

EXERCISE 9.1

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Prove the following identities:

1. $\sqrt{[(1 - \cos 2x) / (1 + \cos 2x)]} = \tan x$

Solution:

Let us consider LHS:

$$\sqrt{[(1 - \cos 2x) / (1 + \cos 2x)]}$$

$$\begin{aligned}\text{We know that } \cos 2x &= 1 - 2 \sin^2 x \\ &= 2 \cos^2 x - 1\end{aligned}$$

So,

$$\begin{aligned}\sqrt{[(1 - \cos 2x) / (1 + \cos 2x)]} &= \sqrt{[(1 - (1 - 2 \sin^2 x)) / (1 + (2 \cos^2 x - 1))]} \\ &= \sqrt{[(1 - 1 + 2 \sin^2 x) / (1 + 2 \cos^2 x - 1)]} \\ &= \sqrt{[2 \sin^2 x / 2 \cos^2 x]} \\ &= \sin x / \cos x \\ &= \tan x \\ &= \text{RHS}\end{aligned}$$

Hence proved.

2. $\sin 2x / (1 - \cos 2x) = \cot x$

Solution:

Let us consider LHS:

$$\sin 2x / (1 - \cos 2x)$$

$$\text{We know that } \cos 2x = 1 - 2 \sin^2 x$$

$$\sin 2x = 2 \sin x \cos x$$

So,

$$\begin{aligned}\sin 2x / (1 - \cos 2x) &= (2 \sin x \cos x) / (1 - (1 - 2 \sin^2 x)) \\ &= (2 \sin x \cos x) / (1 - 1 + 2 \sin^2 x) \\ &= [2 \sin x \cos x / 2 \sin^2 x] \\ &= \cos x / \sin x \\ &= \cot x \\ &= \text{RHS}\end{aligned}$$

Hence proved.

3. $\sin 2x / (1 + \cos 2x) = \tan x$

Solution:

Let us consider LHS:

$$\sin 2x / (1 + \cos 2x)$$

$$\begin{aligned}\text{We know that } \cos 2x &= 1 - 2 \sin^2 x \\ &= 2 \cos^2 x - 1\end{aligned}$$

$$\sin 2x = 2 \sin x \cos x$$

So,

$$\begin{aligned} \sin 2x / (1 + \cos 2x) &= [2 \sin x \cos x / (1 + (2\cos^2 x - 1))] \\ &= [2 \sin x \cos x / (1 + 2\cos^2 x - 1)] \\ &= [2 \sin x \cos x / 2 \cos^2 x] \\ &= \sin x / \cos x \\ &= \tan x \\ &= \text{RHS} \end{aligned}$$

Hence proved.

4. $\sqrt{2 + \sqrt{2 + 2 \cos 4x}} = 2 \cos x, 0 < x < \frac{\pi}{4}$

Solution:

Let us consider LHS:

$$\begin{aligned} \sqrt{2 + \sqrt{2 + 2 \cos 4x}} &= \sqrt{2 + \sqrt{2 + 2(2 \cos^2 2x - 1)}} \\ \{\text{since, } \cos 2x &= 2 \cos^2 x - 1 \Rightarrow \cos 4x = 2 \cos^2 2x - 1\} \\ &= \sqrt{2 + \sqrt{2 + 4 \cos^2 2x - 2}} \\ &= \sqrt{2 + \sqrt{4 \cos^2 2x}} \\ &= \sqrt{2 + 2 \cos 2x} \\ &= \sqrt{2 + 2(2 \cos^2 x - 1)} \quad \{\text{since, } \cos 2x = 2 \cos^2 x - 1\} \\ &= \sqrt{2 + 4 \cos^2 x - 2} \\ &= \sqrt{4 \cos^2 x} \\ &= 2 \cos x \\ &= \text{RHS} \end{aligned}$$

Hence proved.

