The areas of these rectangles are proportional to their corresponding frequencies.

EXAMPLE-5. The marks obtained by 16 students of a class in an examination are given below:-

Marks Obtained	Frequency
10-20	1
20-30	4
30-40	3
40-50	2
50-60	6



Plot a histogram for this data.

SOLUTION : We follow the following steps for plotting the histogram:-

STEP-1 Take a graph paper, draw two perpendicular axis on it and show them as the X and the Y axis.

STEP-2 Along the horizontal axis we will show the class values (marks obtained). Here we have used 1 *square* = 10 marks.

STEP-3 On the perpendicular axis we shall display the frequency (the number of students with those marks). Here 1 *square* = 10 student.

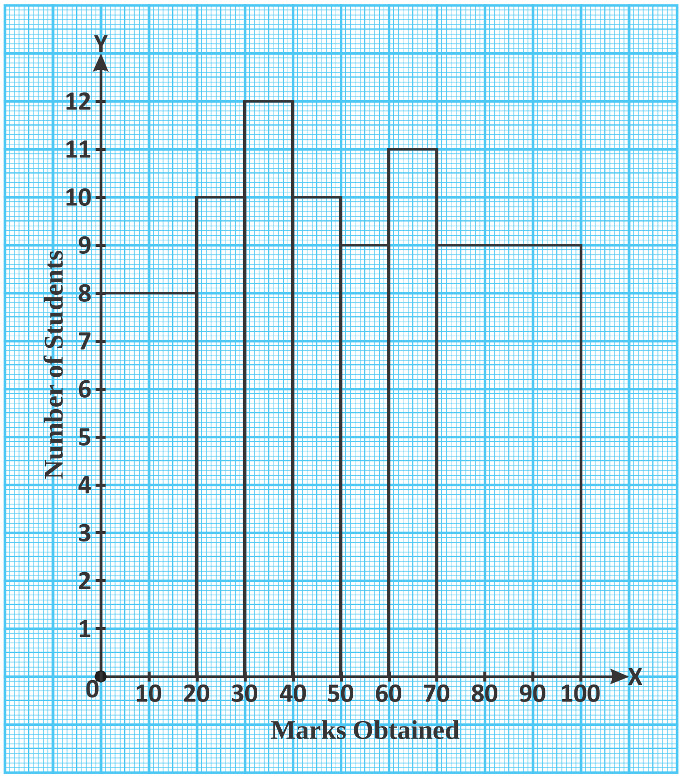
In this way we get the desired histogram.

Histogram for Unequal Class Intervel

Now we consider this different situation. The marks obtained out of 100 by students of a class in science are the following:-

TABLE - 7

Mark Obtained	0-20	20-30	30-40	40-50	50-60	60-70	70 or more than 70
Number of Students	08	10	12	10	09	11	09



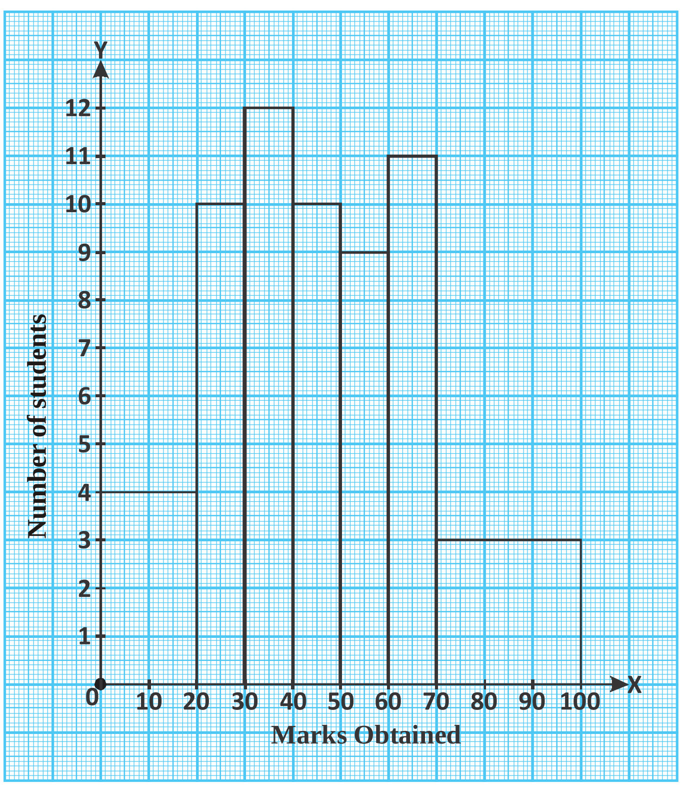
It is clear from the following table that number of students getting less that 20 is 8 and of those getting more than 70 is 9. The data is presented in unequal class intervals. The first interval has a size 20, the last has a size 30 and the rest have class size of 10. A student makes the histogram of the above data as shown in the adjoining diagram. Is this depiction correct?

The class interval is unequal here. To make a bar diagram we would have to change them to equal class intervals, for example in the first class

If the class interval is 20 then the height of the rectangle/bar is 8

$\therefore$ The area under the curve therefore is

$$= \frac{8}{20} \times 10 = 4$$



Similarly, for the last class, the interval is 30 and hence the length of the rectangle/bar would be $= \frac{9}{30} \times 10 = 3$ and since the other class intervals are 10 only we do not have to make any changes in them. We can therefore change the heights of the rectangles in the following way.

