EXERCISE 10.1 PAGE NO: 10.12

1. If in a ΔABC, ∠A = 45º, ∠B = 60º, and ∠C = 75º; find the ratio of its sides.

Solution:

Given: In ΔABC, ∠A = 45º, ∠B = 60º, and ∠C = 75º

By using the sine rule, we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Now by substituting the values we get,

$$\frac{a}{\sin 45°} = \frac{b}{\sin 60°} = \frac{c}{\sin 75°}$$

$$\frac{a}{\sin 45°} = \frac{b}{\sin 60°} = \frac{c}{\sin(30° + 45°)}$$

$$\frac{a}{\sin 45°} = \frac{b}{\sin 60°} = \frac{c}{\sin 30° \cos 45° + \sin 45° \cos 30°}$$

We know, sin (a + b) = sin a cos b + sin b cos a

Now by substituting the corresponding values, we get,

$$\frac{a}{\frac{1}{\sqrt{2}}} = \frac{b}{\frac{\sqrt{3}}{2}} = \frac{c}{\frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}}$$

$$\frac{a}{\frac{1}{\sqrt{2}}} = \frac{b}{\frac{\sqrt{3}}{2}} = \frac{c}{\frac{1 + \sqrt{3}}{2\sqrt{2}}}$$

$$a : b : c = \frac{1}{\sqrt{2}} : \frac{\sqrt{3}}{2} : \frac{1 + \sqrt{3}}{2\sqrt{2}}$$

Multiply the above expression by 2√2, we get

a: b: c = 2: √6: (1+√3)

Hence the ratio of the sides of the given triangle is a: b: c = 2: √6: (1+√3)

2. If in any ΔABC, ∠C = 105º, ∠B = 45º, a = 2, then find b.

Solution:

Given: In ΔABC, ∠C = 105º, ∠B = 45º, a = 2

We know in a triangle,

∠A + ∠B + ∠C = 180°

∠A = 180° - ∠B - ∠C

Substituting the given values, we get
∠A = 180° - 45° - 105°
∠A = 30°
By using the sine rule, we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Now by substituting the corresponding values we get,

$$\frac{2}{\sin 30°} = \frac{b}{\sin 45°}$$

Substitute the equivalent values of the sine, we get

$$\frac{2}{\frac{1}{2}} = \frac{b}{\frac{1}{\sqrt{2}}}$$

$4 = b\sqrt{2}$
$b = 4/\sqrt{2}$
$= 2\sqrt{2}$
Hence the value of b is $2\sqrt{2}$ units.

3. In ΔABC, if a = 18, b = 24 and c = 30 and ∠C = 90°, find sin A, sin B and sin C.
Solution:
Given: In ΔABC, a = 18, b = 24 and c = 30 and ∠C = 90°
By using the sine rule, we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Now by substituting the given values we get,

$$\frac{18}{\sin A} = \frac{30}{\sin 90°}$$

$$\sin A = \frac{18 \times \sin 90°}{30}$$

$$\sin A = \frac{18 \times 1}{30}$$

$$\sin A = \frac{3}{5}$$

Similarly,

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Substitute the given values, we get

$$\frac{24}{\sin B} = \frac{30}{\sin 90°}$$

$$\sin B = \frac{24 \times \sin 90°}{30}$$

$$\sin B = \frac{24 \times 1}{30}$$

$$\sin B = \frac{4}{5}$$

And given, ∠C = 90°, so sin C = sin 90° = 1.

Hence the values of sin A = 3/5, sin B = 4/5 and sin C = 1 respectively.

In any triangle ABC, prove the following:

4. $$\frac{a-b}{a+b} = \frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)}$$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{a}{\sin A} = k$$

So, a = k sin A

Similarly, b = k sin B

And c = k sin C

We know,

a - b = k (sin A - sin B)

a + b = k (sin A + sin B)

Now let us consider LHS:

$$\frac{a-b}{a+b} = \frac{k(\sin A - \sin B)}{k(\sin A + \sin B)}$$

$$= \frac{(\sin A - \sin B)}{(\sin A + \sin B)} \ldots (i)$$

We know,

Sin A – sin B = 2 sin (A-B)/2 cos (A+B)/2

Sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2

Substituting the above formulas in equation (i), we get

$$\frac{a-b}{a+b} = \frac{\left(2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)\right)}{\left(2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right)}$$

