EXERCISE 3.4 PAGE NO: 3.25

1. Let * be a binary operation on Z defined by a * b = a + b – 4 for all a, b ∈ Z.
(i) Show that * is both commutative and associative.
(ii) Find the identity element in Z
(iii) Find the invertible element in Z.

Solution:
(i) First we have to prove commutativity of *
Let a, b ∈ Z. then,
a * b = a + b – 4
= b + a – 4
= b * a
Therefore,
a * b = b * a, ∀ a, b ∈ Z
Thus, * is commutative on Z.
Now we have to prove associativity of Z.
Let a, b, c ∈ Z. then,
a * (b * c) = a * (b + c - 4)
= a + b + c -4 – 4
= a + b + c – 8
(a * b) * c = (a + b – 4) * c
= a + b – 4 + c – 4
= a + b + c – 8
Therefore,
a * (b * c) = (a * b) * c, for all a, b, c ∈ Z
Thus, * is associative on Z.

(ii) Let e be the identity element in Z with respect to * such that
a * e = a = e * a ∀ a ∈ Z
a * e = a and e * a = a, ∀ a ∈ Z
a + e – 4 = a and e + a – 4 = a, ∀ a ∈ Z
e = 4, ∀ a ∈ Z
Thus, 4 is the identity element in Z with respect to *.

(iii) Let a ∈ Z and b ∈ Z be the inverse of a. Then,

a * b = e = b * a
a * b = e and b * a = e
a + b – 4 = 4 and b + a – 4 = 4
b = 8 – a ∈ Z
Thus, 8 – a is the inverse of a ∈ Z

2. Let * be a binary operation on Q_0 (set of non-zero rational numbers) defined by a * b = (3ab/5) for all a, b ∈ Q_0. Show that * is commutative as well as associative. Also, find its identity element, if it exists.

Solution:
First we have to prove commutativity of *
Let a, b ∈ Q_0
a * b = (3ab/5)
= (3ba/5)
= b * a
Therefore, a * b = b * a, for all a, b ∈ Q_0
Now we have to prove associativity of *
Let a, b, c ∈ Q_0
a * (b * c) = a * (3bc/5)
= [a (3 bc/5)] /5
= 3 abc/25
(a * b) * c = (3 ab/5) * c
= [(3 ab/5) c]/ 5
= 3 abc /25
Therefore a * (b * c) = (a * b) * c, for all a, b, c ∈ Q_0
Thus * is associative on Q_0
Now we have to find the identity element
Let e be the identity element in Z with respect to * such that
a * e = a = e * a ∀ a ∈ Q_0
a * e = a and e * a = a, ∀ a ∈ Q_0
3ae/5 = a and 3ea/5 = a, ∀ a ∈ Q_0
e = 5/3 ∀ a ∈ Q_0 [because a is not equal to 0]
Thus, 5/3 is the identity element in Q_0 with respect to *.

3. Let * be a binary operation on Q – {-1} defined by a * b = a + b + ab for all a, b ∈ Q – {-1}. Then,

(i) Show that * is both commutative and associative on Q – {-1}
(ii) Find the identity element in Q – {-1}
(iii) Show that every element of Q – {-1} is invertible. Also, find inverse of an arbitrary element.

Solution:
(i) First we have to check commutativity of *
Let a, b ∈ Q – {-1}
Then a * b = a + b + ab
= b + a + ba
= b * a
Therefore,
a * b = b * a, ∀ a, b ∈ Q – {-1}
Now we have to prove associativity of *
Let a, b, c ∈ Q – {-1}, Then,
a * (b * c) = a * (b + c + b c)
= a + (b + c + b c) + a (b + c + b c)
= a + b + c + b c + a b + a c + a b c
(a * b) * c = (a + b + a b) * c
= a + b + a b + c + (a + b + a b) c
= a + b + a b + c + a c + b c + a b c
Therefore,
a * (b * c) = (a * b) * c, ∀ a, b, c ∈ Q – {-1}
Thus, * is associative on Q – {-1}.

(ii) Let e be the identity element in I^+ with respect to * such that
a * e = a = e * a, ∀ a ∈ Q – {-1}
a * e = a and e * a = a, ∀ a ∈ Q – {-1}
a + e + ae = a and e + a + ea = a, ∀ a ∈ Q – {-1}
e + ae = 0 and e + ea = 0, ∀ a ∈ Q – {-1}
e (1 + a) = 0 and e (1 + a) = 0, ∀ a ∈ Q – {-1}
e = 0, ∀ a ∈ Q – {-1} [because a not equal to -1]
Thus, 0 is the identity element in Q – {-1} with respect to *.

(iii) Let a ∈ Q – {-1} and b ∈ Q – {-1} be the inverse of a. Then,
a * b = e = b * a
a * b = e and b * a = e

a + b + ab = 0 and b + a + ba = 0
b (1 + a) = - a Q – {-1}
b = -a/1 + a Q – {-1} [because a not equal to -1]
Thus, -a/1 + a is the inverse of a ∈ Q – {-1}

4. Let A = R_0 × R, where R_0 denote the set of all non-zero real numbers. A binary operation 'O' is defined on A as follows: (a, b) O (c, d) = (ac, bc + d) for all (a, b), (c, d) ∈ R_0 × R.
(i) Show that 'O' is commutative and associative on A
(ii) Find the identity element in A
(iii) Find the invertible element in A.

Solution:
(i) Let X = (a, b) and Y = (c, d) ∈ A, ∀ a, c ∈ R_0 and b, d ∈ R
Then, X O Y = (ac, bc + d)
And Y O X = (ca, da + b)
Therefore,
X O Y = Y O X, ∀ X, Y ∈ A
Thus, O commutative on A.
Now we have to check associativity of O
Let X = (a, b), Y = (c, d) and Z = (e, f), ∀ a, c, e ∈ R_0 and b, d, f ∈ R
X O (Y O Z) = (a, b) O (ce, de + f)
= (ace, bce + de + f)
(X O Y) O Z = (ac, bc + d) O (e, f)
= (ace, (bc + d) e + f)
= (ace, bce + de + f)
Therefore, X O (Y O Z) = (X O Y) O Z, ∀ X, Y, Z ∈ A

(ii) Let E = (x, y) be the identity element in A with respect to O, ∀ x ∈ R_0 and y ∈ R
Such that,
X O E = X = E O X, ∀ X ∈ A
X O E = X and EOX = X
(ax, bx +y) = (a, b) and (xa, ya + b) = (a, b)
Considering (ax, bx + y) = (a, b)
ax = a
x = 1
And bx + y = b

y = 0 [since x = 1]
Considering (xa, ya + b) = (a, b)
xa = a
x = 1
And ya + b = b
y = 0 [since x = 1]
Therefore (1, 0) is the identity element in A with respect to O.

(iii) Let F = (m, n) be the inverse in A ∀ m ∈ R_0 and n ∈ R
X O F = E and F O X = E
(am, bm + n) = (1, 0) and (ma, na + b) = (1, 0)
Considering (am, bm + n) = (1, 0)
am = 1
m = 1/a
And bm + n = 0
n = -b/a [since m = 1/a]
Considering (ma, na + b) = (1, 0)
ma = 1
m = 1/a
And na + b = 0
n = -b/a
Therefore the inverse of (a, b) ∈ A with respect to O is (1/a, -1/a)