Length of the rectangle

$$= \frac{\text{Frequency}}{\text{Width of that class}} \times \begin{matrix}\text{Minimum class} \\ \text{width in the data}\end{matrix}$$

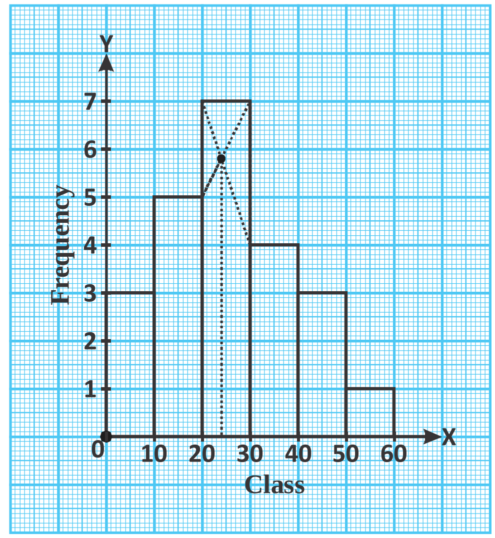
In this way for each class we find the height of the rectangle with a class size of 10. Therefore, the corrected histogram with modified lengths of rectangles would be as given below.

Graphical Method to Locate Mode

The histogram of data with exclusive classes can be used to find out mode as well, for example:-

TABLE - 8

Class	Frequency
0 - 10	3
10 - 20	5
20 - 30	7
30 - 40	4
40 - 50	3
50 - 60	1



Step-1 Make the histogram for the above data.

Step-2 The rectangle with the maximum height is taken to be the modal class for the data. Join the upper right cornerof the model class to the right edge of previous rectangle and join the upper left corner of the model class to the left edge of the next rectangle.

Step-3 Draw a perpendicular line to the X axis from the intersecting point.

Step-4 The point at which the perpendicular line meets the X axis is the mode of the data and it is 24, hence the mode of the data is 24.

Frequency Polygon

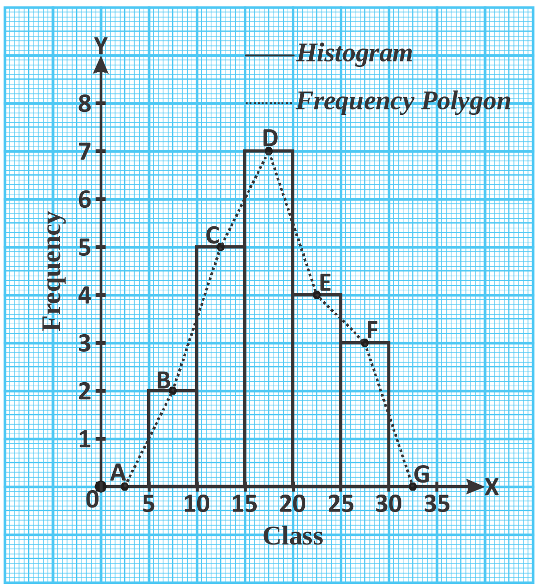
Another way to depict a classified frequency distribution is to make a frequency polygon. To construct a frequency polygon we make a bar/ rectangle on each class interval and the middle points of the upper sides of the rectangles are joined using straight lines. This diagram has many sides and hence is called frequency polygon. Frequency polygons are made in two ways.

1. Using histograms
2. By direct method

1. Constructing Frequency Polygon using a Histogram

TABLE - 9

Class	5-10	10-15	15-20	20-25	25-30
Frequency	2	5	7	4	3



There are the following steps in this method:

Step-1 Make a histogram from the frequency descriptions.

Step-2 Mark the mid points B, C, D, E, F on the upper edge of each rectangle. Join these mid points sequentially using straight lines.

Step-3 Now take the class intervals before and after the data distribution. Namely here for example, the intervals 0-5 and 30-35. Mark the mid points of these intervals on the X axis. These would be at 2.5 and 32.5 respectively. Call these A and G respectively. Join B to A and F to G. The required polygon is ABCDEFG..

Think and Discuss

The areas of the frequency polygon and of the histogram are the same, Why?
(***Hint*** **:** Use the property of congruency of triangles)

2. Constructing Frequency Polygon by Direct Method

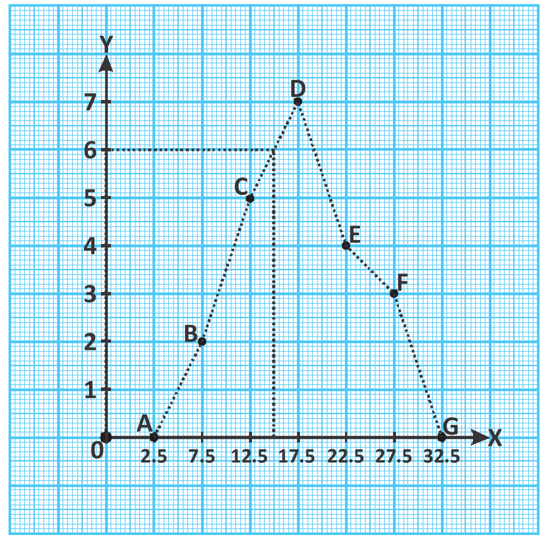
TABLE - 10

Class	5-10	10-15	15-20	20-25	25-30
Frequency	2	5	7	4	3

This method has the following steps:-

Step-1 First find the class marks of each class.

