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1. Let *A* be the set of all human beings in a town at a particular time. Determine whether of the following relation is reflexive, symmetric and transitive:
(i) R = {(x, y): x and y work at the same place}
(ii) R = {(x, y): x and y live in the same locality}
(iii) R = {(x, y): x is wife of y}
(iv) R = {(x, y): x is father of y}

Solution:
(i) Given R = {(x, y): x and y work at the same place}
Now we have to check whether the relation is reflexive:
Let x be an arbitrary element of R.
Then, x ∈R
⇒ x and x work at the same place is true since they are the same.
⇒(x, x) ∈R [condition for reflexive relation]
So, R is a reflexive relation.
Now let us check Symmetric relation:
Let (x, y) ∈R
⇒x and y work at the same place [given]
⇒y and x work at the same place
⇒(y, x) ∈R
So, R is a symmetric relation.
Transitive relation:
Let (x, y) ∈R and (y, z) ∈R.
Then, x and y work at the same place. [Given]
y and z also work at the same place. [(y, z) ∈R]
⇒ x, y and z all work at the same place.
⇒x and z work at the same place.
⇒ (x, z) ∈R
So, R is a transitive relation.
Hence R is reflexive, symmetric and transitive.

(ii) Given R = {(x, y): x and y live in the same locality}
Now we have to check whether the relation R is reflexive, symmetric and transitive.
Let x be an arbitrary element of R.

Then, x ∈R
It is given that x and x live in the same locality is true since they are the same.
So, R is a reflexive relation.
Symmetry:
Let (x, y) ∈ R
⇒ x and y live in the same locality [given]
⇒ y and x live in the same locality
⇒ (y, x) ∈ R
So, R is a symmetric relation.
Transitivity:
Let (x, y) ∈R and (y, z) ∈R.
Then,
x and y live in the same locality and y and z live in the same locality
⇒ x, y and z all live in the same locality
⇒ x and z live in the same locality
⇒ (x, z) ∈ R
So, R is a transitive relation.
Hence R is reflexive, symmetric and transitive.

(iii) Given R = {(x, y): x is wife of y}
Now we have to check whether the relation R is reflexive, symmetric and transitive.
First let us check whether the relation is reflexive:
Let x be an element of R.
Then, x is wife of x cannot be true.
⇒ (x, x) ∉R
So, R is not a reflexive relation.
Symmetric relation:
Let (x, y) ∈R
⇒ x is wife of y
⇒ x is female and y is male
⇒ y cannot be wife of x as y is husband of x
⇒ (y, x) ∉R
So, R is not a symmetric relation.
Transitive relation:
Let (x, y) ∈R, but (y, z) ∉R
Since x is wife of y, but y cannot be the wife of z, y is husband of x.
⇒ x is not the wife of z

⇒(x, z) ∈R
So, R is a transitive relation.

(iv) Given R = {(x, y): x is father of y}
Now we have to check whether the relation R is reflexive, symmetric and transitive.
Reflexivity:
Let x be an arbitrary element of R.
Then, x is father of x cannot be true since no one can be father of himself.
So, R is not a reflexive relation.
Symmetry:
Let (x, y) ∈R
⇒ x is father of y
⇒ y is son/daughter of x
⇒ (y, x) ∉R
So, R is not a symmetric relation.
Transitivity:
Let (x, y) ∈R and (y, z) ∈R.
Then, x is father of y and y is father of z
⇒ x is grandfather of z
⇒ (x, z) ∉R
So, R is not a transitive relation.

2. Three relations R_1, R_2 and R_3 are defined on a set A = {a, b, c} as follows:
R_1 = {(a, a), (a, b), (a, c), (b, b), (b, c), (c, a), (c, b), (c, c)}
R_2 = {(a, a)}
R_3 = {(b, c)}
R_4 = {(a, b), (b, c), (c, a)}.
Find whether or not each of the relations R_1, R_2, R_3, R_4 on *A* is (i) reflexive (ii) symmetric and (iii) transitive.

Solution:
(i) Consider R_1
Given R_1 = {(a, a), (a, b), (a, c), (b, b), (b, c), (c, a), (c, b), (c, c)}
Now we have check R_1 is reflexive, symmetric and transitive
Reflexive:
Given (a, a), (b, b) and (c, c) ∈ R_1
So, R_1 is reflexive.

