EXERCISE 8.1 PAGE NO: 8.6

1. Express each of the following as the sum or difference of sines and cosines:
(i) 2 sin 3x cos x
(ii) 2 cos 3x sin 2x
(iii) 2 sin 4x sin 3x
(iv) 2 cos 7x cos 3x
Solution:
(i) 2 sin 3x cos x
By using the formula,
2 sin A cos B = sin (A + B) + sin (A – B)
2 sin 3x cos x = sin (3x + x) + sin (3x – x)
= sin (4x) + sin (2x)
= sin 4x + sin 2x

(ii) 2 cos 3x sin 2x
By using the formula,
2 cos A sin B = sin (A + B) – sin (A – B)
2 cos 3x sin 2x = sin (3x + 2x) – sin (3x – 2x)
= sin (5x) – sin (x)
= sin 5x – sin x

(iii) 2 sin 4x sin 3x
By using the formula,
2 sin A sin B = cos (A – B) – cos (A + B)
2 sin 4x sin 3x = cos (4x – 3x) – cos (4x + 3x)
= cos (x) – cos (7x)
= cos x – cos 7x

(iv) 2 cos 7x cos 3x
By using the formula,
2 cos A cos B = cos (A + B) + cos (A – B)
2 sin 3x cos x = cos (7x + 3x) + cos (7x – 3x)
= cos (10x) + cos (4x)
= cos 10x + cos 4x

2. Prove that:
(i) 2 sin 5π/12 sin π/12 = 1/2
(ii) 2 cos 5π/12 cos π/12 = 1/2

(iii) 2 sin 5π/12 cos π/12 = (√3 + 2)/2

Solution:

(i) 2 sin 5π/12 sin π/12 = 1/2

By using the formula,

2 sin A sin B = cos (A – B) – cos (A + B)

2 sin 5π/12 sin π/12 = cos (5π/12 - π/12) – cos (5π/12 + π/12)

= cos (4π/12) – cos (6π/12)

= cos (π/3) – cos (π/2)

= cos (180°/3) – cos (180°/2)

= cos 60° - cos 90°

= 1/2 – 0

= 1/2

Hence Proved.

(ii) 2 cos 5π/12 cos π/12 = 1/2

By using the formula,

2 cos A cos B = cos (A + B) + cos (A - B)

2 cos 5π/12 cos π/12 = cos (5π/12 + π/12) + cos (5π/12 - π/12)

= cos (6π/12) + cos (4π/12)

= cos (π/2) + cos (π/3)

= cos (180°/2) + cos (180°/3)

= cos 90° + cos 60°

= 0 + 1/2

= 1/2

Hence Proved.

(iii) 2 sin 5π/12 cos π/12 = (√3 + 2)/2

By using the formula,

2 sin A cos B = sin (A + B) + sin (A - B)

2 sin 5π/12 cos π/12 = sin (5π/12 + π/12) + sin (5π/12 - π/12)

= sin (6π/12) + sin (4π/12)

= sin (π/2) + sin (π/3)

= sin (180°/2) + sin (180°/3)

= sin 90° + sin 60°

= 1 + √3

= (2 + √3)/2

= (√3 + 2)/2

Hence Proved.

3. show that:
(i) sin 50° cos 85° = (1 - √2sin 35°)/2√2
(ii) sin 25° cos 115° = 1/2 {sin 140° - 1}
Solution:
(i) sin 50° cos 85° = (1 - √2sin 35°)/2√2
By using the formula,
2 sin A cos B = sin (A + B) + sin (A – B)
sin A cos B = [sin (A + B) + sin (A – B)] / 2
sin 50° cos 85° = [sin(50° + 85°) + sin (50° – 85°)] / 2
= [sin (135°) + sin (-35°)] / 2
= [sin (135°) - sin (35°)] / 2 (since, sin (-x) = -sin x)
= [sin (180° – 45°) – sin 35°] / 2
= [sin 45° – sin 35°] / 2
= [(1/√2) – sin 35°] / 2
= [(1 – sin 35°)/√2] / 2
= (1 – sin 35°) / 2√2
Hence proved.

(ii) sin 25° cos 115° = 1/2 {sin 140° - 1}
By using the formula,
2 sin A cos B = sin (A + B) + sin (A – B)
sin A cos B = [sin (A + B) + sin (A – B)] / 2
sin 20° cos 115° = [sin(25° + 115°) + sin (25° – 115°)] / 2
= [sin (140°) + sin (-90°)] / 2
= [sin (140°) - sin (90°)] / 2 (since, sin (-x) = -sin x)
= 1/2 {sin 140° - 1}
Hence proved.

4. Prove that:
4 cos x cos (π/3 + x) cos (π/3 - x) = cos 3x
Solution:
Let us consider LHS:
4 cos x cos (π/3 + x) cos (π/3 - x) = 2 cos x (2 cos (π/3 + x) cos (π/3 - x))
By using the formula,
2 cos A cos B = cos (A + B) + cos (A – B)
2 cos x (2 cos (π/3+x) cos (π/3 - x)) = 2 cos x (cos (π/3+x + π/3-x) + cos (π/3+x - π/3+ x))
= 2 cos x (cos (2π/3) + cos (2x))
= 2 cos x {cos 120° + cos 2x}
= 2 cos x {cos (180° - 60°) + cos 2x}

= 2 cos x (cos 2x - cos 60°) (since, {cos (180° - A) = - cos A})
= 2 cos 2x cos x – 2 cos x cos 60°
= (cos (x + 2x) + cos (2x - x)) – (2cos x)/2
= cos 3x + cos x – cos x
= cos 3x
= RHS

Hence Proved.

EXERCISE 8.2

1. Express each of the following as the product of sines and cosines:
(i) sin 12x + sin 4x
(ii) sin 5x – sin x
(iii) cos 12x + cos 8x
(iv) cos 12x – cos 4x
(v) sin 2x + cos 4x
Solution:
(i) sin 12x + sin 4x
By using the formula,
sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2
sin 12x + sin 4x = 2 sin (12x + 4x)/2 cos (12x - 4x)/2
= 2 sin 16x/2 cos 8x/2
= 2 sin 8x cos 4x

(ii) sin 5x – sin x
By using the formula,
sin A - sin B = 2 cos (A+B)/2 sin (A-B)/2
sin 5x - sin x = 2 cos (5x + x)/2 sin (5x - x)/2
= 2 cos 6x/2 sin 4x/2
= 2 cos 3x sin 2x

(iii) cos 12x + cos 8x
By using the formula,
cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2
cos 12x + cos 8x = 2 cos (12x + 8x)/2 cos (12x - 8x)/2
= 2 cos 20x/2 cos 4x/2
= 2 cos 10x cos 2x

(iv) cos 12x – cos 4x
By using the formula,
cos A - cos B = -2 sin (A+B)/2 sin (A-B)/2
cos 12x - cos 4x = -2 sin (12x + 4x)/2 sin (12x - 4x)/2
= -2 sin 16x/2 sin 8x/2
= -2 sin 8x sin 4x

(v) sin 2x + cos 4x
sin 2x + cos 4x = sin 2x + sin (90^o – 4x)

By using the formula,
sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2
sin 2x + sin (90° – 4x) = 2 sin (2x + 90° - 4x)/2 cos (2x – 90° + 4x)/2
= 2 sin (90° – 2x)/2 cos (6x – 90°)/2
= 2 sin (45° - x) cos (3x – 45°)
= 2 sin (45° - x) cos {-(45° - 3x)} (since, {cos (-x) = cos x})
= 2 sin (45° - x) cos (45° - 3x)
= 2 sin (π/4 - x) cos (π/4 - 3x)

2. Prove that :
(i) sin 38° + sin 22° = sin 82°
(ii) cos 100° + cos 20° = cos 40°
(iii) sin 50° + sin 10° = cos 20°
(iv) sin 23° + sin 37° = cos 7°
(v) sin 105° + cos 105° = cos 45°
(vi) sin 40° + sin 20° = cos 10°
Solution:
(i) sin 38° + sin 22° = sin 82°
Let us consider LHS:
sin 38° + sin 22°
By using the formula,
sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2
sin 38° + sin 22° = 2 sin (38° + 22°)/2 cos (38° – 22°)/2
= 2 sin 60°/2 cos 16°/2
= 2 sin 30° cos 8°
= 2 × 1/2 × cos 8°
= cos 8°
= cos (90° - 82°)
= sin 82° (since, {cos (90° - A) = sin A})
= RHS
Hence Proved.

