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1. Differentiate x^2 with respect to x^3.

Solution:

Let $u = x^2$ and $v = x^3$.

We have to differentiate u with respect to v that is find $\frac{du}{dv}$.

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}(x^2)$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{du}{dx} = 2x^{2-1}$$

$$\therefore \frac{du}{dx} = 2x$$

Now, on differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(x^3)$$

$$\Rightarrow \frac{dv}{dx} = 3x^{3-1}$$

$$\therefore \frac{dv}{dx} = 3x^2$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{2x}{3x^2}$$

$$\therefore \frac{du}{dv} = \frac{2}{3x}$$

Thus, $\frac{du}{dv} = \frac{2}{3x}$

2. Differentiate log (1 +x^2) with respect to $\tan^{-1}$ x.

Solution:

Let u = log (1 + x^2) and v = $\tan^{-1}$x.

We have to differentiate u with respect to v that is find $\frac{du}{dv}$.

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}[\log(1 + x^2)]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$

$$\Rightarrow \frac{du}{dx} = \frac{1}{1+x^2}\frac{d}{dx}(1 + x^2)$$

Now by using chain rule, we get

$$\Rightarrow \frac{du}{dx} = \frac{1}{1 + x^2}\left[\frac{d}{dx}(1) + \frac{d}{dx}(x^2)\right]$$

However, $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = \frac{1}{1 + x^2}[0 + 2x^{2-1}]$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{1 + x^2}[2x]$$

$$\therefore \frac{du}{dx} = \frac{2x}{1 + x^2}$$

Now, on differentiating v with respect to x, we get

$$\therefore \frac{dv}{dx} = \frac{1}{1 + x^2}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{2x}{1+x^2}}{\frac{1}{1+x^2}}$$

$$\Rightarrow \frac{du}{dv} = \frac{2x}{1+x^2} \times (1+x^2)$$

$$\therefore \frac{du}{dv} = 2x$$

Thus, $\frac{du}{dv} = 2x$

3. Differentiate (log x)^x with respect to log x.

Solution:

Let u = (log x)^x and v = log x.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have u = (log x)^x

Taking log on both sides, we get

Log u = log (log x)^x

⇒ Log u = x × log (log x) [∵ log a^m = m × log a]

On differentiating both the sides with respect to x, we get

$$\frac{d}{du}(\log u) \times \frac{du}{dx} = \frac{d}{dx}[x \times \log(\log x)]$$

We know that (u v)′ = vu′ + uv′

$$\Rightarrow \frac{d}{du}(\log u) \times \frac{du}{dx} = \log(\log x)\frac{d}{dx}(x) + x\frac{d}{dx}[\log(\log x)]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$ and $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{1}{u} \times \frac{du}{dx} = \log(\log x) \times 1 + x\left[\frac{1}{\log x}\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \log(\log x) + \frac{x}{\log x}\frac{d}{dx}(\log x)$$

But, u = $(\log x)^x$ and $\frac{d}{dx}(\log x) = \frac{1}{x}$

$$\Rightarrow \frac{1}{(\log x)^x}\frac{du}{dx} = \log(\log x) + \frac{x}{\log x} \times \frac{1}{x}$$

$$\Rightarrow \frac{1}{(\log x)^x}\frac{du}{dx} = \log(\log x) + \frac{1}{\log x}$$

$$\therefore \frac{du}{dx} = (\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right]$$

Now, on differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(\log x)$$

$$\therefore \frac{dv}{dx} = \frac{1}{x}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{(\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right]}{\frac{1}{x}}$$

$$\Rightarrow \frac{du}{dv} = x(\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right]$$

$$\Rightarrow \frac{du}{dv} = x(\log x)^x\left[\frac{\log(\log x)\log x + 1}{\log x}\right]$$

$$\Rightarrow \frac{du}{dv} = \frac{x(\log x)^x}{\log x}[\log(\log x)\log x + 1]$$

$$\therefore \frac{du}{dv} = x(\log x)^{x-1}[1 + \log x \log(\log x)]$$

Thus, $\frac{du}{dv} = x(\log x)^{x-1}[1 + \log x \log(\log x)]$

4. Differentiate $\sin^{-1} \sqrt{(1-x^2)}$ with respect to $\cos^{-1}x$, if
(i) $x \in (0, 1)$
(ii) $x \in (-1, 0)$

Solution:

(i) Given $\sin^{-1} \sqrt{(1-x^2)}$

Let $u = \sin^{-1}\sqrt{1 - x^2}$ and $v = \cos^{-1}x$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\sqrt{1 - x^2}$

By substituting $x = \cos\theta$, we have

$u = \sin^{-1}\sqrt{1 - (\cos\theta)^2}$

$\Rightarrow u = \sin^{-1}\sqrt{1 - \cos^2\theta}$

$\Rightarrow u = \sin^{-1}\sqrt{\sin^2\theta}$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow u = \sin^{-1}(\sin\theta)$

Given $x \in (0, 1)$

However, $x = \cos\theta$.

