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Find dy/dx, when

1. x = at² and y = 2 at

Solution:

Given that $x = at^2$, $y = 2at$

Now by differentiating $x = at^2$ with respect to t we get

$$\frac{dx}{dt} = \frac{d(at^2)}{dt} = 2at$$

Again by differentiating y = 2at with respect to t we get

$$\frac{dy}{dt} = \frac{d(2at)}{dt} = 2a$$

Therefore,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

2. x = a (θ + sin θ) and y = a (1 – cos θ)

Solution:

Given that $x = at^2$, $y = 2at$

Now by differentiating $x = at^2$ with respect to t we get

$$\frac{dx}{dt} = \frac{d(at^2)}{dt} = 2at$$

Again by differentiating y = 2at with respect to t we get

$$\frac{dy}{dt} = \frac{d(2at)}{dt} = 2a$$

Therefore,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

x = a (θ + sin θ)

Differentiating it with respect to θ,

$\frac{dx}{d\theta} = a(1 + \cos\theta)$ ...... (1)

And,

y = a (1- cos θ)

Differentiating it with respect to θ,

$$\frac{dy}{d\theta} = a(0 + \sin\theta)$$

$\frac{dy}{d\theta} = a\sin\theta$ ...... (2)

Using equation (1) and (2),

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{a\sin\theta}{a(1 - \cos\theta)}$$

$$= \frac{\frac{2\sin\theta}{2}\frac{(\cos\theta)}{2}}{\frac{2\sin^2\theta}{2}},$$

{Since, $1 - \cos\theta = \frac{2\sin^2\theta}{2}$ }

$$= \frac{dy}{dx} = \frac{\tan\theta}{2}$$

3. x = a cos θ and y = b sin θ

Solution:

Given x = a cos θ and y = b sin θ

Now by differentiating x with respect to θ we get,

$$\frac{dx}{d\theta} = \frac{d(a\cos\theta)}{d\theta} = -a\sin\theta$$

Again by differentiating y with respect to θ we get,

$$\frac{dy}{d\theta} = \frac{d(b\sin\theta)}{d\theta} = b\cos\theta$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$$

4. $x = a\,e^{\theta}(\sin\theta - \cos\theta)$, $y = a\,e^{\theta}(\sin\theta + \cos\theta)$

Solution:

Given that $x = a\,e^{\theta}(\sin\theta - \cos\theta)$

Differentiating it with respect to θ

$$\frac{dx}{d\theta} = a[e^{\theta}\frac{d(\sin\theta - \cos\theta)}{d\theta} + (\sin\theta - \cos\theta)\frac{d(e^{\theta})}{d\theta}]$$

$= a\,[e^{\theta}(\cos\theta + \sin\theta) + (\sin\theta - \cos\theta)\,e^{\theta}]$

$\frac{dx}{d\theta} = a[2e^{\theta}\sin\theta]$ …… (1)

And also given that, $y = a\,e^{\theta}(\sin\theta + \cos\theta)$

Differentiating it with respect to θ,

$$\frac{dy}{d\theta} = a[e^{\theta}\frac{d(\sin\theta + \cos\theta)}{d\theta} + (\sin\theta + \cos\theta)\frac{d(e^{\theta})}{d\theta}]$$

$= a\,[e^{\theta}(\cos\theta - \sin\theta) + (\sin\theta + \cos\theta)\,e^{\theta}]$

$\frac{dy}{d\theta} = a[2e^{\theta}\cos\theta]$ …… (2)

Dividing equation (2) by equation (1),

$$\frac{dy}{dx} = \frac{a\left(2e^{\theta}\cos\theta\right)}{a(2e^{\theta}\sin\theta)}$$

$$\frac{dy}{dx} = \cot\theta$$

5. x = b sin² θ and y = a cos² θ

Solution:

