

## EXERCISE 10.1

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1. Show that  $f(x) = |x - 3|$  is continuous but not differentiable at  $x = 3$ .

**Solution:**

$$\text{Given } f(x) = |x - 3|$$

Therefore we can write given function as,

$$f(x) = \begin{cases} -(x - 3), & x < 3 \\ x - 3, & x \geq 3 \end{cases}$$

$$\text{But } f(3) = 3 - 3 = 0$$

$$\text{LHL} = \lim_{x \rightarrow 3^-} f(x)$$

$$= \lim_{h \rightarrow 0} f(3 - h)$$

$$= \lim_{h \rightarrow 0} 3 - (3 - h)$$

$$= \lim_{h \rightarrow 0} 0$$

Now consider,

$$\text{RHL} = \lim_{x \rightarrow 3^+} f(x)$$

$$= \lim_{h \rightarrow 0} f(3 + h)$$

$$= \lim_{h \rightarrow 0} 3 + h - 3$$

$$= 0$$

$$\text{LHL} = \text{RHL} = f(3)$$

Since,  $f(x)$  is continuous at  $x = 3$

$$\begin{aligned}
 \text{LHD at } x = 3 &= \lim_{x \rightarrow 3^-} \frac{f(x) - f(3)}{x - 3} \\
 &= \lim_{h \rightarrow 0^-} \frac{f(3-h) - f(3)}{3-h-3} \\
 &= \lim_{h \rightarrow 0^-} \frac{3 - (3-h) - 0}{-h} \\
 &= \lim_{h \rightarrow 0^-} \frac{h}{-h} \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 \text{RHD at } x = 3 &= \lim_{x \rightarrow 3^+} \frac{f(x) - f(3)}{x - 3} \\
 &= \lim_{h \rightarrow 0^+} \frac{f(3+h) - f(3)}{3+h-3} \\
 &= \lim_{h \rightarrow 0^+} \frac{3 + h - 3 - 0}{h} \\
 &= \lim_{h \rightarrow 0^+} \frac{h}{h} \\
 &= 1
 \end{aligned}$$

LHD at  $x = 3 \neq$  RHD at  $x = 3$

Hence,  $f(x)$  is continuous but not differentiable at  $x = 3$ .

**2. Show that  $f(x) = x^{1/3}$  is not differentiable at  $x = 0$ .**

**Solution:**

For differentiability,

LHD (at  $x = 0$ ) = RHD (at  $x = 0$ )

$$\begin{aligned}
 (\text{LHD at } x = 0) &= \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} \\
 &= \lim_{h \rightarrow 0^-} \frac{f(0-h) - f(0)}{0-h-0}
 \end{aligned}$$

$$= \lim_{h \rightarrow 0^-} \frac{(-h)^{\frac{1}{3}} - 0}{-h}$$

$$= \lim_{h \rightarrow 0^-} \frac{(-h)^{\frac{1}{3}}}{-h}$$

$$= \lim_{h \rightarrow 0^-} \frac{(-1)^{\frac{1}{3}}(h)^{\frac{1}{3}}}{(-1)h}$$

$$= \lim_{h \rightarrow 0} (-1)^{\frac{-2}{3}} h^{\frac{-2}{3}}$$

= Not defined

$$\text{(RHD at } x = 3) = \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{h \rightarrow 0^+} \frac{f(0 + h) - f(0)}{0 + h - 0}$$

$$= \lim_{h \rightarrow 0^+} \frac{(h)^{\frac{1}{3}} - 0}{+h}$$

$$= \lim_{h \rightarrow 0^+} \frac{(h)^{\frac{1}{3}}}{+h}$$

$$= \lim_{h \rightarrow 0} h^{\frac{-2}{3}}$$

= Not defined

Since, LHD and RHD does not exist at  $x = 0$

Hence,  $f(x)$  is not differentiable at  $x = 0$

3. Show that  $f(x) = \begin{cases} 12x - 13, & \text{if } x \leq 3 \\ 2x^2 + 5, & \text{if } x > 3 \end{cases}$  is differentiable at  $x = 3$ . Also, find  $f'(3)$

**Solution:**

Now we have to check differentiability of given function at  $x = 3$

That is LHD (at  $x = 3$ ) = RHD (at  $x = 3$ )

$$\begin{aligned}
 (\text{LHD at } x = 3) &= \lim_{x \rightarrow 3^-} \frac{f(x) - f(3)}{x - 3} \\
 &= \lim_{h \rightarrow 0^-} \frac{f(3-h) - f(3)}{3-h-3} \\
 &= \lim_{h \rightarrow 0^-} \frac{[12(3-h) - 13] - [12(3) - 13]}{-h} \\
 &= \lim_{h \rightarrow 0^-} \frac{36 - 12h - 13 - 36 + 13}{-h} \\
 &= \lim_{h \rightarrow 0^-} \frac{-12h}{-h} \\
 &= 12
 \end{aligned}$$

$$\begin{aligned}
 (\text{RHD at } x = 3) &= \lim_{x \rightarrow 3^+} \frac{f(x) - f(3)}{x - 3} \\
 &= \lim_{h \rightarrow 0^+} \frac{f(3+h) - f(3)}{3+h-3} \\
 &= \lim_{h \rightarrow 0^+} \frac{[2(3+h^2) + 5] - [12(3) - 13]}{3+h-3} \\
 &= \lim_{h \rightarrow 0^+} \frac{18 + 12h + 2h^2 + 5 - 36 + 13}{h} \\
 &= \lim_{h \rightarrow 0^+} \frac{2h^2 + 12h}{h} \\
 &= \lim_{h \rightarrow 0^+} \frac{h(2h + 12)}{h} \\
 &= 12
 \end{aligned}$$

Since,  $(\text{LHD at } x = 3) = (\text{RHD at } x = 3)$   
 Hence,  $f(x)$  is differentiable at  $x = 3$ .

