BYJU'S The Learning App

Quadratic Equation Practice Problems

1. If α and β are the roots of a quadratic equation $px^2 + qx + r = 0$, then find the quadratic equation whose roots are, $\alpha^2 + \beta^2$ and $1/\alpha^2 + 1/\beta^2$.

Solution:

From the quadratic equation $px^2 + qx + r = 0$,

α + β = - qp . . . . (1)

αβ = rp . . . . (2)

Now, the product of roots $\alpha^2 + \beta^2$ and $1/\alpha^2 + 1/\beta^2$:

$$= (\alpha^2 + \beta^2)\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right) = \frac{(\alpha^2 + \beta^2)^2}{\alpha^2 \beta^2}$$

i.e. $$\frac{(\alpha^2 + \beta^2)^2}{\alpha^2 \beta^2} = \frac{(\alpha + \beta)^2 - 2(\alpha\beta)}{(\alpha\beta)^2} = \frac{\left(\frac{p^2}{q^2}\right) - 2\left(\frac{r}{q}\right)}{\left(\frac{r^2}{q^2}\right)}$$ [Using Equation (1) and (2)]

Now, The Sum of roots $\alpha^2 + \beta^2$ and $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$:

$$\alpha^2 + \beta^2 + \left(\frac{\beta^2 + \alpha^2}{\alpha^2 \beta^2}\right) = (\alpha + \beta)^2 - 2\alpha\beta + \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$$

i.e. $$(\alpha + \beta)^2 - 2\alpha\beta + \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2} = \frac{q^2}{p^2} - \frac{2r}{p} + \frac{\frac{q^2}{p^2} - \frac{2r}{p}}{\frac{r^2}{p^2}}$$

$$= (q^2 - 2pr)\left(\frac{p^2 + r^2}{r^2 p^2}\right)$$ [Using Equation (1) and (2)]

Hence, the required quadratic Equation is: x^2 – (Sum of Zeros)x + (Product of Zeros) = 0

i.e. $$x^2 - \left[(q^2 - 2pr)\left(\frac{p^2 + r^2}{r^2 p^2}\right)\right]x + \frac{\left(\frac{p^2}{q^2}\right) - 2\left(\frac{r}{q}\right)}{\left(\frac{r^2}{q^2}\right)} = 0$$

Therefore, $(prx)^2 - [(q^2 - 2pr)(p^2 + r^2)] + (p^2 - 2rq) = 0$ is the required quadratic equation.

2. If α and β are roots of the quadratic equation $x^2 - x + 1 = 0$, then, find the value of $\alpha^{2009} + \beta^{2009}$.

Solved Examples on Quadratic Equations

Solution:

The roots of the quadratic equation $x^2 - x + 1 = 0$ are given by:

$$(\alpha, \beta) = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}}{2}$$

Therefore, $\alpha = \frac{1}{2} + \frac{\sqrt{3}\,i}{2}$ and, $\alpha = \frac{1}{2} - \frac{\sqrt{3}\,i}{2}$

i.e. $\alpha = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ and, $\alpha = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$

Since, $z^p = \cos(p\,\theta) + i\sin(p\,\theta)$ [De Moivre's Theorem]

Therefore, $\alpha^{2009} = z^p = \cos(2009\,\theta) + i\sin(2009\,\theta)$

And, $\beta^{2009} = z^p = \cos(2009\,\theta) - i\sin(2009\,\theta)$

i.e. $\alpha^{2009} + \beta^{2009} = 2\cos 2009\frac{\pi}{3}$

$= 2\cos\left[668\,\pi + \pi + \frac{2\,\pi}{3}\right] = 2\cos\left[\pi + \frac{2\,\pi}{3}\right]$

$= -2\cos\frac{2\,\pi}{3} = -2\left(\frac{-1}{2}\right) = 1$

Therefore, the value of $\alpha^{2009} + \beta^{2009} = 1$.