Symmetric:
We see that the ordered pairs obtained by interchanging the components of R_1 are also in R_1.
So, R_1 is symmetric.
Transitive:
Here, $(a, b) \in R_1$, $(b, c) \in R_1$ and also $(a, c) \in R_1$
So, R_1 is transitive.

(ii) Consider R_2
Given $R_2 = \{(a, a)\}$
Reflexive:
Clearly $(a, a) \in R_2$.
So, R_2 is reflexive.
Symmetric:
Clearly $(a, a) \in R$
$\Rightarrow (a, a) \in R$.
So, R_2 is symmetric.
Transitive:
R_2 is clearly a transitive relation, since there is only one element in it.

(iii) Consider R_3
Given $R_3 = \{(b, c)\}$
Reflexive:
Here,$(b, b) \notin R_3$ neither $(c, c) \notin R_3$
So, R_3 is not reflexive.
Symmetric:
Here, $(b, c) \in R_3$, but $(c, b) \notin R_3$
So, R_3 is not symmetric.
Transitive:
Here, R_3 has only two elements.
Hence, R_3 is transitive.

(iv) Consider R_4
Given $R_4 = \{(a, b), (b, c), (c, a)\}$.
Reflexive:
Here, $(a, a) \notin R_4$, $(b, b) \notin R_4$ $(c, c) \notin R_4$
So, R_4 is not reflexive.

Symmetric:
Here, $(a, b) \in R_4$, but $(b, a) \notin R_4$.
So, R_4 is not symmetric
Transitive:
Here, $(a, b) \in R_4$, $(b, c) \in R_4$, but $(a, c) \notin R_4$
So, R_4 is not transitive.

3. Test whether the following relation R_1, R_2, and R_3 are (i) reflexive (ii) symmetric and (iii) transitive:
(i) R_1 on Q_0 defined by $(a, b) \in R_1 \Leftrightarrow a = 1/b$.
(ii) R_2 on Z defined by $(a, b) \in R_2 \Leftrightarrow |a – b| \le 5$
(iii) R_3 on R defined by $(a, b) \in R_3 \Leftrightarrow a^2 – 4ab + 3b^2 = 0$.

Solution:
(i) Given R_1 on Q_0 defined by $(a, b) \in R_1 \Leftrightarrow a = 1/b$.
Reflexivity:
Let a be an arbitrary element of R_1.
Then, $a \in R_1$
$\Rightarrow a \neq 1/a$ for all $a \in Q_0$
So, R_1 is not reflexive.
Symmetry:
Let $(a, b) \in R_1$
Then, $(a, b) \in R_1$
Therefore we can write 'a' as $a = 1/b$
$\Rightarrow b = 1/a$
$\Rightarrow (b, a) \in R_1$
So, R_1 is symmetric.
Transitivity:
Here, $(a, b) \in R_1$ and $(b, c) \in R_2$
$\Rightarrow a = 1/b$ and $b = 1/c$
$\Rightarrow a = 1/ (1/c) = c$
$\Rightarrow a \neq 1/c$
$\Rightarrow (a, c) \notin R_1$
So, R_1 is not transitive.

(ii) Given R_2 on Z defined by $(a, b) \in R_2 \Leftrightarrow |a – b| \le 5$
Now we have check whether R_2 is reflexive, symmetric and transitive.

Reflexivity:
Let a be an arbitrary element of R_2.
Then, $a \in R_2$
On applying the given condition we get,
$\Rightarrow |a-a| = 0 \leq 5$
So, R_1 is reflexive.
Symmetry:
Let $(a, b) \in R_2$
$\Rightarrow |a-b| \leq 5$ [Since, $|a-b| = |b-a|$]
$\Rightarrow |b-a| \leq 5$
$\Rightarrow (b, a) \in R_2$
So, R2 is symmetric.
Transitivity:
Let $(1, 3) \in R_2$ and $(3, 7) \in R_2$
$\Rightarrow |1-3| \leq 5$ and $|3-7| \leq 5$
But $|1-7| \nleq 5$
$\Rightarrow (1, 7) \notin R_2$
So, R_2 is not transitive.

