EXERCISE 9.2 PAGE NO: 9.36

Prove that:

1. $\sin 5x = 5 \sin x – 20 \sin^3 x + 16 \sin^5 x$

Solution:

Let us consider LHS:

sin 5x

Now,

$\sin 5x = \sin (3x + 2x)$

But we know,

$\sin (x + y) = \sin x \cos y + \cos x \sin y$…..(i)

So,

$\sin 5x = \sin 3x \cos 2x + \cos 3x \sin 2x$

$= \sin (2x + x) \cos 2x + \cos (2x + x) \sin 2x$……..(ii)

And

$\cos (x + y) = \cos x \cos y – \sin x \sin y$……(iii)

Now substituting equation (i) and (iii) in equation (ii), we get

$\sin 5x = (\sin 2x \cos x + \cos 2x \sin x) \cos 2x + (\cos 2x \cos x – \sin 2x \sin x) \sin 2x$

$= \sin 2x \cos 2x \cos x + \cos^2 2x \sin x + (\sin 2x \cos 2x \cos x – \sin^2 2x \sin x)$

$= 2\sin 2x \cos 2x \cos x + \cos^2 2x \sin x – \sin^2 2x \sin x$ …….(iv)

Now $\sin 2x = 2\sin x \cos x$………(v)

And $\cos 2x = \cos^2x – \sin^2x$……..(vi)

Substituting equation (v) and (vi) in equation (iv), we get

$\sin 5x = 2(2\sin x \cos x) (\cos^2x – \sin^2x) \cos x + (\cos^2x – \sin^2x)^2 \sin x – (2\sin x \cos x)^2 \sin x$

$= 4(\sin x \cos^2 x) ([1– \sin^2x] – \sin^2x) + ([1–\sin^2x] – \sin^2x)^2 \sin x – (4\sin^2 x \cos^2 x)\sin x$

(as $\cos^2x + \sin^2x = 1 \Rightarrow \cos^2x = 1– \sin^2x$)

$\sin 5x = 4(\sin x [1 – \sin^2x]) (1 – 2\sin^2x) + (1 – 2\sin^2x)^2 \sin x – 4\sin^3 x [1 – \sin^2x]$

$= 4\sin x (1 – \sin^2x) (1 – 2\sin^2 x) + (1 – 4\sin^2x + 4\sin^4x) \sin x – 4\sin^3 x + 4\sin^5x$

$= (4\sin x – 4\sin^3x) (1 – 2\sin^2x) + \sin x – 4\sin^3x + 4\sin^5x – 4\sin^3 x + 4\sin^5x$

$= 4\sin x – 8\sin^3x – 4\sin^3x + 8\sin^5x + \sin x – 8\sin^3x + 8\sin^5x$

$= 5\sin x – 20\sin^3x + 16\sin^5x$

= RHS

Hence proved.

2. $4 (\cos^3 10^o + \sin^3 20^o) = 3 (\cos 10^o + \sin 20^o)$

Solution:

Let us consider LHS:

$4(\cos^3 10^o + \sin^3 20^o)$

We know that, $\sin 60^o = \sqrt{3}/2 = \cos 30^o$

$\sin 30^o = \cos 60^o = 1/2$

So,

$\sin(3\times 20^o) = \cos(3\times 10^o)$

$3\sin 20° - 4\sin^3 20° = 4\cos^3 10° - 3\cos 10°$

(we know, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ and $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$)

So,

$4(\cos^3 10° + \sin^3 20°) = 3(\sin 20° + \cos 10°)$

= RHS

Hence proved.

