EXERCISE

SHORT ANSWER TYPE

1. **For a positive integer n, find the value of $(1 – i)^n (1 – 1/i)^n$.**

Solution:

According to the question,

We have,

$$(1-i)^n\left(1-\frac{1}{i}\right)^n = (1-i)^n\left(1-\frac{1}{i^2}\right)^n$$

$= (1 – i)^n (1 + i)^n$

$= (1 – i^2)^n$

$= 2^n$

Therefore, $(1 – i)^n (1 – 1/i)^n = 2^n$

2. Evaluate

$$\sum_{n=1}^{13}\left(i^n + i^{n+1}\right),$$

where n ϵ N.

Solution:

According to the question,

We have,

$$\sum_{n=1}^{13}(i^n + i^{n+1}) = \sum_{n=1}^{13}(1+i)i^n$$

$$= (1 + i)\,(1 + i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + i^8 + i^9 + i^{10} + i^{11} + i^{12} + i^{13})$$

$$= (1+i)\frac{i(i^{13}-1)}{i-1}$$

$$= (1+i)\frac{i(i-1)}{i-1}$$

$= (1 + i)\, i$

$= i + i^2$

$= i – 1$

$$\therefore \sum_{n=1}^{13}(i^n + i^{n+1}) = i - 1$$

3. If

$$\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x + iy$$

, then find (x, y).

Solution:

According to the question,

We have,

$$x + iy = \left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3$$
$$= \left(\frac{(1+i)^2}{1-i^2}\right)^3 - \left(\frac{(1-i)^2}{1-i^2}\right)^3$$
$$= \left(\frac{1+2i+i^2}{1+1}\right)^3 - \left(\frac{1-2i+i^2}{1+1}\right)^3$$
$$= \left(\frac{2i}{2}\right)^3 - \left(\frac{-2i}{2}\right)^3$$
$= i^3 – (-i^3)$
$= 2i^3$
$= 0 – 2i$
Thus, $(x, y) = (0, -2)$

4. If $\frac{(1+i)^2}{2-i} = x + iy$ **, then find the value of x + y.**

Solution:

According to the question,
We have,
$$x + iy = \frac{(1+i)^2}{2-i}$$
$$= \frac{1+2i+i^2}{2-i}$$
$$= \frac{2i}{2-i}$$
Rationalizing the denominator,
$$= \frac{2i(2+i)}{(2-i)(2+i)}$$
$$= \frac{4i+2i^2}{4-i^2}$$
$$= \frac{4i-2}{4+1}$$
$$= \frac{-2}{5} + \frac{4i}{5}$$
Thus,
$$x = -\frac{2}{5}, y = \frac{4}{5}$$
Hence,
$$x + y = -\frac{2}{5} + \frac{4}{5} = \frac{2}{5}$$

5. If

$$\left(\frac{1-i}{1+i}\right)^{100} = a + ib;$$ **then find (a, b).**

Solution:

According to the question,

We have,

$$a + ib = \left(\frac{1-i}{1+i}\right)^{100}$$

$$= \left[\frac{1-i}{1+i}.\frac{1-i}{1-i}\right]^{100}$$

$$= \left[\frac{(1-i)^2}{1-i^2}\right]^{100}$$

$$= \left(\frac{1-2i+i^2}{1+1}\right)^{100}$$

$$= \left(-\frac{2i}{2}\right)^{100}$$

$= (i^4)^{25}$

= 1

Hence, (a, b) = (1, 0)

6. **If a = cos θ + i sin θ, find the value of** $\frac{1+a}{1-a}$.

Solution:

According to the question,

We have,

a = cos θ + i sin θ

$$\Rightarrow \frac{1+a}{1-a} = \frac{(1+\cos\theta) + i\sin\theta}{(1-\cos\theta) - i\sin\theta}$$

$$= \frac{2\cos^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2} - i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right)}$$

$$= \frac{i\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}{\sin\frac{\theta}{2}\left(i\sin\frac{\theta}{2} - i^2\cos\frac{\theta}{2}\right)}$$

$$= \frac{i\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}{\sin\frac{\theta}{2}\left(i\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)}$$

$$= i\cot\frac{\theta}{2}$$

7. If (1 + i)z = (1 – i) $\bar{z}$, then show that z = i $\bar{z}$.

Solution:

According to the question,

We have,

$(1 + i) z = (1 - i) \bar{z}$

$\Rightarrow z = \frac{1 - i}{1 + i}\bar{z}$

Rationalizing the denominator,

We get,

$= \frac{(1 - i)(1 - i)}{(1 + i)(1 - i)}\bar{z}$

$= \frac{(1 - i)^2}{(1 - i^2)}\bar{z}$

$= \frac{1 - 2i + i^2}{1 + 1}\bar{z}$

$= \frac{1 - 2i - 1}{2}\bar{z}$

$= -i\bar{z}$

Hence proved.