(ii) cos 100° + cos 20° = cos 40°
Let us consider LHS:
cos 100° + cos 20°
By using the formula,
cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2
cos 100° + cos 20° = 2 cos (100° + 20°)/2 cos (100° – 20°)/2
= 2 cos 120°/2 cos 80°/2

= 2 cos 60° cos 4°
= 2 × 1/2 × cos 40°
= cos 40°
= RHS
Hence Proved.

(iii) sin 50° + sin 10° = cos 20°
Let us consider LHS:
sin 50° + sin 10°
By using the formula,
sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2
sin 50° + sin 10° = 2 sin (50° + 10°)/2 cos (50° – 10°)/2
= 2 sin 60°/2 cos 40°/2
= 2 sin 30° cos 20°
= 2 × 1/2 × cos 20°
= cos 20°
= RHS
Hence Proved.

(iv) sin 23° + sin 37° = cos 7°
Let us consider LHS:
sin 23° + sin 37°
By using the formula,
sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2
sin 23° + sin 37° = 2 sin (23° + 37°)/2 cos (23° – 37°)/2
= 2 sin 60°/2 cos -14°/2
= 2 sin 30° cos -7°
= 2 × 1/2 × cos -7°
= cos 7° (since, {cos (-A) = cos A})
= RHS
Hence Proved.

(v) sin 105° + cos 105° = cos 45°
Let us consider LHS: sin 105° + cos 105°
sin 105° + cos 105° = sin 105° + sin (90° – 105°) [since, {sin (90° - A) = cos A}]
= sin 105° + sin (-15°)
= sin 105° – sin 15° [{sin(-A) = - sin A}]
By using the formula,
Sin A – sin B = 2 cos (A+B)/2 sin (A-B)/2

sin 105° – sin 15° = 2 cos (105° + 15°)/2 sin (105° – 15°)/2
= 2 cos 120°/2 sin 90°/2
= 2 cos 60° sin 45°
= 2 × 1/2 × 1/√2
= 1/√2
= cos 45°
= RHS
Hence proved.

(vi) sin 40° + sin 20° = cos 10°
Let us consider LHS:
sin 40° + sin 20°
By using the formula,
sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2
sin 40° + sin 20° = 2 sin (40° + 20°)/2 cos (40° – 20°)/2
= 2 sin 60°/2 cos 20°/2
= 2 sin 30° cos 10°
= 2 × 1/2 × cos 10°
= cos 10°
= RHS
Hence Proved.

3. Prove that:
(i) cos 55° + cos 65° + cos 175° = 0
(ii) sin 50° – sin 70° + sin 10° = 0
(iii) cos 80° + cos 40° – cos 20° = 0
(iv) cos 20° + cos 100° + cos 140° = 0
(v) sin 5π/18 – cos 4π/9 = √3 sin π/9
(vi) cos π/12 – sin π/12 = 1/√2
(vii) sin 80° – cos 70° = cos 50°
(viii) sin 51° + cos 81° = cos 21°
Solution:
(i) cos 55° + cos 65° + cos 175° = 0
Let us consider LHS:
cos 55° + cos 65° + cos 175°
By using the formula,
cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2
cos 55° + cos 65° + cos 175° = 2 cos (55° + 65°)/2 cos (55° – 65°) + cos (180° – 5°)
= 2 cos 120°/2 cos (-10°)/2 – cos 5° (since, {cos (180° - A) = - cos A})

= 2 cos 60° cos (-5°) – cos 5° (since, {cos (-A) = cos A})
= 2 × 1/2 × cos 5° – cos 5°
= 0
= RHS
Hence Proved.

(ii) sin 50° – sin 70° + sin 10° = 0
Let us consider LHS:
sin 50° – sin 70° + sin 10°
By using the formula,
sin A – sin B = 2 cos (A+B)/2 sin (A-B)/2
sin 50° – sin 70° + sin 10° = 2 cos (50° + 70°)/2 sin (50° – 70°) + sin 10°
= 2 cos 120°/2 sin (-20°)/2 + sin 10°
= 2 cos 60° (- sin 10°) + sin 10° [since,{sin (-A) = -sin (A)}]
= 2 × 1/2 × - sin 10° + sin 10°
= 0
= RHS
Hence proved.

(iii) cos 80° + cos 40° – cos 20° = 0
Let us consider LHS:
cos 80° + cos 40° – cos 20°
By using the formula,
cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2
cos 80° + cos 40° – cos 20° = 2 cos (80° + 40°)/2 cos (80° – 40°) – cos 20°
= 2 cos 120°/2 cos 40°/2 – cos 20°
= 2 cos 60° cos 20° – cos 20°
= 2 × 1/2 × cos 20° – cos 20°
= 0
= RHS
Hence Proved.

(iv) cos 20° + cos 100° + cos 140° = 0
Let us consider LHS:
cos 20° + cos 100° + cos 140°
By using the formula,
cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2
cos 20° + cos 100° + cos 140° = 2 cos (20° + 100°)/2 cos (20° – 100°) + cos (180° – 40°)
= 2 cos 120°/2 cos (-80°)/2 – cos 40° (since, {cos (180° - A) = - cos A})

= 2 cos 60° cos (-40°) – cos 40° (since, {cos (-A) = cos A})
= 2 × 1/2 × cos 40^o – cos 40^o
= 0
= RHS

Hence Proved.

(v) sin 5π/18 – cos 4π/9 = √3 sin π/9

Let us consider LHS:

sin 5π/18 – cos 4π/9 = sin 5π/18 – sin (π/2 - 4π/9) (since, cos A = sin (90^o - A))
= sin 5π/18 – sin (9π - 8π)/18
= sin 5π/18 – sin π/18

By using the formula,

sin A – sin B = 2 cos (A+B)/2 sin (A-B)/2

$$sin\frac{5\pi}{18} - sin\frac{\pi}{18} = 2cos\left(\frac{\frac{5\pi}{18}+\frac{\pi}{18}}{2}\right)sin\left(\frac{\frac{5\pi}{18}-\frac{\pi}{18}}{2}\right)$$

= 2 cos (6π/36) sin (4π/36)
= 2 cos π/6 sin π/9
= 2 cos 30^o sin π/9
= 2 × √3/2 × sin π/9
= √3 sin π/9
= RHS

Hence proved.

(vi) cos π/12 – sin π/12 = 1/√2

Let us consider LHS:

cos π/12 – sin π/12 = sin (π/2 – π/12) – sin π/12 (since, cos A = sin(90^o - A))
= sin (6π - 5π)/12 – sin π/12
= sin 5π/12 – sin π/12

By using the formula,

sin A – sin B = 2 cos (A+B)/2 sin (A-B)/2

$$sin\frac{5\pi}{12} - sin\frac{\pi}{12} = 2cos\left(\frac{\frac{5\pi}{12}+\frac{\pi}{12}}{2}\right)sin\left(\frac{\frac{5\pi}{12}-\frac{\pi}{12}}{2}\right)$$

= 2 cos (6π/24) sin (4π/24)
= 2 cos π/4 sin π/6
= 2 cos 45^o sin 30^o
= 2 × 1/√2 × 1/2
= 1/√2
= RHS

Hence proved.

(vii) sin 80° – cos 70° = cos 50°
sin 80° = cos 50° + cos 70°
So, now let us consider RHS
cos 50° + cos 70°
By using the formula,
cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2
cos 50° + cos 70° = 2 cos (50° + 70°)/2 cos (50° – 70°)/2
= 2 cos 120°/2 cos (-20°)/2
= 2 cos 60° cos (-10°)
= 2 × 1/2 × cos 10° (since, cos (-A) = cos A)
= cos 10°
= cos (90° - 80°)
= sin 80° (since, cos (90° - A) = sin A)
= LHS
Hence Proved.