Class	Frequency	Mid-Point
5-10	2	7.5
10-15	5	12.5
15-20	7	17.5
20-25	4	22.5
25-30	3	27.5



Step-2 We will plot the mid points on the X-axis and the frequencies onthe Y-axis. Mark the mid points on the X-axis.

Step-3 Mark the points B, C, D, E, F using the corresponding frequency.

Step-4 Mark on the X-axis the midpoint 2.5 (point A) of the class 0-5, that lies before the first class and the midpoint 32.5 (point G) of the class 30-35, which is after the first class. We have marked them on the X-axis because their frequencies are zero.

Step-5 Join all the marked mid points sequentially.

Step-6 The figure ABCDEFG obtained is the frequency polygon.

Note : Frequency polygon shows the increase or fall in the value of the frequency. Using this we can estimate the value of the frequency for any particular point. For example for 15 on the X-axis the corresponding frequency would be 6.

Difference between a Histogram and a Frequency Polygon

Histogram	Frequency Polygon
1. Shows the frequency using bars.	1. More useful since it indicates the increase or fall in frequency. Frequency is shown as a polygon in this.
2. Frequencies are assumed to be distributed over the entire interval.	2. Frequencies of the class are assumed to be located at the midpoint.
3. It can be constructed for equal or unequal class widths.	3. This can be made only for equal class widths.

Ogive or Cumulative Frequency Curve

The graphical depiction of the class and its associated cumulative frequency is called cumulative frequency curve or Ogive. The graph of the cumulative frequency curve is plotted in a manner similar to the way frequency polygon is made.

There are two ways to plot the cumulative frequency curve:

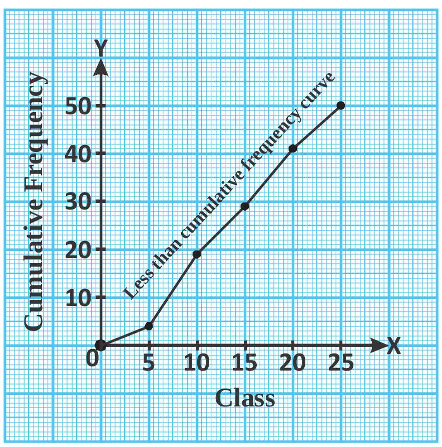
1. "Less than" method 2. "More than" method

1. **"Less than" method**: The cumulative frequency of a class is the sum of the frequency of that class and frequencies of all the classes before it.

Based on following table let us try to plot the "less than" cumulative curve:-

TABLE - 11

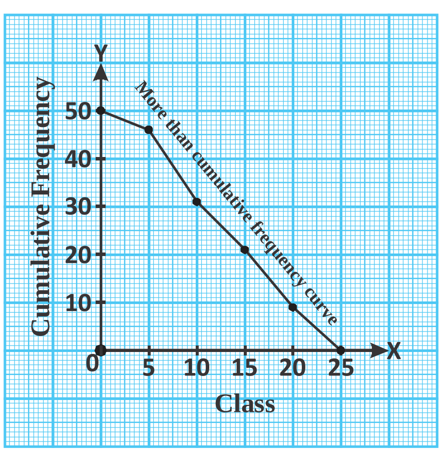
Class	0-5	5-10	10-15	15-20	20-25
Frequency	4	15	10	12	09



SOLUTION : Cumulative frequency for "less than"

Class	Frequency	Cumulative Frequency
less than 0	0	0
less than 5	4	4
less than 10	15	19=(15+4)
less than 15	10	29=(10+15+4)
less than 20	12	41=(12+10+15+4)
less than 25	9	50=(9+12+10+15+4)

2. **"More than" method** : The cumulative frequency of the class is the sum of the frequency of that class and frequencies of all the classes following it.



Based on following table let us try to plot the "more than" cumulative curve:-

Clas	Cumulative Frequency
More than 0	50 = (4+15+10+12+9+0)
More than 5	46 = (15+10+12+9+0)
More than 10	31 = (10+12+9+0)
More than 15	21 = (12+9+0)
More than 20	9 = (9+0)
More than 25	0

Importance of Cumulative Frequency Curve or Ogive

Cumulative Frequency Curve or Ogive is used in study of data in many ways.

For example:-

1. When you want to extrapolate the data to one below or one above the given data.
2. Ogive is used for comparative study.
3. Ogive is used to find the measures of central tendency like- median, first quartile, third quartile etc.
4. We can also find out the value of the variable which is included in the special cumulative frequency.

Exercise - 16.2

1. Choose the correct alternative from the following:-

(i) In an inclusive class:-

a) Both the limits are included in different classes

b) Both the limits are included in the same class

c) It is not decided in which class the limits are included

d) None of the above

(ii) Method to make the cumulative frequency table is:-

a) Less than b) More than

c) Both a and b d) None of the above

(iii) Using the histogram we can find out

a) Mode b) Median

c) Both a and b d) None of the above

(iv) Using the cumulative frequency curve we can find out:-

a) Mode b) Median

c) Both a and b d) None of the above

(v) The width of the rectangle in a histogram depends on:-

a) Class interval of the class b) Class mark

c) On the class frequency d) On all of the above

2. The time taken (in second) by 25 students in doing a question is as follows:-

16, 20, 26, 27, 28, 30, 33, 37, 38, 40, 42, 43, 46,
46, 46, 48, 49, 50, 53, 58, 59, 60, 64, 52, 20

(i) Make frequency table of this data using a class interval of 10 second.

(ii) Make a histogram depicting this frequency distribution.