8. If z = x + iy, then show that $z\bar{z} + 2(z + \bar{z}) + b = 0$ where b∈R, representing z in the complex plane is a circle.

Solution:

According to the question,

We have,

z = x + iy

$\Rightarrow \bar{z} = x - iy$

Now, we also have,

$z\bar{z} + 2(z + \bar{z}) + b = 0$

$\Rightarrow (x + iy)(x - iy) + 2(x + iy + x - iy) + b = 0$

$\Rightarrow x^2 + y^2 + 4x + b = 0$

The equation obtained represents the equation of a circle.

9. If the real part of ($\bar{z}$ + 2)/ ($\bar{z}$ – 1) is 4, then show that the locus of the point representing z in the complex plane is a circle.

Solution:

According to the question,

Let z = x + iy

Now,

$\frac{\bar{z}+2}{\bar{z}-1} = \frac{x-iy+2}{x-iy-1}$

$= \frac{[(x+2)-iy][(x-1)+iy]}{[(x-1)-iy][(x-1)+iy]}$

$= \frac{(x-1)(x+2)+y^2+i[(x+2)y-(x-1)y]}{(x-1)^2+y^2}$

According to the question, we have, real part = 4.

$\Rightarrow \frac{(x-1)(x+2)+y^2}{(x-1)^2+y^2} = 4$

$\Rightarrow x^2 + x - 2 + y^2 = 4(x^2 - 2x + 1 + y^2)$

$\Rightarrow 3x^2 + 3y^2 - 9x + 6 = 0$

The equation obtained represents the equation of a circle.

Hence, locus of z is a circle.

10. Show that the complex number z, satisfying the condition arg ((z-1)/(z+1)) = π/4 lies on a circle.

Solution:

According to the question,

Let z = x + iy

arg ((z-1)/(z+1)) = π/4

⇒ arg (z – 1) – arg (z + 1) = π/4

⇒ arg (x + iy – 1) – arg (x + iy + 1) = π/4

⇒ arg (x – 1 + iy) – arg (x + 1 + iy) = π/4

$\Rightarrow \tan^{-1}\frac{y}{x-1} + \tan^{-1}\frac{y}{x+1} = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}\left[\frac{\frac{y}{x-1} - \frac{y}{x+1}}{1 + (\frac{y}{x-1})(\frac{y}{x+1})}\right] = \frac{\pi}{4}$

$\Rightarrow \frac{y(x+1-x+1)}{x^2-1+y^2} = \tan\frac{\pi}{4}$

$\Rightarrow \frac{2y}{x^2+y^2-1} = 1$

$\Rightarrow x^2 + y^2 - 1 = 2y$

$\Rightarrow x^2 + y^2 - 2y - 1 = 0$

The equation obtained represents the equation of a circle.

11. **Solve that equation |z| = z + 1 + 2i.**

Solution:

According to the question,

We have,

|z| = z + 1 + 2i

Substituting z = x + iy, we get,

⇒ |x + iy| = x + iy + 1 + 2i

We know that,
$|z| = \sqrt{(x^2 + y^2)}$
$\sqrt{(x^2 + y^2)} = (x + 1) + i(y + 2)$
Comparing real and imaginary parts,
We get,
$\sqrt{(x^2 + y^2)} = (x + 1)$
And $0 = y + 2$
$\Rightarrow y = -2$
Substituting the value of y in $\sqrt{(x^2 + y^2)} = (x + 1)$,
We get,
$\Rightarrow x^2 + (-2)^2 = (x + 1)^2$
$\Rightarrow x^2 + 4 = x^2 + 2x + 1$
Hence, $x = 3/2$
Hence, $z = x + iy$
$= 3/2 – 2i$

LONG ANSWER TYPE

12. If |z + 1| = z + 2 (1 + i), then find z.
Solution:

According to the question,
We have,
$|z + 1| = z + 2 (1 + i)$
Substituting $z = x + iy$, we get,
$\Rightarrow |x + iy + 1| = x + iy + 2 (1 + i)$
We know,
$|z| = \sqrt{(x^2 + y^2)}$
$\sqrt{((x + 1)^2 + y^2)} = (x + 2) + i(y + 1)$
Comparing real and imaginary parts,
$\Rightarrow \sqrt{((x + 1)^2 + y^2)} = x + 2$
And $0 = y + 2$
$\Rightarrow y = -2$
Substituting the value of y in $\sqrt{((x + 1)^2 + y^2)} = x + 2$,
$\Rightarrow (x + 1)^2 + (-2)^2 = (x + 2)^2$
$\Rightarrow x^2 + 2x + 1 + 4 = x^2 + 4x + 4$
$\Rightarrow 2x = 1$
Hence, $x = ½$
Hence, $z = x + iy$
$= ½ – 2i$