(viii) sin 51° + cos 81° = cos 21°
Let us consider LHS:
sin 51° + cos 81° = sin 51° + sin (90° – 81°)
= sin 51° + sin 9° (since, sin (90° - A) = cos A)
By using the formula,
sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2
sin 51° + sin 9° = 2 sin (51° + 9°)/2 cos (51° – 9°)/2
= 2 sin 60°/2 cos 42°/2
= 2 sin 30° cos 21°
= 2 × 1/2 × cos 21°
= cos 21°
= RHS
Hence proved.

4. Prove that:
(i) cos (3π/4 + x) – cos (3π/4 – x) = -√2 sin x
(ii) cos (π/4 + x) + cos (π/4 - x) = √2 cos x
Solution:
(i) cos (3π/4 + x) – cos (3π/4 – x) = -√2 sin x
Let us consider LHS:
cos (3π/4 + x) – cos (3π/4 – x)

By using the formula,
cos A – cos B = -2 sin (A+B)/2 sin (A-B)/2
cos (3π/4 + x) – cos (3π/4 – x) = -2 sin (3π/4 + x + 3π/4 - x)/2 sin (3π/4 + x - 3π/4 + x)/2
= -2 sin (6π/4)/2 sin 2x/2
= -2 sin 6π/8 sin x
= -2 sin 3π/4 sin x
= -2 sin (π – π/4) sin x
= -2 sin π/4 sin x (since, (π-A) = sin A)
= -2 × 1/√2 × sin x
= -√2 sin x
= RHS
Hence proved.

(ii) cos (π/4 + x) + cos (π/4 - x) = √2 cos x
Let us consider LHS:
cos (π/4 + x) + cos (π/4 - x)
By using the formula,
cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2
cos (π/4 + x) + cos (π/4 - x) = 2 cos (π/4 + x + π/4 - x)/2 cos (π/4 + x - π/4 + x)/2
= 2 cos (2π/4)/2 cos 2x/2
= 2 cos 2π/8 cos x
= 2 sin π/4 cos x
= 2 × 1/√2 × cos x
= √2 cos x
= RHS
Hence proved.

5. Prove that:
(i) sin 65° + cos 65° = √2 cos 20°
(ii) sin 47° + cos 77° = cos 17°
Solution:
(i) sin 65° + cos 65° = √2 cos 20°
Let us consider LHS:
sin 65° + cos 65° = sin 65° + sin (90° – 65°)
= sin 65° + sin 25°
By using the formula,
sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2
sin 65° + sin 25° = 2 sin (65° + 25°)/2 cos (65° – 25°)/2
= 2 sin 90°/2 cos 40°/2

$= 2 \sin 45^o \cos 20^o$
$= 2 \times 1/\sqrt{2} \times \cos 20^o$
$= \sqrt{2} \cos 20^o$
= RHS
Hence proved.

(ii) $\sin 47^o + \cos 77^o = \cos 17^o$
Let us consider LHS:
$\sin 47^o + \cos 77^o = \sin 47^o + \sin (90^o – 77^o)$
$= \sin 47^o + \sin 13^o$
By using the formula,
sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2
$\sin 47^o + \sin 13^o = 2 \sin (47^o + 13^o)/2 \cos (47^o – 13^o)/2$
$= 2 \sin 60^o/2 \cos 34^o/2$
$= 2 \sin 30^o \cos 17^o$
$= 2 \times 1/2 \times \cos 17^o$
$= \cos 17^o$
= RHS
Hence proved.

6. Prove that:
(i) cos 3A + cos 5A + cos 7A + cos 15A = 4 cos 4A cos 5A cos 6A
(ii) cos A + cos 3A + cos 5A + cos 7A = 4 cos A cos 2A cos 4A
(iii) sin A + sin 2A + sin 4A + sin 5A = 4 cos A/2 cos 3A/2 sin 3A
(iv) sin 3A + sin 2A – sin A = 4 sin A cos A/2 cos 3A/2
(v) $\mathbf{\cos 20^o \cos 100^o + \cos 100^o \cos 140^o – \cos 140^o \cos 200^o = - 3/4}$
(vi) sin x/2 sin 7x/2 + sin 3x/2 sin 11x/2 = sin 2x sin 5x
(vii) cos x cos x/2 – cos 3x cos 9x/2 = sin 4x sin 7x/2
Solution:
(i) cos 3A + cos 5A + cos 7A + cos 15A = 4 cos 4A cos 5A cos 6A
Let us consider LHS:
cos 3A + cos 5A + cos 7A + cos 15A
So now,
(cos 5A + cos 3A) + (cos 15A + cos 7A)
By using the formula,
cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2
(cos 5A + cos 3A) + (cos 15A + cos 7A)
= [2 cos (5A+3A)/2 cos (5A-3A)/2] + [2 cos (15A+7A)/2 cos (15A-7A)/2]
= [2 cos 8A/2 cos 2A/2] + [2 cos 22A/2 cos 8A/2]

= [2 cos 4A cos A] + [2 cos 11A cos 4A]
= 2 cos 4A (cos 11A + cos A)
Again by using the formula,
cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2
2 cos 4A (cos 11A + cos A) = 2 cos 4A [2 cos (11A+A)/2 cos (11A-A)/2]
= 2 cos 4A [2 cos 12A/2 cos 10A/2]
= 2 cos 4A [2 cos 6A cos 5A]
= 4 cos 4A cos 5A cos 6A
= RHS
Hence proved.

(ii) cos A + cos 3A + cos 5A + cos 7A = 4 cos A cos 2A cos 4A
Let us consider LHS:
cos A + cos 3A + cos 5A + cos 7A
So now,
(cos 3A + cos A) + (cos 7A + cos 5A)
By using the formula,
cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2
(cos 3A + cos A) + (cos 7A + cos 5A)
= [2 cos (3A+A)/2 cos (3A-A)/2] + [2 cos (7A+5A)/2 cos (7A-5A)/2]
= [2 cos 4A/2 cos 2A/2] + [2 cos 12A/2 cos 2A/2]
= [2 cos 2A cos A] + [2 cos 6A cos A]
= 2 cos A (cos 6A + cos 2A)
Again by using the formula,
cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2
2 cos A (cos 6A + cos 2A) = 2 cos A [2 cos (6A+2A)/2 cos (6A-2A)/2]
= 2 cos A [2 cos 8A/2 cos 4A/2]
= 2 cos A [2 cos 4A cos 2A]
= 4 cos A cos 2A cos 4A
= RHS
Hence proved.

(iii) sin A + sin 2A + sin 4A + sin 5A = 4 cos A/2 cos 3A/2 sin 3A
Let us consider LHS:
sin A + sin 2A + sin 4A + sin 5A
So now,
(sin 2A + sin A) + (sin 5A + sin 4A)
By using the formula,
sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2

(sin 2A + sin A) + (sin 5A + sin 4A) =

= [2 sin (2A+A)/2 cos (2A-A)/2] + [2 sin (5A+4A)/2 cos (5A-4A)/2]

= [2 sin 3A/2 cos A/2] + [2 sin 9A/2 cos A/2]

= 2 cos A/2 (sin 9A/2 + sin 3A/2)

Again by using the formula,

sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2

2 cos A/2 (sin 9A/2 + sin 3A/2) = 2 cos A/2 [2 sin (9A/2 + 3A/2)/2 cos (9A/2 – 3A/2)/2]

= 2 cos A/2 [2 sin ((9A+3A)/2)/2 cos ((9A-3A)/2)/2]

= 2 cos A/2 [2 sin 12A/4 cos 6A/4]

= 2 cos A/2 [2 sin 3A cos 3A/2]

= 4 cos A/2 cos 3A/2 sin 3A

= RHS

Hence proved.