5. $[1 - \cos 2x + \sin 2x] / [1 + \cos 2x + \sin 2x] = \tan x$

Solution:

Let us consider LHS:

$$[1 - \cos 2x + \sin 2x] / [1 + \cos 2x + \sin 2x]$$

$$\begin{aligned} \text{We know that, } \cos 2x &= 1 - 2 \sin^2 x \\ &= 2 \cos^2 x - 1 \end{aligned}$$

$$\sin 2x = 2 \sin x \cos x$$

So,

$$\begin{aligned} &= \frac{1 - (1 - 2 \sin^2 x) + 2 \sin x \cos x}{1 + (2 \cos^2 x - 1) + 2 \sin x \cos x} \\ &= \frac{1 - 1 + 2 \sin^2 x + 2 \sin x \cos x}{1 + 2 \cos^2 x - 1 + 2 \sin x \cos x} \\ &= \frac{2 \sin^2 x + 2 \sin x \cos x}{2 \cos^2 x + 2 \sin x \cos x} \\ &= \frac{2 \sin x (\sin x + \cos x)}{2 \cos x (\cos x + \sin x)} \\ &= \frac{\sin x}{\cos x} \end{aligned}$$

$$= \tan x$$

$$= \text{RHS}$$

Hence proved.

$$6. [\sin x + \sin 2x] / [1 + \cos x + \cos 2x] = \tan x$$

Solution:

Let us consider LHS:

$$[\sin x + \sin 2x] / [1 + \cos x + \cos 2x]$$

We know that, $\cos 2x = \cos^2 x - \sin^2 x$

$$\sin 2x = 2 \sin x \cos x$$

So,

$$\begin{aligned} \frac{\sin x + \sin 2x}{1 + \cos x + \cos 2x} &= \frac{\sin x + 2 \sin x \cos x}{1 + \cos x + (2 \cos^2 x - 1)} \\ &= \frac{\sin x + 2 \sin x \cos x}{1 + \cos x + 2 \cos^2 x - 1} \\ &= \frac{\sin x + 2 \sin x \cos x}{\cos x + 2 \cos^2 x} \\ &= \frac{\sin x (1 + 2 \cos x)}{\cos x (1 + 2 \cos x)} \\ &= \frac{\sin x}{\cos x} \\ &= \tan x \end{aligned}$$

= RHS

Hence proved.

$$7. \cos 2x / (1 + \sin 2x) = \tan (\pi/4 - x)$$

Solution:

Let us consider LHS:

$$\cos 2x / (1 + \sin 2x)$$

We know that, $\cos 2x = \cos^2 x - \sin^2 x$

$$\sin 2x = 2 \sin x \cos x$$

So,

$$\begin{aligned} \frac{\cos 2x}{1 + \sin 2x} &= \frac{\cos^2 x - \sin^2 x}{1 + 2 \sin x \cos x} \\ &= \frac{(\cos x - \sin x)(\cos x + \sin x)}{\sin^2 x + \cos^2 x + 2 \sin x \cos x} \\ (\text{since, } a^2 - b^2 &= (a - b)(a + b) \text{ \& } \sin^2 x + \cos^2 x = 1) \\ &= \frac{(\cos x - \sin x)(\cos x + \sin x)}{(\sin x + \cos x)^2} \\ (\text{since, } a^2 + b^2 + 2ab &= (a + b)^2) \\ &= \frac{(\cos x - \sin x)(\cos x + \sin x)}{(\sin x + \cos x)(\sin x + \cos x)} \\ &= \frac{(\cos x - \sin x)}{(\sin x + \cos x)} \end{aligned}$$

Multiplying numerator and denominator by $1/\sqrt{2}$

We get,

$$\begin{aligned} &= \frac{\frac{1}{\sqrt{2}}(\cos x - \sin x)}{\frac{1}{\sqrt{2}}(\sin x + \cos x)} \\ &= \frac{\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)}{\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)} \\ &= \frac{\left(\sin \frac{\pi}{4}\cos x - \cos \frac{\pi}{4}\sin x\right)}{\left(\sin \frac{\pi}{4}\sin x + \cos \frac{\pi}{4}\cos x\right)} \quad (\text{since, } 1/\sqrt{2} = \sin \pi/4) \end{aligned}$$

$$\frac{\sin\left(\frac{\pi}{4} - x\right)}{\cos\left(\frac{\pi}{4} - x\right)}$$

By using the formulas,

$$\sin(A - B) = \sin A \cos B - \sin B \cos A$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$= \tan(\pi/4 - x)$$

$$= \text{RHS}$$

Hence proved.