13. If arg (z – 1) = arg (z + 3i), then find x – 1 : y. where z = x + iy
Solution:

According to the question,
Let $z = x + iy$
Given that,
$\arg (z – 1) = \arg (z + 3i)$
$\Rightarrow \arg (x + iy – 1) = \arg (x + iy + 3i)$

⇒ arg (x – 1 + iy) = arg (x + I (y) = π/4

$$\Rightarrow \tan^{-1}\frac{y}{x-1} = \tan^{-1}\frac{y+3}{x}$$

$$\Rightarrow \frac{y}{x-1} = \frac{y+3}{x}$$

⇒ xy = xy – y + 3x – 3

⇒ 3x – 3 = y

⇒ (x – 1)/y = 1/3

Hence, (x – 1): y = 1: 3

14. Show that |(z – 2) / (z – 3)| = 2 represents a circle. Find its centre and radius.

Solution:

According to the question,

We have,

|(z – 2) / (z – 3)| = 2

Substituting z = x + iy, we get

⇒ |(x + iy – 2) / (x + iy – 3)| = 2

⇒ |x – 2 + iy| = 2 |x – 3 + iy|

⇒ $\sqrt{((x-2)^2 + y^2)} = 2\sqrt{((x-3)^2 + y^2)}$

⇒ $x^2 - 4x + 4 + y^2 = 4(x^2 - 6x + 9 + y^2)$

⇒ $3x^2 + 3y^2 - 20x + 32 = 0$

$$\Rightarrow x^2 + y^2 - \frac{20}{3}x + \frac{32}{3} = 0$$

$$\Rightarrow \left(x - \frac{10}{3}\right)^2 + y^2 + \frac{32}{3} - \frac{100}{9} = 0$$

$$\Rightarrow \left(x - \frac{10}{3}\right)^2 + (y-0)^2 = \frac{4}{9}$$

Therefore, centre of circle is (10/3, 0) and radius is 4/9 or 2/3.

15. If (z – 1)/(z + 1) is a purely imaginary number (z ≠ –1), then find the value of |z|.

Solution:

According to the question,

Let z = x + iy

Now,

$\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1}$

$$= \frac{[(x-1)+iy][(x+1)-iy]}{[(x+1)+iy][(x+1)-iy]}$$

$$= \frac{(x^2-1)+y^2+i[(x+1)y-(x-1)y]}{(x+1)^2+y^2}$$

According to the question, we have,

$\frac{z-1}{z+1}$ is purely imaginary.

$$\Rightarrow \frac{(x^2-1)+y^2}{(x+1)^2+y^2} = 0$$

$\Rightarrow x^2 - 1 + y^2 = 0$
$\Rightarrow x^2 + y^2 = 1$
$\Rightarrow \sqrt{(x^2 + y^2)} = 1$
Hence, $|z| = 1$

16. z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$ and arg (z_1) + arg $(z_2) = \pi$, then show that $z_1 = -\overline{z}_2$.
Solution:

According to the question,
Let $z_1 = |z_1|$ (cos θ_1 + I sin θ_1) and $z_2 = |z_2|$ (cos θ_2 + I sin θ_2)
Given that $|z_1| = |z_2|$
And arg (z_1) + arg $(z_2) = \pi$
$\Rightarrow \theta_1 + \theta_2 = \pi$
$\Rightarrow \theta_1 = \pi - \theta_2$
Now, $z_1 = |z_2|$ (cos $(\pi - \theta_2)$ + I sin $(\pi - \theta_2)$)
$\Rightarrow z_1 = |z_2|$ (-cos θ_2 + I sin θ_2)
$\Rightarrow z_1 = -|z_2|$ (cos θ_2 – I sin θ_2)
$\Rightarrow z_1 = -$ [$|z_2|$ (cos θ_2 – I sin θ_2)]
Hence, $z_1 = -\overline{z}_2$
Hence proved.

