

EXERCISE 9.1

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Prove the following identities:

1. $\sqrt{[(1 - \cos 2x) / (1 + \cos 2x)]} = \tan x$

Solution:

Let us consider LHS:

$$\sqrt{[(1 - \cos 2x) / (1 + \cos 2x)]}$$

$$\begin{aligned}\text{We know that } \cos 2x &= 1 - 2 \sin^2 x \\ &= 2 \cos^2 x - 1\end{aligned}$$

So,

$$\begin{aligned}\sqrt{[(1 - \cos 2x) / (1 + \cos 2x)]} &= \sqrt{[(1 - (1 - 2\sin^2 x)) / (1 + (2\cos^2 x - 1))]} \\ &= \sqrt{[(1 - 1 + 2\sin^2 x) / (1 + 2\cos^2 x - 1)]} \\ &= \sqrt{[2 \sin^2 x / 2 \cos^2 x]} \\ &= \sin x / \cos x \\ &= \tan x \\ &= \text{RHS}\end{aligned}$$

Hence proved.

2. $\sin 2x / (1 - \cos 2x) = \cot x$

Solution:

Let us consider LHS:

$$\sin 2x / (1 - \cos 2x)$$

$$\text{We know that } \cos 2x = 1 - 2 \sin^2 x$$

$$\sin 2x = 2 \sin x \cos x$$

So,

$$\begin{aligned}\sin 2x / (1 - \cos 2x) &= (2 \sin x \cos x) / (1 - (1 - 2\sin^2 x)) \\ &= (2 \sin x \cos x) / (1 - 1 + 2\sin^2 x) \\ &= [2 \sin x \cos x / 2 \sin^2 x] \\ &= \cos x / \sin x \\ &= \cot x \\ &= \text{RHS}\end{aligned}$$

Hence proved.

3. $\sin 2x / (1 + \cos 2x) = \tan x$

Solution:

Let us consider LHS:

$$\sin 2x / (1 + \cos 2x)$$

$$\begin{aligned}\text{We know that } \cos 2x &= 1 - 2 \sin^2 x \\ &= 2 \cos^2 x - 1\end{aligned}$$

$$\sin 2x = 2 \sin x \cos x$$

So,

$$\begin{aligned} \sin 2x / (1 + \cos 2x) &= [2 \sin x \cos x / (1 + (2\cos^2 x - 1))] \\ &= [2 \sin x \cos x / (1 + 2\cos^2 x - 1)] \\ &= [2 \sin x \cos x / 2 \cos^2 x] \\ &= \sin x / \cos x \\ &= \tan x \\ &= \text{RHS} \end{aligned}$$

Hence proved.

4. $\sqrt{2 + \sqrt{2 + 2 \cos 4x}} = 2 \cos x, 0 < x < \frac{\pi}{4}$

Solution:

Let us consider LHS:

$$\begin{aligned} \sqrt{2 + \sqrt{2 + 2 \cos 4x}} &= \sqrt{2 + \sqrt{2 + 2(2 \cos^2 2x - 1)}} \\ \{\text{since, } \cos 2x &= 2 \cos^2 x - 1 \Rightarrow \cos 4x = 2 \cos^2 2x - 1\} \\ &= \sqrt{2 + \sqrt{2 + 4 \cos^2 2x - 2}} \\ &= \sqrt{2 + \sqrt{4 \cos^2 2x}} \\ &= \sqrt{2 + 2 \cos 2x} \\ &= \sqrt{2 + 2(2 \cos^2 x - 1)} \quad \{\text{since, } \cos 2x = 2 \cos^2 x - 1\} \\ &= \sqrt{2 + 4 \cos^2 x - 2} \\ &= \sqrt{4 \cos^2 x} \\ &= 2 \cos x \\ &= \text{RHS} \end{aligned}$$

Hence proved.

5. $[1 - \cos 2x + \sin 2x] / [1 + \cos 2x + \sin 2x] = \tan x$

Solution:

Let us consider LHS:

$$[1 - \cos 2x + \sin 2x] / [1 + \cos 2x + \sin 2x]$$

$$\begin{aligned} \text{We know that, } \cos 2x &= 1 - 2 \sin^2 x \\ &= 2 \cos^2 x - 1 \end{aligned}$$

$$\sin 2x = 2 \sin x \cos x$$

So,

$$\begin{aligned} &= \frac{1 - (1 - 2 \sin^2 x) + 2 \sin x \cos x}{1 + (2 \cos^2 x - 1) + 2 \sin x \cos x} \\ &= \frac{1 - 1 + 2 \sin^2 x + 2 \sin x \cos x}{1 + 2 \cos^2 x - 1 + 2 \sin x \cos x} \\ &= \frac{2 \sin^2 x + 2 \sin x \cos x}{2 \cos^2 x + 2 \sin x \cos x} \\ &= \frac{2 \sin x (\sin x + \cos x)}{2 \cos x (\cos x + \sin x)} \\ &= \frac{\sin x}{\cos x} \end{aligned}$$

$$= \tan x$$

$$= \text{RHS}$$

Hence proved.

$$6. [\sin x + \sin 2x] / [1 + \cos x + \cos 2x] = \tan x$$

Solution:

Let us consider LHS:

$$[\sin x + \sin 2x] / [1 + \cos x + \cos 2x]$$

We know that, $\cos 2x = \cos^2 x - \sin^2 x$

$$\sin 2x = 2 \sin x \cos x$$

So,

$$\begin{aligned} \frac{\sin x + \sin 2x}{1 + \cos x + \cos 2x} &= \frac{\sin x + 2 \sin x \cos x}{1 + \cos x + (2 \cos^2 x - 1)} \\ &= \frac{\sin x + 2 \sin x \cos x}{1 + \cos x + 2 \cos^2 x - 1} \\ &= \frac{\sin x + 2 \sin x \cos x}{\cos x + 2 \cos^2 x} \\ &= \frac{\sin x (1 + 2 \cos x)}{\cos x (1 + 2 \cos x)} \\ &= \frac{\sin x}{\cos x} \\ &= \tan x \end{aligned}$$

= RHS

Hence proved.

$$7. \cos 2x / (1 + \sin 2x) = \tan (\pi/4 - x)$$

Solution:

Let us consider LHS:

$$\cos 2x / (1 + \sin 2x)$$

We know that, $\cos 2x = \cos^2 x - \sin^2 x$

$$\sin 2x = 2 \sin x \cos x$$

So,

$$\begin{aligned} \frac{\cos 2x}{1 + \sin 2x} &= \frac{\cos^2 x - \sin^2 x}{1 + 2 \sin x \cos x} \\ &= \frac{(\cos x - \sin x)(\cos x + \sin x)}{\sin^2 x + \cos^2 x + 2 \sin x \cos x} \\ (\text{since, } a^2 - b^2 &= (a - b)(a + b) \text{ \& } \sin^2 x + \cos^2 x = 1) \\ &= \frac{(\cos x - \sin x)(\cos x + \sin x)}{(\sin x + \cos x)^2} \end{aligned}$$

$$\begin{aligned} (\text{since, } a^2 + b^2 + 2ab &= (a + b)^2) \\ &= \frac{(\cos x - \sin x)(\cos x + \sin x)}{(\sin x + \cos x)(\sin x + \cos x)} \\ &= \frac{(\cos x - \sin x)}{(\sin x + \cos x)} \end{aligned}$$