3. If α and β are roots of a quadratic equation $x^2 + bx - c = 0$ and γ and δ are roots of a quadratic equation $x^2 + bx + s = 0$, where q and r ≠ 0, then find the value of (α - γ)(α - δ)/(β - γ)(β - δ).

Solution:

From the given quadratic equations α + β = -b and γ + δ = -b . . . . (1)

Therefore, α + β = γ + δ . . . . (2)

And, αβ = -c and γδ = s . . . . (3)

Now,

(α - γ) (α - δ) = α^2 - (γ + δ)α + γδ

= α^2 - (α + β)α + γδ [From Equation (2)]

$= -\alpha\beta + \gamma\delta = c + s$ [From Equation (3)]

Therefore, $(\alpha - \gamma)(\alpha - \delta) = c + s$

Similarly, $(\beta - \gamma)(\beta - \delta) = c + s$

Therefore, the value of $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta) = 1$.

4. If α and β are roots of a quadratic equation $px^2 + qx + r = 0$ and γ and δ are roots of a quadratic equation $gx^2 + hx + i = 0$, then find the equation whose roots are $(\alpha\gamma + \beta\delta)$ and $(\alpha\delta + \beta\gamma)$.

Solution:

From the above quadratic equations $\alpha + \beta = -qp$ and $\alpha\beta = -rp$ . . . . (1)

And, $\gamma + \delta = -hg$ and $\gamma\delta = ig$ . . . . (2)

Now, the product and sum of roots $(\alpha\gamma + \beta\delta)$ and $(\alpha\delta + \beta\gamma)$:

Product $= (\alpha\gamma + \beta\delta)(\alpha\delta + \beta\gamma)$

$= \alpha^2\gamma\delta + \alpha\beta\gamma^2 + \alpha\beta\delta^2 + \beta^2\gamma\delta = \gamma\delta(\alpha^2 + \beta^2) + \alpha\beta(\gamma^2 + \delta^2)$

$= \gamma\delta(\alpha^2 + \beta^2) + \alpha\beta(\gamma^2 + \delta^2)$

$= \gamma\delta[(\alpha + \beta)^2 - 2(\alpha\beta)] + \alpha\beta[(\gamma + \delta)^2 - 2(\gamma\delta)]$

Now, on substituting the values of Equation (1) and (2), we will g

$$= \frac{i}{g}\left(\frac{-q}{p}\right)^2 - 2\left(\frac{r}{p}\right) + \frac{r}{p}\left(\frac{-h}{g}\right)^2 - 2\left(\frac{i}{g}\right)$$

$$= \frac{i\,g\,q^2 - 2\,g^2\,p\,r + r\,p\,h^2 - 2\,i\,g\,p^2}{g^2\,p^2}$$

$$= \frac{i}{g}\left[\left(\frac{-q}{p}\right)^2 - 2\left(\frac{r}{p}\right)\right] + \frac{r}{p}\left[\left(\frac{-h}{g}\right)^2 - 2\left(\frac{i}{g}\right)\right]$$

$$= \frac{q^2\,i}{g\,p^2} - \frac{2\,r\,i}{g\,p} + \frac{h^2\,r}{p\,g^2} - \frac{2\,i\,r}{p\,g}$$

$$= \frac{q^2\,i\,g - 4\,r\,i\,g\,p + h^2\,r\,p}{g^2\,p^2}$$

Sum = $(\alpha\gamma + \beta\delta) + (\alpha\delta + \beta\gamma)$

$= \alpha\gamma + \beta\delta + \alpha\delta + \beta\gamma = \alpha(\gamma + \delta) + \beta(\gamma + \delta)$

$= (\alpha + \beta)(\gamma + \delta) = \frac{-q}{p} \cdot \frac{-h}{g} = \frac{q\,h}{p\,g}$ [From Equation (1) and (2)]

The quadratic Equation is given by: x^2 – (Sum of Roots)x + (Product of Roots) = 0

i.e. $\mathbf{x^2} - \left(\frac{q\,h}{p\,g}\right)\mathbf{x} + \frac{q^2\,i\,g - 4\,r\,i\,g\,p + h^2\,r\,p}{g^2\,p^2} = \mathbf{0}$

Therefore, $g^2p^2x^2$ - (pgqh)x + q^2ig - 4rigp + h^2rp = 0 is the required quadratic equation.