17. If $|z_1| = 1$ ($z_1 \neq -1$) and $z_2 = (z_1 - 1) / (z + 1)$, then show that the real part of z_2 is zero.
Solution:

According to the question,
Let $z_1 = x + iy$

$\Rightarrow |z_1| = \sqrt{x^2 + y^2} = 1$

$z_2 = \frac{z_1 - 1}{z_1 + 1} = \frac{x + iy - 1}{x + iy + 1}$

$= \frac{[(x-1) + iy][(x+1) - iy]}{[(x+1) + iy][(x+1) - iy]}$

$= \frac{(x^2 - 1) + y^2 + i[(x+1)y - (x-1)y]}{(x+1)^2 + y^2}$

$= \frac{x^2 + y^2 - 1 + 2iy}{(x+1)^2 + y^2}$

Since $x^2 + y^2 = 1$

$= \frac{1 - 1 + 2iy}{(x+1)^2 + y^2}$

$= 0 + \frac{2iy}{(x+1)^2 + y^2}$

Therefore, the real part of z_2 is zero.

18. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then find $\arg(z_1/z_4) + \arg(z_2/z_3)$.
Solution:

According to the question,
We have,
z_1 and z_2 are conjugate complex numbers.
The negative side of the real axis
$= r_1 (\cos \theta_1 - i \sin \theta_1)$
$= r_1 [\cos (-\theta_1) + I \sin (-\theta_1)]$
Similarly, $z_3 = r_2 (\cos \theta_2 - i \sin \theta_2)$
$\Rightarrow z_4 = r_2 [\cos (-\theta_2) + I \sin (-\theta_2)]$

$\Rightarrow \arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right) = \arg(z_1) - \arg(z_4) + \arg(z_2) - \arg(z_3)$

$= \theta_1 – (-\theta_2) + (-\theta_1) – \theta_2$
$= \theta_1 + \theta_2 – \theta_1 – \theta_2$
$= 0$
$\Rightarrow \arg(z_1/z_4) + \arg(z_2/z_3) = 0$

19. If $|z_1| = |z_2| = ….. = |z_n| = 1$, then show that $|z_1 + z_2 + z_3 + …. + z_n| = | 1/z_1 + 1/z_2 + 1/z_3 + … + 1/z_n|$
Solution:

According to the question,
We have,
$|z_1| = |z_2| = … = |z_n| = 1$
$\Rightarrow |z_1|^2 = |z_2|^2 = … = |z_n|^2 = 1$
$\Rightarrow z_1 \overline{z}_1 = z_2 \overline{z}_2 = z_3 \overline{z}_3 = … = z_n \overline{z}_n = 1$

$\Rightarrow z_1 = \frac{1}{\overline{z_1}}, z_2 = \frac{1}{\overline{z_2}}, \ldots, z_n = \frac{1}{\overline{z_n}}$

Now,

$\Rightarrow |z_1 + z_2 + z_3 + z_4 + \cdots + z_n| = \left|\frac{z_1\overline{z_1}}{\overline{z_1}} + \frac{z_2\overline{z_2}}{\overline{z_2}} + \frac{z_3\overline{z_3}}{\overline{z_3}} + \cdots + \frac{z_n\overline{z_n}}{\overline{z_n}}\right|$

$= \left|\frac{1}{\overline{z_1}} + \frac{1}{\overline{z_2}} + \frac{1}{\overline{z_3}} + \cdots + \frac{1}{\overline{z_n}}\right|$

$= \left|\overline{\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \cdots + \frac{1}{z_n}}\right|$

$= \left|\frac{1}{z_1} + \frac{1}{z_2} + \cdots + \frac{1}{z_n}\right|$

Hence proved.

20. If for complex numbers z_1 and z_2, arg (z_1) – arg (z_2) = 0, then show that $|z_1 – z_2| = |z_1| – |z_2|$.

Solution:

According to the question,

Let $z_1 = |z_1| (\cos \theta_1 + I \sin \theta_1)$ and $z_2 = |z_2| (\cos \theta_2 + I \sin \theta_2)$

We have,

arg (z_1) - arg (z_2) = 0

$\Rightarrow \theta_1 - \theta_2 = 0$

$\Rightarrow \theta_1 = \theta_2$

We also have,

$z_2 = |z_2| (\cos \theta_1 + I \sin \theta_1)$

$\Rightarrow z_1 – z_2 = ((|z_1|\cos \theta_1 - |z_2| \cos \theta_1) + i (|z_1| \sin \theta_1 - |z_2| \sin \theta_1))$

$\Rightarrow |z_1 - z_2| = \sqrt{(|z_1|\cos\theta_1 - |z_2|\cos\theta_1)^2 + (|z_1|\sin\theta_1 - |z_2|\sin\theta_1)^2}$

$= \sqrt{|z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos^2\theta_1 - 2|z_1||z_2|\sin^2\theta_1}$

$= \sqrt{|z_1|^2 + |z_2|^2 - 2|z_1||z_2|[\cos^2\theta_1 + \sin^2\theta_1]}$

We know that $\cos^2 \theta + \sin^2 \theta = 1$

$= \sqrt{|z_1|^2 + |z_2|^2 - 2|z_1||z_2|}$

$= \sqrt{(|z_1| - |z_2|)^2}$

Hence, $|z_1 – z_2| = |z_1| - |z_2|$

Hence proved.