Given that $x = b \sin^2 \theta$

Now by differentiating above equation with respect to θ, we get

$$\frac{dx}{d\theta} = \frac{d(b\sin^2\theta\ )}{d\theta} = 2b\sin\theta\cos\theta$$

And also given that $y = a \cos^2 \theta$

Now by differentiating above equation with respect to θ, we get

$$\frac{dy}{d\theta} = d(a\cos^2\theta) = -2a\cos\theta\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{2a\cos\theta\sin\theta}{2b\sin\theta\cos\theta} = -\frac{a}{b}$$

6. x = a (1 – cos θ) and y = a (θ + sin θ) at θ = π/2

Solution:

Given x = a (1 – cos θ)

Differentiate x with respect to θ, we get

$$\frac{dx}{d\theta} = \frac{d[a(1-\cos\theta)]}{d\theta} = a(\sin\theta)$$

And also given that y = a (θ + sin θ)

Differentiate x with respect to θ, we get

$$\frac{dy}{d\theta} = \frac{d(\theta + \sin\theta)}{d\theta} = a(1 + \cos\theta)$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a(1 + \cos\theta)}{a(\sin\theta)}\bigg|\left(\theta = \frac{\pi}{2}\right)$$

$$= \frac{a(1 + 0)}{a} = 1$$

7. $x = \frac{e^t + e^{-t}}{2}$ and $y = \frac{e^t - e^{-t}}{2}$

Solution:

Given $x = \frac{e^t + e^{-t}}{2}$

Differentiating above equation with respect to t

$$\frac{dx}{dt} = \frac{1}{2}\left[\frac{d(e^t)}{dt} + \frac{d(e^{-t})}{dt}\right]$$

$$= \frac{1}{2}\left[e^t + e^{-t}\frac{d(-t)}{dt}\right]$$

$\frac{dx}{dt} = \frac{1}{2}(e^t - e^{-t}) = y$ ...... (1)

And also given that $y = \frac{e^t - e^{-t}}{2}$

Differentiating above equation with respect to t,

$$\frac{dy}{dt} = \frac{1}{2}\left[\frac{d(e^t)}{dt} - \frac{d(e^{-t})}{dt}\right]$$

$$= \frac{1}{2}\left[e^{t-} - e^{-t}\frac{d(-t)}{dt}\right]$$

$$= \frac{1}{2}\left(e^{t} - e^{-t}(-1)\right)$$

$$\frac{dy}{dt} = \frac{e^{\theta}+e^{\theta}}{2} = x \text{ ...... (2)}$$

Dividing equation (2) by (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{x}{y}$$

$$\frac{dy}{dx} = \frac{x}{y}$$

8. $x = \frac{3at}{1+t^2}$ *and* $y = \frac{3at^2}{1+t^2}$

Solution:

Given $x = \frac{3at}{1+t^2}$

Differentiating above equation with respect to t using quotient rule,

$$\frac{dx}{dt} = \left[\frac{\left((1+t^2)\frac{d(3at)}{dt} - 3at\frac{d(1+t^2)}{dt}\right)}{(1+t^2)^2}\right]$$

$$= \left[\frac{(1+t^2)(3a) - 3at(2t)}{(1+t^2)^2}\right]$$

$$= \left[\frac{(3a) + 3at^2 - 6at^2}{(1+t^2)^2}\right]$$

$$= \left[\frac{3a - 3at^2}{(1+t^2)^2}\right]$$

$$\frac{dx}{dt} = \frac{3a(1-t^2)}{(1+t^2)^2} \text{ ...... (1)}$$

And also given that $y = \frac{3at^2}{1+t^2}$

Differentiating above equation with respect to t using quotient rule

$$\frac{dy}{dx} = \left[\frac{(1+t^2)\frac{d(3at^2)}{dt} - 3at^2\frac{d(1+t^2)}{dt}}{(1+t^2)^2}\right]$$

$$\frac{dy}{dt} = \left[\frac{(1+t^2)(6at) - (3at^2)(2t)}{(1+t^2)^2}\right]$$

$$= \left[\frac{6at + 6at^3 - 6at^3}{(1+t^2)^2}\right]$$

$\frac{dy}{dt} = \frac{6at}{(1+t^2)^2}$ ..... (2)