Upon rearranging we get,

$$= \frac{\left(\sin\left(\frac{A-B}{2}\right)\right)}{\left(\cos\left(\frac{A-B}{2}\right)\right)} \times \frac{\cos\left(\frac{A+B}{2}\right)}{\sin\left(\frac{A+B}{2}\right)}$$

$$= \frac{\left(\tan\left(\frac{A-B}{2}\right)\right)}{1} \times \frac{1}{\tan\left(\frac{A+B}{2}\right)}$$

$$= \frac{\left(\tan\left(\frac{A-B}{2}\right)\right)}{\left(\tan\left(\frac{A+B}{2}\right)\right)}$$

= RHS

Hence proved.

5. (a - b) cos C/2 = C sin (A - B)/2

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{a}{\sin A} = k$$

So, a = k sin A

Similarly, b = k sin B

We know,

a - b = k (sin A - sin B) …. (i)

Now let us consider LHS:

$$(a - b)\cos\frac{C}{2}$$

Substituting equation (i) in above equation, we get

$$(k(\sin A - \sin B))\cos\frac{C}{2} \ldots (ii)$$

We know,

Sin A – sin B = 2 sin (A-B)/2 cos (A+B)/2

Substituting the above formulas in equation (ii), we get

$$(a - b)\cos\frac{C}{2} = \left(k\left(2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)\right)\right)\cos\frac{C}{2}$$

$$= \left(k\left(2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)\right)\right)\cos\frac{(\pi - (A+B))}{2}$$

$$= \left(2k\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)\right)\sin\left(\frac{A+B}{2}\right)$$ [since, cos(π/2 - A) = sin A]

Upon rearranging we get,

$$= k\sin\left(\frac{A-B}{2}\right)\left(2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A+B}{2}\right)\right)$$

We know, sin A = 2 cos (A/2) sin (A/2)

So the above equation becomes,

$$= k\sin\left(\frac{A-B}{2}\right)(\sin(A+B))$$

$$= k\sin\left(\frac{A-B}{2}\right)(\sin(\pi - C))$$ [since, π = A+B+C, where, A+B = π-C]

$$= k\sin(C)\sin\left(\frac{A-B}{2}\right)$$ [since, sin (π - A) = sin A]

From the sine rule,

$$\frac{c}{\sin C} = k \Rightarrow c = k\sin C$$

So the above equation becomes,

$$= c\sin\left(\frac{A-B}{2}\right)$$

= RHS

Hence proved.

6. $\frac{c}{a-b} = \frac{\tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right)}{\tan\left(\frac{A}{2}\right) - \tan\left(\frac{B}{2}\right)}$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{a}{\sin A} = k$$

So, a = k sin A

Similarly, b = k sin B

And c = k sin C … (i)

We know,

a - b = k (sin A - sin B) …. (ii)

Now let us consider LHS:

$$\frac{c}{a-b}$$

Substituting equation (i) and (ii) in above equation, we get

$$\frac{k\sin C}{k(\sin A - \sin B)} = \frac{\sin C}{(\sin A - \sin B)} \text{ .... (iii)}$$

By applying half angle rule,

$$\sin C = 2\sin\frac{C}{2}\cos\frac{C}{2} \text{ … (iv)}$$

And we know,

Sin A – sin B = 2 sin (A-B)/2 cos (A+B)/2 … (v)

Substituting the above equations (iv) and (v) in equation (iii), we get

$$\frac{c}{(a-b)} = \frac{2\sin\frac{C}{2}\cos\frac{C}{2}}{2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)}$$

$$= \frac{\sin\left(\frac{\pi-(A+B)}{2}\right)\cos\left(\frac{C}{2}\right)}{\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)}$$ [since, π = A+B+C, where, C = π – (A+B)]

$$= \frac{\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{C}{2}\right)}{\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)}$$ [since, sin (π/2 - A) = cos A]

Upon simplification we get,

$$= \frac{\cos\left(\frac{C}{2}\right)}{\sin\left(\frac{A-B}{2}\right)}$$

$$= \frac{\cos\left(\frac{\pi-(A+B)}{2}\right)}{\sin\left(\frac{A-B}{2}\right)}$$ [since, π = A+B+C, where, C = π – (A+B)]

$$= \frac{\sin\left(\frac{(A+B)}{2}\right)}{\sin\left(\frac{A-B}{2}\right)}$$ … (vi) [since, cos (π/2 - A) = sin A]