(iii) Given R_3 on R defined by $(a, b) \in R_3 \Leftrightarrow a^2 - 4ab + 3b^2 = 0$.
Now we have check whether R_2 is reflexive, symmetric and transitive.
Reflexivity:
Let a be an arbitrary element of R_3.
Then, $a \in R_3$
$\Rightarrow a^2 - 4a \times a + 3a^2 = 0$
So, R_3 is reflexive
Symmetry:
Let $(a, b) \in R_3$
$\Rightarrow a^2-4ab+3b^2=0$
But $b^2-4ba+3a^2 \neq 0$ for all $a, b \in R$
So, R_3 is not symmetric.
Transitivity:
Let $(1, 2) \in R_3$ and $(2, 3) \in R_3$
$\Rightarrow 1 - 8 + 6 = 0$ and $4 - 24 + 27 = 0$
But $1 - 12 + 9 \neq 0$
So, R_3 is not transitive.

4. Let A = {1, 2, 3}, and let R_1 = {(1, 1), (1, 3), (3, 1), (2, 2), (2, 1), (3, 3)}, R_2 = {(2, 2), (3, 1), (1, 3)}, R_3 = {(1, 3), (3, 3)}. Find whether or not each of the relations R_1, R_2, R_3 on A is (i) reflexive (ii) symmetric (iii) transitive.

Solution:

Consider R_1
Given R_1 = {(1, 1), (1, 3), (3, 1), (2, 2), (2, 1), (3, 3)}
Reflexivity:
Here, (1, 1), (2, 2), (3, 3) ∈R
So, R_1 is reflexive.
Symmetry:
Here, (2, 1) ∈ R_1,
But (1, 2) ∉ R_1
So, R_1 is not symmetric.
Transitivity:
Here, (2, 1) ∈R_1 and (1, 3) ∈R_1,
But (2, 3) ∉R_1
So, R_1 is not transitive.
Now consider R_2
Given R_2 = {(2, 2), (3, 1), (1, 3)}
Reflexivity:
Clearly, (1, 1) and (3, 3) ∉R_2
So, R_2 is not reflexive.
Symmetry:
Here, (1, 3) ∈ R_2 and (3, 1) ∈ R_2
So, R_2 is symmetric.
Transitivity:
Here, (1, 3) ∈ R_2 and (3, 1) ∈ R_2
But (3, 3) ∉R_2
So, R_2 is not transitive.
Consider R_3
Given R_3 = {(1, 3), (3, 3)}
Reflexivity:
Clearly, (1, 1) ∉ R_3
So, R_3 is not reflexive.
Symmetry:
Here, (1, 3) ∈ R_3, but (3, 1) ∉ R_3

So, R_3 is not symmetric.
Transitivity:
Here, $(1, 3) \in R_3$ and $(3, 3) \in R_3$
Also, $(1, 3) \in R_3$
So, R_3 is transitive.

5. The following relation is defined on the set of real numbers.
(i) aRb if $a - b > 0$
(ii) aRb iff $1 + a b > 0$
(iii) aRb if $|a| \leq b$.
Find whether relation is reflexive, symmetric or transitive.

Solution:
(i) Consider aRb if $a - b > 0$
Now for this relation we have to check whether it is reflexive, transitive and symmetric.
Reflexivity:
Let a be an arbitrary element of R.
Then, $a \in R$
But $a - a = 0 \not> 0$
So, this relation is not reflexive.
Symmetry:
Let $(a, b) \in R$
$\Rightarrow a - b > 0$
$\Rightarrow -(b - a) > 0$
$\Rightarrow b - a < 0$
So, the given relation is not symmetric.
Transitivity:
Let $(a, b) \in R$ and $(b, c) \in R$.
Then, $a - b > 0$ and $b - c > 0$
Adding the two, we get
$a - b + b - c > 0$
$\Rightarrow a - c > 0$
$\Rightarrow (a, c) \in R$.
So, the given relation is transitive.

(ii) Consider aRb iff $1 + a b > 0$
Now for this relation we have to check whether it is reflexive, transitive and symmetric.

Reflexivity:
Let a be an arbitrary element of R.
Then, a ∈ R
⇒ $1 + a \times a > 0$
i.e. $1 + a^2 > 0$ [Since, square of any number is positive]
So, the given relation is reflexive.
Symmetry:
Let (a, b) ∈ R
⇒ $1 + a b > 0$
⇒ $1 + b a > 0$
⇒ (b, a) ∈ R
So, the given relation is symmetric.
Transitivity:
Let (a, b) ∈R and (b, c) ∈R
⇒ $1 + a b > 0$ and $1 + b c > 0$
But $1+ ac \ngtr 0$
⇒ (a, c) ∉ R
So, the given relation is not transitive.