Multiplying numerator and denominator by $1/\sqrt{2}$

We get,

$$\begin{aligned} &= \frac{\frac{1}{\sqrt{2}}(\cos x - \sin x)}{\frac{1}{\sqrt{2}}(\sin x + \cos x)} \\ &= \frac{\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)}{\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)} \\ &= \frac{\left(\sin \frac{\pi}{4}\cos x - \cos \frac{\pi}{4}\sin x\right)}{\left(\sin \frac{\pi}{4}\sin x + \cos \frac{\pi}{4}\cos x\right)} \quad (\text{since, } 1/\sqrt{2} = \sin \pi/4) \end{aligned}$$

$$\frac{\sin\left(\frac{\pi}{4} - x\right)}{\cos\left(\frac{\pi}{4} - x\right)}$$

By using the formulas,

$$\begin{aligned}\sin(A - B) &= \sin A \cos B - \sin B \cos A \\ \cos(A - B) &= \cos A \cos B + \sin A \sin B \\ &= \tan(\pi/4 - x) \\ &= \text{RHS}\end{aligned}$$

Hence proved.

8. $\cos x / (1 - \sin x) = \tan(\pi/4 + x/2)$

Solution:

Let us consider LHS:

$$\cos x / (1 - \sin x)$$

We know that, $\cos 2x = \cos^2 x - \sin^2 x$

$$\cos x = \cos^2 x/2 - \sin^2 x/2$$

$$\sin 2x = 2 \sin x \cos x$$

$$\sin x = 2 \sin x/2 \cos x/2$$

So,

$$\begin{aligned}\frac{\cos x}{1 - \sin x} &= \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{1 - 2 \sin \frac{x}{2} \cos \frac{x}{2}} \\ &= \frac{\left(\cos \frac{x}{2} - \sin \frac{x}{2}\right)\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}}\end{aligned}$$

(By using the formula, $a^2 - b^2 = (a - b)(a + b)$ & $\sin^2 x + \cos^2 x = 1$)

$$= \frac{\left(\cos \frac{x}{2} - \sin \frac{x}{2}\right)\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)}{\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)^2}$$

(By using the formula, $a^2 + b^2 + 2ab = (a + b)^2$)

$$\begin{aligned}&= \frac{\left(\cos \frac{x}{2} - \sin \frac{x}{2}\right)\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)}{\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)} \\ &= \frac{\left(\cos \frac{x}{2} - \sin \frac{x}{2}\right)}{\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)}\end{aligned}$$

$$= \frac{\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)}{\left(\sin \frac{x}{2} - \cos \frac{x}{2}\right)}$$

Let us multiply numerator and denominator by $1/\sqrt{2}$

We get,

$$\begin{aligned} &= \frac{\frac{1}{\sqrt{2}}\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)}{\frac{1}{\sqrt{2}}\left(\sin \frac{x}{2} - \cos \frac{x}{2}\right)} \\ &= \frac{\left(\frac{1}{\sqrt{2}}\cos \frac{x}{2} + \frac{1}{\sqrt{2}}\sin \frac{x}{2}\right)}{\left(\frac{1}{\sqrt{2}}\sin \frac{x}{2} - \frac{1}{\sqrt{2}}\cos \frac{x}{2}\right)} \\ &= \frac{\left(\sin \frac{\pi}{4}\cos \frac{x}{2} + \cos \frac{\pi}{4}\sin \frac{x}{2}\right)}{\left(\sin \frac{\pi}{4}\sin \frac{x}{2} - \cos \frac{\pi}{4}\cos \frac{x}{2}\right)} \quad (\text{since, } 1/\sqrt{2} = \sin \pi/4) \\ &= \frac{\sin\left(\frac{\pi}{4} - x\right)}{\cos\left(\frac{\pi}{4} - x\right)} \\ &= \tan(\pi/4 - x) \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$9. \cos^2 \frac{\pi}{8} + \cos^2 \frac{3\pi}{8} + \cos^2 \frac{5\pi}{8} + \cos^2 \frac{7\pi}{8} = 2$$

Solution:

Let us consider LHS:

$$\cos^2 \frac{\pi}{8} + \cos^2 \frac{3\pi}{8} + \cos^2 \frac{5\pi}{8} + \cos^2 \frac{7\pi}{8}$$

We know that $\cos 2x = 2\cos^2 x - 1$

$$\cos 2x + 1 = 2\cos^2 x$$

$$\cos^2 x = (\cos 2x + 1)/2$$

So,

$$\cos^2 \frac{\pi}{8} + \cos^2 \frac{3\pi}{8} + \cos^2 \frac{5\pi}{8} + \cos^2 \frac{7\pi}{8}$$

$$\begin{aligned}
 &= \frac{1 + \cos \frac{2\pi}{8}}{2} + \frac{1 + \cos \frac{6\pi}{8}}{2} + \frac{1 + \cos \frac{10\pi}{8}}{2} + \frac{1 + \cos \frac{14\pi}{8}}{2} \\
 &= \frac{1 + \cos \frac{2\pi}{8}}{2} + \frac{1 + \cos \left(\pi - \frac{2\pi}{8} \right)}{2} + \frac{1 + \cos \left(\pi + \frac{2\pi}{8} \right)}{2} + \frac{1 + \cos \left(2\pi - \frac{2\pi}{8} \right)}{2} \\
 &\left\{ \because \pi - \frac{2\pi}{8} = \frac{6\pi}{8}; \pi + \frac{2\pi}{8} = \frac{10\pi}{8}; 2\pi - \frac{2\pi}{8} = \frac{14\pi}{8} \right\} \\
 &= \frac{1 + \cos \frac{2\pi}{8}}{2} + \frac{1 - \cos \frac{2\pi}{8}}{2} + \frac{1 - \cos \frac{2\pi}{8}}{2} + \frac{1 + \cos \frac{2\pi}{8}}{2} \\
 &\{ \text{we know, } \cos(\pi - A) = -\cos A, \cos(\pi + A) = -\cos A \text{ \& } \cos(2\pi - A) = \cos A \} \\
 &= 2 \times \frac{1 + \cos \frac{2\pi}{8}}{2} + 2 \times \frac{1 - \cos \frac{2\pi}{8}}{2} \\
 &= 1 + \cos \frac{2\pi}{8} + 1 - \cos \frac{2\pi}{8} \\
 &= 2 \\
 &= \text{RHS} \\
 &\text{Hence proved.}
 \end{aligned}$$