We know,

Sin (A + B)/2 = sin (A/2 + B/2) = sin A/2 cos B/2 + cos A/2 sin B/2

Sin (A - B)/2 = sin (A/2 - B/2) = sin A/2 cos B/2 - cos A/2 sin B/2

Substituting the above equations in equation (vi) we get,

$$= \frac{\sin\frac{A}{2}\cos\frac{B}{2} + \cos\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\sin\frac{A}{2}\cos\frac{B}{2} - \cos\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}$$

Let us divide the numerator and denominator by cos A/2 cos B/2, we get

$$= \frac{\frac{\sin\frac{A}{2}\cos\frac{B}{2} + \cos\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}}{\frac{\sin\frac{A}{2}\cos\frac{B}{2} - \cos\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}}$$

$$= \frac{\frac{\sin\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} + \frac{\cos\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}}{\frac{\sin\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} - \frac{\cos\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}}$$

Upon simplification we get,

$$= \frac{\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} + \frac{\sin\left(\frac{B}{2}\right)}{\cos\frac{B}{2}}}{\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} - \frac{\sin\left(\frac{B}{2}\right)}{\cos\frac{B}{2}}}$$

$$= \frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{\tan\frac{A}{2} - \tan\frac{B}{2}}$$

= RHS

Hence proved.

7. $\frac{c}{a+b} = \frac{1-\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)}{1+\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)}$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{a}{\sin A} = k$$

So, a = k sin A

Similarly, b = k sin B

And c = k sin C … (i)

We know,

a + b = k (sin A + sin B) …. (ii)

Now let us consider LHS:

$$\frac{c}{a+b}$$

Substituting equation (i) and (ii) in above equation, we get

$$\frac{k\sin C}{k(\sin A + \sin B)} = \frac{\sin C}{(\sin A + \sin B)} \quad \text{.... (iii)}$$

By applying half angle rule,

$$\sin C = 2\sin\frac{C}{2}\cos\frac{C}{2}$$ … (iv)

And we know,

Sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2 … (v)

Substituting the above equations (iv) and (v) in equation (iii), we get

$$\frac{c}{(a+b)} = \frac{2\sin\frac{C}{2}\cos\frac{C}{2}}{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}$$

$$= \frac{\sin\left(\frac{\pi-(A+B)}{2}\right)\cos\left(\frac{\pi-(A+B)}{2}\right)}{\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}$$ [Since, π = A+B+C, where, C= π – (A+B)]

$$= \frac{\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A+B}{2}\right)}{\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}$$ [Since, sin (π/2 - A) = cos A, cos (π/2 - A) = sin A]

Upon simplification we get,

$$= \frac{\cos\left(\frac{(A+B)}{2}\right)}{\cos\left(\frac{A-B}{2}\right)}$$ … (vi)

We know,

cos (A + B)/2 = cos (A/2 + B/2) = cos A/2 cos B/2 + sin A/2 sin B/2

cos (A - B)/2 = cos (A/2 - B/2) = cos A/2 cos B/2 - sin A/2 sin B/2

Substituting the above equations in equation (vi) we get,

$$= \frac{\cos\frac{A}{2}\cos\frac{B}{2} + \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2} - \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}$$

Let us divide the numerator and denominator by cos A/2 cos B/2, we get

$$= \frac{\frac{\cos\frac{A}{2}\cos\frac{B}{2} + \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}}{\frac{\cos\frac{A}{2}\cos\frac{B}{2} - \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}}$$

$$= \frac{\frac{\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} + \frac{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}}{\frac{\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} - \frac{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}}$$

Upon simplification we get,

$$= \frac{1 + \frac{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}}{1 - \frac{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}}$$

$$= \frac{1 + \tan\frac{A}{2}\tan\frac{B}{2}}{1 - \tan\frac{A}{2}\tan\frac{B}{2}}$$

= RHS

Hence proved.

