JEE Main 2020 Paper

Date of Exam: **8th January 2020 (Shift 1)**
Time: **9:30 A.M. to 12:30 P.M.**
Subject: **Mathematics**

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1. For which of the following ordered pairs (μ, δ), the system of linear equations
$x + 2y + 3z = 1$
$3x + 4y + 5z = \mu$
$4x + 4y + 4z = \delta$
is inconsistent?

a. $(4,\ 6)$
b. $(3,\ 4)$
c. $(1,\ 0)$
d. $(4,\ 3)$

Answer: (d)

Solution:

$$D = \begin{vmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 4 & 4 & 4 \end{vmatrix}$$

$R_3 \to R_3 - 2R_1 + 2R_2$

$$D = \begin{vmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

For inconsistent system, one of $D_x,\ D_y,\ D_z$ should not be equal to 0

$$D_x = \begin{vmatrix} \mu & 4 & 5 \\ 1 & 2 & 3 \\ \delta & 4 & 4 \end{vmatrix} \qquad D_y = \begin{vmatrix} 3 & \mu & 5 \\ 1 & 1 & 3 \\ 4 & \delta & 4 \end{vmatrix}$$

$$D_z = \begin{vmatrix} 3 & 4 & \mu \\ 1 & 2 & 1 \\ 4 & 4 & \delta \end{vmatrix}$$

For inconsistent system, $2\mu \neq \delta + 2$

$\therefore$ The system will be inconsistent for $\mu = 4,\ \delta = 3$.

2. Let $y = y(x)$ be a solution of the differential equation, $\sqrt{1-x^2}\,\frac{dy}{dx} + \sqrt{1-y^2} = 0, |x| < 1$. If $y\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}$, then $y\left(\frac{-1}{\sqrt{2}}\right)$ is equal to :

a. $-\frac{1}{\sqrt{2}}$
b. $-\frac{\sqrt{3}}{2}$
c. $\frac{1}{\sqrt{2}}$
d. $\frac{\sqrt{3}}{2}$

Answer: (c)

Solution:

$\sqrt{1-x^2}\frac{dy}{dx}+\sqrt{1-y^2}=0$

$\Rightarrow \frac{dy}{\sqrt{1-y^2}}+\frac{dx}{\sqrt{1-x^2}}=0$

$\Rightarrow \sin^{-1} y+\sin^{-1} x=c$

If $x=\frac{1}{2}, y=\frac{\sqrt{3}}{2}$ then,

$\sin^{-1}\frac{\sqrt{3}}{2}+\sin^{-1}\frac{1}{2}=c$

$\therefore \frac{\pi}{3}+\frac{\pi}{6}=c \Rightarrow c=\frac{\pi}{2}$

$\Rightarrow \sin^{-1} y=\frac{\pi}{2}-\sin^{-1} x=\cos^{-1} x$

$\therefore \sin^{-1} y=\cos^{-1}\frac{1}{\sqrt{2}}$

$\Longrightarrow \sin^{-1} y=\frac{\pi}{4}$

$\Longrightarrow y\left(\frac{1}{\sqrt{2}}\right)=\frac{1}{\sqrt{2}}$

3. If a, b and c are the greatest values of ${}^{19}C_p, {}^{20}C_q, {}^{21}C_r$ respectively, then :

a. $\frac{a}{11}=\frac{b}{22}=\frac{c}{42}$

b. $\frac{a}{10}=\frac{b}{11}=\frac{c}{42}$

c. $\frac{a}{11}=\frac{b}{22}=\frac{c}{21}$

d. $\frac{a}{10}=\frac{b}{11}=\frac{c}{21}$

Answer: (a)

Solution:

We know that, ${}^nC_r$ is maximum when $r=\begin{cases}\frac{n}{2}, & n \text{ is even}\\ \frac{n+1}{2} \text{ or } \frac{n-1}{2}, & n \text{ is odd}\end{cases}$