$\Rightarrow \cos\theta \in (0, 1)$

$\Rightarrow \theta \in \left(0, \frac{\pi}{2}\right)$

Hence, $u = \sin^{-1}(\sin\theta) = \theta$.

$\Rightarrow$ u = $\cos^{-1}x$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}(\cos^{-1}x)$$

We know $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$

$$\therefore \frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Now, on differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(\cos^{-1}x)$$

$$\therefore \frac{dv}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

We have, $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{-\frac{1}{\sqrt{1-x^2}}}{-\frac{1}{\sqrt{1-x^2}}}$$

$$\Rightarrow \frac{du}{dv} = -\frac{1}{\sqrt{1-x^2}} \times \left(-\sqrt{1-x^2}\right)$$

$$\therefore \frac{du}{dv} = 1$$

Thus, $\frac{du}{dv} = 1$

(ii) Given $\sin^{-1} \sqrt{(1-x^2)}$

Let $u = \sin^{-1}\sqrt{1-x^2}$ and v = $\cos^{-1}x$.

Now we have to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\sqrt{1 - x^2}$

By substituting x = cos θ, we get

$u = \sin^{-1}\sqrt{1 - (\cos\theta)^2}$

$\Rightarrow u = \sin^{-1}\sqrt{1 - \cos^2\theta}$

$\Rightarrow u = \sin^{-1}\sqrt{\sin^2\theta}$ [∵ $\sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow u = \sin^{-1}(\sin\theta)$

Given x ϵ (–1, 0)

However, x = cos θ.

⇒ Cos θ ϵ (–1, 0)

$\Rightarrow \theta \in \left(\frac{\pi}{2}, \pi\right)$

Hence, $u = \sin^{-1}(\sin\theta) = \pi - \theta$.

$\Rightarrow u = \pi - \cos^{-1}x$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}(\pi - \cos^{-1}x)$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(\pi) - \frac{d}{dx}(\cos^{-1}x)$$

We know $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 0 - \left(-\frac{1}{\sqrt{1 - x^2}}\right)$$

$$\therefore \frac{du}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

Now, on differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(\cos^{-1} x)$$

$$\therefore \frac{dv}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{1}{\sqrt{1-x^2}}}{-\frac{1}{\sqrt{1-x^2}}}$$

$$\Rightarrow \frac{du}{dv} = \frac{1}{\sqrt{1-x^2}} \times \left(-\sqrt{1-x^2}\right)$$

$$\therefore \frac{du}{dv} = -1$$

Thus, $\frac{du}{dv} = -1$

5. $Differentiate\ \sin^{-1}\left(4x\sqrt{1-4x^2}\right)\ with\ respect\ to\ \sqrt{1-4x^2}\ if,$

(i) $x \epsilon \left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$

(ii) $x \epsilon \left(\frac{1}{2\sqrt{2}}, \frac{1}{2}\right)$

(iii) $x \epsilon \left(-\frac{1}{2}, -\frac{1}{2\sqrt{2}}\right)$

Solution:

(i) Let

$u = \sin^{-1}\left(4x\sqrt{1-4x^2}\right)$ And $v = \sqrt{1-4x^2}$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\left(4x\sqrt{1-4x^2}\right)$

$\Rightarrow u = \sin^{-1}\left(4x\sqrt{1-(2x)^2}\right)$

By substituting 2x = cos θ, we have

$u = \sin^{-1}\left(2\cos\theta\sqrt{1-(\cos\theta)^2}\right)$

$\Rightarrow u = \sin^{-1}\left(2\cos\theta\sqrt{1-(\cos\theta)^2}\right)$

$\Rightarrow u = \sin^{-1}\left(2\cos\theta\sqrt{\sin^2\theta}\right)$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow u = \sin^{-1}(2\cos\theta\sin\theta)$

$\Rightarrow u = \sin^{-1}(\sin 2\theta)$

Given $x \in \left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$

However, $2x = \cos\theta \Rightarrow x = \frac{\cos\theta}{2}$

$\Rightarrow \frac{\cos\theta}{2} \in \left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$

$\Rightarrow \cos\theta \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\Rightarrow \theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

$\Rightarrow 2\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Hence, $u = \sin^{-1}(\sin 2\theta) = \pi - 2\theta$.