(iii) Use the histogram to make a frequency polygon.

3. The length of 30 leaves of a plant is given in milimetre below:-

Length of leaves (in *mm.*)	111-120	121-130	131-140	141-150	151-160	161-170
Number of leaves	3	5	7	9	4	2

(i) Make a histogram to depict the frequency distribution.
(***Hint:*** Make the class intervals continuous)

(ii) Make a frequency polygon using the direct method.

(iii) In which class the number of leaves is the largest?

4. The number of runs made by two teams A and B in a 10 over (60 ball) match is given below:-

Number of Balls	Team A	Team B
1-6	3	6
7-12	2	6
13-18	7	1
19-24	8	10
25-30	3	6
31-36	8	5
37-42	5	3
43-48	11	4
49-54	6	7
55-60	2	11

Use the above data make a frequency polygon on one graph paper.

(***Hint:*** First make the class intervals continuous)

5. The frequency distribution of marks obtained by 100 students is as follows:-

Marks obtained	0-10	10-20	20-30	30-40	40-50	50-60	Total
Number of students	7	10	23	51	6	3	100

Make a cumulative frequency curve using the above data.

6. Make the frequency table and the cumulative frequency curve for the following data:-

Marks Obtained	Number of Students
less than 10	3
less than 20	8
less than 30	12
less than 40	19
less than 50	31
less than 60	42
less than 70	60

7. The marks obtained by two groups of students in a test are given below:-

Class Interval	Group A	Group B
50-52	4	8
47-49	10	9
44-46	15	10
41-43	18	14
38-40	20	12
35-37	12	17
32-34	13	22
Total	**92**	**92**

Make frequency polygons for each of these groups on one graph paper.

What Have We Learnt

1. Observation collected for a specific purpose is known as data.
2. When the number of observation is more we use tally marks to find the frequency.
3. The number of times a particular observation occurs in a data is called the frequency of that observation.

4. The table showing the frequencies of different observations in the data is called frequency distribution table or the frequency table.

5. When the number of observations is very large then we organise data in groups. These groups are called classes and the data is referred to as classified data.

6. The histogram is a graphical representation of classified data. In this rectangle is drawn for each group with X-axis showing the class width and heights on the Y-axis corresponding to the frequencies.

7. If we mark points using the mid-points of the classes as the X-coordinates and the corresponding frequencies of the classes as the Y-coordinates, then the polygon made joining these points is called the frequency polygon.

8. When we find the total number of observations up to the lower limit of all the class intervals, we obtained the ascending cumulative frequency.

9. If in a classified frequency distribution the class intervals are different, that the rectangles in the bar graphs would have to be constructed using frequency density.

$$\text{Frequency density} = \frac{\text{Class frequency}}{\text{Class width}} \times \text{Lower limit of the data}$$

10. For the same data, the area of frequency polygon and cumulative frequency graph are equal.

Whsx cr i c

Exercise - 1.1

1. Right pair

Bharti Krishnateertha	– Vedic Mathematics
Varahamihira	– Panch Siddhanta
Brahmagupta	– Brahmasphoot Siddhanta
Bhaskaracharya	– Siddhanta Shiromani
Aaryabhatt	– Aaryabhatiya

2. (1) b (2) 10^{53}
3. Aaryabhatt, Aaryabhatiya
4. Beej Ganit
5. 16

Exercise - 1.2

2. 6, 2, 7, 9, 7, 3, 1, 2, 4, 4,

3. 2, 3, 5, 7

Exercise - 1.3

1. 736	2. 2288	3. 3404	4. 3025
5. 39483	6. 96048	7. 311472	8. 367836

Exercise - 1.4

1. 5643	2. 43775622	3. 86913	4. 34499655
5. 432	6. 9447543		

Exercise - 1.5

1. 221	2. 616	3. 1224	4. 9009	5. 1225
6. 2016	7. 5616	8. 9021	9. 11024	10. 11025

11. 42021 12. 164025 13. 255016 14. 366021 15. 497024

16. 819025 17. 38021 18. 87016 19. 156016 20. 245025

Exercise - 1.6

1. 169 2. 10608 3. 11130 4. 9212

5. 12444 6. 10272 7. 1016048 8. 972052 9. 1013968.

Exercise - 1.7

225, 625, 1225, 2025, 3025, 5625, 7225, 9025, 11025, 13225.

Exercise - 1.8

(1) 1156 (2) 361 (3) 2916 (4) 4096 (5) 8464.

Exercise - 1.9

114, 196, 10404, 11025, 11664, 8836, 992016

Exercise - 1.10

(1) 97 (2) 87 (3) 91 (4) 57

Exercise - 1.11

(1) $4x^2 + 21x + 5$ (2) $12x^2 + 18xy + 6y^2$ (3) $x^2 - 9y^2$

(4) $x^2 + 4x + 16$ (5) $x^4 + 5x^3 + 11x^2 + 11x + 4$

(6) $6x^4 + 17x^2y - 2x^2 + 12y^2 - y - 20$

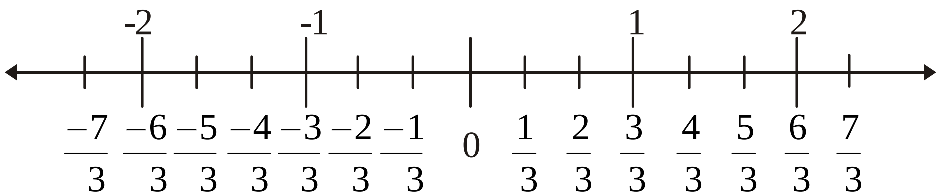
Exercise - 2.1

1. (i) 1 (ii) Zero (iii) Zero (**Note :-** There can be several examples of (i) and (iii))