$$8. \cos x / (1 - \sin x) = \tan(\pi/4 + x/2)$$

Solution:

Let us consider LHS:

$$\cos x / (1 - \sin x)$$

We know that, $\cos 2x = \cos^2 x - \sin^2 x$

$$\cos x = \cos^2 x/2 - \sin^2 x/2$$

$$\sin 2x = 2 \sin x \cos x$$

$$\sin x = 2 \sin x/2 \cos x/2$$

So,

$$\begin{aligned} \frac{\cos x}{1 - \sin x} &= \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{1 - 2 \sin \frac{x}{2} \cos \frac{x}{2}} \\ &= \frac{\left(\cos \frac{x}{2} - \sin \frac{x}{2}\right)\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}} \end{aligned}$$

(By using the formula, $a^2 - b^2 = (a - b)(a + b)$ & $\sin^2 x + \cos^2 x = 1$)

$$= \frac{\left(\cos \frac{x}{2} - \sin \frac{x}{2}\right)\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)}{\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)^2}$$

(By using the formula, $a^2 + b^2 + 2ab = (a + b)^2$)

$$\begin{aligned} &= \frac{\left(\cos \frac{x}{2} - \sin \frac{x}{2}\right)\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)}{\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)} \\ &= \frac{\left(\cos \frac{x}{2} - \sin \frac{x}{2}\right)}{\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)} \end{aligned}$$

$$= \frac{\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)}{\left(\sin \frac{x}{2} - \cos \frac{x}{2}\right)}$$

Let us multiply numerator and denominator by $1/\sqrt{2}$

We get,

$$\begin{aligned} &= \frac{\frac{1}{\sqrt{2}}\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)}{\frac{1}{\sqrt{2}}\left(\sin \frac{x}{2} - \cos \frac{x}{2}\right)} \\ &= \frac{\left(\frac{1}{\sqrt{2}}\cos \frac{x}{2} + \frac{1}{\sqrt{2}}\sin \frac{x}{2}\right)}{\left(\frac{1}{\sqrt{2}}\sin \frac{x}{2} - \frac{1}{\sqrt{2}}\cos \frac{x}{2}\right)} \\ &= \frac{\left(\sin \frac{\pi}{4}\cos \frac{x}{2} + \cos \frac{\pi}{4}\sin \frac{x}{2}\right)}{\left(\sin \frac{\pi}{4}\sin \frac{x}{2} - \cos \frac{\pi}{4}\cos \frac{x}{2}\right)} \quad (\text{since, } 1/\sqrt{2} = \sin \pi/4) \\ &= \frac{\sin\left(\frac{\pi}{4} - x\right)}{\cos\left(\frac{\pi}{4} - x\right)} \\ &= \tan(\pi/4 - x) \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$9. \cos^2 \frac{\pi}{8} + \cos^2 \frac{3\pi}{8} + \cos^2 \frac{5\pi}{8} + \cos^2 \frac{7\pi}{8} = 2$$

Solution:

Let us consider LHS:

$$\cos^2 \frac{\pi}{8} + \cos^2 \frac{3\pi}{8} + \cos^2 \frac{5\pi}{8} + \cos^2 \frac{7\pi}{8}$$

We know that $\cos 2x = 2\cos^2 x - 1$

$$\cos 2x + 1 = 2\cos^2 x$$

$$\cos^2 x = (\cos 2x + 1)/2$$

So,

$$\cos^2 \frac{\pi}{8} + \cos^2 \frac{3\pi}{8} + \cos^2 \frac{5\pi}{8} + \cos^2 \frac{7\pi}{8}$$