(iv) sin 3A + sin 2A – sin A = 4 sin A cos A/2 cos 3A/2

Let us consider LHS:

sin 3A + sin 2A – sin A

So now,

(sin 3A – sin A) + sin 2A

By using the formula,

sin A – sin B = 2 cos (A+B)/2 sin (A-B)/2

(sin 3A – sin A) + sin 2A = 2 cos (3A + A)/2 sin (3A - A)/2 + sin 2A

= 2 cos 4A/2 sin 2A/2 + sin 2A

We know that, sin 2A = 2 sin A cos A

= 2 cos 2A Sin A + 2 sin A cos A

= 2 sin A (cos 2A + cos A)

Again by using the formula,

cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2

2 sin A (cos 2A + cos A) = 2 sin A [2 cos (2A+A)/2 cos (2A-A)/2]

= 2 sin A [2 cos 3A/2 cos A/2]

= 4 sin A cos A/2 cos 3A/2

= RHS

Hence proved.

(v) cos 20^o cos 100^o + cos 100^o cos 140^o – cos 140^o cos 200^o = - 3/4

Let us consider LHS:

cos 20^o cos 100^o + cos 100^o cos 140^o – cos 140^o cos 200^o =

We shall multiply and divide by 2 we get,

= 1/2 [2 cos 100^o cos 20^o + 2 cos 140^o cos 100^o – 2 cos 200^o cos 140^o]

We know that 2 cos A cos B = cos (A+B) + cos (A-B)
So,

= 1/2 [cos (100^o + 20^o) + cos (100^o - 20^o) + cos (140^o + 100^o) + cos (140^o - 100^o) – cos (200^o + 140^o) – cos (200^o - 140^o)]]

= 1/2 [cos 120^o + cos 80^o + cos 240^o + cos 40^o – cos 340^o – cos 60^o]

= 1/2 [cos (90^o + 30^o) + cos 80^o + cos (180^o + 60^o) + cos 40^o – cos (360^o – 20^o) – cos 60^o]

We know, cos (180^o + A) = - cos A, cos (90^o + A) = - sin A, cos (360^o - A) = cos A
So,

= 1/2 [- sin 30^o + cos 80^o – cos 60^o + cos 40^o – cos 20^o – cos 60^o]

= 1/2 [- sin 30^o + cos 80^o + cos 40^o – cos 20^o – 2 cos 60^o]

Again by using the formula,
cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2

= 1/2 [- sin 30^o + 2 cos (80^o+40^o)/2 cos (80^o-40^o)/2 – cos 20^o – 2 × 1/2]

= 1/2 [- sin 30^o + 2 cos 120^o/2 cos 40^o/2 – cos 20^o - 1]

= 1/2 [- sin 30^o + 2 cos 60^o cos 20^o – cos 20^o - 1]

= 1/2 [- 1/2 + 2×1/2×cos 20^o – cos 20^o - 1]

= 1/2 [-1/2 + cos 20^o – cos 20^o - 1]

= 1/2 [-1/2 -1]

= 1/2 [-3/2]

= -3/4

= RHS

Hence proved.

(vi) sin x/2 sin 7x/2 + sin 3x/2 sin 11x/2 = sin 2x sin 5x
Let us consider LHS:
sin x/2 sin 7x/2 + sin 3x/2 sin 11x/2 =
We shall multiply and divide by 2 we get,

= 1/2 [2 sin 7x/2 sin x/2 + 2 sin 11x/2 sin 3x/2]

We know that 2 sin A sin B = cos (A-B) - cos (A+B)
So,

= 1/2 [cos (7x/2 – x/2) – cos (7x/2 + x/2) + cos (11x/2 – 3x/2) – cos (11x/2 + 3x/2)]

= 1/2 [cos (7x-x)/2 – cos (7x+x)/2 + cos (11x-3x)/2 – cos (11x+3x)/2]

= 1/2 [cos 6x/2 – cos 8x/2 + cos 8x/2 – cos 14x/2]

= 1/2 [cos 3x – cos 7x]

= - 1/2 [cos 7x – cos 3x]

Again by using the formula,
cos A - cos B = -2 sin (A+B)/2 sin (A-B)/2

= -1/2 [-2 sin (7x+3x)/2 sin (7x-3x)/2]

= -1/2 [-2 sin 10x/2 sin 4x/2]
= -1/2 [-2 sin 5x sin 2x]
= -2/-2 sin 5x sin 2x
= sin 2x sin 5x
= RHS
Hence proved.

(vii) cos x cos x/2 – cos 3x cos 9x/2 = sin 4x sin 7x/2
Let us consider LHS:
cos x cos x/2 – cos 3x cos 9x/2 =
We shall multiply and divide by 2 we get,
= 1/2 [2 cos x cos x/2 – 2 cos 9x/2 cos 3x]
We know that 2 cos A cos B = cos (A+B) + cos (A-B)
So,
= 1/2 [cos (x + x/2) + cos (x – x/2) – cos (9x/2 + 3x) – cos (9x/2 – 3x)]
= 1/2 [cos (2x+x)/2 + cos (2x-x)/2 – cos (9x+6x)/2 – cos (9x-6x)/2]
= 1/2 [cos 3x/2 + cos x/2 – cos 15x/2 – cos 3x/2]
= 1/2 [cos x/2 – cos 15x/2]
= - 1/2 [cos 15x/2 – cos x/2]
Again by using the formula,
cos A - cos B = -2 sin (A+B)/2 sin (A-B)/2
= - 1/2 [-2 sin (15x/2 + x/2)/2 sin (15x/2 – x/2)/2]
= -1/2 [-2 sin (16x/2)/2 sin (14x/2)/2]
= -1/2 [-2 sin 16x/4 sin 7x/2]
= - 1/2 [-2 sin 4x sin 7x/2]
= -2/-2 [sin 4x sin 7x/2]
= sin 4x sin 7x/2
= RHS
Hence proved.

7. Prove that:

$$(i)\ \frac{\sin A+\sin 3A}{\cos A-\cos 3A}=\cot A$$

$$(ii)\frac{\sin 9A-\sin 7A}{\cos 7A-\cos 9A}=\cot 8A$$

$$(iii)\frac{\sin A-\sin B}{\cos A+\cos B}=\tan\frac{A-B}{2}$$

$(iv) \frac{sin A + sin B}{sin A - sin B} = tan(\frac{A+B}{2}) cot(\frac{A-B}{2})$

$(v) \frac{cos A + cos B}{cos B - cos A} = cot(\frac{A+B}{2}) cot(\frac{A-B}{2})$

Solution:

$(i) \ \frac{sin\ A + sin\ 3A}{cos\ A - cos\ 3A} = cot\ A$

Solution:

Let us consider LHS:

$\frac{sin\ A + sin\ 3A}{cos\ A - cos\ 3A}$

By using the formulas,

Sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2

Cos A – cos B = - 2 sin (A+B)/2 sin (A-B)/2

So now,

$$\frac{sin\ 3A + sin\ A}{cos\ A - cos\ 3A} = \frac{2(sin\frac{A+3A}{2}\ cos\frac{3A-A}{2})}{-2(sin\frac{A+3A}{2}\ sin\frac{A-3A}{2})}$$

$$= -\frac{sin\frac{4A}{2}\ cos\frac{2A}{2}}{sin\frac{4A}{2}\ sin\frac{-2A}{2}}$$

$$= -\frac{cos\ A}{-sin A}\ (since, sin(-A) = -sin A)$$

$$= \frac{cos\ A}{sin\ A}$$

= cot A

= RHS

Hence proved.