$\Rightarrow u = \pi - 2\cos^{-1}(2x)$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}[\pi - 2\cos^{-1}(2x)]$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(\pi) - \frac{d}{dx}[2\cos^{-1}(2x)]$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(\pi) - 2\frac{d}{dx}[\cos^{-1}(2x)]$$

We know $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 0 - 2\left[-\frac{1}{\sqrt{1-(2x)^2}}\frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{2}{\sqrt{1-4x^2}}\left[\frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{2}{\sqrt{1-4x^2}}\left[2\frac{d}{dx}(x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}}\frac{d}{dx}(x)$$

However, $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}} \times 1$$

$$\therefore \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}}$$

Now, we have $v = \sqrt{1-4x^2}$

On differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}\left(\sqrt{1-4x^2}\right)$$

$$\Rightarrow \frac{dv}{dx} = \frac{d}{dx}(1-4x^2)^{\frac{1}{2}}$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2}(1-4x^2)^{\frac{1}{2}-1}\frac{d}{dx}(1-4x^2)$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2}(1-4x^2)^{-\frac{1}{2}}\left[\frac{d}{dx}(1)-\frac{d}{dx}(4x^2)\right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1-4x^2}}\left[\frac{d}{dx}(1)-4\frac{d}{dx}(x^2)\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1-4x^2}}[0-4(2x^{2-1})]$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1-4x^2}}[-8x]$$

$$\therefore \frac{dv}{dx} = -\frac{4x}{\sqrt{1-4x^2}}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{4}{\sqrt{1-4x^2}}}{-\frac{4x}{\sqrt{1-4x^2}}}$$

$$\Rightarrow \frac{du}{dv} = \frac{4}{\sqrt{1-4x^2}} \times \left(-\frac{\sqrt{1-x^2}}{4x}\right)$$

$$\therefore \frac{du}{dv} = -\frac{1}{x}$$

Thus, $\frac{du}{dv} = -\frac{1}{x}$

(ii) Let

$u = \sin^{-1}\left(4x\sqrt{1-4x^2}\right)$ and $v = \sqrt{1-4x^2}$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\left(4x\sqrt{1-4x^2}\right)$

$\Rightarrow u = \sin^{-1}\left(4x\sqrt{1-(2x)^2}\right)$

By substituting 2x = cos θ, we have

$u = \sin^{-1}\left(2\cos\theta\sqrt{1-(\cos\theta)^2}\right)$

$\Rightarrow u = \sin^{-1}\left(2\cos\theta\sqrt{1-(\cos\theta)^2}\right)$

$\Rightarrow u = \sin^{-1}\left(2\cos\theta\sqrt{\sin^2\theta}\right)$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow u = \sin^{-1}(2\cos\theta\sin\theta)$

$\Rightarrow u = \sin^{-1}(\sin 2\theta)$

Given $x \in \left(\frac{1}{2\sqrt{2}}, \frac{1}{2}\right)$

However, 2x = cos θ ⇒ $x = \frac{\cos\theta}{2}$

$\Rightarrow \frac{\cos\theta}{2} \in \left(\frac{1}{2\sqrt{2}}, \frac{1}{2}\right)$

$\Rightarrow \cos\theta \in \left(\frac{1}{\sqrt{2}}, 1\right)$

$\Rightarrow \theta \in \left(0, \frac{\pi}{4}\right)$

$\Rightarrow 2\theta \in \left(0, \frac{\pi}{2}\right)$

Hence, $u = \sin^{-1}(\sin 2\theta) = 2\theta$.

$\Rightarrow u = 2\cos^{-1}(2x)$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}[2\cos^{-1}(2x)]$$

$$\Rightarrow \frac{du}{dx} = 2\frac{d}{dx}[\cos^{-1}(2x)]$$

We know $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 2\left[-\frac{1}{\sqrt{1-(2x)^2}}\frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{du}{dx} = -\frac{2}{\sqrt{1-4x^2}}\left[\frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{du}{dx} = -\frac{2}{\sqrt{1-4x^2}}\left[2\frac{d}{dx}(x)\right]$$

$$\Rightarrow \frac{du}{dx} = -\frac{4}{\sqrt{1-4x^2}}\frac{d}{dx}(x)$$

However, $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{du}{dx} = -\frac{4}{\sqrt{1-4x^2}} \times 1$$

$$\therefore \frac{du}{dx} = -\frac{4}{\sqrt{1-4x^2}}$$

We have $\frac{dv}{dx} = -\frac{4x}{\sqrt{1-4x^2}}$

We know that $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{-\frac{4}{\sqrt{1-4x^2}}}{-\frac{4x}{\sqrt{1-4x^2}}}$$