$$\begin{aligned}
 &= \frac{1 + \cos \frac{2\pi}{8}}{2} + \frac{1 + \cos \frac{6\pi}{8}}{2} + \frac{1 + \cos \frac{10\pi}{8}}{2} + \frac{1 + \cos \frac{14\pi}{8}}{2} \\
 &= \frac{1 + \cos \frac{2\pi}{8}}{2} + \frac{1 + \cos \left(\pi - \frac{2\pi}{8} \right)}{2} + \frac{1 + \cos \left(\pi + \frac{2\pi}{8} \right)}{2} + \frac{1 + \cos \left(2\pi - \frac{2\pi}{8} \right)}{2} \\
 &\left\{ \because \pi - \frac{2\pi}{8} = \frac{6\pi}{8}; \pi + \frac{2\pi}{8} = \frac{10\pi}{8}; 2\pi - \frac{2\pi}{8} = \frac{14\pi}{8} \right\} \\
 &= \frac{1 + \cos \frac{2\pi}{8}}{2} + \frac{1 - \cos \frac{2\pi}{8}}{2} + \frac{1 - \cos \frac{2\pi}{8}}{2} + \frac{1 + \cos \frac{2\pi}{8}}{2} \\
 &\{ \text{we know, } \cos(\pi - A) = -\cos A, \cos(\pi + A) = -\cos A \text{ \& } \cos(2\pi - A) = \cos A \} \\
 &= 2 \times \frac{1 + \cos \frac{2\pi}{8}}{2} + 2 \times \frac{1 - \cos \frac{2\pi}{8}}{2} \\
 &= 1 + \cos \frac{2\pi}{8} + 1 - \cos \frac{2\pi}{8} \\
 &= 2 \\
 &= \text{RHS} \\
 &\text{Hence proved.}
 \end{aligned}$$

10. $\sin^2 \frac{\pi}{8} + \sin^2 \frac{3\pi}{8} + \sin^2 \frac{5\pi}{8} + \sin^2 \frac{7\pi}{8} = 2$

Solution:

Let us consider LHS:

$$\sin^2 \frac{\pi}{8} + \sin^2 \frac{3\pi}{8} + \sin^2 \frac{5\pi}{8} + \sin^2 \frac{7\pi}{8}$$

We know that, $\cos 2x = 1 - 2\sin^2 x$

$$2\sin^2 x = 1 - \cos 2x$$

$$\sin^2 x = (1 - \cos 2x)/2$$

So,

$$\begin{aligned}
 &= \frac{1 - \cos \frac{2\pi}{8}}{2} + \frac{1 - \cos \frac{6\pi}{8}}{2} + \frac{1 - \cos \frac{10\pi}{8}}{2} + \frac{1 - \cos \frac{14\pi}{8}}{2} \\
 &= \frac{1 - \cos \frac{2\pi}{8}}{2} + \frac{1 - \cos \left(\pi - \frac{2\pi}{8} \right)}{2} + \frac{1 - \cos \left(\pi + \frac{2\pi}{8} \right)}{2} + \frac{1 - \cos \left(2\pi - \frac{2\pi}{8} \right)}{2}
 \end{aligned}$$

$$\left\{ \because \pi - \frac{2\pi}{8} = \frac{6\pi}{8}; \pi + \frac{2\pi}{8} = \frac{10\pi}{8}; 2\pi - \frac{2\pi}{8} = \frac{14\pi}{8} \right\}$$

$$= \frac{1 - \cos \frac{2\pi}{8}}{2} + \frac{1 - \left(-\cos \frac{2\pi}{8}\right)}{2} + \frac{1 - \left(-\cos \frac{2\pi}{8}\right)}{2} + \frac{1 - \cos \frac{2\pi}{8}}{2}$$

{we know, $\cos(\pi - A) = -\cos A$, $\cos(\pi + A) = -\cos A$ & $\cos(2\pi - A) = \cos A$ }

$$= \frac{1 - \cos \frac{2\pi}{8}}{2} + \frac{1 + \cos \frac{2\pi}{8}}{2} + \frac{1 + \cos \frac{2\pi}{8}}{2} + \frac{1 - \cos \frac{2\pi}{8}}{2}$$

$$= 2 \times \frac{1 - \cos \frac{2\pi}{8}}{2} + 2 \times \frac{1 + \cos \frac{2\pi}{8}}{2}$$

$$= 1 - \cos \frac{2\pi}{8} + 1 + \cos \frac{2\pi}{8}$$

$$= 2$$

$$= \text{RHS}$$

Hence proved.

