EXERCISE 3.1 PAGE NO: 3.4

1. Determine whether the following operation define a binary operation on the given set or not:
(i) '*' on N defined by a * b = a^b for all a, b ∈ N.
(ii) 'O' on Z defined by a O b = a^b for all a, b ∈ Z.
(iii) '*' on N defined by a * b = a + b - 2 for all a, b ∈ N
(iv) '$\times_6$' on S = {1, 2, 3, 4, 5} defined by a $\times_6$ b = Remainder when a b is divided by 6.
(v) '$+_6$' on S = {0, 1, 2, 3, 4, 5} defined by a $+_6$ b

$$= \begin{cases} a+b, \; if \; a+b<6 \\ a+b-6, \; if \; a+b \geq 6 \end{cases}$$

(vi) '⊙' on N defined by a ⊙ b= $a^b + b^a$ for all a, b ∈ N
(vii) '*' on Q defined by a * b = (a – 1)/ (b + 1) for all a, b ∈ Q

Solution:
(i) Given '*' on N defined by a * b = a^b for all a, b ∈ N.
Let a, b ∈ N. Then,
a^b ∈ N [∵ $a^b \neq 0$ and a, b is positive integer]
⇒ a * b ∈ N
Therefore,
a * b ∈ N, ∀ a, b ∈ N
Thus, * is a binary operation on N.

(ii) Given 'O' on Z defined by a O b = a^b for all a, b ∈ Z.
Both a = 3 and b = -1 belong to Z.
⇒ a * b = 3^{-1}
= 1/3 ∉ Z
Thus, * is not a binary operation on Z.

(iii) Given '*' on N defined by a * b = a + b - 2 for all a, b ∈ N
If a = 1 and b = 1,
a * b = a + b - 2
= 1 + 1 - 2
= 0 ∉ N
Thus, there exist a = 1 and b = 1 such that a * b ∉ N
So, * is not a binary operation on N.

(iv) Given '$\times_6$' on S = {1, 2, 3, 4, 5} defined by a $\times_6$ b = Remainder when a b is divided by 6.
Consider the composition table,

X_6	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

Here all the elements of the table are not in S.
⇒ For a = 2 and b = 3,
a $\times_6$ b = 2 $\times_6$ 3 = remainder when 6 divided by 6 = 0 ≠ S
Thus, $\times_6$ is not a binary operation on S.

(v) Given '$+_6$' on S = {0, 1, 2, 3, 4, 5} defined by a $+_6$ b

$$= \begin{cases} a+b, \; if \; a+b<6 \\ a+b-6, \; if \; a+b \geq 6 \end{cases}$$

Consider the composition table,

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Here all the elements of the table are not in S.
⇒ For a = 2 and b = 3,

$a \times_6 b = 2 \times_6 3$ = remainder when 6 divided by 6 = 0 ≠
Thus, $\times_6$ is not a binary operation on S.

(vi) Given '$\odot$' on N defined by $a \odot b = a^b + b^a$ for all a, b ∈ N
Let a, b ∈ N. Then,
$a^b, b^a \in N$
$\Rightarrow a^b + b^a \in N$ [∵Addition is binary operation on N]
$\Rightarrow a \odot b \in N$
Thus, $\odot$ is a binary operation on N.

(vii) Given '*' on Q defined by a * b = (a – 1)/ (b + 1) for all a, b ∈ Q
If a = 2 and b = -1 in Q,
a * b = (a – 1)/ (b + 1)
= (2 – 1)/ (- 1 + 1)
= 1/0 [which is not defined]
For a = 2 and b = -1
a * b does not belongs to Q
So, * is not a binary operation in Q.

2. Determine whether or not the definition of * given below gives a binary operation. In the event that * is not a binary operation give justification of this.
(i) On Z^+, defined * by a * b = a – b
(ii) On Z^+, define * by a*b = ab
(iii) On R, define * by $a*b = ab^2$
(iv) On Z^+ define * by a * b = |a – b|
(v) On Z^+ define * by a * b = a
(vi) On R, define * by $a * b = a + 4b^2$
Here, Z^+ denotes the set of all non-negative integers.

Solution:
(i) Given On Z^+, defined * by a * b = a – b
If a = 1 and b = 2 in Z^+, then
a * b = a – b
= 1 – 2
= $-1 \notin Z^+$ [because Z^+ is the set of non-negative integers]
For a = 1 and b = 2,
$a * b \notin Z^+$

Thus, * is not a binary operation on Z^+.

(ii) Given Z^+, define * by a*b = a b
Let a, b ∈ Z^+
⇒ a, b ∈ Z^+
⇒ a * b ∈ Z^+
Thus, * is a binary operation on R.

(iii) Given on R, define by a*b = ab^2
Let a, b ∈ R
⇒ a, b^2 ∈ R
⇒ ab^2 ∈ R
⇒ a * b ∈ R
Thus, * is a binary operation on R.

(iv) Given on Z^+ define * by a * b = |a − b|
Let a, b ∈ Z^+
⇒ | a – b | ∈ Z^+
⇒ a * b ∈ Z^+
Therefore,
a * b ∈ Z^+, ∀ a, b ∈ Z^+
Thus, * is a binary operation on Z^+.

(v) Given on Z^+ define * by a * b = a
Let a, b ∈ Z^+
⇒ a ∈ Z^+
⇒ a * b ∈ Z^+
Therefore, a * b ∈ Z^+ ∀ a, b ∈ Z^+
Thus, * is a binary operation on Z^+.

(vi) Given On R, define * by a * b = a + $4b^2$
Let a, b ∈ R
⇒ a, $4b^2$ ∈ R
⇒ a + $4b^2$ ∈ R
⇒ a * b ∈ R
Therefore, a *b ∈ R, ∀ a, b ∈ R
Thus, * is a binary operation on R.

3. Let * be a binary operation on the set I of integers, defined by a * b = 2a + b − 3. Find the value of 3 * 4.

Solution:

Given $a * b = 2a + b - 3$

$3 * 4 = 2(3) + 4 - 3$

$= 6 + 4 - 3$

$= 7$

4. Is * defined on the set {1, 2, 3, 4, 5} by a * b = LCM of a and b a binary operation? Justify your answer.

Solution:

LCM	1	2	3	4	5
1	1	2	3	4	5
2	2	2	6	4	10
3	3	5	3	12	15
4	4	4	12	4	20
5	5	10	15	20	5

In the given composition table, all the elements are not in the set {1, 2, 3, 4, 5}.

If we consider a = 2 and b = 3, a * b = LCM of a and b = 6 $\notin$ {1, 2, 3, 4, 5}.

Thus, * is not a binary operation on {1, 2, 3, 4, 5}.

5. Let S = {a, b, c}. Find the total number of binary operations on *S*.

Solution:

Number of binary operations on a set with n elements is n^{n^2}

Here, S = {a, b, c}

Number of elements in S = 3

Number of binary operations on a set with 3 elements is 3^{3^2}