$(ii) \frac{sin\ 9A - sin\ 7A}{cos 7A - cos\ 9A} = cot 8A$

Solution:

Let us consider LHS:

$\frac{sin\ 9A - sin\ 7A}{cos 7A - cos\ 9A}$

By using the formulas,

Sin A - sin B = 2 cos (A+B)/2 sin (A-B)/2
Cos A – cos B = - 2 sin (A+B)/2 sin (A-B)/2
So now,

$$\frac{sin9A - sin7A}{cos7A - cos9A} = \frac{2(cos\frac{9A+7A}{2}sin\frac{9A-7A}{2})}{-2(sin\frac{7A+9A}{2}sin\frac{7A-9A}{2})}$$

$$= -\frac{(cos\frac{16A}{2}sin\frac{2A}{2})}{(sin\frac{16A}{2}\ sin\frac{(-2A)}{2})}$$

$$= -\frac{(cos8A\ sin\ A)}{(-sin\ 8A\ sin\ A)}(since,\ sin(-A) = -sinA)$$

$$= \frac{cos\ 8A}{sin\ 8A}$$

= cot A
= RHS

Hence proved.

$$(iii)\frac{sinA - sinB}{cosA + cosB} = tan\frac{A - B}{2}$$

Solution:
Let us consider LHS:

$$\frac{sinA - sinB}{cosA + cosB}$$

By using the formulas,
Sin A - sin B = 2 cos (A+B)/2 sin (A-B)/2
Cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2
So now,

$$\frac{sin\ A - sinB}{cos\ A + cosB} = \frac{2(cos\ \frac{A+B}{2}\ sin\ \frac{A-B}{2})}{2(cos\ \frac{A+B}{2}\ cos\ \frac{A-B}{2})}$$

$$= \frac{sin(\frac{A-B}{2})}{cos(\frac{A-B}{2})}$$

$$= tan\frac{A - B}{2}$$

= RHS

Hence proved.

$(iv) \frac{sinA + sinB}{sinA - sinB} = tan(\frac{A+B}{2})cot(\frac{A-B}{2})$

Solution:

Let us consider LHS:

$\frac{sinA + sinB}{sinA - sinB}$

By using the formulas,

Sin A - sin B = 2 cos (A+B)/2 sin (A-B)/2

Sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2

So now,

$$\frac{sinA + sinB}{sinA - sinB} = \frac{2(sin\ \frac{A+B}{2}\ cos\ \frac{A-B}{2})}{2(cos\ \frac{A+B}{2}\ sin\ \frac{A-B}{2})}$$

$$= \frac{sin\frac{A+B}{2}\ cos\frac{A-B}{2}}{cos\frac{A+B}{2}\ sin\frac{A-B}{2}}$$

$$= tan(\frac{A+B}{2})cot(\frac{A-B}{2})$$

= RHS

Hence proved.

$(v) \frac{cosA + cosB}{cosB - cosA} = cot(\frac{A+B}{2})cot(\frac{A-B}{2})$

Solution:

Let us consider LHS:

$\frac{cosA + cosB}{cosB - cosA}$

By using the formulas,

cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2

cos A - cos B = -2 sin (A+B)/2 sin (A-B)/2

So now,

$$\frac{cosA + cosB}{cosB - cosA} = \frac{2(cos\frac{A+B}{2}\ cos\ \frac{A-B}{2})}{-2(sin\ \frac{A+B}{2}\ sin\ \frac{B-A}{2})}$$

$$= -\frac{cos\ \frac{A+B}{2}\ cos\ \frac{A-B}{2}}{sin\ \frac{A+B}{2}\ sin(-\frac{A-B}{2})}$$

$$= -\frac{cos\,\frac{A+B}{2}\,cos\,\frac{A-B}{2}}{-sin\,\frac{A+B}{2}\,sin(\frac{A-B}{2})}(since, sin(-x) = -sinx)$$

$$= cot(\frac{A+B}{2})cot(\frac{A-B}{2})$$

= RHS

Hence proved.

8. Prove that:

$$\text{(i)}\frac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} = \tan 3A$$

$$\text{(ii)}\frac{\cos 3A + 2\cos 5A + \cos 7A}{\cos A + 2\cos 3A + \cos 5A} = \frac{\cos 5A}{\cos 3A}$$

$$\text{(iii)}\frac{\cos 4A + \cos 3A + \cos 2A}{\sin 4A + \sin 3A + \sin 2A} = \cot 3A$$

$$\text{(iv)}\frac{\sin 3A + \sin 5A + \sin 7A + \sin 9A}{\cos 3A + \cos 5A + \cos 7A + \cos 9A} = \tan 6A$$

$$\text{(v)}\frac{\sin 5A - \sin 7A + \sin 8A - \sin 4A}{\cos 4A + \cos 7A - \cos 5A - \cos 8A} = \cot 6A$$

$$\text{(vi)}\frac{\sin 5A \cos 2A - \sin 6A \cos A}{\sin A \sin 2A - \cos 2A \cos 3A} = \tan A$$

$$\text{(vii)}\frac{\sin 11A \sin A + \sin 7A \sin 3A}{\cos 11A \sin A + \cos 7A \sin 3A} = \tan 8A$$

$$\text{(viii)}\frac{\sin 3A \cos 4A - \sin A \cos 2A}{\sin 4A \sin A + \cos 6A \cos A} = \tan 2A$$

$$\text{(ix)}\frac{\sin A \sin 2A + \sin 3A \sin 6A}{\sin A \cos 2A + \sin 3A \cos 6A} = \tan 5A$$

$$\text{(x)}\frac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \frac{\sin 3A}{\sin 5A}$$

(xi) $\frac{\sin(\theta+\phi) - 2\sin\theta + \sin(\theta-\phi)}{\cos(\theta+\phi) - 2\cos\theta + \cos(\theta-\phi)} = \tan\theta$

Solution:

(i) $\frac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} = \tan 3A$

Let us consider LHS:

$\frac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A}$

By using the formulas,

sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2

cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2

So now,

$$\frac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} = \frac{(\sin 5A + \sin A) + \sin 3A}{(\cos 5A + \cos A) + \cos 3A}$$

$$= \frac{\left(2\sin\frac{5A+A}{2}\cos\frac{5A-A}{2}\right) + \sin 3A}{\left(2\cos\frac{5A+A}{2}\cos\frac{5A-A}{2}\right) + \cos 3A}$$

$$= \frac{\left(2\sin\frac{6A}{2}\cos\frac{4A}{2}\right) + \sin 3A}{\left(2\cos\frac{6A}{2}\cos\frac{4A}{2}\right) + \cos 3A}$$

$$= \frac{(2\sin 3A\cos 2A) + \sin 3A}{(2\cos 3A\cos 2A) + \cos 3A}$$

$$= \frac{\sin 3A(2\cos 2A + 1)}{\cos 3A(2\cos 2A + 1)}$$

= tan 3A

= RHS

Hence proved.

(ii) $\frac{\cos 3A + 2\cos 5A + \cos 7A}{\cos A + 2\cos 3A + \cos 5A} = \frac{\cos 5A}{\cos 3A}$

Let us consider LHS:

$\frac{\cos 3A + 2\cos 5A + \cos 7A}{\cos A + 2\cos 3A + \cos 5A}$

By using the formula,
cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2
So now,

$$\frac{\cos 3A + 2\cos 5A + \cos 7A}{\cos A + 2\cos 3A + \cos 5A} = \frac{(\cos 7A + \cos 3A) + 2\cos 5A}{(\cos 5A + \cos A) + 2\cos 3A}$$

$$= \frac{\left(2\cos\frac{7A + 3A}{2}\cos\frac{7A - 3A}{2}\right) + 2\cos 5A}{\left(2\cos\frac{5A + A}{2}\cos\frac{5A - A}{2}\right) + 2\cos 3A}$$

$$= \frac{\left(2\cos\frac{10A}{2}\cos\frac{4A}{2}\right) + 2\cos 5A}{\left(2\cos\frac{6A}{2}\cos\frac{4A}{2}\right) + 2\cos 3A}$$

$$= \frac{(2\cos 5A\cos 2A) + 2\cos 5A}{(2\cos 3A\cos 2A) + 2\cos 3A}$$

$$= \frac{2\cos 5A(\cos 2A + 1)}{2\cos 3A(\cos 2A + 1)}$$

$$= \frac{\cos 5A}{\cos 3A}$$

= RHS

Hence proved.