11. $(\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2 = 4 \cos^2 (\alpha - \beta)/2$

Solution:

Let us consider LHS:

$$(\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2$$

Upon expansion, we get,

$$\begin{aligned} (\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2 &= \cos^2 \alpha + \cos^2 \beta + 2 \cos \alpha \cos \beta + \sin^2 \alpha + \sin^2 \beta + 2 \sin \alpha \sin \beta \\ &= 2 + 2 \cos \alpha \cos \beta + 2 \sin \alpha \sin \beta \\ &= 2 (1 + \cos \alpha \cos \beta + \sin \alpha \sin \beta) \\ &= 2 (1 + \cos (\alpha - \beta)) \text{ [since, } \cos (A - B) = \cos A \cos B + \sin A \sin B \text{]} \\ &= 2 (1 + 2 \cos^2 (\alpha - \beta)/2 - 1) \text{ [since, } \cos 2x = 2 \cos^2 x - 1 \text{]} \\ &= 2 (2 \cos^2 (\alpha - \beta)/2) \\ &= 4 \cos^2 (\alpha - \beta)/2 \\ &= \text{RHS} \end{aligned}$$

Hence Proved.

12. $\sin^2 (\pi/8 + x/2) - \sin^2 (\pi/8 - x/2) = 1/\sqrt{2} \sin x$

Solution:

Let us consider LHS:

$$\sin^2 (\pi/8 + x/2) - \sin^2 (\pi/8 - x/2)$$

we know, $\sin^2 A - \sin^2 B = \sin (A+B) \sin (A-B)$

so,

$$\begin{aligned}\sin^2 (\pi/8 + x/2) - \sin^2 (\pi/8 - x/2) &= \sin (\pi/8 + x/2 + \pi/8 - x/2) \sin (\pi/8 + x/2 - (\pi/8 - x/2)) \\ &= \sin (\pi/8 + \pi/8) \sin (\pi/8 + x/2 - \pi/8 + x/2) \\ &= \sin \pi/4 \sin x \\ &= 1/\sqrt{2} \sin x \text{ [since, since } \pi/4 = 1/\sqrt{2}] \\ &= \text{RHS}\end{aligned}$$

Hence proved.

13. $1 + \cos^2 2x = 2 (\cos^4 x + \sin^4 x)$

Solution:

Let us consider LHS:

$$1 + \cos^2 2x$$

We know, $\cos 2x = \cos^2 x - \sin^2 x$

$$\cos^2 x + \sin^2 x = 1$$

so,

$$\begin{aligned}1 + \cos^2 2x &= (\cos^2 x + \sin^2 x)^2 + (\cos^2 x - \sin^2 x)^2 \\ &= (\cos^4 x + \sin^4 x + 2 \cos^2 x \sin^2 x) + (\cos^4 x + \sin^4 x - 2 \cos^2 x \sin^2 x) \\ &= \cos^4 x + \sin^4 x + \cos^4 x + \sin^4 x \\ &= 2 \cos^4 x + 2 \sin^4 x \\ &= 2 (\cos^4 x + \sin^4 x) \\ &= \text{RHS}\end{aligned}$$

Hence proved.

14. $\cos^3 2x + 3 \cos 2x = 4 (\cos^6 x - \sin^6 x)$

Solution:

Let us consider RHS:

$$4 (\cos^6 x - \sin^6 x)$$

Upon expansion we get,

$$\begin{aligned}4 (\cos^6 x - \sin^6 x) &= 4 [(\cos^2 x)^3 - (\sin^2 x)^3] \\ &= 4 (\cos^2 x - \sin^2 x) (\cos^4 x + \sin^4 x + \cos^2 x \sin^2 x)\end{aligned}$$

By using the formula,

$$a^3 - b^3 = (a-b) (a^2 + b^2 + ab)$$

$$= 4 \cos 2x (\cos^4 x + \sin^4 x + \cos^2 x \sin^2 x + \cos^2 x \sin^2 x - \cos^2 x \sin^2 x)$$

We know, $\cos 2x = \cos^2 x - \sin^2 x$

So,

$$\begin{aligned}&= 4 \cos 2x (\cos^4 x + \sin^4 x + 2 \cos^2 x \sin^2 x - \cos^2 x \sin^2 x) \\ &= 4 \cos 2x [(\cos^2 x)^2 + (\sin^2 x)^2 + 2 \cos^2 x \sin^2 x - \cos^2 x \sin^2 x]\end{aligned}$$