$$\Rightarrow \frac{du}{dv} = -\frac{4}{\sqrt{1-4x^2}} \times \left(-\frac{\sqrt{1-x^2}}{4x}\right)$$

$$\therefore \frac{du}{dv} = \frac{1}{x}$$

Thus, $\frac{du}{dv} = \frac{1}{x}$

(iii) Let

$u = \sin^{-1}\left(4x\sqrt{1-4x^2}\right)$ And $v = \sqrt{1-4x^2}$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\left(4x\sqrt{1-4x^2}\right)$

$$\Rightarrow u = \sin^{-1}\left(4x\sqrt{1-(2x)^2}\right)$$

By substituting 2x = cos θ, we have

$$u = \sin^{-1}\left(2\cos\theta\sqrt{1-(\cos\theta)^2}\right)$$

$$\Rightarrow u = \sin^{-1}\left(2\cos\theta\sqrt{1-(\cos\theta)^2}\right)$$

$\Rightarrow u = \sin^{-1}\left(2\cos\theta\sqrt{\sin^2\theta}\right)$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow u = \sin^{-1}(2\cos\theta\sin\theta)$

$\Rightarrow u = \sin^{-1}(\sin 2\theta)$

Given $x \in \left(-\frac{1}{2}, -\frac{1}{2\sqrt{2}}\right)$

However, $2x = \cos\theta \Rightarrow x = \frac{\cos\theta}{2}$

$$\Rightarrow \frac{\cos\theta}{2} \in \left(-\frac{1}{2}, -\frac{1}{2\sqrt{2}}\right)$$

$\Rightarrow \cos\theta \in \left(-1, -\frac{1}{\sqrt{2}}\right)$

$\Rightarrow \theta \in \left(\frac{3\pi}{4}, \pi\right)$

$\Rightarrow 2\theta \in \left(\frac{3\pi}{2}, 2\pi\right)$

Hence, $u = \sin^{-1}(\sin 2\theta) = 2\pi - 2\theta$.

$\Rightarrow u = 2\pi - 2\cos^{-1}(2x)$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}[2\pi - 2\cos^{-1}(2x)]$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(2\pi) - \frac{d}{dx}[2\cos^{-1}(2x)]$$

$$\Rightarrow \frac{du}{dx} = 2\frac{d}{dx}(\pi) - 2\frac{d}{dx}[\cos^{-1}(2x)]$$

We know $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 0 - 2\left[-\frac{1}{\sqrt{1-(2x)^2}}\frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{2}{\sqrt{1-4x^2}}\left[\frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{2}{\sqrt{1-4x^2}}\left[2\frac{d}{dx}(x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}}\frac{d}{dx}(x)$$

However, $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}} \times 1$$

$$\therefore \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}}$$

We have $\frac{dv}{dx} = -\frac{4x}{\sqrt{1-4x^2}}$

We know that $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{4}{\sqrt{1-4x^2}}}{-\frac{4x}{\sqrt{1-4x^2}}}$$

$$\Rightarrow \frac{du}{dv} = \frac{4}{\sqrt{1-4x^2}} \times \left(-\frac{\sqrt{1-x^2}}{4x}\right)$$

$$\therefore \frac{du}{dv} = -\frac{1}{x}$$

Thus, $\frac{du}{dv} = -\frac{1}{x}$

6. *Differentiate* $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ *with respect to* $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, *if* $-1 < x < 1,\ x \neq 0.$

Solution:

Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

By substituting x = tan θ, we have

$$u = \tan^{-1}\left(\frac{\sqrt{1+(\tan\theta)^2}-1}{\tan\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\sqrt{\sec^2\theta-1}}{\tan\theta}\right) \quad [\because \sec^2\theta - \tan^2\theta = 1]$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{1-\cos\left(2\times\frac{\theta}{2}\right)}{\sin\left(2\times\frac{\theta}{2}\right)}\right)$$

But, $\cos 2\theta = 1 - 2\sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.

$$\Rightarrow u = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

Given $-1 < x < 1 \Rightarrow x \in (-1, 1)$

However, $x = \tan\theta$

$\Rightarrow$ Tan $\theta \in (-1, 1)$

$$\Rightarrow \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$$

$$\Rightarrow \frac{\theta}{2} \in \left(-\frac{\pi}{8}, \frac{\pi}{8}\right)$$

Hence, $u = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$

$$\Rightarrow u = \frac{1}{2}\tan^{-1} x$$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{1}{2}\tan^{-1} x\right)$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{2}\frac{d}{dx}(\tan^{-1} x)$$