4. How many real roots does the equation $x^7 + 14x^5 + 16x^3 + 30x - 560 = 0$ will have?

Solution:

Since the given equation has 1 sign change so it can have a maximum 1 positive real roots.
Now for f(-x), the equation has no sign change i.e. f (-x) = $x^7 + 14x^5 + 16x^3 + 30x - 560 = 0$.
Hence, the equation will have zero negative real root.

5. When pr = 2(q + s), where p, q, r, s are real numbers, show that at least one of the equations

x^2 + px + q and x^2 + rx + s = 0 has real roots.

Solution:

For at least one of the given equations to have real roots means one of their discriminant must be non-negative. The given equations are;

f(α) = 0 + px + q = 0 . . . . . (i)

f(α) = 0 + rx + s = 0 . . . . . . (ii)

Consider D_1 and D_2 be the discriminant of equations (i) and (ii) respectively,

∴ $D_1 + D_2 = p^2 - 4q + r^2 - 4s$

$= p^2 + r^2 - 4(q + s) = p^2 + r^2 - 2pr$

= (p – r)2 > 0 [because p and r are real]

Since, at least one of D_1 and D_2 must be non-negative. Hence, at least one of the given equation has real roots.

6. Find the quadratic equation where one of the roots is 1/(2 + √5).

Solution:

If one root is (α + √β) then other one will be (α - √β)

Given $\alpha = \frac{1}{2+\sqrt{5}}$

Multiplying the numerator and denominator by $2 - \sqrt{5}$, we get

$$= \frac{2-\sqrt{5}}{(2+\sqrt{5})(2-\sqrt{5})} = \sqrt{5} - 2$$

Then the other root of quadratic equation $x^2 + px + q = 0$ will be -2 - √5,Now, α + β = -4 and αβ = −1

Thus, the required quadratic equation is:

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$, or, $x^2 + 4x - 1 = 0$.

7. If α and αn are the roots of the quadratic equation $ax^2 + bx + c = 0$, then show that:

$$(ac^n)^{\frac{1}{n+1}} + (a^n c)^{\frac{1}{n+1}} + b = 0.$$

Solution

By using the sum and product of roots formulae we can prove this.

Give that α and αn are the roots.

$$\alpha^n = \frac{c}{a}$$

$$\Rightarrow \alpha = \left(\frac{c}{a}\right)^{\frac{1}{n+1}}$$

$$And\, \alpha + \alpha^n = \frac{-b}{a}$$

$$\Rightarrow \left(\frac{c}{a}\right)^{\frac{1}{n+1}} + \left(\frac{c}{a}\right)^{\frac{1}{n+1}} = \frac{-b}{a}$$

Or $(ca^n)^{\frac{1}{n+1}} + (c^n a)^{\frac{1}{n+1}} + b = 0.$

8. Solve for x when;

$$\log_{10}\left(\sqrt{\log_{10} x}\right) = \log_{(x^2)} x : x > 1$$

Solution:

By using the formula

$$\log_a M^x = x\log_a . M\, and \log_b a = \frac{\log_{10} a}{\log_{10} b}$$

$$\log_{10}\left(\sqrt{\log_{10} x}\right) = \log_{(x^2)}$$

x = $\frac{\log_{10} x}{\log_{10} x^2} = \frac{1}{2}$

Let, y = $\log_{10}$x, then, 1/2 = $\log_{10} \sqrt{y}$

Therefore, y = 10. Thus, x = 10^{10}.