(iii) Consider aRb if $|a| \le b$.
Now for this relation we have to check whether it is reflexive, transitive and symmetric.
Reflexivity:
Let a be an arbitrary element of R.
Then, a ∈ R [Since, $|a|=a$]
⇒ $|a| \nless a$
So, R is not reflexive.
Symmetry:
Let (a, b) ∈ R
⇒ $|a| \le b$
⇒ $|b| \nleq a$ for all a, b ∈ R
⇒ (b, a) ∉ R
So, R is not symmetric.
Transitivity:
Let (a, b) ∈ R and (b, c) ∈ R
⇒ $|a| \le b$ and $|b| \le c$
Multiplying the corresponding sides, we get
$|a| \times |b| \le b c$

⇒ |a| ≤ c
⇒ (a, c) ∈ R
Thus, R is transitive.

6. Check whether the relation R defined in the set {1, 2, 3, 4, 5, 6} as R = {(a, b): b = a + 1} is reflexive, symmetric or transitive.

Solution:
Given R = {(a, b): b = a + 1}
Now for this relation we have to check whether it is reflexive, transitive and symmetric
Reflexivity:
Let a be an arbitrary element of R.
Then, a = a + 1 cannot be true for all a ∈ A.
⇒ (a, a) ∉ R
So, R is not reflexive on A.
Symmetry:
Let (a, b) ∈ R
⇒ b = a + 1
⇒ −a = −b + 1
⇒ a = b − 1
Thus, (b, a) ∉ R
So, R is not symmetric on A.
Transitivity:
Let (1, 2) and (2, 3) ∈ R
⇒ 2 = 1 + 1 and 3
2 + 1 is true.
But 3 ≠ 1+1
⇒ (1, 3) ∉ R
So, R is not transitive on A.

7. Check whether the relation R on R defined as R = {(a, b): $a \le b^3$} is reflexive, symmetric or transitive.

Solution:
Given R = {(a, b): $a \le b^3$}
It is observed that (1/2, 1/2) in R as $1/2 > (1/2)^3 = 1/8$
∴ R is not reflexive.

Now,
(1, 2) ∈ R (as $1 < 2^3 = 8$)
But,
(2, 1) ∉ R (as $2 > 1^3 = 1$)
∴ R is not symmetric.
We have (3, 3/2), (3/2, 6/5) in "R as" $3 < (3/2)^3$ and $3/2 < (6/5)^3$
But (3, 6/5) ∉ R as $3 > (6/5)^3$
∴ R is not transitive.
Hence, R is neither reflexive, nor symmetric, nor transitive.

8. Prove that every identity relation on a set is reflexive, but the converse is not necessarily true.

Solution:
Let A be a set.
Then, Identity relation IA=I_A is reflexive, since (a, a) ∈ A ∀a
The converse of it need not be necessarily true.
Consider the set A = {1, 2, 3}
Here,
Relation R = {(1, 1), (2, 2) , (3, 3), (2, 1), (1, 3)} is reflexive on A.
However, R is not an identity relation.

9. If A = {1, 2, 3, 4} define relations on A which have properties of being
(i) Reflexive, transitive but not symmetric
(ii) Symmetric but neither reflexive nor transitive.
(iii) Reflexive, symmetric and transitive.

Solution:
(i) The relation on A having properties of being reflexive, transitive, but not symmetric is
R = {(1, 1), (2, 2), (3, 3), (4, 4), (2, 1)}
Relation R satisfies reflexivity and transitivity.
⇒ (1, 1), (2, 2), (3, 3) ∈ R
And (1, 1), (2, 1) ∈ R ⇒ (1, 1) ∈ R
However, (2, 1) ∈ R, but (1, 2) ∉ R

(ii) The relation on A having properties of being reflexive, transitive, but not symmetric is

R = {(1, 1), (2, 2), (3, 3), (4, 4), (2, 1)}
Relation R satisfies reflexivity and transitivity.
⇒ (1, 1), (2, 2), (3, 3) ∈ R
And (1, 1), (2, 1) ∈ R ⇒ (1, 1) ∈ R
However, (2, 1) ∈ R, but (1, 2) ∉ R

(iii) The relation on A having properties of being symmetric, reflexive and transitive is
R = {(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1)}
The relation R is an equivalence relation on A.