(iii) $\frac{\cos 4A + \cos 3A + \cos 2A}{\sin 4A + \sin 3A + \sin 2A} = \cot 3A$

Let us consider LHS:

$$\frac{\cos 4A + \cos 3A + \cos 2A}{\sin 4A + \sin 3A + \sin 2A}$$

By using the formula,
cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2
sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2
So now,

$$\frac{\cos 4A + \cos 3A + \cos 2A}{\sin 4A + \sin 3A + \sin 2A} = \frac{(\cos 4A + \cos 2A) + \cos 3A}{(\sin 4A + \sin 2A) + \sin 3A}$$

$$= \frac{\left(2\cos\frac{4A+2A}{2}\cos\frac{4A-2A}{2}\right) + \cos 3A}{\left(2\sin\frac{4A+2A}{2}\cos\frac{4A-2A}{2}\right) + \sin 3A}$$

$$= \frac{\left(2\cos\frac{6A}{2}\cos\frac{2A}{2}\right) + \cos 3A}{\left(2\sin\frac{6A}{2}\cos\frac{2A}{2}\right) + \sin 3A}$$

$$= \frac{(2\cos 3A\cos A) + \cos 3A}{(2\sin 3A\cos A) + \sin 3A}$$

$$= \frac{\cos 3A\,(2\cos A + 1)}{\sin 3A\,(2\cos A + 1)}$$

= cot 3A

= RHS

Hence proved.

(iv) $\frac{\sin 3A + \sin 5A + \sin 7A + \sin 9A}{\cos 3A + \cos 5A + \cos 7A + \cos 9A} = \tan 6A$

Let us consider LHS:

$$\frac{\sin 3A + \sin 5A + \sin 7A + \sin 9A}{\cos 3A + \cos 5A + \cos 7A + \cos 9A}$$

By using the formulas,

cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2

sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2

So now,

$$\frac{\sin 3A + \sin 5A + \sin 7A + \sin 9A}{\cos 3A + \cos 5A + \cos 7A + \cos 9A}$$

When we rearrange we get,

$$= \frac{(\sin 9A + \sin 3A) + (\sin 7A + \sin 5A)}{(\cos 9A + \cos 3A) + (\cos 7A + \cos 5A)}$$

$$= \frac{\left(2\sin\frac{9A+3A}{2}\cos\frac{9A-3A}{2}\right) + \left(2\sin\frac{7A+5A}{2}\cos\frac{7A-5A}{2}\right)}{\left(2\cos\frac{9A+3A}{2}\cos\frac{9A-3A}{2}\right) + \left(2\cos\frac{7A+5A}{2}\cos\frac{7A-5A}{2}\right)}$$

$$= \frac{\left(2\sin\frac{12A}{2}\cos\frac{6A}{2}\right) + \left(2\sin\frac{12A}{2}\cos\frac{2A}{2}\right)}{\left(2\cos\frac{12A}{2}\cos\frac{6A}{2}\right) + \left(2\cos\frac{12A}{2}\cos\frac{2A}{2}\right)}$$

$$= \frac{(2\sin 6A\cos 3A) + (2\sin 6A\cos A)}{(2\cos 6A\cos 3A) + (2\cos 6A\cos A)}$$

$$= \frac{2\sin 6A\,(\cos 3A + \cos A)}{2\cos 6A\,(\cos 3A + \cos A)}$$

$$= \frac{\sin 6A}{\cos 6A}$$

= tan 6A

= RHS

Hence proved.

(v) $\frac{\sin 5A - \sin 7A + \sin 8A - \sin 4A}{\cos 4A + \cos 7A - \cos 5A - \cos 8A} = \cot 6A$

Let us consider LHS:

$$\frac{\sin 5A - \sin 7A + \sin 8A - \sin 4A}{\cos 4A + \cos 7A - \cos 5A - \cos 8A}$$

By using the formulas,

cos A - cos B = -2 sin (A+B)/2 sin (A-B)/2

sin A - sin B = 2 cos (A+B)/2 sin (A-B)/2

So now,

$$\frac{\sin 5A - \sin 7A + \sin 8A - \sin 4A}{\cos 4A + \cos 7A - \cos 5A - \cos 8A} = \frac{-(\sin 7A - \sin 5A) + (\sin 8A - \sin 4A)}{(\cos 7A - \cos 5A) - (\cos 8A - \cos 4A)}$$

When we rearrange we get,

$$= \frac{-\left(2\cos\frac{7A + 5A}{2}\sin\frac{7A - 5A}{2}\right) + \left(2\cos\frac{8A + 4A}{2}\sin\frac{8A - 4A}{2}\right)}{\left(-2\sin\frac{7A + 5A}{2}\sin\frac{7A - 5A}{2}\right) - \left(-2\sin\frac{8A + 4A}{2}\sin\frac{8A - 4A}{2}\right)}$$

$$= \frac{-\left(2\cos\frac{12A}{2}\sin\frac{2A}{2}\right) + \left(2\cos\frac{12A}{2}\sin\frac{4A}{2}\right)}{\left(-2\sin\frac{12A}{2}\sin\frac{2A}{2}\right) - \left(-2\sin\frac{12A}{2}\sin\frac{4A}{2}\right)}$$

$$= \frac{-(2\cos 6A\sin A) + (2\cos 6A\sin 2A)}{-(2\sin 6A\sin A) + (2\sin 6A\sin 2A)}$$

$$= \frac{2\cos 6A\,(-\sin A + \sin 2A)}{2\sin 6A\,(-\sin A + \sin 2A)}$$

$$= \frac{\cos 6A}{\sin 6A}$$

= cot 6A

= RHS

Hence proved.

(vi) $\frac{\sin 5A \cos 2A - \sin 6A \cos A}{\sin A \sin 2A - \cos 2A \cos 3A} = \tan A$

Let us consider LHS:

$\frac{\sin 5A \cos 2A - \sin 6A \cos A}{\sin A \sin 2A - \cos 2A \cos 3A}$

Let us multiply and divide the above expression by 2 we get,

$\frac{\sin 5A \cos 2A - \sin 6A \cos A}{\sin A \sin 2A - \cos 2A \cos 3A} = \frac{(2\sin 5A \ \cos 2A) - (2\sin 6A \cos A)}{(2\sin 2A \sin A) - (2\cos 3A \cos 3A)}$

We know that, 2 sin A sin B = cos (A – B) – cos (A + B),

2 sin A cos B = sin (A + B) + sin (A – B) and

2 cos A cos B = cos (A + B) + cos (A – B)

So now,

$= \frac{\{\sin(5A + 2A) + \sin(5A - 2A)\} - \{\sin(6A + A) + \sin(6A - A)\}}{\{\cos(2A - A) - \cos(2A + A)\} - \{\cos(3A + 2A) + \cos(3A - 2A)\}}$

$= \frac{\{\sin 7A + \sin 3A\} - \{\sin 7A + \sin 5A\}}{\{\cos A - \cos 3A\} - \{\cos 5A + \cos A\}}$

$= \frac{\sin 7A + \sin 3A - \sin 7A - \sin 5A}{\cos A - \cos 3A - \cos 5A - \cos A}$

$= \frac{\sin 3A - \sin 5A}{-(\cos 5A + \cos 3A)}$

$= \frac{-(\sin 5A - \sin 3A)}{-(\cos 5A + \cos 3A)}$

$= \frac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A}$

By using the formulas,

sin A – sin B = 2 cos (A+B)/2 sin (A-B)/2

cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2

so,

$$= \frac{2\cos\frac{5A+3A}{2}\sin\frac{5A-3A}{2}}{2\cos\frac{5A+3A}{2}\cos\frac{5A-3A}{2}}$$

$$= \frac{2\cos\frac{8A}{2}\sin\frac{2A}{2}}{2\cos\frac{8A}{2}\cos\frac{2A}{2}}$$

$$= \frac{\sin A}{\cos A}$$

= tan A

= RHS

Hence proved.

$$\text{(vii)} \frac{\sin 11A \sin A + \sin 7A \sin 3A}{\cos 11A \sin A + \cos 7A \sin 3A} = \tan 8A$$