10. $\sin^2 \frac{\pi}{8} + \sin^2 \frac{3\pi}{8} + \sin^2 \frac{5\pi}{8} + \sin^2 \frac{7\pi}{8} = 2$

Solution:

Let us consider LHS:

$$\sin^2 \frac{\pi}{8} + \sin^2 \frac{3\pi}{8} + \sin^2 \frac{5\pi}{8} + \sin^2 \frac{7\pi}{8}$$

We know that, $\cos 2x = 1 - 2\sin^2 x$

$$2\sin^2 x = 1 - \cos 2x$$

$$\sin^2 x = (1 - \cos 2x)/2$$

So,

$$\begin{aligned}
 &= \frac{1 - \cos \frac{2\pi}{8}}{2} + \frac{1 - \cos \frac{6\pi}{8}}{2} + \frac{1 - \cos \frac{10\pi}{8}}{2} + \frac{1 - \cos \frac{14\pi}{8}}{2} \\
 &= \frac{1 - \cos \frac{2\pi}{8}}{2} + \frac{1 - \cos \left(\pi - \frac{2\pi}{8} \right)}{2} + \frac{1 - \cos \left(\pi + \frac{2\pi}{8} \right)}{2} + \frac{1 - \cos \left(2\pi - \frac{2\pi}{8} \right)}{2}
 \end{aligned}$$

$$\left\{ \because \pi - \frac{2\pi}{8} = \frac{6\pi}{8}; \pi + \frac{2\pi}{8} = \frac{10\pi}{8}; 2\pi - \frac{2\pi}{8} = \frac{14\pi}{8} \right\}$$

$$= \frac{1 - \cos \frac{2\pi}{8}}{2} + \frac{1 - \left(-\cos \frac{2\pi}{8}\right)}{2} + \frac{1 - \left(-\cos \frac{2\pi}{8}\right)}{2} + \frac{1 - \cos \frac{2\pi}{8}}{2}$$

{we know, $\cos(\pi - A) = -\cos A$, $\cos(\pi + A) = -\cos A$ & $\cos(2\pi - A) = \cos A$ }

$$= \frac{1 - \cos \frac{2\pi}{8}}{2} + \frac{1 + \cos \frac{2\pi}{8}}{2} + \frac{1 + \cos \frac{2\pi}{8}}{2} + \frac{1 - \cos \frac{2\pi}{8}}{2}$$

$$= 2 \times \frac{1 - \cos \frac{2\pi}{8}}{2} + 2 \times \frac{1 + \cos \frac{2\pi}{8}}{2}$$

$$= 1 - \cos \frac{2\pi}{8} + 1 + \cos \frac{2\pi}{8}$$

$$= 2$$

$$= \text{RHS}$$

Hence proved.

11. $(\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2 = 4 \cos^2 (\alpha - \beta)/2$

Solution:

Let us consider LHS:

$$(\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2$$

Upon expansion, we get,

$$\begin{aligned} (\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2 &= \cos^2 \alpha + \cos^2 \beta + 2 \cos \alpha \cos \beta + \sin^2 \alpha + \sin^2 \beta + 2 \sin \alpha \sin \beta \\ &= 2 + 2 \cos \alpha \cos \beta + 2 \sin \alpha \sin \beta \\ &= 2 (1 + \cos \alpha \cos \beta + \sin \alpha \sin \beta) \\ &= 2 (1 + \cos (\alpha - \beta)) \text{ [since, } \cos (A - B) = \cos A \cos B + \sin A \sin B] \\ &= 2 (1 + 2 \cos^2 (\alpha - \beta)/2 - 1) \text{ [since, } \cos 2x = 2 \cos^2 x - 1] \\ &= 2 (2 \cos^2 (\alpha - \beta)/2) \\ &= 4 \cos^2 (\alpha - \beta)/2 \\ &= \text{RHS} \end{aligned}$$

Hence Proved.

12. $\sin^2 (\pi/8 + x/2) - \sin^2 (\pi/8 - x/2) = 1/\sqrt{2} \sin x$

Solution:

Let us consider LHS:

$$\sin^2 (\pi/8 + x/2) - \sin^2 (\pi/8 - x/2)$$

we know, $\sin^2 A - \sin^2 B = \sin (A+B) \sin (A-B)$

so,

$$\begin{aligned}\sin^2 (\pi/8 + x/2) - \sin^2 (\pi/8 - x/2) &= \sin (\pi/8 + x/2 + \pi/8 - x/2) \sin (\pi/8 + x/2 - (\pi/8 - x/2)) \\ &= \sin (\pi/8 + \pi/8) \sin (\pi/8 + x/2 - \pi/8 + x/2) \\ &= \sin \pi/4 \sin x \\ &= 1/\sqrt{2} \sin x \text{ [since, since } \pi/4 = 1/\sqrt{2}] \\ &= \text{RHS}\end{aligned}$$

Hence proved.

13. $1 + \cos^2 2x = 2 (\cos^4 x + \sin^4 x)$

Solution:

Let us consider LHS:

$$1 + \cos^2 2x$$

We know, $\cos 2x = \cos^2 x - \sin^2 x$

$$\cos^2 x + \sin^2 x = 1$$

so,

$$\begin{aligned}1 + \cos^2 2x &= (\cos^2 x + \sin^2 x)^2 + (\cos^2 x - \sin^2 x)^2 \\ &= (\cos^4 x + \sin^4 x + 2 \cos^2 x \sin^2 x) + (\cos^4 x + \sin^4 x - 2 \cos^2 x \sin^2 x) \\ &= \cos^4 x + \sin^4 x + \cos^4 x + \sin^4 x \\ &= 2 \cos^4 x + 2 \sin^4 x \\ &= 2 (\cos^4 x + \sin^4 x) \\ &= \text{RHS}\end{aligned}$$

Hence proved.