8. $\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{a}{\sin A} = k$$

So, a = k sin A

Similarly, b = k sin B

And c = k sin C … (i)

We know,

a + b = k (sin A + sin B) …. (ii)

Now let us consider LHS:

$$\frac{a+b}{c}$$

Substituting equation (i) and (ii) in above equation, we get

$$\frac{k(\sin A+\sin B)}{k(\sin C)}=\frac{(\sin A+\sin B)}{(\sin C)} \quad \text{.... (iii)}$$

By applying half angle rule,

$$\sin C=2\sin\frac{C}{2}\cos\frac{C}{2} \quad \text{... (iv)}$$

And we know,

Sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2 … (v)

Substituting the above equations (iv) and (v) in equation (iii), we get

$$\frac{a+b}{c}=\frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)}$$

$$=\frac{\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)\cos\left(\frac{\pi-(A+B)}{2}\right)}$$ [Since, π = A+B+C, where, C= π – (A+B)]

$$=\frac{\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)\sin\left(\frac{A+B}{2}\right)}$$ [Since, sin (π/2 - A) = cos A, cos (π/2 - A) = sin A]

Upon simplification we get,

$$=\frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)}$$

= RHS

Hence proved.

9. $\sin\left(\frac{B-C}{2}\right)=\frac{b-c}{a}\cos\frac{A}{2}$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$$

$$\frac{c}{\sin C} = k$$

So, c = k sin C
Similarly, b = k sin B
We know,
b – c = k (sin B - sin C) …. (i)
Now let us consider RHS:

$$\frac{b-c}{a}\cos\frac{A}{2}$$

Substituting equation (i) in above equation, we get

$$\frac{\left(k(\sin B - \sin C)\right)}{k\sin A}\cos\frac{A}{2} = \frac{(\sin B - \sin C)}{\sin A}\cos\frac{A}{2} \quad \text{.... (ii)}$$

And we know,
Sin B - sin C = 2 sin (B-C)/2 cos (B+C)/2 … (iii)
Substituting the above equation (iii) in equation (ii), we get

$$\frac{b-c}{a}\cos\frac{A}{2} = \frac{2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)}{\sin A}\cos\left(\frac{\pi-(B+C)}{2}\right)$$ [Since, π = A+B+C, where, C= π – (A+B)]

$$= \frac{2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)}{\sin A}\sin\left(\frac{(B+C)}{2}\right)$$ [Since, cos (π/2 - A) = sin A]

Upon rearranging the above equation we get,

$$= \frac{\sin\left(\frac{B-C}{2}\right)\left(2\sin\left(\frac{(B+C)}{2}\right)\cos\left(\frac{B+C}{2}\right)\right)}{\sin A}$$

We know sin A = 2 cos (A/2) sin (A/2)
So,

$$= \frac{\sin\left(\frac{B-C}{2}\right)(\sin(B+C))}{\sin A}$$

$$= \frac{\sin\left(\frac{B-C}{2}\right)(\sin(\pi - A))}{\sin A}$$ [Since, π = A+B+C, where, A+B = π-C]

$$= \frac{\sin\left(\frac{B-C}{2}\right)\sin A}{\sin A}$$ [Since, sin (π-A) = sin A]

Upon simplification we get,

$$= \sin\left(\frac{B-C}{2}\right)$$

= LHS

Hence proved.

10. $\frac{a^2 - c^2}{b^2} = \frac{\sin(A-C)}{\sin(A+C)}$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{c}{\sin C} = k$$

So, c = k sin C

Similarly, a = k sin A

And b = k sin B

So, a – c = k (sin A – sin C) … (i)

We know,

Now let us consider LHS:

$$\frac{a^2 - c^2}{b^2}$$

Substituting the values in the above equation, we get

$$\frac{(k\sin A)^2 - (k\sin C)^2}{(k\sin B)^2} = \frac{k^2(\sin^2 A - \sin^2 C)}{k^2\sin^2 B}$$ .... (ii)

And we know,

$\sin^2 A - \sin^2 C = \sin(A+C)\sin(A-C)$… (iii)

Substituting the above equation (iii) in equation (ii), we get

$$\frac{a^2 - c^2}{b^2} = \frac{\sin(A + C)\sin(A - C)}{\sin^2(\pi - (A + C))}$$ [Since, π = A+B+C, where, C= π – (A+B)]

$$= \frac{\sin(A + C)\sin(A - C)}{\sin^2((A + C))}$$ [Since, sin (π - A) = sin A]

$$= \frac{\sin(A - C)}{\sin(A + C)}$$

= RHS

Hence proved.