We know $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$$\Rightarrow \frac{du}{dx} = \frac{1}{2} \times \frac{1}{1+x^2}$$

$$\therefore \frac{du}{dx} = \frac{1}{2(1+x^2)}$$

Now, we have $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

By substituting x = tan θ, we have

$$v = \sin^{-1}\left(\frac{2\tan\theta}{1+(\tan\theta)^2}\right)$$

$$\Rightarrow v = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$\Rightarrow v = \sin^{-1}\left(\frac{2\tan\theta}{\sec^2\theta}\right)$$ [∵ $\sec^2\theta - \tan^2\theta = 1$]

$$\Rightarrow v = \sin^{-1}\left(\frac{2 \times \frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos^2\theta}}\right)$$

$$\Rightarrow v = \sin^{-1}\left(2 \times \frac{\sin\theta}{\cos\theta} \times \cos^2\theta\right)$$

$\Rightarrow$ v = $\sin^{-1}(2\sin\theta\cos\theta)$

But, $\sin 2\theta = 2\sin\theta\cos\theta$

$\Rightarrow$ v = $\sin^{-1}(\sin 2\theta)$

However, $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \Rightarrow 2\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Hence, v = $\sin^{-1}(\sin 2\theta) = 2\theta$

$\Rightarrow$ v = $2\tan^{-1}x$

On differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(2\tan^{-1}x)$$

$$\Rightarrow \frac{dv}{dx} = 2\frac{d}{dx}(\tan^{-1}x)$$

We know $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$$\Rightarrow \frac{dv}{dx} = 2 \times \frac{1}{1+x^2}$$

$$\therefore \frac{dv}{dx} = \frac{2}{1+x^2}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{1}{2(1+x^2)}}{\frac{2}{1+x^2}}$$

$$\Rightarrow \frac{du}{dv} = \frac{1}{2(1+x^2)} \times \frac{1+x^2}{2}$$

$$\therefore \frac{du}{dv} = \frac{1}{4}$$

Thus, $\frac{du}{dv} = \frac{1}{4}$

7. $Differentiate \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ *with respect to* $\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$, *if*,

(i) x ∈ (0, 1/ √2)

(ii) x ∈ (1/√2, 1)

Solution:

(i) Let

$u = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ And $v = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$.

We have to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

By substituting x = sin θ, we have

$$u = \sin^{-1}\left(2\sin\theta\sqrt{1-(\sin\theta)^2}\right)$$

$$\Rightarrow u = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$$

$\Rightarrow u = \sin^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow u = \sin^{-1}(2\sin\theta\cos\theta)$

$\Rightarrow u = \sin^{-1}(\sin 2\theta)$

Now, we have $v = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

By substituting x = sin θ, we have

$$v = \sec^{-1}\left(\frac{1}{\sqrt{1-(\sin\theta)^2}}\right)$$

$\Rightarrow v = \sec^{-1}\left(\frac{1}{\sqrt{1-\sin^2\theta}}\right)$

$\Rightarrow v = \sec^{-1}\left(\frac{1}{\sqrt{\cos^2\theta}}\right)$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow v = \sec^{-1}\left(\frac{1}{\cos\theta}\right)$

$\Rightarrow v = \sec^{-1}(\sec\theta)$

Given $x \in \left(0, \frac{1}{\sqrt{2}}\right)$

However, $x = \sin\theta$

$\Rightarrow \sin\theta \in \left(0, \frac{1}{\sqrt{2}}\right)$

$\Rightarrow \theta \in \left(0, \frac{\pi}{4}\right)$

$\Rightarrow 2\theta \in \left(0, \frac{\pi}{2}\right)$

Hence, $u = \sin^{-1}(\sin 2\theta) = 2\theta$.

$\Rightarrow u = 2\sin^{-1}(x)$

On differentiating u with respect to x, we get

$\frac{du}{dx} = \frac{d}{dx}(2\sin^{-1}x)$

$\Rightarrow \frac{du}{dx} = 2\frac{d}{dx}(\sin^{-1}x)$

We know $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \frac{du}{dx} = 2 \times \frac{1}{\sqrt{1-x^2}}$

$\therefore \frac{du}{dx} = \frac{2}{\sqrt{1-x^2}}$

We have $\theta \in \left(0, \frac{\pi}{4}\right)$

Hence, $v = \sec^{-1}(\sec\theta) = \theta$

$\Rightarrow v = \sin^{-1}x$

On differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(\sin^{-1}x)$$

We know $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$$\therefore \frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{2}{\sqrt{1-x^2}}}{\frac{1}{\sqrt{1-x^2}}}$$

$$\Rightarrow \frac{du}{dv} = \frac{2}{\sqrt{1-x^2}} \times \sqrt{1-x^2}$$

$$\therefore \frac{du}{dv} = 2$$

Thus, $\frac{du}{dv} = 2$

(ii) Let

$u = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ And $v = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$.