14. $\cos^3 2x + 3 \cos 2x = 4 (\cos^6 x - \sin^6 x)$

Solution:

Let us consider RHS:

$$4 (\cos^6 x - \sin^6 x)$$

Upon expansion we get,

$$\begin{aligned}4 (\cos^6 x - \sin^6 x) &= 4 [(\cos^2 x)^3 - (\sin^2 x)^3] \\ &= 4 (\cos^2 x - \sin^2 x) (\cos^4 x + \sin^4 x + \cos^2 x \sin^2 x)\end{aligned}$$

By using the formula,

$$a^3 - b^3 = (a-b) (a^2 + b^2 + ab)$$

$$= 4 \cos 2x (\cos^4 x + \sin^4 x + \cos^2 x \sin^2 x + \cos^2 x \sin^2 x - \cos^2 x \sin^2 x)$$

We know, $\cos 2x = \cos^2 x - \sin^2 x$

So,

$$\begin{aligned}&= 4 \cos 2x (\cos^4 x + \sin^4 x + 2 \cos^2 x \sin^2 x - \cos^2 x \sin^2 x) \\ &= 4 \cos 2x [(\cos^2 x)^2 + (\sin^2 x)^2 + 2 \cos^2 x \sin^2 x - \cos^2 x \sin^2 x]\end{aligned}$$

We know, $a^2 + b^2 + 2ab = (a + b)^2$

$$= 4 \cos 2x [(1)^2 - 1/4 (4 \cos^2 x \sin^2 x)]$$

$$= 4 \cos 2x [(1)^2 - 1/4 (2 \cos x \sin x)^2]$$

We know, $\sin 2x = 2 \sin x \cos x$

$$= 4 \cos 2x [(1)^2 - 1/4 (\sin 2x)^2]$$

$$= 4 \cos 2x (1 - 1/4 \sin^2 2x)$$

We know, $\sin^2 x = 1 - \cos^2 x$

$$= 4 \cos 2x [1 - 1/4 (1 - \cos^2 2x)]$$

$$= 4 \cos 2x [1 - 1/4 + 1/4 \cos^2 2x]$$

$$= 4 \cos 2x [3/4 + 1/4 \cos^2 2x]$$

$$= 4 (3/4 \cos 2x + 1/4 \cos^3 2x)$$

$$= 3 \cos 2x + \cos^3 2x$$

$$= \cos^3 2x + 3 \cos 2x$$

$$= \text{LHS}$$

Hence proved.

15. $(\sin 3x + \sin x) \sin x + (\cos 3x - \cos x) \cos x = 0$

Solution:

Let us consider LHS:

$$(\sin 3x + \sin x) \sin x + (\cos 3x - \cos x) \cos x$$

$$= (\sin 3x) (\sin x) + \sin^2 x + (\cos 3x) (\cos x) - \cos^2 x$$

$$= [(\sin 3x) (\sin x) + (\cos 3x) (\cos x)] + (\sin^2 x - \cos^2 x)$$

$$= [(\sin 3x) (\sin x) + (\cos 3x) (\cos x)] - (\cos^2 x - \sin^2 x)$$

$$= \cos (3x - x) - \cos 2x$$

We know, $\cos 2x = \cos^2 x - \sin^2 x$

$\cos A \cos B + \sin A \sin B = \cos(A - B)$

So,

$$= \cos 2x - \cos 2x$$

$$= 0$$

$$= \text{RHS}$$

Hence Proved.

16. $\cos^2 (\pi/4 - x) - \sin^2 (\pi/4 - x) = \sin 2x$

Solution:

Let us consider LHS:

$$\cos^2 (\pi/4 - x) - \sin^2 (\pi/4 - x)$$

We know, $\cos^2 A - \sin^2 A = \cos 2A$

So,

$$\cos^2 (\pi/4 - x) - \sin^2 (\pi/4 - x) = \cos 2 (\pi/4 - x)$$

$$= \cos (\pi/2 - 2x)$$

$$= \sin 2x \text{ [since, } \cos (\pi/2 - A) = \sin A]$$

$$= \text{RHS}$$

Hence proved.

$$17. \cos 4x = 1 - 8 \cos^2 x + 8 \cos^4 x$$

Solution:

Let us consider LHS:

$$\cos 4x$$

$$\text{We know, } \cos 2x = 2 \cos^2 x - 1$$

So,

$$\begin{aligned} \cos 4x &= 2 \cos^2 2x - 1 \\ &= 2(2 \cos^2 x - 1)^2 - 1 \\ &= 2[(2 \cos^2 x)^2 + 1^2 - 2 \times 2 \cos^2 x] - 1 \\ &= 2(4 \cos^4 x + 1 - 4 \cos^2 x) - 1 \\ &= 8 \cos^4 x + 2 - 8 \cos^2 x - 1 \\ &= 8 \cos^4 x + 1 - 8 \cos^2 x \\ &= \text{RHS} \end{aligned}$$

Hence Proved.

$$18. \sin 4x = 4 \sin x \cos^3 x - 4 \cos x \sin^3 x$$

Solution:

Let us consider LHS:

$$\sin 4x$$

$$\text{We know, } \sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

So,

$$\begin{aligned} \sin 4x &= 2 \sin 2x \cos 2x \\ &= 2 (2 \sin x \cos x) (\cos^2 x - \sin^2 x) \\ &= 4 \sin x \cos x (\cos^2 x - \sin^2 x) \\ &= 4 \sin x \cos^3 x - 4 \sin^3 x \cos x \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$19. 3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) = 13$$

Solution:

Let us consider LHS:

$$3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x)$$

$$\text{We know, } (a + b)^2 = a^2 + b^2 + 2ab$$

$$(a - b)^2 = a^2 + b^2 - 2ab$$

$$a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$$

So,

$$\begin{aligned} 3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) &= 3\{(\sin x - \cos x)^2\}^2 + 6\{(\sin x)^2 + (\cos x)^2 + 2\sin x \cos x\} + 4\{(\sin^2 x)^3 + (\cos^2 x)^3\} \\ &= 3\{(\sin x)^2 + (\cos x)^2 - 2\sin x \cos x\}^2 + 6(1 + 2\sin x \cos x) + 4\{(\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x)\} \end{aligned}$$

$$= 3(1 - 2\sin x \cos x)^2 + 6(1 + 2\sin x \cos x) + 4\{(1)(\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x)\}$$

We know, $\sin^2 x + \cos^2 x = 1$

So,

$$\begin{aligned} &= 3\{1^2 + (2\sin x \cos x)^2 - 4\sin x \cos x\} + 6(1 + 2\sin x \cos x) + 4\{(\sin^2 x)^2 + (\cos^2 x)^2 + 2\sin^2 x \cos^2 x - 3\sin^2 x \cos^2 x\} \\ &= 3\{1 + 4\sin^2 x \cos^2 x - 4\sin x \cos x\} + 6(1 + 2\sin x \cos x) + 4\{(\sin^2 x + \cos^2 x)^2 - 3\sin^2 x \cos^2 x\} \\ &= 3 + 12\sin^2 x \cos^2 x - 12\sin x \cos x + 6 + 12\sin x \cos x + 4\{(1)^2 - 3\sin^2 x \cos^2 x\} \\ &= 9 + 12\sin^2 x \cos^2 x + 4(1 - 3\sin^2 x \cos^2 x) \\ &= 9 + 12\sin^2 x \cos^2 x + 4 - 12\sin^2 x \cos^2 x \\ &= 13 \\ &= \text{RHS} \end{aligned}$$

Hence proved.