Dividing equation (2) by (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6at}{(1+t^2)^2} \times \frac{3a(1-t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dx} = \frac{2t}{1-t^2}$$

9. x = a (cos θ + θ sin θ) and y = a (sin θ – θ cos θ)

Solution:

Given x = a (cos θ + θ sin θ)

Now differentiating x with respect to θ

$$\frac{dx}{d\theta} = a\left[\frac{d}{d\theta}\cos\theta + \frac{d}{d\theta}(\theta\sin\theta)\right]$$

$$= a\left[-\sin\theta + \frac{\theta d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(\theta)\right]$$

$$= a[-\sin\theta + \theta\cos\theta + \sin\theta] = a\theta\cos\theta$$

And also given y = a (sin θ – cos θ),

Now differentiating x with respect to θ

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\theta\cos\theta)\right]$$

$$= a\left[\cos\theta - \left\{\frac{\theta d}{d\theta}(\cos\theta) + \cos\theta\frac{d}{d\theta}(\theta)\right\}\right]$$

$$= a[\cos\theta + \theta\sin\theta - \cos\theta]$$

$= a\ \theta \sin \theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

10. $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right)$ *and* $y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$

Solution:

Given $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right)$

Differentiating x with respect to θ using the product rule,

$$\frac{dx}{d\theta} = e^{\theta}\frac{d}{d\theta}\left(\theta + \frac{1}{\theta}\right) + \left(\theta + \frac{1}{\theta}\right)\frac{d}{d\theta}(e^{\theta})$$

$$= e^{\theta}\left(1 - \frac{1}{\theta^2}\right) + \frac{\theta^2 + 1}{\theta}(e^{\theta})$$

$$= e^{\theta}\left(1 - \frac{1}{\theta^2} + \frac{\theta^2 + 1}{\theta}\right)$$

$$= e^{\theta}\left(\frac{\theta^2 - 1 + \theta^3 + \theta}{\theta^2}\right)$$

$\frac{dx}{d\theta} = e^{\theta}\left(\frac{\theta^3+\theta^2+\theta-1}{\theta^2}\right)$ …… (1)

And also given that, $y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$

Differentiating y with respect to θ using the product rule,

$$\frac{dy}{d\theta} = e^{-\theta}\frac{d}{d\theta}\left(\theta - \frac{1}{\theta}\right) + \left(\theta - \frac{1}{\theta}\right)\frac{d}{d\theta}(e^{-\theta})$$

$$= e^{-\theta}\left(1 + \frac{1}{\theta^2}\right) + \left(\theta - \frac{1}{\theta}\right)e^{-\theta}\frac{d}{d\theta}(-\theta)$$

$$= e^{-\theta}\left(1 + \frac{1}{\theta^2}\right) + \left(\theta - \frac{1}{\theta}\right)e^{-\theta}(-1)$$

$$\frac{dy}{d\theta} = e^{-\theta}\left(1 + \frac{1}{\theta^2} - \theta + \frac{1}{\theta}\right)$$

$$= e^{-\theta}\left(\frac{\theta^2 + 1 - \theta^3 + \theta}{\theta^2}\right)$$

$\frac{dy}{d\theta} = e^{-\theta}\left(\frac{-\theta^3+\theta^2+\theta+1}{\theta^2}\right)$ …… (2)

Divide equation (2) by (1)

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = e^{-\theta}\left(\frac{-\theta^3 + \theta^2 + \theta + 1}{\theta^2}\right) \times \frac{1}{e^{\theta}\left(\frac{\theta^3 + \theta^2 + \theta - 1}{\theta^2}\right)}$$

$$= e^{-2\theta}\left(\frac{-\theta^3 + \theta^2 + \theta + 1}{\theta^3 + \theta^2 + \theta - 1}\right)$$

$$11.\ x = \frac{2t}{1+t^2}\ and\ y = \frac{1-t^2}{1+t^2}$$

Solution:

Given, $x = \frac{2t}{1+t^2}$

Differentiating x with respect to t using quotient rule,

$$\frac{dx}{dt}=\left[\frac{(1+t^2)\frac{d}{dt}(2t)-2t\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right]$$

$$=\left[\frac{(1+t^2)(2)-2t(2t)}{(1+t^2)^2}\right]$$

$$=\left[\frac{2+2t^2-4t^2}{(1+t^2)^2}\right]$$

$$=\left[\frac{2-2t^2}{(1+t^2)^2}\right]$$

$\frac{dx}{dt}=\left[\frac{2-2t^2}{(1+t^2)^2}\right]$ ...... (1)

And also given that, $y=\frac{1-t^2}{1+t^2}$

Differentiating y with respect to t using quotient rule,

$$\frac{dy}{dt}=\left[\frac{(1+t^2)\frac{d}{dt}(1-t^2)-(1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right]$$

$$=\left[\frac{(1+t^2)(-2t)-(1-t^2)(2t)}{(1+t^2)^2}\right]$$

$$=\left[\frac{-2t-2t^3-2t+2t^3}{(1+t^2)^2}\right]$$

$\frac{dy}{dt}=\left[\frac{-4t}{(1+t^2)^2}\right]$ ...... (2)

Dividing equation (2) by (1),

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\left[\frac{-4t}{(1+t^2)^2}\right]\times\frac{1}{\left[\frac{2-2t^2}{(1+t^2)^2}\right]}$$

$$=-\frac{2t}{1-t^2}$$

$$\frac{dy}{dx} = -\frac{x}{y}\ [\text{since}, \frac{x}{y} = \frac{2t}{1+t^2} \times \frac{1+t^2}{1-t^2} = \frac{2t}{1-t^2}\ ]$$

12. $x = \cos^{-1}\frac{1}{\sqrt{1+t^2}}$ *and* $y = \sin^{-1}\frac{1}{\sqrt{1+t^2}}$, $t \epsilon R$

Solution:

Given $x = \cos^{-1}\frac{1}{\sqrt{1+t^2}}$

Differentiating x with respect to t using chain rule,

$$\frac{dx}{dt} = -\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1+t^2}}\right)^2}}\frac{d}{dt}\left(\frac{1}{\sqrt{1+t^2}}\right)$$

$$= -\frac{1}{\sqrt{1-\frac{1}{1+t^2}}}\left\{-\frac{1}{2(1+t^2)^{\frac{3}{2}}}\right\}\frac{d}{dt}(1+t^2)$$

$$= -\frac{(1+t^2)^{\frac{1}{2}}}{\sqrt{(1+t^2-1)}}\left\{-\frac{1}{2(1+t^2)^{\frac{3}{2}}}\right\}(2t)$$

$$= -\frac{t}{\sqrt{t^2} \times (1+t^2)}$$

$\frac{dx}{dt} = -\frac{1}{1+t^2}$ ...... (1)

Also given that, $y = \sin^{-1}\frac{1}{\sqrt{1+t^2}}$

Differentiating y with respect to t using chain rule,

$$\frac{dy}{dt} = \frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1+t^2}}\right)^2}}\frac{d}{dt}\left(\frac{1}{\sqrt{1+t^2}}\right)$$

$$= \frac{1}{\sqrt{1-\frac{1}{1+t^2}}}\left\{-\frac{1}{2(1+t^2)^{\frac{3}{2}}}\right\}\frac{d}{dt}(1+t^2)$$

$$= \frac{(1+t^2)^{\frac{1}{2}}}{\sqrt{(1+t^2-1)}}\left\{-\frac{1}{2(1+t^2)^{\frac{3}{2}}}\right\}(2t)$$

$$= \frac{t}{\sqrt{t^2}\times(1+t^2)}$$

$\frac{dy}{dt} = -\frac{1}{1+t^2}$ ...... (2)