2. $\frac{25}{4}, \frac{26}{4}, \frac{27}{4}$ 3. $\frac{25}{42}, \frac{26}{42}, \frac{27}{42}, \frac{28}{42}, \frac{29}{42}$

4. $\frac{33}{48}, \frac{34}{48}, \frac{35}{48}$ 5. Any four numbers between $\frac{-4}{5}, \frac{-3}{5}, \frac{-2}{5} \ldots\ldots \frac{8}{5}, \frac{9}{5}$

6. Any three numbers between $\frac{21}{40} \ldots\ldots\ldots\ldots \frac{35}{40}$

7. 

8. (i) 25.2 (Terminating decimal)
(ii) 20.9375 (Terminating decimal)
(iii) 3.14285714... (Non terminating recurring)
(iv) –39.33... (Non terminating recurring)

9. (i) $\frac{53}{100}$ (ii) $\frac{84}{5}$ (iii) $\frac{421}{4}$ (iv) $\frac{184}{25}$

10. (i) $\frac{70}{99}$ (ii) $\frac{277}{33}$ (iii) $\frac{563}{180}$ (iv) $\frac{5120}{999}$

Exercise - 2.2

1. (i) $18\sqrt{3}$ (ii) $3\sqrt{7}$

2. $7\sqrt{2}+\sqrt{5}$ 3. $3\sqrt{7}-8\sqrt{5}$

4. (i) 1 (ii) 20 (iii) $7+3\sqrt{6}$ (iv) $10-2\sqrt{21}$

5. (i) $\frac{\sqrt{5}}{5}$ (ii) $\frac{\sqrt{6}}{3}$ (iii) $\frac{\sqrt{7}}{1}$

6. (i) $a=\frac{13}{11};\ b=\frac{4}{11}$ (ii) $a=4,\ b=1$

7. (i) $20+3\sqrt{3}$ (ii) $16-\sqrt{3}$ 8. 6

Exercise - 3.1

1. (i) 32 (ii) 81 (iii) 16

2. (i) 8 (ii) 625 (iii) –9765625

3. (i) $128t^3$ (ii) -1

5. (i) 8.52×10^{-12} (ii) 8.02×10^{15} (iii) 4.196×10^{10}

6. (i) 0.00000502 (ii) 0.00000007 (iii) 1000010000

7. (i) 7×10^{-6} meter (ii) 1.2756×10^{7} (iii) 8×10^{-2} meter

Exercise - 3.2

1. (i) 4 (ii) 3 (iii) 5

2. (i) 529 (ii) 1,331 (iii) 441 (iv) $\frac{1}{15}$

3. (i) $\left(\frac{25}{9}\right)^{\frac{4}{3}}$ (ii) $\left(\frac{13}{2}\right)^{\frac{1}{3}}$ (iii) $\frac{1}{3}$ (iv) 243

4. (i) $\sqrt[9]{512} < \sqrt[3]{64} < \sqrt{81}$ (ii) $\sqrt[4]{625} < \sqrt[3]{343} < \sqrt{100}$

   (iii) $\sqrt[5]{243} < \sqrt[3]{216} < \sqrt{64}$ (iv) $\sqrt[7]{128} < \sqrt[4]{256} < \sqrt[3]{1000}$

5. (i) Surd (ii) No (iii) Surd

   (iv) No (v) Surd

Exercise - 4.1

1. (i) It is polynomial because the power of variable is whole number.

   (ii) It is not a polynomial, because we writes $z+\frac{3}{z}$ as $z+3z^{-1}$

   i.e. the power of z is not a whole number.

   (iii) It is not a polynomial, because we writes $\sqrt{y}+2y+3$ as $y^{\frac{1}{2}}+2y+3$

   i.e. the power of y is $\frac{1}{2}$ which is not the whole number.

   (iv) It is polynomial, because the power of variable is a whole number.

   (v) It is polynomial, because the power of variable is a whole number.

2. (i) 5 (ii) 3 (iii) -5 (iv) $\frac{1}{2}$ (v) $\frac{1}{4}$

3. (i) $\frac{1}{5}$, 5 (ii) $\sqrt{2}$, 7 (iii) 0, 2

4. (i) $x^4 + 3$, $3x^4 + 4$ etc. (ii) $y^6 + 3y + 3$, $t^6 + 3t - 5$ etc.
(iii) $3x^5$, z^5, $11z^5$ etc.