We have to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

By substituting x = sin θ, we have

$$u = \sin^{-1}\left(2\sin\theta\sqrt{1-(\sin\theta)^2}\right)$$

$$\Rightarrow u = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$$

$$\Rightarrow u = \sin^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$$ [∵ $\sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow u = \sin^{-1}(2\sin\theta\cos\theta)$

$\Rightarrow u = \sin^{-1}(\sin 2\theta)$

Now, we have $v = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

By substituting x = sin θ, we have

$$v = \sec^{-1}\left(\frac{1}{\sqrt{1-(\sin\theta)^2}}\right)$$

$$\Rightarrow v = \sec^{-1}\left(\frac{1}{\sqrt{1-\sin^2\theta}}\right)$$

$$\Rightarrow v = \sec^{-1}\left(\frac{1}{\sqrt{\cos^2\theta}}\right)$$ [∵ $\sin^2\theta + \cos^2\theta = 1$]

$$\Rightarrow v = \sec^{-1}\left(\frac{1}{\cos\theta}\right)$$

$$\Rightarrow v = \sec^{-1}(\sec\theta)$$

Given $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$

However, x = sin θ

$$\Rightarrow \sin\theta \in \left(\frac{1}{\sqrt{2}}, 1\right)$$

$$\Rightarrow \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$$

$$\Rightarrow 2\theta \in \left(\frac{\pi}{2}, \pi\right)$$

Hence, $u = \sin^{-1}(\sin 2\theta) = \pi - 2\theta$.

$\Rightarrow u = \pi - 2\sin^{-1}(x)$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}(\pi - 2\sin^{-1}x)$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(\pi) - \frac{d}{dx}(2\sin^{-1}x)$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(\pi) - 2\frac{d}{dx}(\sin^{-1}x)$$

We know $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 0 - 2 \times \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{du}{dx} = \frac{-2}{\sqrt{1-x^2}}$$

We have $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Hence, $v = \sec^{-1}(\sec\theta) = \theta$

$\Rightarrow v = \sin^{-1}x$

On differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(\sin^{-1}x)$$

We know $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$$\therefore \frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{-\frac{2}{\sqrt{1-x^2}}}{\frac{1}{\sqrt{1-x^2}}}$$

$$\Rightarrow \frac{du}{dv} = -\frac{2}{\sqrt{1-x^2}} \times \sqrt{1-x^2}$$

$$\therefore \frac{du}{dv} = -2$$

Thus, $\frac{du}{dv} = -2$

8. Differentiate (cos x)$^{\sin x}$ with respect to (sin x)$^{\cos x}$.

Solution:

Let u = (cos x)$^{\sin x}$ and v = (sin x)$^{\cos x}$.

We have to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have u = (cos x)$^{\sin x}$

Taking log on both sides, we get

Log u = log (cos x)$^{\sin x}$

⇒ Log u = (sin x) × log (cos x) [∵ log a^m = m × log a]

On differentiating both the sides with respect to x, we get

$$\frac{d}{du}(\log u) \times \frac{du}{dx} = \frac{d}{dx}[\sin x \times \log(\cos x)]$$

We know that (uv)' = vu' + uv'

$$\Rightarrow \frac{d}{du}(\log u) \times \frac{du}{dx} = \log(\cos x)\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}[\log(\cos x)]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$ and $\frac{d}{dx}(\sin x) = \cos x$

$$\Rightarrow \frac{1}{u} \times \frac{du}{dx} = \log(\cos x) \times \cos x + \sin x \left[\frac{1}{\cos x}\frac{d}{dx}(\cos x)\right]$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \cos x \log(\cos x) + \frac{\sin x}{\cos x}\frac{d}{dx}(\cos x)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \cos x \log(\cos x) + \tan x \frac{d}{dx}(\cos x)$$

We know $\frac{d}{dx}(\cos x) = -\sin x$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \cos x \log(\cos x) + \tan x(-\sin x)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \cos x \log(\cos x) - \tan x \sin x$$

But, $u = (\cos x)^{\sin x}$

$$\Rightarrow \frac{1}{(\cos x)^{\sin x}}\frac{du}{dx} = \cos x \log(\cos x) - \tan x \sin x$$

$$\therefore \frac{du}{dx} = (\cos x)^{\sin x}[\cos x \log(\cos x) - \tan x \sin x]$$