Therefore, $\max\left({}^{19}C_p\right)={}^{19}C_9=a$

$\max\left({}^{20}C_q\right)={}^{20}C_{10}=b$

$\max({}^{21}C_r)={}^{21}C_{11}=c$

$$\therefore \frac{a}{{}^{19}C_9} = \frac{b}{\frac{20}{10} \times {}^{19}C_9} = \frac{c}{\frac{21}{11} \times \frac{20}{10} \times {}^{19}C_9}$$

$$\Rightarrow \frac{a}{1} = \frac{b}{2} = \frac{c}{\frac{42}{11}}$$

$$\Rightarrow \frac{a}{11} = \frac{b}{22} = \frac{c}{42}$$

4. Which of the following is a tautology?

a. $(P \wedge (P \longrightarrow Q)) \longrightarrow Q$
b. $P \wedge (P \vee Q)$
c. $(Q \rightarrow (P \ \wedge (P \rightarrow Q))$
d. $P \vee (P \wedge Q$

Answer: (a)

Solution:

$(P \wedge (P \longrightarrow Q)) \longrightarrow Q$

$= (P \wedge (\backsim P \vee Q)) \longrightarrow Q$

$= [(P \wedge \sim P) \vee (P \wedge Q)] \longrightarrow Q$

$= P \wedge Q \longrightarrow Q$

$= \sim (P \wedge Q) \vee Q$

$= \sim P \vee \sim Q \vee Q$

$= T$

5. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be such that for all $x \in \mathbf{R}$, $(2^{1+x} + 2^{1-x}), f(x)$ and $(3^x + 3^{-x})$ are in A.P., then the minimum value of $f(x)$ is :

a. 0
b. 4
c. 3
d. 2

Answer: (c)

Solution:

$2^{1-x} + 2^{1+x}, f(x), 3^x + 3^{-x}$ are in A.P.

$$\therefore f(x) = \frac{3^x + 3^{-x} + 2^{1+x} + 2^{1-x}}{2} = \frac{(3^x + 3^{-x})}{2} + \frac{2^{1+x} + 2^{1-x}}{2}$$

Applying A.M. ≥ G.M. inequality, we get

$$\frac{(3^x + 3^{-x})}{2} \geq \sqrt{3^x . 3^{-x}}$$

$\Rightarrow \frac{(3^x+3^{-x})}{2} \geq 1 \quad \ldots (1)$

Also, Applying A.M. ≥ G.M. inequality, we get

$$\frac{2^{1+x}+2^{1-x}}{2} \geq \sqrt{2^{1+x}.2^{1-x}}$$

$$\Rightarrow \frac{2^{1+x}+2^{1-x}}{2} \geq 2 \quad \ldots (2)$$

Adding (1) and (2), we get

$f(x) \geq 1+2=3$

Thus, minimum value of $f(x)$ is 3.

6. The locus of a point which divides the line segment joining the point $(0,-1)$ and a point on the parabola, $x^2=4y$, internally in the ratio $1:2$, is :

a. $9x^2-12y=8$

b. $4x^2-3y=2$

c. $x^2-3y=2$

d. $9x^2-3y=2$

Answer: (a)

Solution:

Let point P be $(2t, t^2)$ and Q be (h, k).

$$h=\frac{2t}{3}, k=\frac{-2+t^2}{3}$$

Now, eliminating t from the above equations we get:

$$3k+2=\left(\frac{3h}{2}\right)^2$$

Replacing h and k by x and y, we get the locus of the curve as $9x^2-12y=8$.