Dividing equation (2) by (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{1}{1+t^2}\times -\frac{1+t^2}{1}$$

$$\frac{dy}{dx} = 1$$

13. $x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$

Solution:

Given $x = \frac{1-t^2}{1+t^2}$

Differentiating x with respect to t using quotient rule,

$$\frac{dx}{dt} = \left[\frac{(1+t^2)\frac{d}{dt}(1-t^2)-(1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right]$$

$$= \left[\frac{(1+t^2)(-2t)-(1-t^2)(2t)}{(1+t^2)^2}\right]$$

$$= \left[\frac{-2t - 2t^3 - 2t + 2t^3}{(1+t^2)^2}\right]$$

$$\frac{dx}{dt} = \left[\frac{-4t}{(1+t^2)^2}\right] \text{...... (1)}$$

And also given that, $y = \frac{2t}{1+t^2}$

Differentiating y with respect to t using quotient rule,

$$\frac{dy}{dt} = \left[\frac{(1+t^2)\frac{d}{dt}(2t) - (2t)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right]$$

$$= \left[\frac{(1+t^2)(2) - (2t)(2t)}{(1+t^2)^2}\right]$$

$$= \left[\frac{2 + 2t^2 - 4t^2}{(1+t^2)^2}\right]$$

$$\frac{dy}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \text{...... (2)}$$

Divide equation (2) by (1) so,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{1}{\frac{-4t}{(1+t^2)^2}}$$

$$\frac{dy}{dx} = \frac{2(1-t^2)}{-4t}$$

14. $If\ x = 2\cos\theta - \cos 2\theta\ and\ y = 2\sin\theta - sin2\theta,\ prove\ that\ \frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right).$

Solution:
Given x = 2cos θ – cos 2θ

Differentiating x with respect to θ using chain rule,

$$\frac{dx}{d\theta} = 2(-\sin\theta) - (-\sin2\theta)\frac{d}{d\theta}(2\theta)$$

$$= -2\sin\theta + 2\sin2\theta$$

$\frac{dx}{d\theta} = 2(\sin2\theta - \sin\theta)$ ...... (1)

And also given that, y = 2sin θ – sin 2θ

Differentiating y with respect to θ using chain rule,

$$\frac{dy}{d\theta} = 2\cos\theta - \cos2\theta\frac{d}{d\theta}(2\theta)$$

$$= 2\cos\theta - \cos2\theta(2)$$

$$= 2\cos\theta - 2\cos2\theta$$

$\frac{dy}{d\theta} = 2(\cos\theta - \cos2\theta)$ ...... (2)

Dividing equation (2) by equation (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2(\cos\theta - \cos2\theta)}{2(\sin2\theta - \sin\theta)}$$

$$= \frac{(\cos\theta - \cos2\theta)}{(\sin2\theta - \sin\theta)}$$

$$\frac{dy}{dx} = \frac{-2\sin\left(\frac{\theta + 2\theta}{2}\right)\sin\left(\frac{\theta - 2\theta}{2}\right)}{2\cos\left(\frac{\theta + 2\theta}{2}\right)\sin\left(\frac{2\theta - \theta}{2}\right)}$$

$$\left[\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)\right]$$

$$= -\frac{\sin\left(\frac{3\theta}{2}\right)\left(\sin\left(-\frac{\theta}{2}\right)\right)}{\cos\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}$$

$$= -\frac{\sin\left(\frac{3\theta}{2}\right)\left(-\sin\frac{\theta}{2}\right)}{\cos\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}$$

$$= \frac{\sin\left(\frac{3\theta}{2}\right)}{\cos\left(\frac{3\theta}{2}\right)}$$

$$\frac{dy}{dx} = \tan(\frac{3\theta}{2})$$