11. b sin B – c sin C = a sin (B – C)

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{c}{\sin C} = k$$

So, c = k sin C

Similarly, a = k sin A

And b = k sin B

We know,

Now let us consider LHS:

b sin B – c sin C

Substituting the values of b and c in the above equation, we get

k sin B sin B – k sin C sin C = k ($\sin^2$ B – $\sin^2$ C) ……….(i)

We know,

Sin^2 B – $\sin^2$ C = sin (B + C) sin (B – C),

Substituting the above values in equation (i), we get

k ($\sin^2$ B – $\sin^2$ C) = k (sin (B + C) sin (B - C)) [since, π = A + B + C ⇒ B + C = π –A]

The above equation becomes,

= k (sin (π –A) sin (B - C)) [since, sin (π - θ) = sin θ]

= k (sin (A) sin (B - C))

From sine rule, a = k sin A, so the above equation becomes,

= a sin (B - C)

= RHS

Hence proved.

12. $a^2 \sin (B - C) = (b^2 - c^2) \sin A$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{c}{\sin C} = k$$

So, c = k sin C

Similarly, a = k sin A

And b = k sin B

We know,

Now let us consider RHS:

$(b^2 - c^2) \sin A$ …

Substituting the values of b and c in the above equation, we get

$(b^2 - c^2) \sin A = [(k \sin B)^2 - (k \sin C)^2] \sin A$

$= k^2 (\sin^2 B - \sin^2 C) \sin A$………. (i)

We know,

$\text{Sin}^2 B - \sin^2 C = \sin (B + C) \sin (B - C)$,

Substituting the above values in equation (i), we get

$= k^2 (\sin (B + C) \sin (B - C)) \sin A$ [since, $\pi = A + B + C \Rightarrow B + C = \pi - A$]

$= k^2 (\sin (\pi - A) \sin (B - C)) \sin A$

$= k^2 (\sin (A) \sin (B - C)) \sin A$ [since, $\sin (\pi - \theta) = \sin \theta$]

Rearranging the above equation we get

$= (k \sin (A))(\sin (B - C)) (k \sin A)$

From sine rule, a = k sin A, so the above equation becomes,

$= a^2 \sin (B - C)$

= RHS

Hence proved.

13. $\frac{\sqrt{\sin A} - \sqrt{\sin B}}{\sqrt{\sin A} + \sqrt{\sin B}} = \frac{a + b - 2\sqrt{ab}}{a - b}$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\sin A = \frac{a}{k}, \sin B = \frac{b}{k}, \sin C = \frac{c}{k}$$

Let us consider LHS,

$$\frac{\sqrt{\sin A}-\sqrt{\sin B}}{\sqrt{\sin A}+\sqrt{\sin B}}$$

Let us multiply and divide the above expression by $\frac{\sqrt{\sin A}-\sqrt{\sin B}}{\sqrt{\sin A}-\sqrt{\sin B}}$ we get,

$$\frac{\sqrt{\sin A}-\sqrt{\sin B}}{\sqrt{\sin A}+\sqrt{\sin B}}\times\frac{\sqrt{\sin A}-\sqrt{\sin B}}{\sqrt{\sin A}-\sqrt{\sin B}}=\frac{(\sqrt{\sin A}-\sqrt{\sin B})^2}{(\sqrt{\sin A})^2-(\sqrt{\sin B})^2}$$

$$=\frac{(\sqrt{\sin A})^2+(\sqrt{\sin B})^2-(2\sqrt{\sin A}\times\sqrt{\sin B})}{\sin A-\sin B}$$

$$=\frac{\sin A+\sin B-(2\sqrt{\sin A\times\sin B})}{\sin A-\sin B}$$

Substituting the values of a and b from sine rule in the above equation, we get

$$=\frac{\frac{a}{k}+\frac{b}{k}-\left(2\sqrt{\frac{a}{k}\times\frac{b}{k}}\right)}{\frac{a}{k}-\frac{b}{k}}$$

$$=\frac{\frac{1}{k}(a+b-2\sqrt{ab})}{\frac{1}{k}(a-b)}$$

$$=\frac{a+b-2\sqrt{ab}}{a-b}$$

= RHS

Hence proved.