20. $2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1 = 0$

Solution:

Let us consider LHS:

$$2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1$$

We know, $(a + b)^2 = a^2 + b^2 + 2ab$

$$a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$$

So,

$$\begin{aligned} 2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1 &= 2\{(\sin^2 x)^3 + (\cos^2 x)^3\} - 3\{(\sin^2 x)^2 + (\cos^2 x)^2\} + 1 \\ &= 2\{(\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x)\} - 3\{(\sin^2 x)^2 + (\cos^2 x)^2 + 2\sin^2 x \cos^2 x - 2\sin^2 x \cos^2 x\} + 1 \\ &= 2\{(1)(\sin^4 x + \cos^4 x + 2\sin^2 x \cos^2 x - 3\sin^2 x \cos^2 x)\} - 3\{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x\} + 1 \end{aligned}$$

We know, $\sin^2 x + \cos^2 x = 1$

$$2\sin^2 x \cos^2 x\} + 1$$

$$= 2\{(\sin^2 x + \cos^2 x)^2 - 3 \sin^2 x \cos^2 x\} - 3\{(1)^2 -$$

$$= 2\{(1)^2 - 3 \sin^2 x \cos^2 x\} - 3(1 - 2\sin^2 x \cos^2 x) + 1$$

$$= 2(1 - 3 \sin^2 x \cos^2 x) - 3 + 6 \sin^2 x \cos^2 x + 1$$

$$= 2 - 6 \sin^2 x \cos^2 x - 2 + 6 \sin^2 x \cos^2 x$$

$$= 0$$

$$= \text{RHS}$$

Hence proved.

21. $\cos^6 x - \sin^6 x = \cos 2x (1 - \frac{1}{4} \sin^2 2x)$

Solution:

Let us consider LHS:

$$\cos^6 x - \sin^6 x$$

$$\text{We know, } (a + b)^2 = a^2 + b^2 + 2ab$$

$$a^3 - b^3 = (a - b)(a^2 + b^2 + ab)$$

So,

$$\begin{aligned} \cos^6 x - \sin^6 x &= (\cos^2 x)^3 - (\sin^2 x)^3 \\ &= (\cos^2 x - \sin^2 x)(\cos^4 x + \sin^4 x + \cos^2 x \sin^2 x) \end{aligned}$$

$$\text{We know, } \cos 2x = \cos^2 x - \sin^2 x$$

So,

$$\begin{aligned} &= \cos 2x [(\cos^2 x)^2 + (\sin^2 x)^2 + 2 \cos^2 x \sin^2 x - \cos^2 x \sin^2 x] \\ &= \cos 2x [(\cos^2 x)^2 + (\sin^2 x)^2 - \frac{1}{4} \times 4 \cos^2 x \sin^2 x] \end{aligned}$$

$$\text{We know, } \sin^2 x + \cos^2 x = 1$$

So,

$$= \cos 2x [(1)^2 - \frac{1}{4} \times (2 \cos x \sin x)^2]$$

$$\text{We know, } \sin 2x = 2 \sin x \cos x$$

So,

$$\begin{aligned} &= \cos 2x [1 - \frac{1}{4} \times (\sin 2x)^2] \\ &= \cos 2x [1 - \frac{1}{4} \times \sin^2 2x] \\ &= \text{RHS} \end{aligned}$$

Hence proved.

22. $\tan(\pi/4 + x) + \tan(\pi/4 - x) = 2 \sec 2x$

Solution:

Let us consider LHS:

$$\tan(\pi/4 + x) + \tan(\pi/4 - x)$$

We know,

$$\tan (A+B) = (\tan A + \tan B)/(1 - \tan A \tan B)$$

$$\tan (A-B) = (\tan A - \tan B)/(1 + \tan A \tan B)$$

So,

$$\tan\left(\frac{\pi}{4} + x\right) + \tan\left(\frac{\pi}{4} - x\right) = \frac{\tan\frac{\pi}{4} + \tan x}{1 - \tan\frac{\pi}{4}\tan x} + \frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}$$

We know, $\tan \pi/4 = 1$

So,

$$\begin{aligned} &= \frac{1 + \tan x}{1 - \tan x} + \frac{1 - \tan x}{1 + \tan x} \\ &= \frac{(1 + \tan x)^2 + (1 - \tan x)^2}{(1 - \tan x)(1 + \tan x)} \end{aligned}$$

We know, $(a - b)(a + b) = a^2 - b^2$;

$$(a + b)^2 = a^2 + b^2 + 2ab \text{ \&}$$

$$(a - b)^2 = a^2 + b^2 - 2ab$$

So,

$$\begin{aligned} &= \frac{1^2 + \tan^2 x + 2 \tan x + 1^2 + \tan^2 x - 2 \tan x}{1^2 - \tan^2 x} \\ &= \frac{1 + \tan^2 x + 1 + \tan^2 x}{1 - \tan^2 x} \\ &= \frac{2(1 + \tan^2 x)}{1 - \tan^2 x} \end{aligned}$$

We know, $\tan x = \sin x/\cos x$

So,

$$\begin{aligned} &= \frac{2\left(1 + \left(\frac{\sin x}{\cos x}\right)^2\right)}{1 - \left(\frac{\sin x}{\cos x}\right)^2} \\ &= \frac{2\left(1 + \frac{\sin^2 x}{\cos^2 x}\right)}{1 - \frac{\sin^2 x}{\cos^2 x}} \\ &= \frac{2\left(\frac{\cos^2 x + \sin^2 x}{\cos^2 x}\right)}{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}} \end{aligned}$$

We know, $\cos^2 x + \sin^2 x = 1$ & $\cos 2x = \cos^2 x - \sin^2 x$

So,

$$\begin{aligned} &= \frac{2 \left(\frac{1}{\cos^2 x} \right)}{\frac{\cos 2x}{\cos^2 x}} \\ &= \frac{2}{\cos 2x} \\ &= 2 \sec 2x \text{ (since, } 1/\cos 2x = \sec 2x) \\ &= \text{RHS} \end{aligned}$$

Hence proved.