7. For $a>0$, let the curves $C_1: y^2=ax$ and $C_2: x^2=ay$ intersect at origin O and a point P. Let the line $x=b$ $(0<b<a)$ intersect the chord OP and the x-axis at points Q and R, respectively. If the line $x=b$ bisects the area bounded by the curves, C_1 and C_2, and the area of $\Delta OQR=\frac{1}{2}$, then $'a'$ satisfies the equation :

a. $x^6-12x^3+4=0$

b. $x^6-12x^3-4=0$

c. $x^6+6x^3-4=0$

d. $x^6-6x^3+4=0$

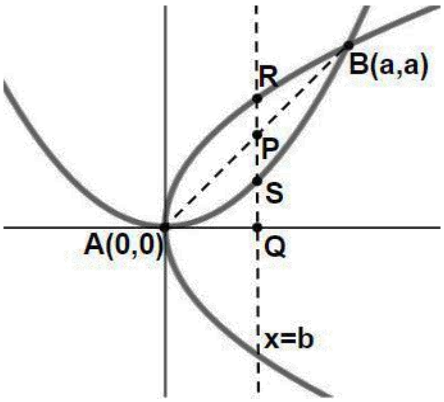
Answer: (a)

Solution:

Given, $ar(\Delta APQ)=\frac{1}{2}$

$$\Rightarrow \frac{1}{2}\times b\times b=\frac{1}{2}$$

$$\Rightarrow b=1$$



As per the question

$$\Rightarrow \int_0^1\left(\sqrt{ax}-\frac{x^2}{a}\right)dx=\frac{1}{2}\int_0^a\left(\sqrt{ax}-\frac{x^2}{a}\right)dx$$
$$\Rightarrow \frac{2}{3}\sqrt{a}-\frac{1}{3a}=\frac{a^2}{6}$$
$$\Rightarrow 2a\sqrt{a}-1=\frac{a^3}{2}$$
$$\Rightarrow 4a\sqrt{a}=2+a^3$$
$$\Rightarrow 16a^3=4+a^6+4a^3$$
$$\Rightarrow a^6-12a^3+4=0.$$

8. The inverse function of $f(x)=\frac{8^{2x}-8^{-2x}}{8^{2x}+8^{-2x}}$, $x\in(-1,1)$, is

a. $\frac{1}{4}(\log_8 e)\log_e\left(\frac{1-x}{1+x}\right)$

b. $\frac{1}{4}(\log_8 e)\log_e\left(\frac{1+x}{1-x}\right)$

c. $\frac{1}{4}\log_e\left(\frac{1+x}{1-x}\right)$

d. $\frac{1}{4}\log_e\left(\frac{1-x}{1+x}\right)$

Answer: (b)

Solution:

$$f(x)=\frac{8^{2x}-8^{-2x}}{8^{2x}+8^{-2x}}=\frac{8^{4x}-1}{8^{4x}+1}$$

Put $y=\frac{8^{4x}-1}{8^{4x}+1}$

Applying componendo-dividendo on both sides

$$\frac{y+1}{y-1}=\frac{2\times 8^{4x}}{-2}$$

$$\frac{y+1}{y-1}=-8^{4x}\Rightarrow 8^{4x}=\frac{1+y}{1-y}$$

$$\Rightarrow x=\frac{1}{4}\log_8\left(\frac{1+y}{1-y}\right)$$

$$f^{-1}(x)=\frac{1}{4}\log_8\left(\frac{1+x}{1-x}\right)=\frac{1}{4}\log_8 e\left(\log_e\frac{1+x}{1-x}\right)$$

9. $\lim_{x\to 0}\left(\frac{3x^2+2}{7x^2+2}\right)^{\frac{1}{x^2}}$ is equal to :

a. e

b. $\frac{1}{e^2}$

c. $\frac{1}{e}$

d. e^2

Answer: (b)

Solution:

Let $L=\lim_{x\to 0}\left(\frac{3x^2+2}{7x^2+2}\right)^{\frac{1}{x^2}}$ [Intermediate form 1^∞]

$$\therefore L=e^{\lim_{x\to 0}\frac{1}{x^2}\left(\frac{3x^2+2}{7x^2+2}-1\right)}$$

$$=e^{\lim_{x\to 0}\frac{1}{x^2}\left(-\frac{4x^2}{7x^2+2}\right)}$$

$$=\frac{1}{e^2}$$

10. Let $f(x)=(\sin(\tan^{-1}x)+\ \sin(\cot^{-1}x))^2-1$, where $|x|>1$.
If $\frac{dy}{dx}=\frac{1}{2}\frac{d}{dx}(\sin^{-1}(f(x)))$ and $y(\sqrt{3})=\frac{\pi}{6}$, then $y(-\sqrt{3})$ is equal to :

a. $\frac{\pi}{3}$

b. $\frac{2\pi}{3}$

c. $-\frac{\pi}{6}$

d. $\frac{5\pi}{6}$

Answer: (Bonus)

Solution:

$f(x) = [\sin(\tan^{-1} x) + \sin(\cot^{-1} x)]^2 - 1$

Put $\tan^{-1} x = \phi$, where $\phi \in \left(-\frac{\pi}{2}, -\frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$= [\sin(\tan^{-1} x) + \sin(\cot^{-1} x)]^2 - 1 = [\sin\phi + \cos\phi]^2 - 1$

$= 1 + 2\sin\phi\cos\phi - 1 = \sin 2\phi = \frac{2x}{1+x^2}$

It is given that $\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}(\sin^{-1}(f(x)))$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{1+x^2}$, for $|x| > 1$

$|x| > 1 \Rightarrow x > 1$ and $x < -1$

To get the value of $y(-\sqrt{3})$, we have to integrate the value of $\frac{dy}{dx}$. To integrate the expression, the interval should be continuous. Therefore, we have to integrate the expression in both the intervals.

$\Rightarrow y = -\tan^{-1} x + C_1$, for $x > 1$ and $y = -\tan^{-1} x + C_2$, for $x < -1$

For $x > 1$, $C_1 = \frac{\pi}{2} \because y(\sqrt{3}) = \frac{\pi}{6}$ is given.

But C_2 can't be determined as no other information is given for $x < -1$. Therefore, all the options can be true as C_2 can't be determined.

11. If the equation, $x^2 + bx + 45 = 0$ $(b \in \boldsymbol{R})$ has conjugate complex roots and they satisfy $|z + 1| = 2\sqrt{10}$, then :

a. $b^2 + b = 12$

b. $b^2 - b = 42$

c. $b^2 - b = 30$

d. $b^2 + b = 72$

Answer: (c)

Solution:

Given $x^2 + bx + 45 = 0, b \in \boldsymbol{R}$, let roots of the equation be $p \pm iq$

Then, sum of roots $= 2p = -b$

Product of roots $= p^2 + q^2 = 45$

As $p \pm iq$ lies on $|z + 1| = 2\sqrt{10}$, we get

$(p + 1)^2 + q^2 = 40$

$\Rightarrow p^2 + q^2 + 2p + 1 = 40$

$\Rightarrow 45 - b + 1 = 40$

$\Rightarrow b = 6$

$\Rightarrow b^2 - b = 30.$

12. The mean and standard deviation (s.d.) of 10 observations are 20 and 2 respectively. Each of these 10 observations is multiplied by p and then reduced by q, where $p \neq 0$ and $q \neq 0$. If the new mean and standard deviation become half of their original values, then q is equal to:

a. -20 b. -5

c. 10 d. -10

Answer: (a)

Solution:

If mean $\bar{x}$ is multiplied by p and then q is subtracted from it,

then new mean $\bar{x}' = p\bar{x} - q$

$\therefore \bar{x}' = \frac{1}{2}\bar{x}$ and $\bar{x} = 10$

$\Rightarrow 10 = 20p - q \quad \ldots (1)$

If standard deviation is multiplied by p, new standard deviation (σ') is $|p|$ times of the initial standard deviation (σ).