5. (i) 3 (ii) 9 (iii) 4
(iv) 3 (v) 1 (vi) 0

6. Constant (vi), (x), (xii) Linear (iv), (ix), (xi)
Binomial (iii), (vii), (viii) Trinomial (i), (ii), (v)

Exercise - 4.2

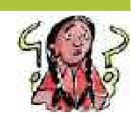
1. (i) -2 (ii) 4 (iii) -56

2. (i) 2, 3, 36, 119 (ii) -1, 0, 3, 8
(iii) $\frac{1}{3}, \frac{4}{3}, \frac{7}{3}, \frac{16}{3}$ (iv) 1, 0, 9, 28 (v) 2, 1, 4, 5

3. (i) Yes (ii) Yes (iii) No (iv) Yes
(v) No (vi) Yes (vii) Yes

4. (i) -6 (ii) 6 (iii) 0 (iv) 0 (v) $\frac{-d}{c}$
(vi) 2 (vii) $-\frac{3}{2}$ (viii) $\sqrt{5}$ (ix) $\frac{4}{3}$

Exercise - 4.3



1. (i) $5x^2 + 5x + 6$ (ii) $3p^3 + 8p^2 - 7p + 11$ (iii) $2x^3 + 7x^2 - 8x - 1$

2. (i) $4y^2 + 9y$ power 2 (ii) $-r^2 + 4r + 17$ power 2
(iii) $3x^3 + 4x^2 + 8$ power 3

3. (i) $7t^3 - 4t^2 + 5t$ (ii) $-4p^3 + 7p^2 + 8p - 12$ (iii) $7z^3 - 10z^2 + 9z + 17$

4. $2x^4 + 2x^3 - 3x^2 + 11x + 4$ 5. 5

6. $-x^3 + 4x^2 - x - 2$ 7. $-u^7 + 4u^6 - 4u^2 - u - 6$

8. $-3y^2 - y + 1$ 9. $-t^3 - 2t - 11$

10. (i) $21x^3 + 34x^2 + 11x + 4$ (ii) $15x^5 - 45x^4 + 36x^3 - 18x^2 + 12x$
(iii) p^7 & $5p^5 + p^4 + 3p^3$ &$5p^2 + 3$

11. $2x^6 + 14x^4 + 3x^3 - 21x - 9$ 12. 5

Exercise - 5.1

1. (i) $x + (x + 1) = 11$ (ii) $2y + y = 30$ (iii) $2z + 2z + z = 40$
   (iv) $2[(w + 3) + w] = 15$ (v) $2x + 3x + 4x = 18$

2. (i) $x = 3$ (ii) $p = \frac{23}{5}$ (iii) $x = 2$
   (iv) $t = -\frac{1}{9}$ (v) $z = 1$ (vi) $x = \frac{2}{7}$

3. (i) If we add 3 in a certain number, we get 27.
   (ii) Sneha's age is half at Rajiv's age and sum of both their ages is 18 years.
   (iii) If we add 2 in any number, then divide the sum by same number, we get 30.

4. (i) $\frac{-6}{5}$ (ii) $k = 4$ (iii) $p = 5$
   (iv) $x = -\frac{68}{25}$ (v) $m = \frac{7}{5}$ (vi) $t = 2$
   (vii) $x = \frac{27}{10}$ (viii) $x = \frac{35}{33}$ (ix) $y = -8$

Exercise - 5.2

1.	6 meter	2.	15 cm, 10 cm	3.	$35^{o}, 50^{o}, 95^{o}$
4.	3, 5, 7	5.	20 cm	6.	35
7.	28 and 20 years	8.	15	9.	80, 120
10.	180, 185, 190	11.	Rohit = 39 years, Pradeep = 17 years		
12.	22 $km / hour$				

Exercise - 6.1

1. (i) A = 6, B = 9 (ii) X = 5, Y = 6
   (iii) L = 0, M=1, N = 8 (iv) Z = 5
   (v) X = 1, Y = 4 (vi) P = 4, Q = 7
   (vii) M = 7, L = 4

Exercise - 6.2

1. Multiple of 5 = 9560, 205, 800, Multiple of 10 = 9560, 800

2.	1, 4, 7	3.	P = 0 or 5

4. A = 7, B = 2; A = 3, B = 6; A = 2, B = 7; A = 6, B = 3

5.	(i)	Yes	(ii)	No	(iii)	Yes	(iv)	Yes
6.	(i)	Yes	(ii)	No	(iii)	Yes	(iv)	Yes
7.	(i)	Yes	(ii)	Yes	(iii)	Yes	(iv)	No

Exercise - 7.1

1.	Rs. 15800	2.	2 : 3	3.	Rs. 190 and 20%
4.	Rs. 8050	5.	Rs. 139100	6.	Rs. 1900
7.	Equal	8.	(i) Rs. 210	(ii)	Rs. 157.50
9.	Rs. 5000	10.	Rs. 9500		

Exercise - 7.2

1. (i) Compound interest = Rs. 3248.70, Principal = Rs. 10248.70
   (ii) Compound interest = Rs.1590, Principal = Rs. 7840
   (iii) Compound interest = Rs. 2522, Principal = Rs. 18522
2. Amount = Rs. 175616, interest amount = Rs. 50616

3.	Rs. 16125	4.	Rs. 3.15	5.	Rs. 80
6.	Rs. 374.59	7.	Rs. 741.90	8.	Rs. 544.69
9.	Rs. 30250	10.	Rs. 609	11.	Rs. 6000
12.	Rs. 16000	13.	4%	14.	5%
15.	3 years	16.	$1\frac{1}{2}$ years	17.	Rs. 16000

Exercise - 7.3

1.	Rs. 33.80	2.	33.8%	3.	Rs. 673.11
4.	Rs. 8302.075	5.	Rs. 639.35	6.	Rs. 6250
7.	Rs. 4840	8.	Rs. 1313.30	9.	Rs. 12696

Exercise - 8.1

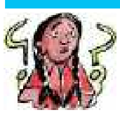
1. (i) $\sin A = \frac{4}{5}, \cos C = \frac{4}{5}, \tan A = \frac{4}{3}$