We know, $a^2 + b^2 + 2ab = (a + b)^2$

$$= 4 \cos 2x [(1)^2 - 1/4 (4 \cos^2 x \sin^2 x)]$$

$$= 4 \cos 2x [(1)^2 - 1/4 (2 \cos x \sin x)^2]$$

We know, $\sin 2x = 2 \sin x \cos x$

$$= 4 \cos 2x [(1)^2 - 1/4 (\sin 2x)^2]$$

$$= 4 \cos 2x (1 - 1/4 \sin^2 2x)$$

We know, $\sin^2 x = 1 - \cos^2 x$

$$= 4 \cos 2x [1 - 1/4 (1 - \cos^2 2x)]$$

$$= 4 \cos 2x [1 - 1/4 + 1/4 \cos^2 2x]$$

$$= 4 \cos 2x [3/4 + 1/4 \cos^2 2x]$$

$$= 4 (3/4 \cos 2x + 1/4 \cos^3 2x)$$

$$= 3 \cos 2x + \cos^3 2x$$

$$= \cos^3 2x + 3 \cos 2x$$

$$= \text{LHS}$$

Hence proved.

15. $(\sin 3x + \sin x) \sin x + (\cos 3x - \cos x) \cos x = 0$

Solution:

Let us consider LHS:

$$(\sin 3x + \sin x) \sin x + (\cos 3x - \cos x) \cos x$$

$$= (\sin 3x) (\sin x) + \sin^2 x + (\cos 3x) (\cos x) - \cos^2 x$$

$$= [(\sin 3x) (\sin x) + (\cos 3x) (\cos x)] + (\sin^2 x - \cos^2 x)$$

$$= [(\sin 3x) (\sin x) + (\cos 3x) (\cos x)] - (\cos^2 x - \sin^2 x)$$

$$= \cos (3x - x) - \cos 2x$$

We know, $\cos 2x = \cos^2 x - \sin^2 x$

$\cos A \cos B + \sin A \sin B = \cos(A - B)$

So,

$$= \cos 2x - \cos 2x$$

$$= 0$$

$$= \text{RHS}$$

Hence Proved.

16. $\cos^2 (\pi/4 - x) - \sin^2 (\pi/4 - x) = \sin 2x$

Solution:

Let us consider LHS:

$$\cos^2 (\pi/4 - x) - \sin^2 (\pi/4 - x)$$

We know, $\cos^2 A - \sin^2 A = \cos 2A$

So,

$$\cos^2 (\pi/4 - x) - \sin^2 (\pi/4 - x) = \cos 2 (\pi/4 - x)$$

$$= \cos (\pi/2 - 2x)$$

$$\begin{aligned} &= \sin 2x \text{ [since, } \cos(\pi/2 - A) = \sin A] \\ &= \text{RHS} \end{aligned}$$

Hence proved.

17. $\cos 4x = 1 - 8 \cos^2 x + 8 \cos^4 x$

Solution:

Let us consider LHS:

$$\cos 4x$$

$$\text{We know, } \cos 2x = 2 \cos^2 x - 1$$

So,

$$\begin{aligned} \cos 4x &= 2 \cos^2 2x - 1 \\ &= 2(2 \cos^2 2x - 1)^2 - 1 \\ &= 2[(2 \cos^2 2x)^2 + 1^2 - 2 \times 2 \cos^2 2x] - 1 \\ &= 2(4 \cos^4 2x + 1 - 4 \cos^2 2x) - 1 \\ &= 8 \cos^4 2x + 2 - 8 \cos^2 2x - 1 \\ &= 8 \cos^4 2x + 1 - 8 \cos^2 2x \\ &= \text{RHS} \end{aligned}$$

Hence Proved.

18. $\sin 4x = 4 \sin x \cos^3 x - 4 \cos x \sin^3 x$

Solution:

Let us consider LHS:

$$\sin 4x$$

$$\text{We know, } \sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

So,

$$\begin{aligned} \sin 4x &= 2 \sin 2x \cos 2x \\ &= 2 (2 \sin x \cos x) (\cos^2 x - \sin^2 x) \\ &= 4 \sin x \cos x (\cos^2 x - \sin^2 x) \\ &= 4 \sin x \cos^3 x - 4 \sin^3 x \cos x \\ &= \text{RHS} \end{aligned}$$

Hence proved.