Let us consider LHS:

$$\frac{\sin 11A \sin A + \sin 7A \sin 3A}{\cos 11A \sin A + \cos 7A \sin 3A}$$

Let us multiply and divide the above expression by 2 we get,

$$\frac{\sin 11A \sin A + \sin 7A \sin 3A}{\cos 11A \sin A + \cos 7A \sin 3A} = \frac{(2\sin 11A\ \sin A) + (2\sin 7A \sin 3A)}{(2\cos 11A\ \sin A) + (2\cos 7A\ \sin 3A)}$$

We know that, 2 sin A sin B = cos (A – B) – cos (A + B),

2 cos A sin B = sin (A + B) - sin (A – B)

So now,

$$= \frac{\{\cos(11A - A) - \cos(11A + A)\} + \{\cos(7A - 3A) - \cos(7A + 3A)\}}{\{\sin(11A + A) - \sin(11A - A)\} + \{\sin(7A + 3A) - \sin(7A - 3A)\}}$$

$$= \frac{\{\cos 10A - \cos 12A\} + \{\cos 4A - \cos 10A\}}{\{\sin 12A - \sin 10A\} + \{\sin 10A - \sin 4A\}}$$

$$= \frac{\cos 10A - \cos 12A + \cos 4A - \cos 10A}{\sin 12A - \sin 10A + \sin 10A - \sin 4A}$$

$$= \frac{-(\cos 12A - \cos 4A)}{\sin 12A - \sin 4A}$$

By using the formulas,

sin A – sin B = 2 cos (A+B)/2 sin (A-B)/2

cos A – cos B = -2 sin (A+B)/2 sin (A-B)/2 so,

$$= -\frac{-2\sin\frac{12A + 4A}{2}\sin\frac{12A - 4A}{2}}{2\cos\frac{12A + 4A}{2}\sin\frac{12A - 4A}{2}}$$

$$= \frac{\sin\frac{16A}{2}\sin\frac{8A}{2}}{\cos\frac{16A}{2}\sin\frac{8A}{2}}$$

$$= \frac{\sin 8A}{\cos 8A}$$

= tan 8A

= RHS

Hence proved.

$$\text{(viii)}\frac{\sin 3A\cos 4A - \sin A\cos 2A}{\sin 4A\sin A + \cos 6A\cos A} = \tan 2A$$

Let us consider LHS:

$$\frac{\sin 3A\cos 4A - \sin A\cos 2A}{\sin 4A\sin A + \cos 6A\cos A}$$

Let us multiply and divide the above expression by 2 we get,

$$\frac{\sin 3A\cos 4A - \sin A\cos 2A}{\sin 4A\sin A + \cos 6A\cos A} = \frac{(2\sin 3A\cos 4A) - (2\sin A\cos 2A)}{(2\sin 4A\sin A) + (2\cos 6A\cos A)}$$

We know that, 2 sin A sin B = cos (A – B) – cos (A + B),

2 cos A cos B = cos (A + B) + cos (A – B)

2 sin A cos B = sin (A + B) + sin (A – B)

So now,

$$= \frac{\{\sin(3A + 4A) + \sin(3A - 4A)\} - \{\sin(A + 2A) + \sin(A - 2A)\}}{\{\cos(4A - A) - \cos(4A + A)\} + \{\cos(6A + A) + \cos(6A - A)\}}$$

$$= \frac{\{\sin 7A + \sin(-A)\} - \{\sin 3A + \sin(-A)\}}{\{\cos 3A - \cos 5A\} + \{\cos 7A + \cos 5A\}}$$

$$= \frac{\{\sin 7A - \sin A\} - \{\sin 3A - \sin A\}}{\{\cos 3A - \cos 5A\} + \{\cos 7A + \cos 5A\}}$$ (since, (-A) = -sin A)

$$= \frac{\sin 7A - \sin A - \sin 3A + \sin A}{\cos 3A - \cos 5A + \cos 7A + \cos 5A}$$

$= \frac{\sin 7A - \sin 3A}{\cos 7A + \cos 3A}$

By using the formulas,

sin A – sin B = 2 cos (A+B)/2 sin (A-B)/2

cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2

so,

$= \frac{2\cos\frac{7A+3A}{2}\sin\frac{7A-3A}{2}}{2\cos\frac{7A+3A}{2}\cos\frac{7A-3A}{2}}$

$= \frac{\cos\frac{10A}{2}\sin\frac{4A}{2}}{\cos\frac{10A}{2}\cos\frac{4A}{2}}$

$= \frac{\sin 2A}{\cos 2A}$

= tan 2A

= RHS

Hence proved.

(ix) $\frac{\sin A \sin 2A + \sin 3A \sin 6A}{\sin A \cos 2A + \sin 3A \cos 6A} = \tan 5A$

Let us consider LHS:

$\frac{\sin A \sin 2A + \sin 3A \sin 6A}{\sin A \cos 2A + \sin 3A \cos 6A}$

Let us multiply and divide the above expression by 2 we get,

$\frac{\sin A \sin 2A + \sin 3A \sin 6A}{\sin A \cos 2A + \sin 3A \cos 6A} = \frac{(2\sin 2A \ \sin A) + (2\sin 6A \sin 3A)}{(2\sin A\cos 2A) + (2\sin 3A\cos 6A)}$

We know that, 2 sin A sin B = cos (A – B) – cos (A + B),

2 sin A cos B = sin (A + B) + sin (A – B)

So now,

$= \frac{\{\cos(2A - A) - \cos(2A + A)\} + \{\cos(6A - 3A) - \cos(6A + 3A)\}}{\{\sin(A + 2A) + \sin(A - 2A)\} + \{\sin(3A + 6A) + \sin(3A - 6A)\}}$

$= \frac{\{\cos A - \cos 3A\} + \{\cos 3A - \cos 9A\}}{\{\sin 3A + \sin(-A)\} + \{\sin 9A + \sin(-3A)\}}$

$= \frac{\{\cos A - \cos 3A\} + \{\cos 3A - \cos 9A\}}{\{\sin 3A - \sin A\} + \{\sin 9A - \sin 3A\}}$ (since, sin (-A) = -sin A)

$= \frac{\cos A - \cos 3A + \cos 3A - \cos 9A}{\sin 3A - \sin A + \sin 9A - \sin 3A}$

$= \frac{\cos A - \cos 9A}{\sin 9A - \sin A}$

$= \frac{-(\cos 9A - \cos A)}{\sin 9A - \sin A}$

By using the formulas,

sin A – sin B = 2 cos (A+B)/2 sin (A-B)/2

cos A – cos B = -2 sin (A+B)/2 sin (A-B)/2

so,

$= -\frac{-2\sin\frac{9A + A}{2}\sin\frac{9A - A}{2}}{2\cos\frac{9A + A}{2}\sin\frac{9A - A}{2}}$

$= \frac{\sin\frac{10A}{2}\sin\frac{8A}{2}}{\cos\frac{10A}{2}\sin\frac{8A}{2}}$

$= \frac{\sin 5A}{\cos 5A}$

= tan 5A

= RHS

Hence proved.

(x) $\frac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \frac{\sin 3A}{\sin 5A}$

Let us consider LHS:

$\frac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A}$

$\frac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \frac{(\sin 5A + \sin A) + 2\sin 3A}{(\sin 7A + \sin 3A) + 2\sin 5A}$

We know that,

Sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2

So now,

$$= \frac{\left(2\sin\frac{5A+A}{2}\cos\frac{5A-A}{2}\right) + 2\sin 3A}{\left(2\sin\frac{7A+3A}{2}\cos\frac{7A-3A}{2}\right) + 2\sin 5A}$$

$$= \frac{\left(2\sin\frac{6A}{2}\cos\frac{4A}{2}\right) + 2\sin 3A}{\left(2\sin\frac{10A}{2}\cos\frac{4A}{2}\right) + 2\sin 5A}$$

$$= \frac{(2\sin 3A\cos 2A) + 2\sin 3A}{(2\sin 5A\cos 2A) + 2\sin 5A}$$

$$= \frac{2\sin 3A\,(\cos 2A + 1)}{2\sin 5A\,(\cos 2A + 1)}$$

$$= \frac{\sin 3A}{\sin 5A}$$

= RHS

Hence proved.