EXERCISE 9.2

PAGE NO: 9.36

Prove that:

1. $\sin 5x = 5 \sin x - 20 \sin^3 x + 16 \sin^5 x$

Solution:

Let us consider LHS:

$$\sin 5x$$

Now,

$$\sin 5x = \sin (3x + 2x)$$

But we know,

$$\sin (x + y) = \sin x \cos y + \cos x \sin y \dots (i)$$

So,

$$\begin{aligned} \sin 5x &= \sin 3x \cos 2x + \cos 3x \sin 2x \\ &= \sin (2x + x) \cos 2x + \cos (2x + x) \sin 2x \dots (ii) \end{aligned}$$

And

$$\cos (x + y) = \cos x \cos y - \sin x \sin y \dots (iii)$$

Now substituting equation (i) and (iii) in equation (ii), we get

$$\begin{aligned} \sin 5x &= (\sin 2x \cos x + \cos 2x \sin x) \cos 2x + (\cos 2x \cos x - \sin 2x \sin x) \sin 2x \\ &= \sin 2x \cos 2x \cos x + \cos^2 2x \sin x + (\sin 2x \cos 2x \cos x - \sin^2 2x \sin x) \\ &= 2\sin 2x \cos 2x \cos x + \cos^2 2x \sin x - \sin^2 2x \sin x \dots (iv) \end{aligned}$$

$$\text{Now } \sin 2x = 2\sin x \cos x \dots (v)$$

$$\text{And } \cos 2x = \cos^2 x - \sin^2 x \dots (vi)$$

Substituting equation (v) and (vi) in equation (iv), we get

$$\begin{aligned} \sin 5x &= 2(2\sin x \cos x) (\cos^2 x - \sin^2 x) \cos x + (\cos^2 x - \sin^2 x)^2 \sin x - (2\sin x \cos x)^2 \sin x \\ &= 4(\sin x \cos^2 x) ([1 - \sin^2 x] - \sin^2 x) + ([1 - \sin^2 x] - \sin^2 x)^2 \sin x - (4\sin^2 x \cos^2 x) \sin x \end{aligned}$$

$$(\text{as } \cos^2 x + \sin^2 x = 1 \Rightarrow \cos^2 x = 1 - \sin^2 x)$$

$$\begin{aligned} \sin 5x &= 4(\sin x [1 - \sin^2 x]) (1 - 2\sin^2 x) + (1 - 2\sin^2 x)^2 \sin x - 4\sin^3 x [1 - \sin^2 x] \\ &= 4\sin x (1 - \sin^2 x) (1 - 2\sin^2 x) + (1 - 4\sin^2 x + 4\sin^4 x) \sin x - 4\sin^3 x + 4\sin^5 x \\ &= (4\sin x - 4\sin^3 x) (1 - 2\sin^2 x) + \sin x - 4\sin^3 x + 4\sin^5 x - 4\sin^3 x + 4\sin^5 x \\ &= 4\sin x - 8\sin^3 x - 4\sin^3 x + 8\sin^5 x + \sin x - 8\sin^3 x + 8\sin^5 x \\ &= 5\sin x - 20\sin^3 x + 16\sin^5 x \\ &= \text{RHS} \end{aligned}$$

Hence proved.

2. $4 (\cos^3 10^\circ + \sin^3 20^\circ) = 3 (\cos 10^\circ + \sin 20^\circ)$

Solution:

Let us consider LHS:

$$4 (\cos^3 10^\circ + \sin^3 20^\circ)$$

We know that, $\sin 60^\circ = \sqrt{3}/2 = \cos 30^\circ$

$$\sin 30^\circ = \cos 60^\circ = 1/2$$

So,

$$\sin (3 \times 20^\circ) = \cos (3 \times 10^\circ)$$

$$3 \sin 20^\circ - 4 \sin^3 20^\circ = 4 \cos^3 10^\circ - 3 \cos 10^\circ$$

(we know, $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$ and $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$)

So,

$$4(\cos^3 10^\circ + \sin^3 20^\circ) = 3(\sin 20^\circ + \cos 10^\circ)$$

= RHS

Hence proved.

$$3. \cos^3 x \sin 3x + \sin^3 x \cos 3x = 3/4 \sin 4x$$

Solution:

We know that,

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$$

$$\text{So, } 4 \cos^3 \theta = \cos 3\theta + 3 \cos \theta$$

$$\cos^3 \theta = [\cos 3\theta + 3 \cos \theta]/4 \dots\dots (i)$$

Similarly,

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$$

$$4 \sin^3 \theta = 3 \sin \theta - \sin 3\theta$$

$$\sin^3 \theta = [3 \sin \theta - \sin 3\theta]/4 \dots\dots\dots (ii)$$

Now,

Let us consider LHS:

$$\cos^3 x \sin 3x + \sin^3 x \cos 3x$$

Substituting the values from equation (i) and (ii), we get

$$\begin{aligned} \cos^3 x \sin 3x + \sin^3 x \cos 3x &= (\cos 3x + 3 \cos x)/4 \sin 3x + (3 \sin x - \sin 3x)/4 \cos 3x \\ &= 1/4 (\sin 3x \cos 3x + 3 \sin 3x \cos x + 3 \sin x \cos 3x - \sin 3x \cos 3x) \\ &= 1/4 (3(\sin 3x \cos x + \sin x \cos 3x) + 0) \\ &= 1/4 (3 \sin (3x + x)) \\ &= (We know, \sin(x + y) = \sin x \cos y + \cos x \sin y) \\ &= 3/4 \sin 4x \\ &= RHS \end{aligned}$$

Hence proved.

$$4. \sin 5x = 5 \cos^4 x \sin x - 10 \cos^2 x \sin^3 x + \sin^5 x$$

Solution:

Let us consider LHS:

$$\sin 5x$$

Now,

$$\sin 5x = \sin (3x + 2x)$$

But we know,

$$\sin (x + y) = \sin x \cos y + \cos x \sin y \dots (i)$$

So,

$$\begin{aligned} \sin 5x &= \sin 3x \cos 2x + \cos 3x \sin 2x \\ &= \sin (2x + x) \cos 2x + \cos (2x + x) \sin 2x \dots (ii) \end{aligned}$$

And

$$\cos (x + y) = \cos x \cos y - \sin x \sin y \dots (iii)$$

Now substituting equation (i) and (iii) in equation (ii), we get

$$\sin 5x = (\sin 2x \cos x + \cos 2x \sin x) \cos 2x + (\cos 2x \cos x - \sin 2x \sin x) \sin 2x \dots (iv)$$

$$\text{Now } \sin 2x = 2 \sin x \cos x \dots (v)$$

$$\text{And } \cos 2x = \cos^2 x - \sin^2 x \dots (vi)$$

Substituting equation (v) and (vi) in equation (iv), we get

$$\begin{aligned} \sin 5x &= [(2 \sin x \cos x) \cos x + (\cos^2 x - \sin^2 x) \sin x] (\cos^2 x - \sin^2 x) + [(\cos^2 x - \sin^2 x) \cos x - (2 \sin x \cos x) \sin x] (2 \sin x \cos x) \\ &= [2 \sin x \cos^2 x + \sin x \cos^2 x - \sin^3 x] (\cos^2 x - \sin^2 x) + [\cos^3 x - \sin^2 x \cos x - 2 \sin^2 x \cos x] (2 \sin x \cos x) \\ &= \cos^2 x [3 \sin x \cos^2 x - \sin^3 x] - \sin^2 x [3 \sin x \cos^2 x - \sin^3 x] + 2 \sin x \cos^4 x - 2 \sin^3 x \cos^2 x - 4 \sin^3 x \cos^2 x \\ &= 3 \sin x \cos^4 x - \sin^3 x \cos^2 x - 3 \sin^3 x \cos^2 x - \sin^5 x + 2 \sin x \cos^4 x - 2 \sin^3 x \cos^2 x - 4 \sin^3 x \cos^2 x \\ &= 5 \sin x \cos^4 x - 10 \sin^3 x \cos^2 x + \sin^5 x \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$\mathbf{5. \sin 5x = 5 \sin x - 20 \sin^3 x + 16 \sin^5 x}$$