(ii) $\sin A = \frac{5}{13}, \cos C = \frac{5}{13}, \tan A = \frac{5}{12}$

(iii) $\sin A = \frac{12}{13}, \cos C = \frac{12}{13}, \tan A = \frac{12}{5}$

(iv) $\sin A = \frac{12}{15}, \cos C = \frac{12}{15}, \tan A = \frac{12}{9}$

Exercise - 8.2



1. (i) $\sin\theta = \frac{3}{5}, \cos\theta = \frac{4}{5}, \cot\theta = \frac{4}{3}, \sec\theta = \frac{5}{4}, \cos ec\theta = \frac{5}{3}$

(ii) $\cos\theta = \frac{12}{13}, \tan\theta = \frac{5}{12}, \cot\theta = \frac{12}{5}, \sec\theta = \frac{13}{12}, \cos ec\theta = \frac{13}{5}$

(iii) $\sin\alpha = \frac{2\sqrt{2}}{3}, \tan\alpha = \frac{2\sqrt{2}}{1}, \cot\alpha = \frac{1}{2\sqrt{2}}, \sec\alpha = 3, \cos ec\alpha = \frac{3}{2\sqrt{2}}$

(iv) $\sin\theta = \frac{1}{\sqrt{2}}, \cos\theta = \frac{1}{\sqrt{2}}, \tan\theta = 1, \sec\theta = \sqrt{2}, \cos ec\theta = \sqrt{2}$

(v) $\sin A = \frac{4}{5}, \cos A = \frac{3}{5}, \tan A = \frac{4}{3}, \cot A = \frac{3}{4}, \sec A = \frac{5}{3}$

(vi) $\sin\beta = \frac{\sqrt{3}}{2}, \cos\beta = \frac{1}{2}, \tan\beta = \sqrt{3}, \cot\beta = \frac{1}{\sqrt{3}}, co\sec\beta = \frac{2}{\sqrt{3}}$

(vii) $\sin A = \frac{1}{\sqrt{10}}, \cos A = \frac{3}{\sqrt{10}}, \tan A = \frac{1}{3}, \cot A = 3, \sec A = \frac{\sqrt{10}}{3}$

2. $\frac{420}{841}$ 3. $\frac{22}{45}$ 4. $\frac{8}{31}$

5. $\frac{2\sqrt{2}}{3}, 3 + \frac{1}{2\sqrt{2}}, \frac{3}{2\sqrt{2}}$ 6. 1 7. 7 8. $\frac{1}{2}$

Exercise - 8.3

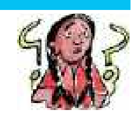
1. (i) c (ii) c

2. (i) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (ii) $\sqrt{\frac{2}{3}}+2$ या $2+\frac{\sqrt{6}}{3}$ या $\frac{6+\sqrt{6}}{3}$ (iii) 3

(iv) $\frac{1}{2-\sqrt{3}}$ (v) 4 (vi) $\frac{\sqrt{3}}{2}$ (vii) $\frac{1}{3}$ (viii) $\frac{3}{2}$

3. (i) False (ii) False (iii) True (iv) True (v) True

Exercise - 8.4



1.	45°	2.	60°	3.	30°	4.	30°
5.	60°	6.	0°	7.	60°	8.	60°
9.	60°	10.	0°	11.	30°		

Exercise - 9.1

1. $x = 130^\circ$, $y = 50^\circ$

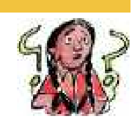
2. $\angle QOT = 30^\circ$ and reflex $\angle ROT = 250^\circ$

3. $z = 126^\circ$

5. If $a+b=c+d$ will be on a line then $\frac{360^\circ}{2}=180^\circ$

7. $\angle ABY = 122^\circ$, reflex $\angle YBX = 302^\circ$

Exercise - 9.2



1. (i) $x = 36^\circ$, $y = 108^\circ$

(ii) $x = 29^\circ$, $y = 87^\circ$

(iii) $x = 95^\circ$, $y = 35^\circ$

2. $x = 126^\circ$ 3. $\angle AGE = 126^\circ$, $\angle GEF = 36^\circ$, $\angle FGE = 54^\circ$

4. 60° 5. $x = 50^\circ$, $y = 77^\circ$ 6. $x = 80^\circ$, $y = 100^\circ$

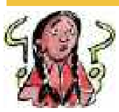
7. (i) $x = 59^\circ$, $y = 60^\circ$ (ii) $x = 40^\circ$, $y = 40^\circ$

(iii) $x = 18^\circ$, $y = 60^\circ$ (iv) $x = 20^\circ$, $y = 63^\circ$

8. (i) 140° (ii) 100° (iii) 250°

10. Strike on mirror and then make the alternate angle $\angle ABC = \angle BCD$, hence $AB \parallel CD$.

Exercise - 9.3



1. $\angle ACB = 17^\circ$, $\angle ABC = 129^\circ$, $\angle DBC = 51^\circ$

2. (i) $x = 40^\circ$, $y = 70^\circ$ (ii) $x = 50^\circ$, $y = 20^\circ$

   (iii) $x = 51^\circ$, $y = 35^\circ$ (iv) $x = 30^\circ$, $y = 75^\circ$

3. $\angle 1 = 60^\circ$ 4. $\angle PRQ = 65^\circ$ 5. $\angle OZY = 32^\circ$, $\angle YOZ = 121^\circ$

6. $\angle DCE = 92^\circ$ 7. $\angle SQT = 60^\circ$ 8. $x = 53^\circ$, $y = 37^\circ$