$\sigma' = |p|\sigma$

$\Rightarrow \frac{1}{2}\sigma = |p|\sigma \Rightarrow |p| = \frac{1}{2}$

If $p = \frac{1}{2}$, $q = 0$

If $p = -\frac{1}{2}$, $q = -20.$

13. If $\int \frac{\cos x}{\sin^3 x (1+\sin^6 x)^{\frac{2}{3}}} dx = f(x)(1+\sin^6 x)^{\frac{1}{\lambda}} + c$, where c is a constant of integration, then $\lambda f\left(\frac{\pi}{3}\right)$ is equal to :

a. $-\frac{9}{8}$ b. $\frac{9}{8}$

c. 2 d. -2

Answer: (d)

Solution:

Let $\sin x = t \Rightarrow \cos x \, dx = dt$

$$\therefore \int \frac{dt}{t^3\,(1+t^6\,)^{\frac{2}{3}}} = \int \frac{dt}{t^7\left(1+\frac{1}{t^6}\right)^{\frac{2}{3}}}$$

Let $1+\frac{1}{t^6} = u \Rightarrow -6t^{-7}dt = du$

$$\Rightarrow \int \frac{dt}{t^7\left(1+\frac{1}{t^6}\right)^{\frac{2}{3}}} = -\frac{1}{6}\int \frac{du}{u^{\frac{2}{3}}} = -\frac{3}{6}u^{\frac{1}{3}} + c = -\frac{1}{2}\left(1+\frac{1}{t^6}\right)^{\frac{1}{3}} + c$$

$$= -\frac{(1+\sin^6 x\,)^{\frac{1}{3}}}{2\sin^2 x} + c = f(x)(1+\sin^6 x)^{\frac{1}{\lambda}}$$

$\therefore \lambda = 3$ and $f(x) = -\frac{1}{2\sin^2 x}$

$\Rightarrow \lambda f\left(\frac{\pi}{3}\right) = -2.$

14. Let A and B be two independent events such that $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{6}$. Then, which of the following is TRUE ?

a. $P(A/(A\cup B)) = \frac{1}{4}$

b. $P(A/B') = \frac{1}{3}$

c. $P(A/B) = \frac{2}{3}$

d. $P(A'/B') = \frac{1}{3}$

Answer: (b)

Solution:

If X and Y are independent events, then

$$P\left(\frac{X}{Y}\right) = \frac{P(X\cap Y)}{P(Y)} = \frac{P(X)P(Y)}{P(Y)} = P(X)$$

Therefore, $P\left(\frac{A}{B}\right) = P(A) = \frac{1}{3} \Longrightarrow P\left(\frac{A}{B'}\right) = P(A) = \frac{1}{3}$

15. If volume of parallelopiped whose coterminous edges are given by $\vec{u} = \hat{\imath}+\hat{\jmath}+\lambda\,\hat{k}$, $\vec{v} == \hat{\imath}+\hat{\jmath}+3\hat{k}$ and $\vec{w} = 2\hat{\imath}+\hat{\jmath}+\hat{k}$ be 1 cu. unit. If θ be the angle between the edges $\vec{u}$ and $\vec{w}$, then, $\cos\theta$ can be :

a. $\frac{7}{6\sqrt{6}}$

b. $\frac{5}{7}$

c. $\frac{7}{6\sqrt{3}}$

d. $\frac{5}{3\sqrt{3}}$

Answer: (c)

Solution:

Volume of parallelepiped = $[\vec{u}\ \ \vec{v}\ \ \vec{w}]$

$\Rightarrow \begin{vmatrix} 1 & 1 & \lambda \\ 1 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = \pm 1$

$\Rightarrow \lambda = 2$ or 4

For $\lambda = 4$,

$\cos\theta = \frac{2+1+4}{\sqrt{6}\sqrt{18}} = \frac{7}{6\sqrt{3}}$.