14. a (sin B – sin C) + b (sin C – sin A) + c (sin A – sin B) = 0

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$$

a = k sin A, b = k sin B, c = k sin C

Let us consider LHS:

a (sin B – sin C) + b (sin C – sin A) + c (sin A – sin B)

Substituting the values of a, b, c from sine rule in above equation, we get a (sin B – sin C) + b (sin C – sin A) + c (sin A – sin B) = k sin A (sin B – sin C) + k sin B (sin C – sin A) + k sin C (sin A – sin B)

= k sin A sin B – k sin A sin C + k sin B sin C – k sin B sin A + k sin C sin A – k sin C sin B

Upon simplification, we get

= 0

= RHS

Hence proved.

15. $\frac{a^2 \sin(B-C)}{\sin A} + \frac{b^2 \sin(C-A)}{\sin B} + \frac{c^2 \sin(A-B)}{\sin C} = 0$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

a = k sin A, b = k sin B, c = k sin C

Let us consider LHS:

$$\frac{a^2 \sin(B-C)}{\sin A} + \frac{b^2 \sin(C-A)}{\sin B} + \frac{c^2 \sin(A-B)}{\sin C}$$

Substituting the values of a, b and c from sine rule in the above equation, we get

$$= \frac{(k \sin A)^2 \sin(B-C)}{\sin A} + \frac{(k \sin B)^2 \sin(C-A)}{\sin B} + \frac{(k \sin C)^2 \sin(A-B)}{\sin C}$$

$$= \frac{k^2 \sin^2 A \sin(B-C)}{\sin A} + \frac{k^2 \sin^2 B \sin(C-A)}{\sin B} + \frac{k^2 \sin^2 C \sin(A-B)}{\sin C}$$

Upon simplification we get,

= k^2 [sin A sin (B - C) + sin B sin (C - A) + sin C sin (A - B)]

We know, sin (A - B) = sin A cos B – cos A sin B

Sin (B - C) = sin B cos C – cos B sin C

Sin (C - A) = sin C cos A – cos C sin A

So the above equation becomes,

= k^2 [sin A (sin B cos C – cos B sin C) + sin B (sin C cos A – cos C sin A) + sin C (sin A cos B – cos A sin B)]

= k^2 [sin A sin B cos C – sin A cos B sin C + sin B sin C cos A – sin B cos C sin A + sin C sin A cos B – sin C cos A sin B)]

Upon simplification we get,

= 0

= RHS

Hence proved.

EXERCISE 10.2 PAGE NO: 10.25

In any ΔABC, prove the following:

1. In a ΔABC, if a = 5, b = 6 and C = 60°, show that its area is (15√3)/2 sq. units.
Solution:
Given:
In a ΔABC, a = 5, b = 6 and C = 60^o
By using the formula,
Area of ΔABC = 1/2 ab sin θ where, a and b are the lengths of the sides of a triangle and θ is the angle between sides.
So,
Area of ΔABC = 1/2 ab sin θ
= 1/2 × 5 × 6 × sin 60^o
= 30/2 × √3/2
= (15√3)/2 sq. units
Hence proved.

2. In a ΔABC, if a = √2, b = √3 and c = √5 show that its area is1/2 √6 sq. units.
Solution:
Given:
In a ΔABC, a = √2, b = √3 and c = √5
By using the formulas,
We know, cos A = $(b^2 + c^2 – a^2)$/2bc
By substituting the values we get,
= $[(\sqrt{3})^2 + (\sqrt{5})^2 – (\sqrt{2})^2]$ / [2 × √3 × √5]
= 3/√15

We know, Area of ΔABC = 1/2 bc sin A
To find sin A:
Sin A = $\sqrt{(1 – \cos^2 A)}$ [by using trigonometric identity]
= $\sqrt{(1 – (3/\sqrt{15})^2)}$
= √(1- (9/15))
= √(6/15)
Now,
Area of ΔABC = 1/2 bc sin A
= 1/2 × √3 × √5 × √(6/15)
= 1/2 √6 sq. units
Hence proved.