Solution:

Let us consider LHS:

$$\sin 5x$$

Now,

$$\sin 5x = \sin (3x + 2x)$$

But we know,

$$\sin (x + y) = \sin x \cos y + \cos x \sin y \dots (i)$$

So,

$$\begin{aligned} \sin 5x &= \sin 3x \cos 2x + \cos 3x \sin 2x \\ &= \sin (2x + x) \cos 2x + \cos (2x + x) \sin 2x \dots (ii) \end{aligned}$$

And

$$\cos(x + y) = \cos x \cos y - \sin x \sin y \dots\dots(iii)$$

Now substituting equation (i) and (iii) in equation (ii), we get

$$\begin{aligned}\sin 5x &= (\sin 2x \cos x + \cos 2x \sin x) \cos 2x + (\cos 2x \cos x - \sin 2x \sin x) \sin 2x \\ &= \sin 2x \cos 2x \cos x + \cos^2 2x \sin x + (\sin 2x \cos 2x \cos x - \sin^2 2x \sin x) \\ &= 2\sin 2x \cos 2x \cos x + \cos^2 2x \sin x - \sin^2 2x \sin x \dots\dots(iv)\end{aligned}$$

$$\text{Now } \sin 2x = 2\sin x \cos x \dots\dots(v)$$

$$\text{And } \cos 2x = \cos^2 x - \sin^2 x \dots\dots(vi)$$

Substituting equation (v) and (vi) in equation (iv), we get

$$\begin{aligned}\sin 5x &= 2(2\sin x \cos x) (\cos^2 x - \sin^2 x) \cos x + (\cos^2 x - \sin^2 x)^2 \sin x - (2\sin x \cos x)^2 \sin x \\ &= 4(\sin x \cos^2 x) ([1 - \sin^2 x] - \sin^2 x) + ([1 - \sin^2 x] - \sin^2 x)^2 \sin x - (4\sin^2 x \cos^2 x) \sin x\end{aligned}$$

$$(\text{as } \cos^2 x + \sin^2 x = 1 \Rightarrow \cos^2 x = 1 - \sin^2 x)$$

$$\begin{aligned}\sin 5x &= 4(\sin x [1 - \sin^2 x]) (1 - 2\sin^2 x) + (1 - 2\sin^2 x)^2 \sin x - 4\sin^3 x [1 - \sin^2 x] \\ &= 4\sin x (1 - \sin^2 x) (1 - 2\sin^2 x) + (1 - 4\sin^2 x + 4\sin^4 x) \sin x - 4\sin^3 x + 4\sin^5 x \\ &= (4\sin x - 4\sin^3 x) (1 - 2\sin^2 x) + \sin x - 4\sin^3 x + 4\sin^5 x - 4\sin^3 x + 4\sin^5 x \\ &= 4\sin x - 8\sin^3 x - 4\sin^3 x + 8\sin^5 x + \sin x - 8\sin^3 x + 8\sin^5 x \\ &= 5\sin x - 20\sin^3 x + 16\sin^5 x \\ &= \text{RHS}\end{aligned}$$

Hence proved.

$$7. \tan x + \tan\left(\frac{\pi}{3} + x\right) - \tan\left(\frac{\pi}{3} - x\right) = 3 \tan 3x$$

Solution:

Let us consider LHS:

$$\begin{aligned}&\tan x + \tan\left(\frac{\pi}{3} + x\right) - \tan\left(\frac{\pi}{3} - x\right) \\ &= \tan x + \left(\frac{\tan \frac{\pi}{3} + \tan x}{1 - \tan x \tan \frac{\pi}{3}}\right) - \left(\frac{\tan \frac{\pi}{3} - \tan x}{1 + \tan x \tan \frac{\pi}{3}}\right)\end{aligned}$$

We know that,

$$\tan(A + B) = \left(\frac{\tan A + \tan B}{1 - \tan A \tan B}\right) \text{ and } \tan(A - B) = \left(\frac{\tan A - \tan B}{1 + \tan A \tan B}\right)$$

So,

$$\begin{aligned}&= \tan x + \left(\frac{\sqrt{3} + \tan x}{1 - \sqrt{3} \tan x}\right) - \left(\frac{\sqrt{3} - \tan x}{1 + \sqrt{3} \tan x}\right) \\ &= \tan x + \left(\frac{(1 + \sqrt{3} \tan x)(\sqrt{3} + \tan x) - (1 - \sqrt{3} \tan x)(\sqrt{3} - \tan x)}{(1 - \tan x(\sqrt{3})) (1 + \tan x(\sqrt{3}))}\right)\end{aligned}$$

Simplify and cancel the similar terms of different sign in the above expression we get,

$$= \tan x + \left(\frac{(0 + 6 \tan x + 2 \tan x + 0)}{(1 - 3 \tan^2 x)} \right)$$

$$= \tan x + \left(\frac{8 \tan x}{(1 - 3 \tan^2 x)} \right)$$

$$= \left(\frac{\tan x (1 - 3 \tan^2 x) + 8 \tan x}{(1 - 3 \tan^2 x)} \right)$$

$$= \left(\frac{(\tan x - 3 \tan^3 x) + 8 \tan x}{(1 - 3 \tan^2 x)} \right)$$

$$= \left(\frac{9 \tan x - 3 \tan^3 x}{(1 - 3 \tan^2 x)} \right)$$

$$= 3 \left(\frac{3 \tan x - \tan^3 x}{(1 - 3 \tan^2 x)} \right)$$

$$= 3 \tan 3x \text{ (since, } \tan 3x = (3 \tan x - \tan^3 x) / (1 - 3 \tan^2 x) \text{)}$$

= RHS

Hence proved.