16. Let two points be $A(1,\ -1)$ and $B(0,2)$. If a point $\mathrm{P}(x',\ y')$ be such that the area of $\Delta \mathrm{P}AB = 5$ sq. units and it lies on the line, $3x + y - 4\lambda = 0$, then the value of λ is :

a. 4
b. 1
c. -3
d. 3

Answer: (d)

Solution:

Area of triangle is

$A = \frac{1}{2}\begin{vmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ x' & y' & 1 \end{vmatrix} = \pm 5$

$\Rightarrow (2 - y') - x' - 2x' = \pm 10$

$\Rightarrow -3x' - y' + 2 = \pm 10$

$3x' + y' = 12$ or $3x' + y' = -8$

$\Rightarrow \lambda = 3$ or -2

17. The shortest distance between the lines

$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and

$\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ is :

a. $2\sqrt{30}$
b. $\frac{7}{2}\sqrt{30}$
c. 3
d. $3\sqrt{30}$

Answer: (d)

Solution:

$\overrightarrow{AB} = -3\hat{\imath} - 7\hat{\jmath} + 6\hat{k} - \left(3\hat{\imath} + 8\hat{\jmath} + 3\hat{k}\right) = -6\hat{\imath} - 15\hat{\jmath} + 3\hat{k}$

$\vec{p} = \widehat{3\imath} - \hat{\jmath} + \hat{k}$

$\vec{q} = -3\hat{\imath} + 2\hat{\jmath} + 4\hat{k}$

$$\vec{p}\times\vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 1 \\ -3 & 2 & 4 \end{vmatrix} = -6\hat{i} - 15\hat{j} + 9\hat{k}$$

Shortest distance $= \frac{|\overrightarrow{AB}.(\vec{p}\times\vec{q})|}{|\vec{p}\times\vec{q}|} = \frac{|36+225+9|}{\sqrt{36+225+9}} = 3\sqrt{30}.$

18. Let the line $y = mx$ and the ellipse $2x^2 + y^2 = 1$ intersect a point P in the first quadrant. If the normal to this ellipse at P meets the co-ordinate axes at $\left(-\frac{1}{3\sqrt{2}}, 0\right)$ and $(0, \beta)$, then β is equal to :

a. $\frac{2}{\sqrt{3}}$

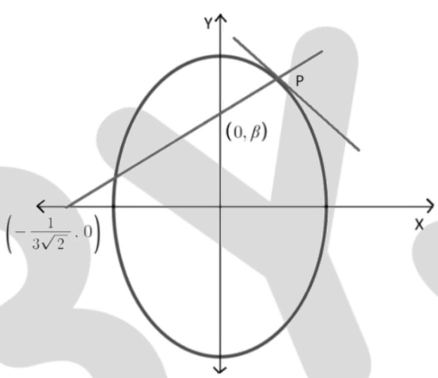
b. $\frac{2}{3}$

c. $\frac{2\sqrt{2}}{3}$

d. $\frac{\sqrt{2}}{3}$

Answer: (d)

Solution:



Let $P \equiv (x_1, y_1)$

$2x^2 + y^2 = 1$ is given equation of ellipse.

$\Rightarrow 4x + 2yy' = 0$

$\Rightarrow y'|_{(x_1,y_1)} = -\frac{2x_1}{y_1}$

Therefore, slope of normal at $P(x_1, y_1)$ is $\frac{y_1}{2x_1}$

Equation of normal at $P(x_1, y_1)$ is

$(y - y_1) = \frac{y_1}{2x_1}(x - x_1)$

It passes through $\left(-\frac{1}{3\sqrt{2}}, 0\right)$

$\Rightarrow -y_1 = \frac{y_1}{2x_1}\left(-\frac{1}{3\sqrt{2}} - x_1\right)$

$\Rightarrow x_1 = \frac{1}{3\sqrt{2}}$

$\Rightarrow y_1 = \frac{2\sqrt{2}}{3}$ as P lies in first quadrant

Since $(0, \beta)$ lies on the normal of the ellipse at point P, hence we get

$\beta = \frac{y_1}{2} = \frac{\sqrt{2}}{3}$

19. If c is a point at which Rolle's theorem holds for the function, $f(x) = \log_e\left(\frac{x^2+\alpha}{7x}\right)$ in the interval $[3, 4]$, where $\alpha \in \mathbf{R}$, then $f''(c)$ is equal to :

a. $-\frac{1}{24}$

b. $\frac{-1}{12}$

c. $\frac{\sqrt{3}}{7}$

d. $\frac{1}{12}$

Answer: (d)