9. If α is a root of the equation $4x^2 + 2x - 1 = 0$, then prove that $4\alpha^3 - 3\alpha$ is the other root.

Solved Examples on Quadratic Equations

Solution:

Consider α, β to be the two roots of the given equation $4x^2 + 2x - 1 = 0$.

Therefore, $\alpha + \beta = -1/2$ and $4\alpha^2 + 2\alpha - 1 = 0$;

$4\alpha^3 - 3\alpha = (4\alpha^2 + 2\alpha - 1)(\alpha - 1/2) - (\alpha + 1/2) = \beta$

Hence $4\alpha^3 - 3\alpha$ is another root.

10. The roots of 1/(x+p) + 1/(x+q) = 1/r are equal in magnitude but opposite in sign, show that p + q = 2r and the product of the roots = $-(p^2 - q^2)/2$

Solution:

By considering α and -α as the roots of the given equation and then by using the sum and product of roots formula we can solve it.

$1/(x+p) + 1/(x+q) = 1/r$ . . . . (i)

$\Rightarrow (x + q + x + p)r = x^2 + (p + q)x + pq$

$\Rightarrow x^2 + (p + q - 2r)x + pq - r(p + q) = 0$.

Since, its roots are equal in magnitude but opposite in sign, therefore, consider roots are α, -α

Sum = $\alpha - \alpha = p + q - 2r$

$\Rightarrow p + q = 2r$

Product of roots = $pq - r(p + q)$

$= pq - (p + q)^2/2 = -(p^2 - q^2)/2$.

11. If α, β are the roots of $x^2 + px + q = 0$. Prove that α/β is a root of $qx^2 + (2q - p^2)x + q = 0$.

Solution:

We need to show that α/β is a root of $ax^2 + (2q - p^2)x + q = 0$

$\Rightarrow q \times \alpha^2/\beta^2 \times (2q - p^2) \times \alpha/\beta + q = 0$

$q\alpha^2+(2q - p^2)\alpha\beta + q\beta^2 = 0$

$q(\alpha^2 + 2\alpha q - \beta^2)p^2\alpha\beta = 0$

$q(\alpha + \beta)^2 - p^2\alpha\beta = 0$

$p^2q - p^2q=0$ which is obviously true.

Solved Examples on Quadratic Equations

12. Find the value of ‘a’ for which $3x^2 + 2(a^2 + 1)x + a^2 - 3a + 2 = 0$ possesses roots with opposite signs.

Solution:

Roots of the given equation are of opposite sign, hence, their product is negative and the discriminant is positive.

Therefore, $a^2 - 3a + 23 < 0$;

$= (a - 2)(a - 1) < 0$ and a (1, 2) and $D > 0$

$4(a^2 + 1) - 4 \times 3(a^3 - 3a + 2) > 0$

This equation will always hold true for $a \in (1, 2)$.

13. If x is real, find the range of the quadratic expression $(x^2 + 14x + 9) \times (x^2 + 2x + 3)^{-1}$

Solution:

Let, $(x^2 + 14x + 9) \times (x^2 + 2x + 3)^{-1} = y$

$x^2 + 14x + 9 = x^2y + 2xy + 3y$

$x^2 (1 - y) + 2(7 - y) + 3(3 - y) = 0$

Hence, $D > 0$

$4(7 - y)^2 - 12(3 - y)(1 - y) > 0$;

$49 + y^2 - 14y - 3(3 - 4y + y^2) > 0$;

$-2y^2 - 2y + 40 > 0$;

$y^2 + y - 20 < 0$;

$(y + 5)(y - 4) < 0$;

∴ the range of the given quadratic expression is $-5 < y < 4$.

14. Find the value of x if $2x + 5 + |x^2 + 4x + 3| = 0$.

Solution:

For $2x + 5 + |x^2 + 4x + 3| = 0$, $2x + 5$ must be less than or equal to zero. And whether $x^2 + 4x + 3$ will be positive or negative depends on the value of x.