3. $\cos^3 x \sin 3x + \sin^3 x \cos 3x = 3/4 \sin 4x$

Solution:

We know that,

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

So, $4\cos^3\theta = \cos 3\theta + 3\cos\theta$

$\cos^3\theta = [\cos 3\theta + 3\cos\theta]/4$ …… (i)

Similarly,

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$4\sin^3\theta = 3\sin\theta - \sin 3\theta$

$\sin^3\theta = [3\sin\theta - \sin 3\theta]/4$ …….. (ii)

Now,

Let us consider LHS:

$\cos^3 x \sin 3x + \sin^3 x \cos 3x$

Substituting the values from equation (i) and (ii), we get

$\cos^3 x \sin 3x + \sin^3 x \cos 3x = (\cos 3x + 3\cos x)/4 \sin 3x + (3\sin x - \sin 3x)/4 \cos 3x$

$= 1/4 (\sin 3x \cos 3x + 3 \sin 3x \text{ cox } x + 3\sin x \cos 3x - \sin 3x \cos 3x)$

$= 1/4 (3(\sin 3x \cos x + \sin x \cos 3x) + 0)$

$= 1/4 (3 \sin (3x + x))$

(We know, $\sin(x + y) = \sin x \cos y + \cos x \sin y$)

$= 3/4 \sin 4x$

= RHS

Hence proved.

4. $\sin 5x = 5 \cos^4 x \sin x - 10 \cos^2 x \sin^3 x + \sin^5 x$

Solution:

Let us consider LHS:

$\sin 5x$

Now,
sin 5x = sin (3x + 2x)
But we know,
Sin (x + y) = sin x cos y + cos x sin y…..(i)
So,
sin 5x = sin 3x cos 2x + cos 3x sin 2x
= sin (2x + x) cos 2x + cos (2x + x) sin 2x……..(ii)
And
cos (x + y) = cos x cos y – sin x sin y……(iii)
Now substituting equation (i) and (iii) in equation (ii), we get
sin 5x = (sin 2x cos x + cos 2x sin x) cos 2x + (cos 2x cos x – sin 2x sin x) sin 2x … (iv)

Now sin 2x = 2sin x cos x………(v)
And cos 2x = $\cos^2x - \sin^2x$………(vi)
Substituting equation (v) and (vi) in equation (iv), we get
sin 5x = $[(2 \sin x \cos x) \cos x + (\cos^2x - \sin^2x) \sin x] (\cos^2x - \sin^2x) + [(\cos^2x - \sin^2x) \cos x - (2 \sin x \cos x) \sin x)] (2 \sin x \cos x)$
= $[2 \sin x \cos^2 x + \sin x \cos^2x - \sin^3x] (\cos^2x - \sin^2x) + [\cos^3x - \sin^2x \cos x - 2 \sin^2 x \cos x] (2 \sin x \cos x)$
= $\cos^2x [3 \sin x \cos^2 x - \sin^3x] - \sin^2x [3 \sin x \cos^2 x - \sin^3x] + 2 \sin x \cos^4x - 2 \sin^3 x \cos^2 x - 4 \sin^3 x \cos^2 x$
= $3 \sin x \cos^4 x - \sin^3x \cos^2x - 3 \sin^3 x \cos^2 x - \sin^5x + 2 \sin x \cos^4x - 2 \sin^3 x \cos^2 x - 4 \sin^3 x \cos^2 x$
= $5 \sin x \cos^4 x - 10\sin^3x\cos^2x + \sin^5x$
= RHS
Hence proved.

5. sin 5x = 5 sin x – 20 $\sin^3$ x + 16 $\sin^5$ x
Solution:
Let us consider LHS:
sin 5x
Now,
sin 5x = sin (3x + 2x)
But we know,
Sin (x + y) = sin x cos y + cos x sin y…..(i)
So,
sin 5x = sin 3x cos 2x + cos 3x sin 2x
= sin (2x + x) cos 2x + cos (2x + x) sin 2x……..(ii)
And

cos (x + y) = cos x cos y – sin x sin y……(iii)

Now substituting equation (i) and (iii) in equation (ii), we get

$\sin 5x = (\sin 2x \cos x + \cos 2x \sin x) \cos 2x + (\cos 2x \cos x – \sin 2x \sin x) \sin 2x$

$= \sin 2x \cos 2x \cos x + \cos^2 2x \sin x + (\sin 2x \cos 2x \cos x – \sin^2 2x \sin x)$

$= 2\sin 2x \cos 2x \cos x + \cos^2 2x \sin x – \sin^2 2x \sin x$ …….(iv)

Now $\sin 2x = 2\sin x \cos x$………(v)