(xi) $\frac{\sin(\theta+\phi) - 2\sin\theta + \sin(\theta-\phi)}{\cos(\theta+\phi) - 2\cos\theta + \cos(\theta-\phi)} = \tan\theta$

Let us consider LHS:

$$\frac{\sin(\theta+\phi) - 2\sin\theta + \sin(\theta-\phi)}{\cos(\theta+\phi) - 2\cos\theta + \cos(\theta-\phi)}$$

$$\frac{\sin(\theta+\phi) - 2\sin\theta + \sin(\theta-\phi)}{\cos(\theta+\phi) - 2\cos\theta + \cos(\theta-\phi)} = \frac{\{\sin(\theta+\Phi) + \sin(\theta-\Phi)\} - 2\sin\theta}{\{\cos(\theta+\Phi) + \cos(\theta-\Phi)\} - 2\cos\theta}$$

We know that,

Sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2

Cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2

So now,

$$= \frac{\left(2\sin\frac{\theta+\Phi+\theta-\Phi}{2}\cos\frac{\theta+\Phi-\theta+\Phi}{2}\right) - 2\sin\theta}{\left(2\cos\frac{\theta+\Phi+\theta-\Phi}{2}\cos\frac{\theta+\Phi-\theta+\Phi}{2}\right) - 2\cos\theta}$$

$$= \frac{\left(2\sin\frac{2\theta}{2}\cos\frac{2\Phi}{2}\right) - 2\sin\theta}{\left(2\cos\frac{2\theta}{2}\cos\frac{2\Phi}{2}\right) - 2\cos\theta}$$

$$= \frac{(2\sin\theta\cos\Phi) - 2\sin\theta}{(2\cos\theta\cos\Phi) - 2\cos\theta}$$

$$= \frac{2\sin\theta(\cos\Phi - 1)}{2\cos\theta(\cos\Phi - 1)}$$

$$= \frac{\sin\theta}{\cos\theta}$$

= tan θ

= RHS

Hence proved.

9. Prove that:

(i) sin α + sin β + sin γ – sin (α + β + γ) = 4 sin (α + β)/2 sin (β + γ)/2 sin (α + γ)/2

(ii) cos (A + B + C) + cos (A – B + C) + cos (A + B – C) + cos (-A + B + C) = 4 cos A cos B cos C

Solution:

(i) sin α + sin β + sin γ – sin (α + β + γ) = 4 sin (α + β)/2 sin (β + γ)/2 sin (α + γ)/2

Let us consider LHS:

sin α + sin β + sin γ – sin (α + β + γ)

By using the formulas,

Sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2

Sin A – sin B = 2 cos (A+B)/2 sin (A-B)/2

$$\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha + \beta + \gamma)$$

$$= (\sin\alpha + \sin\beta) + \{\sin\gamma - \sin(\alpha + \beta + \gamma)\}$$

$$= \left(2\sin\frac{\alpha + \beta}{2}\cos\frac{\alpha - \beta}{2}\right) + \left(2\cos\frac{\gamma + \alpha + \beta + \gamma}{2}\sin\frac{\gamma - \alpha - \beta - \gamma}{2}\right)$$

$$= \left(2\sin\frac{\alpha + \beta}{2}\cos\frac{\alpha - \beta}{2}\right) + \left(2\cos\frac{\alpha + \beta + 2\gamma}{2}\sin\frac{-(\alpha + \beta)}{2}\right)$$

$$= \left(2\sin\frac{\alpha + \beta}{2}\cos\frac{\alpha - \beta}{2}\right) - \left(2\cos\frac{\alpha + \beta + 2\gamma}{2}\sin\frac{\alpha + \beta}{2}\right)$$ (since,sin (-A) =-sin A)

$$= 2\sin\frac{\alpha + \beta}{2}\left(\cos\frac{\alpha - \beta}{2} - \cos\frac{\alpha + \beta + 2\gamma}{2}\right)$$

Again by using the formula,

Cos A – cos B = -2 sin (A+B)/2 sin (A-B)/2

$$= 2\sin\frac{\alpha+\beta}{2}\left(-2\sin\frac{\frac{\alpha-\beta}{2}+\frac{\alpha+\beta+2\gamma}{2}}{2}\sin\frac{\frac{\alpha-\beta}{2}-\frac{\alpha+\beta+2\gamma}{2}}{2}\right)$$

$$= 2\sin\frac{\alpha+\beta}{2}\left(-2\sin\frac{\frac{\alpha-\beta+\alpha+\beta+2\gamma}{2}}{2}\sin\frac{\frac{\alpha-\beta-(\alpha+\beta+2\gamma)}{2}}{2}\right)$$

$$= 2\sin\frac{\alpha+\beta}{2}\left(-2\sin\frac{\frac{2\alpha+2\gamma}{2}}{2}\sin\frac{\frac{\alpha-\beta-\alpha-\beta-2\gamma)}{2}}{2}\right)$$

$$= 2\sin\frac{\alpha+\beta}{2}\left(-2\sin\frac{\frac{2\alpha+2\gamma}{2}}{2}\sin\frac{\frac{-2\beta-2\gamma)}{2}}{2}\right)$$

$$= 2\sin\frac{\alpha+\beta}{2}\left(-2\sin\frac{\frac{2(\alpha+\gamma)}{2}}{2}\sin\frac{\frac{-2(\beta+\gamma)}{2}}{2}\right)$$

$$= 2\sin\frac{\alpha+\beta}{2}\left(-2\sin\frac{\alpha+\gamma}{2}\sin\frac{-(\beta+\gamma)}{2}\right)$$

$$= 2\sin\frac{\alpha+\beta}{2}\left(2\sin\frac{\alpha+\gamma}{2}\sin\frac{\beta+\gamma}{2}\right)$$

$$= 4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\alpha+\gamma}{2}$$

= RHS

Hence proved.

(ii) cos (A + B + C) + cos (A – B + C) + cos (A + B – C) + cos (-A + B + C) = 4 cos A cos B cos C

Let us consider LHS:

cos (A + B + C) + cos (A – B + C) + cos (A + B – C) + cos (-A + B + C)

so,

cos (A + B + C) + cos (A – B + C) + cos (A + B – C) + cos (-A + B + C) =

={cos (A + B + C) + cos (A – B + C)} + {cos (A + B – C) + cos (-A + B + C)}

By using the formula,

Cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2

$$= \left\{2\cos\frac{(A+B+C)+(A-B+C)}{2}\cos\frac{(A+B+C)-(A-B+C)}{2}\right\} + \left\{2\cos\frac{(A+B-C)+(-A+B+C)}{2}\cos\frac{(A+B-C)-(-A+B+C)}{2}\right\}$$

$$= \left\{2\cos\frac{A+B+C+A-B+C}{2}\cos\frac{A+B+C-A+B-C}{2}\right\} + \left\{2\cos\frac{A+B-C-A+B+C}{2}\cos\frac{A+B-C+A-B-C}{2}\right\}$$

$$= \left\{2\cos\frac{2A+2C}{2}\cos\frac{2B}{2}\right\} + \left\{2\cos\frac{2B}{2}\cos\frac{2A-2C}{2}\right\}$$

$$= \left\{2\cos\frac{2(A+C)}{2}\cos\frac{2B}{2}\right\} + \left\{2\cos\frac{2B}{2}\cos\frac{2(A-C)}{2}\right\}$$

= 2 cos (A + C) cos B + 2 cos B cos (A – C)

= 2 cos B {cos (A + C) + cos (A – C)}

By using the formula,

Cos A + cos B = 2 cos (A+B)/2 cos (A-B)/2

$$= 2\cos B\left\{2\cos\frac{A+C+(A-C)}{2}\cos\frac{A+C-(A-C)}{2}\right\}$$

$$= 2\cos B\left\{2\cos\frac{A+C+A-C}{2}\cos\frac{A+C-A+C}{2}\right\}$$

$$= 2\cos B\left\{2\cos\frac{2A}{2}\cos\frac{2C}{2}\right\}$$

= 4 cos A cos B cos C

= RHS

Hence proved.