Solution:

Rolle's theorem is applicable on $f(x)$ in $[3, 4]$

$\Rightarrow f(3) = f(4)$

$\Rightarrow \ln\left(\frac{9+\alpha}{21}\right) = \ln\left(\frac{16+\alpha}{28}\right)$

$\Rightarrow \frac{9+\alpha}{21} = \frac{16+\alpha}{28}$

$\Rightarrow 36 + 4\alpha = 48 + 3\alpha \Rightarrow \alpha = 12$

Now, $f(x) = \ln\left(\frac{x^2+12}{7x}\right) \Rightarrow f'(x) = \frac{7x}{x^2+12} \times \frac{7x\times 2x-(x^2+12)\times 7}{(7x)^2}$

$f'(x) = \frac{x^2 - 12}{x(x^2+12)}$

$f'(c) = 0 \Rightarrow c = 2\sqrt{3}$

$f''(x) = \frac{-x^4 + 48x^2 + 144}{x^2(x^2+12)^2}$

$\therefore f''(c) = \frac{1}{12}$

20. Let $f(x) = x\cos^{-1}(\sin(-|x|))$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then which of the following is true ?
   a. $f'(0) = -\frac{\pi}{2}$
   b. f' is decreasing in $\left(-\frac{\pi}{2}, 0\right)$ and increasing in $\left(0, \frac{\pi}{2}\right)$
   c. f is not differentiable at $x = 0$
   d. f' is increasing in $\left(-\frac{\pi}{2}, 0\right)$ and decreasing in $\left(0, \frac{\pi}{2}\right)$

   Answer: (b)

   Solution:

   $f(x) = x\cos^{-1}(\sin(-|x|))$

   $\Rightarrow f(x) = x\cos^{-1}(-\sin|x|)$

   $\Rightarrow f(x) = x[\pi - \cos^{-1}(\sin|x|)]$

   $\Rightarrow f(x) = x\left[\pi - \left(\frac{\pi}{2} - \sin^{-1}(\sin|x|)\right)\right]$

   $\Rightarrow f(x) = x\left(\frac{\pi}{2} + |x|\right)$

   $$\Rightarrow f(x) = \begin{cases} x\left(\frac{\pi}{2} + x\right), & x \geq 0 \\ x\left(\frac{\pi}{2} - x\right), & x < 0 \end{cases}$$

   $$\Rightarrow f'(x) = \begin{cases} \left(\frac{\pi}{2} + 2x\right), & x \geq 0 \\ \left(\frac{\pi}{2} - 2x\right), & x < 0 \end{cases}$$

   Therefore, $f'(x)$ is decreasing $\left(-\frac{\pi}{2}, 0\right)$ and increasing in $\left(0, \frac{\pi}{2}\right)$.

21. An urn contains 5 red marbles, 4 black marbles and 3 white marbles. Then the number of ways in which 4 marbles can be drawn so that at most three of them are red is _________.

   Answer: (490)

   Solution:

   Number of ways to select at most 3 red balls = $P(0 \text{ red balls}) + P(1 \text{ red ball}) + P(2 \text{ red balls})$
   $+P(3 \text{ red balls})$

   $= {}^7C_4 + {}^5C_1 \times {}^7C_3 + {}^5C_2 \times {}^7C_2 + {}^5C_3 \times {}^7C_1$

   $= 35 + 175 + 210 + 70 = 490$

22. Let the normal at a P on the curve $y^2 - 3x^2 + y + 10 = 0$ intersect the y-axis at $\left(0, \frac{3}{2}\right)$. If m is the slope of the tangent at P to the curve, then $|m|$ is equal to__________.