$2x + 5 + |x^2 + 4x + 3| = 0$

Case - 1: When $x < -3$ or $x > -1$

$x^2 + 4x + 3 + 2x + 5 = 0$

(x + 2) (x + 4) = 0;

x = - 4.

15. Prove that the roots of the equation ax2 + bx + c = 0 are given by $\frac{2c}{-b\mp\sqrt{b^2-4ac}}$

Solution:

We know that the roots of the quadratic equation ax2 + bx + c = 0 are found by $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Therefore, in multiplying and dividing by $-b \mp \sqrt{b^2 - 4ac}$ we can prove the above problem.

ax2 + bx + c = 0

$$\Rightarrow x^2 + \frac{b}{a} \times + \frac{c}{a} = 0 \qquad \Rightarrow \left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2} \;\Rightarrow \left(x - \frac{b}{2a}\right) = \pm\left(\frac{\sqrt{b^2-4ac}}{2a}\right)$$

$$\Rightarrow x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} \Rightarrow x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} \times \frac{-b\mp\sqrt{b^2-4ac}}{-b\mp\sqrt{b^2-4ac}}$$

$$\Rightarrow x = \frac{(-b)^2-(b^2-4ac)}{2a\left(-b\mp\sqrt{b^2-4ac}\right)} \Rightarrow x = \frac{2c}{-b\pm\sqrt{b^2-4ac}}$$

16. Consider the quadratic polynomial f(x) = x^2 - px + q where f(x) = 0 has prime roots. If p + q = 11 and a = $p^2 + q^2$, then find the value of f(a) where a is an odd positive integer.

Solution:

f(x) = x^2 - px + q

∴ α + β = p and αβ = q

Given, p + q = 11

⇒ α + β + αβ = 11

⇒ (α + 1) (β + 1) = 12;

α = 2, β = 3 are the only primes that solve this equation

Therefore, f(x) = (x - 2) (x - 3) = x^2 - 5x + 6

And, p = 5, q = 6

⇒ a = $p^2 + q^2$ = 25+36 = 51;

$f(51) = (51 - 2)(51 - 3) = 49 \times 48 = 3422$.

17. If $y = ax^2 + bx + c$ has no real roots. Prove that $c(a + b + c) > 0$. What can you say about the expression $c(a - b + c)$?

Solution:

Since there are no real roots, y will always be either positive or negative.

Therefore, $f(x_1) . f(x_2) > 0$

$f(0) f(1) > 0$;

$c(a + b + c) > 0$;

Similarly $f(0) f(-1) > 0$

$c(a - b + c) > 0$

18. If α and β are the roots of a quadratic equation $f(x) = x^2 - 2x + 5 = 0$ then form a quadratic equation whose roots are $\alpha^3 + \alpha^2 + 22$ and $\beta^3 + 4\beta^2 - 7\beta + 35$.

Solution:

As α, β are roots of the equation $f(x) = \mathbf{x^2 - 2x + 5 = 0}$, $f(\alpha)$ and $f(\beta)$ will be 0. Therefore, by obtaining the values of $\alpha^3 + \alpha^2 - \alpha + 22$ and $\beta^3 + 4\beta^2 - 7\beta + 35$ we can from the required equation.

From the given equation $\alpha^2 - 2\alpha + 5 = 0$ and $\beta^2 - 2\beta + 5 = 0$

$\Rightarrow \alpha^3 - \alpha^2 - \alpha + 22 = \alpha(\alpha^2 - 2\alpha + 5) + 3\alpha^2 - 6\alpha + 22 = 3(\alpha^2 - 2\alpha + 5) + 7 = 7$

Similarly, $\beta^3 + 4\beta^2 - 7\beta + 35 = \beta(\beta^2 - 2\beta + 5) + 6\beta^2 - 12\beta + 35 = 6(\beta^2 - 2\beta + 5) + 5 = 5$

Therefore, the required quadratic equation is $x^2 - 12x + 35 = 0$.