19. $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) = 13$

Solution:

Let us consider LHS:

$$3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x)$$

$$\text{We know, } (a + b)^2 = a^2 + b^2 + 2ab$$

$$(a - b)^2 = a^2 + b^2 - 2ab$$

$$a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$$

So,

$$\begin{aligned} 3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) &= 3\{(\sin x - \cos x)^2\}^2 + 6\{(\sin x)^2 + (\cos x)^2 + 2\sin x \cos x\} + 4\{(\sin^2 x)^3 + (\cos^2 x)^3\} \\ &= 3\{(\sin x)^2 + (\cos x)^2 - 2\sin x \cos x\}^2 + 6(\sin^2 x + \cos^2 x + 2\sin x \cos x) + 4\{(\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x)\} \end{aligned}$$

$$= 3(1 - 2\sin x \cos x)^2 + 6(1 + 2\sin x \cos x) + 4\{(1)(\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x)\}$$

$$\text{We know, } \sin^2 x + \cos^2 x = 1$$

So,

$$\begin{aligned} &= 3\{1^2 + (2\sin x \cos x)^2 - 4\sin x \cos x\} + 6(1 + 2\sin x \cos x) + 4\{(\sin^2 x)^2 + (\cos^2 x)^2 + 2\sin^2 x \cos^2 x - 3\sin^2 x \cos^2 x\} \\ &= 3\{1 + 4\sin^2 x \cos^2 x - 4\sin x \cos x\} + 6(1 + 2\sin x \cos x) + 4\{(\sin^2 x + \cos^2 x)^2 - 3\sin^2 x \cos^2 x\} \\ &= 3 + 12\sin^2 x \cos^2 x - 12\sin x \cos x + 6 + 12\sin x \cos x + 4\{(1)^2 - 3\sin^2 x \cos^2 x\} \\ &= 9 + 12\sin^2 x \cos^2 x + 4(1 - 3\sin^2 x \cos^2 x) \\ &= 9 + 12\sin^2 x \cos^2 x + 4 - 12\sin^2 x \cos^2 x \\ &= 13 \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$20. 2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1 = 0$$

Solution:

Let us consider LHS:

$$2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1$$

$$\text{We know, } (a + b)^2 = a^2 + b^2 + 2ab$$

$$a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$$

So,

$$\begin{aligned} 2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1 &= 2\{(\sin^2 x)^3 + (\cos^2 x)^3\} - 3\{(\sin^2 x)^2 + (\cos^2 x)^2\} + 1 \\ &= 2\{(\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x)\} - 3\{(\sin^2 x)^2 + (\cos^2 x)^2 + 2\sin^2 x \cos^2 x - 2\sin^2 x \cos^2 x\} + 1 \\ &= 2\{(1)(\sin^4 x + \cos^4 x + 2\sin^2 x \cos^2 x - 3\sin^2 x \cos^2 x)\} - 3\{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x\} + 1 \end{aligned}$$

We know, $\sin^2 x + \cos^2 x = 1$

$$2\sin^2 x \cos^2 x\} + 1$$

$$= 2\{(\sin^2 x + \cos^2 x)^2 - 3 \sin^2 x \cos^2 x\} - 3\{(1)^2 -$$

$$= 2\{(1)^2 - 3 \sin^2 x \cos^2 x\} - 3(1 - 2\sin^2 x \cos^2 x) + 1$$

$$= 2(1 - 3 \sin^2 x \cos^2 x) - 3 + 6 \sin^2 x \cos^2 x + 1$$

$$= 2 - 6 \sin^2 x \cos^2 x - 2 + 6 \sin^2 x \cos^2 x$$

$$= 0$$

$$= \text{RHS}$$

Hence proved.