Now, we have $v = (\sin x)^{\cos x}$

Taking log on both sides, we get

$\text{Log } v = \log (\sin x)^{\cos x}$

$\Rightarrow \text{Log } v = (\cos x) \times \log (\sin x)$ [$\because \log a^m = m \times \log a$]

On differentiating both the sides with respect to x, we get

$$\frac{d}{dv}(\log v) \times \frac{dv}{dx} = \frac{d}{dx}[\cos x \times \log(\sin x)]$$

We know that (uv)' = vu' + uv' (product rule)

$$\Rightarrow \frac{d}{du}(\log u) \times \frac{dv}{dx} = \log(\sin x)\frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}[\log(\sin x)]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$ and $\frac{d}{dx}(\cos x) = -\sin x$

$$\Rightarrow \frac{1}{v} \times \frac{dv}{dx} = \log(\sin x) \times (-\sin x) + \cos x \left[\frac{1}{\sin x}\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = -\sin x \log(\sin x) + \frac{\cos x}{\sin x}\frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = -\sin x \log(\sin x) + \cot x \frac{d}{dx}(\sin x)$$

We know $\frac{d}{dx}(\sin x) = \cos x$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = -\sin x \log(\sin x) + \cot x \times (\cos x)$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = -\sin x \log(\sin x) + \cot x \cos x$$

But, $v = (\sin x)^{\cos x}$

$$\Rightarrow \frac{1}{(\sin x)^{\cos x}}\frac{dv}{dx} = -\sin x \log(\sin x) + \cot x \cos x$$

$$\therefore \frac{dv}{dx} = (\sin x)^{\cos x}[-\sin x \log(\sin x) + \cot x \cos x]$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{(\cos x)^{\sin x}[\cos x \log(\cos x) - \tan x \sin x]}{(\sin x)^{\cos x}[-\sin x \log(\sin x) + \cot x \cos x]}$$

$$\therefore \frac{du}{dv} = \frac{(\cos x)^{\sin x}[\cos x \log(\cos x) - \tan x \sin x]}{(\sin x)^{\cos x}[\cot x \cos x - \sin x \log(\sin x)]}$$

Thus, $\frac{du}{dv} = \frac{(\cos x)^{\sin x}[\cos x \log(\cos x) - \tan x \sin x]}{(\sin x)^{\cos x}[\cot x \cos x - \sin x \log(\sin x)]}$

9. *Differentiate* $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ *with respect to* $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, *if* $0 < x < 1$.

Solution:

Let $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

By substituting x = tan θ, we have

$$u = \sin^{-1}\left(\frac{2\tan\theta}{1+(\tan\theta)^2}\right)$$

$$\Rightarrow u = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$\Rightarrow u = \sin^{-1}\left(\frac{2\tan\theta}{\sec^2\theta}\right)$ [∵ $\sec^2\theta - \tan^2\theta = 1$]

$$\Rightarrow u = \sin^{-1}\left(\frac{2\times\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos^2\theta}}\right)$$

$$\Rightarrow u = \sin^{-1}\left(2\times\frac{\sin\theta}{\cos\theta}\times\cos^2\theta\right)$$

$\Rightarrow u = \sin^{-1}(2\sin\theta\cos\theta)$

But, $\sin 2\theta = 2\sin\theta\cos\theta$

$\Rightarrow u = \sin^{-1}(\sin 2\theta)$

Given $0 < x < 1 \Rightarrow x \in (0, 1)$

However, $x = \tan\theta$

$\Rightarrow \tan\theta \in (0, 1)$

$\Rightarrow \theta \in \left(0, \frac{\pi}{4}\right)$

$\Rightarrow 2\theta \in \left(0, \frac{\pi}{2}\right)$

Hence, $u = \sin^{-1}(\sin 2\theta) = 2\theta$

$\Rightarrow u = 2\tan^{-1}x$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}(2\tan^{-1}x)$$

$$\Rightarrow \frac{du}{dx} = 2\frac{d}{dx}(\tan^{-1}x)$$

We know $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$$\Rightarrow \frac{du}{dx} = 2 \times \frac{1}{1+x^2}$$

$$\therefore \frac{du}{dx} = \frac{2}{1+x^2}$$

Now, we have $v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

By substituting $x = \tan\theta$, we have

$$v = \cos^{-1}\left(\frac{1-(\tan\theta)^2}{1+(\tan\theta)^2}\right)$$

$$\Rightarrow v = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$\Rightarrow v = \cos^{-1}\left(\frac{1-\tan^2\theta}{\sec^2\theta}\right) \quad [\because \sec^2\theta - \tan^2\theta = 1]$$

$$\Rightarrow v = \cos^{-1}\left(\frac{1}{\sec^2\theta} - \frac{\tan^2\theta}{\sec^2\theta}\right)$$

$$\Rightarrow v = \cos^{-1}\left(\frac{1}{\frac{1}{\cos^2\theta}} - \frac{\frac{\sin^2\theta}{\cos^2\theta}}{\frac{1}{\cos^2\theta}}\right)$$