Exercise - 10.1

1. b 2. b 3. c 4. b 5. c

6. (i) R.H.S. (ii) SAS (iii) ASA

   (iv) AAS (v) SSS

7. 25 8. 160 meter (SSS congruency rule)

9. Yes (This is a rectangular shape and the opposite side are equal in rectangle).

10. From ΔABC and ΔACD, BC = AD and AB = CD

    Then AC = CA (common side)

    $\Delta ABC \cong \Delta ACD$

Exercise - 10.2

1. (iii) 2. (i) 3. (iv) 4. (iii) 5. (iii)

6. 30° 7. 23°, acute angled triangle

8. From ΔABC and ΔDEF

   BC = EF, $\angle B = \angle E$, $\angle C = \angle F$

   $\Delta ABC \cong \Delta DEF$

9. From ΔABD and ΔABC

   AD = BC, $\angle A = \angle B$, $AB = BA$

   $\Delta ABD \cong \Delta ABC$

   $\therefore BD = AC$, $\angle ABD = \angle BAC$

Exercise - 10.3

1. $\angle B < \angle A \qquad \therefore AO < BO$ ...... (1)

$\angle C < \angle D \qquad \therefore OD < OC$ ...... (2)

equation (1) + (2)

$\therefore AO + OD < BO + OC$

$AD < BC$

2. $\angle x > \angle y$

$\therefore \angle PMN = 180^\circ - \angle x, \quad \angle PNM = 180^\circ - \angle y$

$\therefore \angle PNM > \angle PMN$ (The larger side opposite to larger angle)

$\therefore PM > NP$

3. $PQ > PR$

$\angle PRQ > \angle PQR$

$\angle QRS > \angle RQS$

$SQ > SR$

4. $\angle PQR = \angle PRQ$

$\therefore \angle PQS > \angle PRQ$

$\therefore PS > PQ$

Exercise - 11.1

2. Angles made at point C and R on a line AB and CD are half of 180° i.e. 90°. These are opposite angles therefore the figure is rectangle.

3. (i) Alternate angle (ii) ABCD is a parallelogram

4. $\angle P = \angle Q = 90^\circ$

$DP = QB$

$AB = DC$

$\therefore \Delta APB \cong \Delta CQD$

Exercise - 14.1

1. 110 cm 2. 55.44 $sq\ cm$ 3. 616 sq meter

4.	1100 meter	5.	9 meter	6.	168 meter
7	Rs. 44,785.71	8.	25.6 cm, 269.5 $sq\ cm$		
9.	$\theta = 60°$	10.	$42\sqrt{2}\ cm$		

Exercise - 15.1

1. 3375 cubic meter   2. 8 cm
3. (i) 4 times   (ii) 9 times   (iii) $n \times n$ times or n^2 times
4. 6 meter
5. (i) cubical peti, 40 $sq\ cm$ (ii) cubical peti, 10 $sq\ cm$
6. 3 days   7. 4000 cubic cm 8. 12288 cubic cm
9. 350 $sq\ cm$ 10. 6 meter   11. 4167 bricks
12. Rs. 4320   13. 16000   14. 6cm, 4:1   15. 7 sheet

Exercise - 16.1

1. Class interval – width of class
   Size of a class – the difference of class mark of two consecutive classes
   Class mark – digit of class
   Class frequency – frequency
   Class limits – higher and lower limit of a class.

2. Inclusive class – the upper limit of a class should not be equal to lower limit of a next class.
   Exclusive class – the upper limit of a class should be lower limit of next class.

3.

Maximum Temp. (Month of August)	**Tally Marks**	**Frequency**
28.5 – 29.5	|	1
29.5 – 30.5	|	1
30.5 – 31.5	||	2
31.5 – 32.5	||	2
32.5 – 33.5	$\cancel{||||}$ ||	7
33.5 – 34.5	||||	4
34.5 – 35.5	||||	4
35.5 – 36.5	||	2
36.5 – 37.5	$\cancel{||||}$ ||	7
37.5 – 38.5	|	1

4.

Distance (in *km*)	Tally Marks	Frequency
0 — 5	卌	5
5 — 10	卌 卌 |	11
10 — 15	卌 卌 ||	12
15 — 20	卌 ||||	9
20 — 25	|	1
25 — 30	|	1
30 — 35	|	1

5.(i)

Marks	Tally Marks	Frequency
0	||	2
1	卌	5
2	卌	5
3	卌 |||	8
4	||||	4
5	卌	5
6	||||	4
7	||||	4
8	卌	5
9	卌 |||	8

(ii) Zero (iii) 3 and 9

6.

Production (in quintals)	10—15	15—20	20—25	25—30	30—35	35—40	40—45	45—50
Frequency	1	7	10	8	7	2	2	3

7.(i)

Hours of TV programme	Tally Marks	Frequency
0 — 5	卌 |||	8
5 — 10	卌 卌	10
10 — 15	卌 |	6
15 — 20	|||	3
20 — 25	|||	3
25 — 30	||||	4
30 — 35	||||	4
35 — 40	||	2

(ii) 0 (iii) 15-20 (iv) 32.5 (v) 13

Exercise - 16.2

1.	(i)	b	(ii)	c	(iii)	c
	(iv)	b	(v)	a		

2.(i)

Time (in second)	Frequency
15 — 25	3
25 — 35	5
35 — 45	5
45 — 55	8
55 — 65	4

(ii) ——— Histogram

(iii) .... Frequency Polygon