EXERCISE 2.3 PAGE NO: 2.54

1. Find fog and gof, if
(i) $f(x) = e^x$, $g(x) = \log_e x$
(ii) $f(x) = x^2$, $g(x) = \cos x$
(iii) $f(x) = |x|$, $g(x) = \sin x$
(iv) $f(x) = x+1$, $g(x) = e^x$
(v) $f(x) = \sin^{-1} x$, $g(x) = x^2$
(vi) $f(x) = x+1$, $g(x) = \sin x$
(vii) $f(x) = x + 1$, $g(x) = 2x + 3$
(viii) $f(x) = c$, $c \in R$, $g(x) = \sin x^2$
(ix) $f(x) = x^2 + 2$, $g(x) = 1 - 1/(1-x)$

Solution:
(i) Given $f(x) = e^x$, $g(x) = \log_e x$
Let f: R → (0, ∞); and g: (0, ∞) → R
Now we have to calculate fog,
Clearly, the range of g is a subset of the domain of f.
fog: (0, ∞) → R
(fog) (x) = f (g (x))
$= f(\log_e x)$
$= \log_e e^x$
= x
Now we have to calculate gof,
Clearly, the range of f is a subset of the domain of g.
⇒ fog: R→ R
(gof) (x) = g (f (x))
$= g(e^x)$
$= \log_e e^x$
= x

(ii) $f(x) = x^2$, $g(x) = \cos x$
f: R→ [0, ∞) ; g: R→[−1, 1]
Now we have to calculate fog,
Clearly, the range of g is not a subset of the domain of f.
⇒ Domain (fog) = {x: x ∈ domain of g and g (x) ∈ domain of f}
⇒ Domain (fog) = x: x ∈ R and cos x ∈ R}

⇒ Domain of (fog) = R
(fog): R→ R
(fog) (x) = f (g (x))
= f (cos x)
= $\cos^2 x$
Now we have to calculate gof,
Clearly, the range of f is a subset of the domain of g.
⇒ fog: R→R
(gof) (x) = g (f (x))
= g (x^2)
= cos x^2

(iii) Given f (x) = |x|, g(x) = sin x
f: R → (0, ∞) ; g : R→[−1, 1]
Now we have to calculate fog,
Clearly, the range of g is a subset of the domain of f.
⇒ fog: R→R
(fog) (x) = f (g (x))
= f (sin x)
= |sin x|
Now we have to calculate gof,
Clearly, the range of f is a subset of the domain of g.
⇒ fog : R→ R
(gof) (x) = g (f (x))
= g (|x|)
= sin |x|

(iv) Given f (x) = x + 1, g(x) = e^x
f: R→R ; g: R → [1, ∞)
Now we have calculate fog:
Clearly, range of g is a subset of domain of f.
⇒ fog: R→R
(fog) (x) = f (g (x))
= f (e^x)
= e^x + 1
Now we have to compute gof,
Clearly, range of f is a subset of domain of g.

⇒ fog: R→R
(gof) (x) = g (f (x))
= g (x+1)
= e^{x+1}

(v) Given f (x) = $\sin^{-1} x$, g(x) = x^2
f: [−1,1]→ [(-π)/2 ,π/2]; g : R → [0, ∞)
Now we have to compute fog:
Clearly, the range of g is not a subset of the domain of f.
Domain (fog) = {x: x ∈ domain of g and g (x) ∈ domain of f}
Domain (fog) = {x: x ∈ R and x^2 ∈ [−1, 1]}
Domain (fog) = {x: x ∈ R and x ∈ [−1, 1]}
Domain of (fog) = [−1, 1]
fog: [−1,1] → R
(fog) (x) = f (g (x))
= f (x^2)
= $\sin^{-1} (x^2)$
Now we have to compute gof:
Clearly, the range of f is a subset of the domain of g.
fog: [−1, 1] → R
(gof) (x) = g (f (x))
= g ($\sin^{-1} x$)
= $(\sin^{-1} x)^2$

(vi) Given f(x) = x+1, g(x) = sin x
f: R→R ; g: R→[−1, 1]
Now we have to compute fog
Set of the domain of f.
⇒ fog: R→ R
(fog) (x) = f (g (x))
= f (sin x)
= sin x + 1
Now we have to compute gof,
Clearly, the range of f is a subset of the domain of g.
⇒ fog: R → R
(gof) (x) = g (f (x))
= g (x+1)

= sin (x+1)

(vii) Given f (x) = x+1, g (x) = 2x + 3

f: R→R ; g: R → R

Now we have to compute fog

Clearly, the range of g is a subset of the domain of f.

⇒ fog: R→ R

(fog) (x) = f (g (x))

= f (2x+3)

= 2x + 3 + 1

= 2x + 4

Now we have to compute gof

Clearly, the range of f is a subset of the domain of g.

⇒ fog: R → R

(gof) (x) = g (f (x))

= g (x+1)

= 2 (x + 1) + 3

= 2x + 5

(viii) Given f (x) = c, g (x) = sin x^2

f: R → {c} ; g: R→ [0, 1]

Now we have to compute fog

Clearly, the range of g is a subset of the domain of f.

fog: R→R

(fog) (x) = f (g (x))

= f (sin x^2)

= c

Now we have to compute gof,

Clearly, the range of f is a subset of the domain of g.