21. $\cos^6 x - \sin^6 x = \cos 2x (1 - \frac{1}{4} \sin^2 2x)$

Solution:

Let us consider LHS:

$$\cos^6 x - \sin^6 x$$

$$\text{We know, } (a + b)^2 = a^2 + b^2 + 2ab$$

$$a^3 - b^3 = (a - b)(a^2 + b^2 + ab)$$

So,

$$\begin{aligned} \cos^6 x - \sin^6 x &= (\cos^2 x)^3 - (\sin^2 x)^3 \\ &= (\cos^2 x - \sin^2 x)(\cos^4 x + \sin^4 x + \cos^2 x \sin^2 x) \end{aligned}$$

$$\text{We know, } \cos 2x = \cos^2 x - \sin^2 x$$

So,

$$\begin{aligned} &= \cos 2x [(\cos^2 x)^2 + (\sin^2 x)^2 + 2 \cos^2 x \sin^2 x - \cos^2 x \sin^2 x] \\ &= \cos 2x [(\cos^2 x)^2 + (\sin^2 x)^2 - \frac{1}{4} \times 4 \cos^2 x \sin^2 x] \end{aligned}$$

$$\text{We know, } \sin^2 x + \cos^2 x = 1$$

So,

$$= \cos 2x [(1)^2 - \frac{1}{4} \times (2 \cos x \sin x)^2]$$

$$\text{We know, } \sin 2x = 2 \sin x \cos x$$

So,

$$\begin{aligned} &= \cos 2x [1 - \frac{1}{4} \times (\sin 2x)^2] \\ &= \cos 2x [1 - \frac{1}{4} \times \sin^2 2x] \\ &= \text{RHS} \end{aligned}$$

Hence proved.

22. $\tan(\pi/4 + x) + \tan(\pi/4 - x) = 2 \sec 2x$

Solution:

Let us consider LHS:

$$\tan(\pi/4 + x) + \tan(\pi/4 - x)$$

We know,

$$\tan (A+B) = (\tan A + \tan B)/(1 - \tan A \tan B)$$

$$\tan (A-B) = (\tan A - \tan B)/(1 + \tan A \tan B)$$

So,

$$\tan\left(\frac{\pi}{4} + x\right) + \tan\left(\frac{\pi}{4} - x\right) = \frac{\tan\frac{\pi}{4} + \tan x}{1 - \tan\frac{\pi}{4}\tan x} + \frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}$$

We know, $\tan \pi/4 = 1$

So,

$$\begin{aligned} &= \frac{1 + \tan x}{1 - \tan x} + \frac{1 - \tan x}{1 + \tan x} \\ &= \frac{(1 + \tan x)^2 + (1 - \tan x)^2}{(1 - \tan x)(1 + \tan x)} \end{aligned}$$

We know, $(a - b)(a + b) = a^2 - b^2$;

$$(a + b)^2 = a^2 + b^2 + 2ab \text{ \&}$$

$$(a - b)^2 = a^2 + b^2 - 2ab$$

So,

$$\begin{aligned} &= \frac{1^2 + \tan^2 x + 2 \tan x + 1^2 + \tan^2 x - 2 \tan x}{1^2 - \tan^2 x} \\ &= \frac{1 + \tan^2 x + 1 + \tan^2 x}{1 - \tan^2 x} \\ &= \frac{2(1 + \tan^2 x)}{1 - \tan^2 x} \end{aligned}$$

We know, $\tan x = \sin x / \cos x$

So,

$$\begin{aligned} &= \frac{2\left(1 + \left(\frac{\sin x}{\cos x}\right)^2\right)}{1 - \left(\frac{\sin x}{\cos x}\right)^2} \\ &= \frac{2\left(1 + \frac{\sin^2 x}{\cos^2 x}\right)}{1 - \frac{\sin^2 x}{\cos^2 x}} \\ &= \frac{2\left(\frac{\cos^2 x + \sin^2 x}{\cos^2 x}\right)}{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}} \end{aligned}$$

We know, $\cos^2 x + \sin^2 x = 1$ & $\cos 2x = \cos^2 x - \sin^2 x$

So,

$$\begin{aligned} &= \frac{2 \left(\frac{1}{\cos^2 x} \right)}{\frac{\cos 2x}{\cos^2 x}} \\ &= \frac{2}{\cos 2x} \\ &= 2 \sec 2x \text{ (since, } 1/\cos 2x = \sec 2x) \\ &= \text{RHS} \end{aligned}$$

Hence proved.