Answer: (4)

Solution:

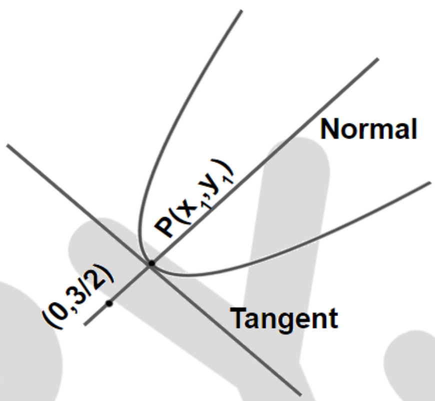
Let co-ordinate of P be (x_1, y_1)

Differentiating the curve w.r.t x

$2yy' - 6x + y' = 0$

Slope of tangent at P

$\Rightarrow \ y' = \frac{6x_1}{1 + 2y_1}$



$\because \ m_{\text{normal}} = \left(\frac{y_1 - \frac{3}{2}}{x_1 - 0}\right)$

$\because \ m_{\text{normal}} \times m_{\text{tangent}} = -1$

$\Rightarrow \frac{\frac{3}{2} - y_1}{-x_1} \times \frac{6x_1}{1+2y_1} = -1$

$\Rightarrow y_1 = 1$

$\Rightarrow x_1 = \pm 2$

Slope of tangent $= \pm\frac{12}{3} = \pm 4$

$\Rightarrow |m| = 4$

23. The least positive value of $'a'$ for which the equation, $2x^2 + (a - 10)x + \frac{33}{2} = 2a$ has real roots is __________.

Answer: (8)

Solution:

$\because 2x^2 + (a-10)x + \frac{33}{2} = 2a, a \in \mathbf{Z}^+$ has real roots

$\Rightarrow D \geq 0 \Rightarrow (a-10)^2 - 4 \times 2 \times \left(\frac{33}{2} - 2a\right) \geq 0$

$\Rightarrow (a-10)^2 - 4(33-4a) \geq 0$

$\Rightarrow a^2 - 4a - 32 \geq 0 \Rightarrow a \in (-\infty, -4] \cup [8, \infty)$

Thus, minimum value of $'a'$ $\forall\, a \in \mathbf{Z}^+$ is 8.

24. The sum $\sum_{k=1}^{20}(1+2+3+.\ldots.+k)$ is ____________.

Answer: (1540)

Solution:

$$= \sum_{k=1}^{20} \frac{k(k+1)}{2}$$

$$= \frac{1}{2}\sum_{k=1}^{20} k^2 + k$$

$$= \frac{1}{2}\left[\frac{20(21)(41)}{6} + \frac{20(21)}{2}\right] = \frac{1}{2}[2870 + 210] = 1540$$

25. The number of all 3×3 matrices A, with entries from the set $\{-1, 0, 1\}$ such that the sum of the diagonal elements of (AA^T) is 3, is __________.

Answer: (672)

Solution:

$tr(AA^T) = 3$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

$tr(AA^T) = a_{11}^2 + a_{12}^2 + a_{13}^2 + a_{21}^2 + a_{22}^2 + a_{23}^2 + a_{31}^2 + a_{32}^2 + a_{33}^2 = 3$

So out of 9 elements (a_{ij})'s, 3 elements must be equal to 1 or -1 and rest elements must be 0.

So, the total possible cases will be

When there is $6(0's)$ and $3(1's)$ then the total possibilities is ${}^9C_6$

For $6(0's)$ and $3(-1's)$ total possibilities is ${}^9C_6$

For $6(0's), 2(1's)$ and $1(-1's)$ total possibilities is ${}^9C_6 \times 3$

For $6(0's), 1(1's)$ and $2(-1's)$ total possibilities is ${}^9C_6 \times 3$

$\therefore$ Total number of cases $= {}^9C_6 \times 8 = 672$