⇒ fog: R→ R

(gof) (x) = g (f (x))

= g (c)

= sin c^2

(ix) Given f (x) = x^2+ 2 and g (x) = 1 - 1 / (1 - x)

f: R → [2, ∞)

For domain of g: 1− x ≠ 0

⇒ $x \neq 1$
⇒ Domain of g = R − {1}
g (x)= 1 - 1/ (1 - x) = (1 – x - 1)/ (1 - x) = (-x)/(1 - x)
For range of g
y = (- x)/ (1 - x)
⇒ y – x y = − x
⇒ y = x y − x
⇒ y = x (y−1)
⇒ x = y/(y - 1)
Range of g = R − {1}
So, g: R − {1} → R − {1}
Now we have to compute fog
Clearly, the range of g is a subset of the domain of f.
⇒ fog: R − {1} → R
(fog) (x) = f (g (x))
= f ((-x)/ (x - 1))
= $((-x)/ (x-1))^2 + 2$
= $(x^2 + 2x^2 + 2 - 4x) / (1 - x)^2$
`= $(3x^2 - 4x + 2)/ (1 - x)^2$
Now we have to compute gof
Clearly, the range of f is a subset of the domain of g.
⇒ gof: R→R
(gof) (x) = g (f (x))
= $g (x^2 + 2)$
= $1 - 1 / (1 - (x^2 + 2))$
= $- 1/ (1 - (x^2 + 2))$
= $(x^2 + 2)/ (x^2 + 1)$

2. Let $f(x) = x^2 + x + 1$ and g(x) = sin x. Show that fog ≠ gof.

Solution:
Given $f(x) = x^2 + x + 1$ and g(x) = sin x
Now we have to prove fog ≠ gof
(fog) (x) = f (g (x))
= f (sin x)
= $\sin^2 x + \sin x + 1$
And (gof) (x) = g (f (x))

$= g(x^2 + x + 1)$
$= \sin(x^2 + x + 1)$
So, fog ≠ gof.

3. If f(x) = |x|, prove that fof = f.

Solution:
Given f(x) = |x|,
Now we have to prove that fof = f.
Consider (fof) (x) = f (f (x))
= f (|x|)
= ||x||
= |x|
= f (x)
So,
(fof) (x) = f (x), ∀x ∈ R
Hence, fof = f

4. If f(x) = 2x + 5 and g(x) = x^2 + 1 be two real functions, then describe each of the following functions:
(i) fog
(ii) gof
(iii) fof
(iv) f^2
Also, show that fof ≠ f^2

Solution:
f(x) and g(x) are polynomials.
⇒ f: R → R and g: R → R.
So, fog: R → R and gof: R → R.
(i) (fog) (x) = f (g (x))
$= f(x^2 + 1)$
$= 2(x^2 + 1) + 5$
$= 2x^2 + 2 + 5$
$= 2x^2 + 7$

(ii) (gof) (x) = g (f (x))

= g (2x +5)
= g $(2x + 5)^2 + 1$
= $4x^2 + 20x + 26$

(iii) (fof) (x) = f (f (x))
= f (2x +5)
= 2 (2x + 5) + 5
= 4x + 10 + 5
= 4x + 15

(iv) f^2 (x) = f (x) x f (x)
= (2x + 5) (2x + 5)
= $(2x + 5)^2$
= $4x^2 + 20x + 25$

5. If f(x) = sin x and g(x) = 2x be two real functions, then describe gof and fog. Are these equal functions?

Solution:
Given f(x) = sin x and g(x) = 2x
We know that
f: R→ [−1, 1] and g: R→ R
Clearly, the range of f is a subset of the domain of g.
gof: R→ R
(gof) (x) = g (f (x))
= g (sin x)
= 2 sin x
Clearly, the range of g is a subset of the domain of f.
fog: R → R
So, (fog) (x) = f (g (x))
= f (2x)
= sin (2x)
Clearly, fog ≠ gof
Hence they are not equal functions.

6. Let f, g, h be real functions given by f(x) = sin x, g (x) = 2x and h (x) = cos x. Prove

that fog = go (f h).

Solution:

Given that f(x) = sin x, g (x) = 2x and h (x) = cos x

We know that f: R→ [−1, 1] and g: R→ R

Clearly, the range of g is a subset of the domain of f.

fog: R → R

Now, (f h) (x) = f (x) h (x) = (sin x) (cos x) = ½ sin (2x)

Domain of f h is R.

Since range of sin x is [-1, 1], −1 ≤ sin 2x ≤ 1

⇒ -1/2 ≤ sin x/2 ≤ 1/2

Range of f h = [-1/2, 1/2]

So, (f h): R → [(-1)/2, 1/2]

Clearly, range of f h is a subset of g.

⇒ go (f h): R → R

⇒ Domains of fog and go (f h) are the same.

So, (fog) (x) = f (g (x))

= f (2x)

= sin (2x)

And (go (f h)) (x) = g ((f h) (x))

= g (sin x cos x)

= 2sin x cos x

= sin (2x)

⇒ (fog) (x) = (go (f h)) (x), ∀x ∈ R

Hence, fog = go (f h)