3. The sides of a triangle are a = 4, b = 6 and c = 8, show that: 8 cos A + 16 cos B + 4 cos C = 17.

Solution:

Given:

Sides of a triangle are a = 4, b = 6 and c = 8

By using the formulas,

Cos A = $(b^2 + c^2 – a^2)/2bc$

Cos B = $(a^2 + c^2 – b^2)/2ac$

Cos C = $(a^2 + b^2 – c^2)/2ab$

So now let us substitute the values of a, b and c we get,

Cos A = $(b^2 + c^2 – a^2)/2bc$

= $(6^2 + 8^2 - 4^2)/2×6×8$

= (36 + 64 - 16)/96

= 84/96

= 7/8

Cos B = $(a^2 + c^2 – b^2)/2ac$

= $(4^2 + 8^2 - 6^2)/2×4×8$

= (16 + 64 - 36)/64

= 44/64

Cos C = $(a^2 + b^2 – c^2)/2ab$

= $(4^2 + 6^2 - 8^2)/2×4×6$

= (16 + 36 - 64)/48

= -12/48

= -1/4

Now considering LHS:

8 cos A + 16 cos B + 4 cos C = 8 × 7/8 + 16 × 44/64 + 4 × (-1/4)

= 7 + 11 – 1

= 17

Hence proved.

4. In a ∆ABC, if a = 18, b = 24, c = 30, find cos A, cos B and cos C

Solution:

Given:

Sides of a triangle are a = 18, b = 24 and c = 30

By using the formulas,

Cos A = $(b^2 + c^2 – a^2)/2bc$

Cos B = $(a^2 + c^2 – b^2)/2ac$
Cos C = $(a^2 + b^2 – c^2)/2ab$
So now let us substitute the values of a, b and c we get,
Cos A = $(b^2 + c^2 – a^2)/2bc$
= $(24^2 + 30^2 - 18^2)/2×24×30$
= 1152/1440
= 4/5

Cos B = $(a^2 + c^2 – b^2)/2ac$
= $(18^2 + 30^2 - 24^2)/2×18×30$
= 648/1080
= 3/5

Cos C = $(a^2 + b^2 – c^2)/2ab$
= $(18^2 + 24^2 - 30^2)/2×18×24$
= 0/864
= 0
∴ cos A = 4/5, cos B = 3/5, cos C = 0

5. For any ΔABC, show that b (c cos A – a cos C) = $c^2 – a^2$
Solution:
Let us consider LHS:
b (c cos A – a cos C)
As LHS contain bc cos A and ab cos C which can be obtained from cosine formulae.
From cosine formula we have:
Cos A = $(b^2 + c^2 – a^2)/2bc$
bc cos A = $(b^2 + c^2 – a^2)/2$ … (i)

Cos C = $(a^2 + b^2 – c^2)/2ab$
ab cos C = $(a^2 + b^2 – c^2)/2$ … (ii)
Now let us subtract equation (i) and (ii) we get,
bc cos A - ab cos C = $(b^2 + c^2 – a^2)/2 - (a^2 + b^2 – c^2)/2$
= $c^2 – a^2$
∴ b (c cos A – a cos C) = $c^2 – a^2$
Hence proved.

6. For any Δ ABC show that c (a cos B – b cos A) = $a^2 – b^2$
Solution:
Let us consider LHS:

c (a cos B – b cos A)
As LHS contain ca cos B and cb cos A which can be obtained from cosine formulae.
From cosine formula we have:
Cos A = $(b^2 + c^2 – a^2)/2bc$
bc cos A = $(b^2 + c^2 – a^2)/2$ … (i)

Cos B = $(a^2 + c^2 – b^2)/2ac$
ac cos B = $(a^2 + c^2 – b^2)/2$ … (ii)
Now let us subtract equation (ii) from (i) we get,
ac cos B - bc cos A = $(a^2 + c^2 – b^2)/2 - (b^2 + c^2 – a^2)/2$
= $a^2 – b^2$
∴ c (a cos B – b cos A) = $a^2 – b^2$
Hence proved.