EXERCISE 9.3

PAGE NO: 9.42

Prove that:

1. $\sin^2 2\pi/5 - \sin^2 \pi/3 = (\sqrt{5} - 1)/8$

Solution:

Let us consider LHS:

$$\sin^2 2\pi/5 - \sin^2 \pi/3 = \sin^2 (\pi/2 - \pi/10) - \sin^2 \pi/3$$

we know, $\sin (90^\circ - A) = \cos A$

$$\text{So, } \sin^2 (\pi/2 - \pi/10) = \cos^2 \pi/10$$

$$\sin \pi/3 = \sqrt{3}/2$$

Then the above equation becomes,

$$= \cos^2 \pi/10 - (\sqrt{3}/2)^2$$

$$\text{We know, } \cos \pi/10 = \sqrt{(10+2\sqrt{5})}/4$$

the above equation becomes,

$$= [\sqrt{(10+2\sqrt{5})}/4]^2 - 3/4$$

$$= [10 + 2\sqrt{5}]/16 - 3/4$$

$$= [10 + 2\sqrt{5} - 12]/16$$

$$= [2\sqrt{5} - 2]/16$$

$$= [\sqrt{5} - 1]/8$$

$$= \text{RHS}$$

Hence proved.

2. $\sin^2 24^\circ - \sin^2 6^\circ = (\sqrt{5} - 1)/8$

Solution:

Let us consider LHS:

$$\sin^2 24^\circ - \sin^2 6^\circ$$

$$\text{we know, } \sin (A + B) \sin (A - B) = \sin^2 A - \sin^2 B$$

Then the above equation becomes,

$$\sin^2 24^\circ - \sin^2 6^\circ = \sin (24^\circ + 6^\circ) \sin (24^\circ - 6^\circ)$$

$$= \sin 30^\circ \sin 18^\circ$$

$$= \sin 30^\circ \times (\sqrt{5} - 1)/4 \text{ [since, } \sin 18^\circ = (\sqrt{5} - 1)/4]$$

$$= 1/2 \times (\sqrt{5} - 1)/4$$

$$= (\sqrt{5} - 1)/8$$

$$= \text{RHS}$$

Hence proved.

3. $\sin^2 42^\circ - \cos^2 78^\circ = (\sqrt{5} + 1)/8$

Solution:

Let us consider LHS:

$$\begin{aligned}\sin^2 42^\circ - \cos^2 78^\circ &= \sin^2 (90^\circ - 48^\circ) - \cos^2 (90^\circ - 12^\circ) \\ &= \cos^2 48^\circ - \sin^2 12^\circ \text{ [since, } \sin (90^\circ - A) = \cos A \text{ and } \cos (90^\circ - A) = \sin A]\end{aligned}$$

We know, $\cos (A + B) \cos (A - B) = \cos^2 A - \sin^2 B$

Then the above equation becomes,

$$\begin{aligned}&= \cos^2 (48^\circ + 12^\circ) \cos (48^\circ - 12^\circ) \\ &= \cos 60^\circ \cos 36^\circ \text{ [since, } \cos 36^\circ = (\sqrt{5} + 1)/4] \\ &= 1/2 \times (\sqrt{5} + 1)/4 \\ &= (\sqrt{5} + 1)/8 \\ &= \text{RHS}\end{aligned}$$

Hence proved.

4. $\cos 78^\circ \cos 42^\circ \cos 36^\circ = 1/8$

Solution:

Let us consider LHS:

$$\cos 78^\circ \cos 42^\circ \cos 36^\circ$$

Let us multiply and divide by 2 we get,

$$\cos 78^\circ \cos 42^\circ \cos 36^\circ = 1/2 (2 \cos 78^\circ \cos 42^\circ \cos 36^\circ)$$

We know, $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$

Then the above equation becomes,

$$\begin{aligned}&= 1/2 (\cos (78^\circ + 42^\circ) + \cos (78^\circ - 42^\circ)) \times \cos 36^\circ \\ &= 1/2 (\cos 120^\circ + \cos 36^\circ) \times \cos 36^\circ \\ &= 1/2 (\cos (180^\circ - 60^\circ) + \cos 36^\circ) \times \cos 36^\circ \\ &= 1/2 (-\cos (60^\circ) + \cos 36^\circ) \times \cos 36^\circ \text{ [since, } \cos (180^\circ - A) = -\cos A] \\ &= 1/2 (-1/2 + (\sqrt{5} + 1)/4) ((\sqrt{5} + 1)/4) \text{ [since, } \cos 36^\circ = (\sqrt{5} + 1)/4] \\ &= 1/2 (\sqrt{5} + 1 - 2)/4 ((\sqrt{5} + 1)/4) \\ &= 1/2 (\sqrt{5} - 1)/4 ((\sqrt{5} + 1)/4) \\ &= 1/2 ((\sqrt{5})^2 - 1^2)/16 \\ &= 1/2 (5 - 1)/16 \\ &= 1/2 (4/16) \\ &= 1/8 \\ &= \text{RHS}\end{aligned}$$

Hence proved.

5. $\cos \pi/15 \cos 2\pi/15 \cos 4\pi/15 \cos 7\pi/15 = 1/16$

Solution:

Let us consider LHS:

$$\cos \pi/15 \cos 2\pi/15 \cos 4\pi/15 \cos 7\pi/15$$

Let us multiply and divide by $2 \sin \pi/15$, we get,

$$= [2 \sin \pi/15 \cos \pi/15] \cos 2\pi/15 \cos 4\pi/15 \cos 7\pi/15 / 2 \sin \pi/15$$

We know, $2\sin A \cos A = \sin 2A$

Then the above equation becomes,

$$= [(\sin 2\pi/15) \cos 2\pi/15 \cos 4\pi/15 \cos 7\pi/15] / 2 \sin \pi/15$$

Now, multiply and divide by 2 we get,

$$= [(2 \sin 2\pi/15 \cos 2\pi/15) \cos 4\pi/15 \cos 7\pi/15] / 2 \times 2 \sin \pi/15$$

We know, $2\sin A \cos A = \sin 2A$

Then the above equation becomes,

$$= [(\sin 4\pi/15) \cos 4\pi/15 \cos 7\pi/15] / 4 \sin \pi/15$$

Now, multiply and divide by 2 we get,

$$= [(2 \sin 4\pi/15 \cos 4\pi/15) \cos 7\pi/15] / 2 \times 4 \sin \pi/15$$

We know, $2\sin A \cos A = \sin 2A$

Then the above equation becomes,

$$= [(\sin 8\pi/15) \cos 7\pi/15] / 8 \sin \pi/15$$

Now, multiply and divide by 2 we get,

$$= [2 \sin 8\pi/15 \cos 7\pi/15] / 2 \times 8 \sin \pi/15$$

We know, $2\sin A \cos B = \sin (A+B) + \sin (A-B)$

Then the above equation becomes,

$$= [\sin (8\pi/15 + 7\pi/15) + \sin (8\pi/15 - 7\pi/15)] / 16 \sin \pi/15$$

$$= [\sin (\pi) + \sin (\pi/15)] / 16 \sin \pi/15$$

$$= [0 + \sin (\pi/15)] / 16 \sin \pi/15$$

$$= \sin (\pi/15) / 16 \sin \pi/15$$

$$= 1/16$$

$$= \text{RHS}$$

Hence proved.