**4. Show that the function  $f$  is defined as follows**

$$f(x) = \begin{cases} 3x - 2, & 0 < x \leq 1 \\ 2x^2 - x, & 1 < x \leq 2 \\ 5x - 4, & x > 2 \end{cases}$$

**Is continuous at  $x = 2$ , but not differentiable thereat.**

**Solution:**

Given

$$f(x) = \begin{cases} 3x - 2, & 0 < x \leq 1 \\ 2x^2 - x, & 1 < x \leq 2 \\ 5x - 4, & x > 2 \end{cases}$$

Now we have to check continuity at  $x = 2$

For continuity,

$$\text{LHL (at } x = 2) = \text{RHL (at } x = 2)$$

$$f(2) = 2(2)^2 - 2$$

$$= 8 - 2 = 6$$

$$\text{LHL} = \lim_{x \rightarrow 2^-} f(x)$$

$$= \lim_{h \rightarrow 0^-} f(2 - h)$$

$$= \lim_{h \rightarrow 0^-} [2(2 - h)^2 - (2 - h)]$$

$$= 8 - 2$$

$$= 6$$

$$\text{RHL} = \lim_{x \rightarrow 2^+} f(x)$$

$$= \lim_{h \rightarrow 0^+} f(2 + h)$$

$$= \lim_{h \rightarrow 0} 5(2 + h) - 4$$

$$= 6$$

Since,  $\text{LHL} = \text{RHL} = f(2)$

Hence,  $F(x)$  is continuous at  $x = 2$

Now we have to differentiability at  $x = 2$

$$(\text{LHD at } x = 2) = \lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{h \rightarrow 0} \frac{f(2-h) - f(2)}{2-h-2}$$

$$= \lim_{h \rightarrow 0} \frac{[2(2-h)^2 - (2-h)] - [8-2]}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{8-8h+2h^2-h-6}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{2h^2-6h}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{h(2h-6)}{-h}$$

$$= \lim_{h \rightarrow 0} (6 - 2h)$$

$$= 6$$

Now consider,

$$(\text{RHD at } x = 2) = \lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{2+h-2}$$

$$= \lim_{h \rightarrow 0} \frac{[5(2+h)-4] - [8-2]}{h}$$

$$= \lim_{h \rightarrow 0} \frac{10+5h-4-6}{h}$$

$$= 5$$

Since,  $(\text{RHD at } x = 2) \neq (\text{LHD at } x = 2)$

Hence,  $f(2)$  is not differentiable at  $x = 2$ .

**5. Discuss the continuity and differentiability of the function  $f(x) = |x| + |x-1|$  in the interval of  $(-1, 2)$ .**

**Solution:**

The given function  $f(x)$  can be defined as

$$f(x) = \begin{cases} x + x + 1, & -1 < x < 0 \\ 1, & 0 \leq x \leq 1 \\ -x - x + 1, & 1 < x < 2 \end{cases}$$

$$f(x) = \begin{cases} 2x + 1, & -1 < x < 0 \\ 1, & 0 \leq x \leq 1 \\ -2x + 1, & 1 < x < 2 \end{cases}$$

We know that a polynomial and a constant function is continuous and differentiable everywhere. So,  $f(x)$  is continuous and differentiable for  $x \in (-1, 0)$  and  $x \in (0, 1)$  and  $(1, 2)$ .

We need to check continuity and differentiability at  $x = 0$  and  $x = 1$ .

Continuity at  $x = 0$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} 2x + 1 = 1$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} 1 = 1$$

$$f(0) = 1$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0)$$

Since,  $f(x)$  is continuous at  $x = 0$

Continuity at  $x = 1$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} 1 = 1$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 1 = 1$$

$$f(1) = 1$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = 1$$

Since,  $f(x)$  is continuous at  $x = 1$

Now we have to check differentiability at  $x = 0$

For differentiability, LHD (at  $x = 0$ ) = RHD (at  $x = 0$ )

Differentiability at  $x = 0$

$$(\text{LHD at } x = 0) = \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \rightarrow 0^-} \frac{2x + 1 - 1}{x - 0}$$

$$= \lim_{x \rightarrow 0^-} \frac{2x}{x}$$

$$= 2$$

$$(\text{RHD at } x = 0) = \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \rightarrow 0^+} \frac{1 - 1}{x}$$

$$= \lim_{x \rightarrow 0^+} \frac{0}{x}$$

$$= 0$$

Since,  $(\text{LHD at } x = 0) \neq (\text{RHD at } x = 0)$

So,  $f(x)$  is differentiable at  $x = 0$ .

Now we have to check differentiability at  $x = 1$

For differentiability,  $\text{LHD (at } x = 1) = \text{RHD (at } x = 1)$

Differentiability at  $x = 1$

$$(\text{LHD at } x = 1) = \lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \rightarrow 1^-} \frac{1 - 1}{x - 1}$$

$$= 0$$

$$(\text{RHD at } x = 1) = \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \rightarrow 1^+} \frac{-2x + 1 - 1}{x - 1}$$

$$= \infty$$

Since,  $f(x)$  is not differentiable at  $x = 1$ .

So,  $f(x)$  is continuous on  $(-1, 2)$  but not differentiable at  $x = 0, 1$