7. For any Δ ABC show that
2 (bc cos A + ca cos B + ab cos C) = $a^2 + b^2 + c^2$
Solution:
Let us consider LHS:
2 (bc cos A + ca cos B + ab cos C)
As LHS contain 2ca cos B, 2ab cos C and 2cb cos A, which can be obtained from cosine formulae.
From cosine formula we have:
Cos A = $(b^2 + c^2 – a^2)/2bc$
2bc cos A = $(b^2 + c^2 – a^2)$ … (i)

Cos B = $(a^2 + c^2 – b^2)/2ac$
2ac cos B = $(a^2 + c^2 – b^2)$… (ii)

Cos C = $(a^2 + b^2 – c^2)/2ab$
2ab cos C = $(a^2 + b^2 – c^2)$ … (iii)
Now let us add equation (i), (ii) and (ii) we get,
2bc cos A + 2ac cos B + 2ab cos C = $(b^2 + c^2 – a^2) + (a^2 + c^2 – b^2) + (a^2 + b^2 – c^2)$
Upon simplification we get,
= $c^2 + b^2 + a^2$
2 (bc cos A + ac cos B + ab cos C) = $a^2 + b^2 + c^2$
Hence proved.

8. For any Δ ABC show that
$(c^2 – a^2 + b^2)$ tan A = $(a^2 – b^2 + c^2)$ tan B = $(b^2 – c^2 + a^2)$ tan C

Solution:
Let us consider LHS:
$(c^2 – a^2 + b^2)$, $(a^2 – b^2 + c^2)$, $(b^2 – c^2 + a^2)$
We know sine rule in Δ ABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

As LHS contain $(c^2 – a^2 + b^2)$, $(a^2 – b^2 + c^2)$ and $(b^2 – c^2 + a^2)$, which can be obtained from cosine formulae.
From cosine formula we have:
$\text{Cos } A = (b^2 + c^2 – a^2)/2bc$
$2bc \cos A = (b^2 + c^2 – a^2)$
Let us multiply both the sides by tan A we get,
$2bc \cos A \tan A = (b^2 + c^2 – a^2) \tan A$
$2bc \cos A (\sin A/\cos A) = (b^2 + c^2 – a^2) \tan A$
$2bc \sin A = (b^2 + c^2 – a^2) \tan A$ … (i)

$\text{Cos } B = (a^2 + c^2 – b^2)/2ac$
$2ac \cos B = (a^2 + c^2 – b^2)$
Let us multiply both the sides by tan B we get,
$2ac \cos B \tan B = (a^2 + c^2 – b^2) \tan B$
$2ac \cos B (\sin B/\cos B) = (a^2 + c^2 – b^2) \tan B$
$2ac \sin B = (a^2 + c^2 – b^2) \tan B$ … (ii)

$\text{Cos } C = (a^2 + b^2 – c^2)/2ab$
$2ab \cos C = (a^2 + b^2 – c^2)$
Let us multiply both the sides by tan C we get,
$2ab \cos C \tan C = (a^2 + b^2 – c^2) \tan C$
$2ab \cos C (\sin C/\cos C) = (a^2 + b^2 – c^2) \tan C$
$2ab \sin C = (a^2 + b^2 – c^2) \tan C$ … (iii)
As we are observing that sin terms are being involved so let's use sine formula.
From sine formula we have,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Let us multiply abc to each of the expression we get,

$$\frac{abc \sin A}{a} = \frac{abc \sin B}{b} = \frac{abc \sin C}{c}$$

$bc \sin A = ac \sin B = ab \sin C$
$2bc \sin A = 2ac \sin B = 2ab \sin C$

∴ From equation (i), (ii) and (iii) we have,
$(c^2 - a^2 + b^2) \tan A = (a^2 - b^2 + c^2) \tan B = (b^2 - c^2 + a^2) \tan C$
Hence proved.

9. For any Δ ABC show that:

$$\frac{c - b \cos A}{b - c \cos A} = \frac{\cos B}{\cos C}$$

Solution:

Let us consider LHS:

$$\frac{c - b \cos A}{b - c \cos A}$$

We can observe that we can get terms c – b cos A and b – c cos A from projection formula

From projection formula we get,

c = a cos B + b cos A

c – b cos A = a cos B …. (i)

And,

b = c cos A + a cos C

b – c cos A = a cos C …. (ii)

Dividing equation (i) by (ii), we get,

$$\frac{c - b \cos A}{b - c \cos A} = \frac{a \cos B}{a \cos C}$$

$$= \frac{\cos B}{\cos C}$$

= RHS

Hence proved.