21. Solve the system of equations Re (z_2) = 0, |z| = 2.

Solution:

According to the question,

We have,

Re (z^2) = 0, |z| = 2

Let z = x + iy.

Then, $|z| = \sqrt{(x^2 + y^2)}$

Given in the question,

$\sqrt{(x^2 + y^2)} = 2$

$\Rightarrow x^2 + y^2 = 4$ … (i)
$z^2 = x^2 + 2ixy - y^2$
$= (x^2 - y^2) + 2ixy$
Now, Re $(z^2) = 0$
$\Rightarrow x^2 - y^2 = 0$ … (ii)
Equating (i) and (ii), we get
$\Rightarrow x^2 = y^2 = 2$
$\Rightarrow x = y = \pm\sqrt{2}$
Hence, $z = x + iy$
$= \pm\sqrt{2} \pm i\sqrt{2}$
$= \sqrt{2} + i\sqrt{2}, \sqrt{2} - i\sqrt{2}, -\sqrt{2} + i\sqrt{2}$ and $-\sqrt{2} - i\sqrt{2}$
Hence, we have four complex numbers.

22. Find the complex number satisfying the equation $z + \sqrt{2}\,|(z + 1)| + i = 0$.
Solution:

According to the question,
We have,
$z + \sqrt{2}\,|(z + 1)| + i = 0$ … (1)
Substituting $z = x + iy$, we get
$\Rightarrow x + iy + \sqrt{2}\,|x + iy + 1| + i = 0$
$$\Rightarrow x + i(1 + y) + \sqrt{2}\left[\sqrt{(x+1)^2 + y^2}\right] = 0$$
$$\Rightarrow x + i(1 + y) + \sqrt{2}\sqrt{(x^2 + 2x + 1 + y^2)} = 0$$
Comparing real and imaginary parts to zero, we get
$$\Rightarrow x + \sqrt{2}\sqrt{x^2 + 2x + 1 + y^2} = 0 \quad \text{… (2)}$$
And,
$y + 1 = 0$
$\Rightarrow y = -1$
Substituting $y = -1$ into equation (2), we get
$$\Rightarrow x + \sqrt{2}\sqrt{x^2 + 2x + 1 + 1} = 0$$
$$\Rightarrow \sqrt{2}\sqrt{x^2 + 2x + 2} = -x$$
$\Rightarrow 2x^2 + 4x + 4 = x^2$
$\Rightarrow x^2 + 4x + 4 = 0$
$\Rightarrow (x + 2)^2 = 0$
$\Rightarrow x = -2$
Hence, $z = x + iy$
$= -2 - i$

23. Write the complex number
$$z = \frac{1 - i}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$$
in polar from.

Solution:

According to the question,
We have,

$$z = \frac{1-i}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$$

$$= \frac{\sqrt{2}\left[\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right]}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$$

$$= \frac{\sqrt{2}\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$$

$$= \sqrt{2}\left[\cos\left(-\frac{\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{4} - \frac{\pi}{3}\right)\right]$$

$$= \sqrt{2}\left[\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right]$$

$$= \sqrt{2}\left[\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right]$$

24. If z and w are two complex numbers such that |zw| = 1 and arg (z) – arg (w) = π/2, then show that $\bar{z}w = -i$.

Solution:

Let z = |z| (cos θ_1 + I sin θ_1) and w = |w| (cos θ_2 + I sin θ_2)
Given |zw| = |z| |w| = 1
Also arg (z) – arg (w) = π/2
⇒ $\theta_1 - \theta_2 = \pi/2$
Now, $\bar{z}$ w = |z| (cos θ_1 - I sin θ_1) |w| (cos θ_2 + I sin θ_2)g | = 1
= |z| |w| (cos ($-\theta_1$) + I sin ($-\theta_1$)) (cos θ_2 + I sin θ_2)
= 1 [cos ($\theta_2 - \theta_1$) + I sin ($\theta_2 - \theta_1$)]
= [cos (-π/2) + I sin (-π/2)]
= 1 [0 – i]
= – i
Hence proved.