And $\cos 2x = \cos^2 x – \sin^2 x$………(vi)

Substituting equation (v) and (vi) in equation (iv), we get

$\sin 5x = 2(2\sin x \cos x) (\cos^2 x – \sin^2 x) \cos x + (\cos^2 x – \sin^2 x)^2 \sin x – (2\sin x \cos x)^2 \sin x$

$= 4(\sin x \cos^2 x) ([1– \sin^2 x] – \sin^2 x) + ([1–\sin^2 x] – \sin^2 x)^2 \sin x – (4\sin^2 x \cos^2 x)\sin x$

(as $\cos^2 x + \sin^2 x = 1 \Rightarrow \cos^2 x = 1– \sin^2 x$)

$\sin 5x = 4(\sin x [1 – \sin^2 x]) (1 – 2\sin^2 x) + (1 – 2\sin^2 x)^2 \sin x – 4\sin^3 x [1 – \sin^2 x]$

$= 4\sin x (1 – \sin^2 x) (1 – 2\sin^2 x) + (1 – 4\sin^2 x + 4\sin^4 x) \sin x – 4\sin^3 x + 4\sin^5 x$

$= (4\sin x – 4\sin^3 x) (1 – 2\sin^2 x) + \sin x – 4\sin^3 x + 4\sin^5 x – 4\sin^3 x + 4\sin^5 x$

$= 4\sin x – 8\sin^3 x – 4\sin^3 x + 8\sin^5 x + \sin x – 8\sin^3 x + 8\sin^5 x$

$= 5\sin x – 20\sin^3 x + 16\sin^5 x$

= RHS

Hence proved.

7. $\tan x + \tan\left(\frac{\pi}{3} + x\right) - \tan\left(\frac{\pi}{3} - x\right) = 3\tan 3x$

Solution:

Let us consider LHS:

$$\tan x + \tan\left(\frac{\pi}{3} + x\right) - \tan\left(\frac{\pi}{3} - x\right)$$

$$= \tan x + \left(\frac{\tan\frac{\pi}{3} + \tan x}{1 - \tan x \tan\frac{\pi}{3}}\right) - \left(\frac{\tan\frac{\pi}{3} - \tan x}{1 + \tan x \tan\frac{\pi}{3}}\right)$$

We know that,

$$\tan(A+B) = \left(\frac{\tan A + \tan B}{1 - \tan A \tan B}\right) \text{ and } \tan(A-B) = \left(\frac{\tan A - \tan B}{1 + \tan A \tan B}\right)$$

So,

$$= \tan x + \left(\frac{\sqrt{3} + \tan x}{1 - \sqrt{3}\tan x}\right) - \left(\frac{\sqrt{3} - \tan x}{1 + \sqrt{3}\tan x}\right)$$

$$= \tan x + \left(\frac{(1 + \sqrt{3}\tan x)(\sqrt{3} + \tan x) - (1 - \sqrt{3}\tan x)(\sqrt{3} - \tan x)}{\left(1 - \tan x\left(\sqrt{3}\right)\right)\left(1 + \tan x\left(\sqrt{3}\right)\right)}\right)$$

Simplify and cancel the similar terms of different sign in the above expression we get,

$$= \tan x + \left(\frac{(0 + 6\tan x + 2\tan x + 0)}{(1 - 3\tan^2 x)}\right)$$

$$= \tan x + \left(\frac{8\tan x}{(1 - 3\tan^2 x)}\right)$$

$$= \left(\frac{\tan x\,(1 - 3\tan^2 x) + 8\tan x}{(1 - 3\tan^2 x)}\right)$$

$$= \left(\frac{(\tan x - 3\tan^3 x) + 8\tan x}{(1 - 3\tan^2 x)}\right)$$

$$= \left(\frac{9\tan x - 3\tan^3 x}{(1 - 3\tan^2 x)}\right)$$

$$= 3\left(\frac{3\tan x - \tan^3 x}{(1 - 3\tan^2 x)}\right)$$

= 3 tan 3x (since, $\tan 3x = (3\tan x - \tan^3 x)/(1-3\tan^2 x)$)

= RHS

Hence proved.