$\Rightarrow v = \cos^{-1}(\cos^2\theta - \sin^2\theta)$

But, $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\Rightarrow$ v = $\cos^{-1}(\cos 2\theta)$

However, $\theta \in \left(0, \frac{\pi}{4}\right) \Rightarrow 2\theta \in \left(0, \frac{\pi}{2}\right)$

Hence, v = $\cos^{-1}(\cos 2\theta) = 2\theta$

$\Rightarrow$ v = $2\tan^{-1}x$

On differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(2\tan^{-1}x)$$

$$\Rightarrow \frac{dv}{dx} = 2\frac{d}{dx}(\tan^{-1}x)$$

We know $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$$\Rightarrow \frac{dv}{dx} = 2 \times \frac{1}{1+x^2}$$

$$\therefore \frac{dv}{dx} = \frac{2}{1+x^2}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{2}{1+x^2}}{\frac{2}{1+x^2}}$$

$$\Rightarrow \frac{du}{dv} = \frac{2}{1+x^2} \times \frac{1+x^2}{2}$$

$$\therefore \frac{du}{dv} = 1$$

Thus, $\frac{du}{dv} = 1$

10. $Differentiate \tan^{-1}\left(\frac{1+ax}{1-ax}\right)$ $with\ respect\ to \sqrt{1+a^2x^2}.$

Solution:

Let $u = \tan^{-1}\left(\frac{1+ax}{1-ax}\right)$ and $v = \sqrt{1+a^2x^2}$.

We have to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \tan^{-1}\left(\frac{1+ax}{1-ax}\right)$

By substituting ax = tan θ, we have

$$u = \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\tan\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\theta\right)\right) \left[\because \tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}\right]$$

$$\Rightarrow u = \frac{\pi}{4}+\theta$$

$$\Rightarrow u = \frac{\pi}{4}+\tan^{-1}(ax)$$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}\left[\frac{\pi}{4}+\tan^{-1}(ax)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}\right)+\frac{d}{dx}[\tan^{-1}(ax)]$$

We know $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 0+\frac{1}{1+(ax)^2}\frac{d}{dx}(ax)$$

$$\Rightarrow \frac{du}{dx} = 0 + \frac{1}{1+(ax)^2}\frac{d}{dx}(ax)$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{1+a^2x^2}\left[a\frac{d}{dx}(x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{a}{1+a^2x^2}\frac{d}{dx}(x)$$

We know $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{du}{dx} = \frac{a}{1+a^2x^2} \times 1$$

$$\therefore \frac{du}{dx} = \frac{a}{1+a^2x^2}$$

Now, we have $v = \sqrt{1+a^2x^2}$

On differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}\left(\sqrt{1+a^2x^2}\right)$$

$$\Rightarrow \frac{dv}{dx} = \frac{d}{dx}(1+a^2x^2)^{\frac{1}{2}}$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2}(1+a^2x^2)^{\frac{1}{2}-1}\frac{d}{dx}(1+a^2x^2)$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2}(1+a^2x^2)^{-\frac{1}{2}}\left[\frac{d}{dx}(1) + \frac{d}{dx}(a^2x^2)\right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1+a^2x^2}}\left[\frac{d}{dx}(1) + a^2\frac{d}{dx}(x^2)\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1+a^2x^2}}[0 + a^2(2x^{2-1})]$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1+a^2x^2}}[2a^2x]$$

$$\therefore \frac{dv}{dx} = \frac{a^2x}{\sqrt{1+a^2x^2}}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{a}{1+a^2x^2}}{\frac{a^2x}{\sqrt{1+a^2x^2}}}$$

$$\Rightarrow \frac{du}{dv} = \frac{a}{1+a^2x^2} \times \frac{\sqrt{1+a^2x^2}}{a^2x}$$

$$\therefore \frac{du}{dv} = \frac{1}{ax\sqrt{1+a^2x^2}}$$

Thus, $\frac{du}{dv} = \frac{1}{ax\sqrt{1+a^2x^2}}$