EXERCISE 11.1 PAGE NO: 11.17

Differentiate the following functions from the first principles:

1. e^{-x}

Solution:

We have to find the derivative of e^{-x} with the first principle method,

So let $f(x) = e^{-x}$

By using the first principle formula, we get,

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$f'(x) = \lim_{h\to 0} \frac{e^{-(x+h)}-e^{-x}}{h}$

$f'(x) = \lim_{h\to 0} \frac{e^{-x}(e^{-h}-1)}{h}$

$f'(x) = \lim_{h\to 0} \frac{e^{-x}(e^{-h}-1)(-1)}{h(-1)}$

[By using $\lim_{x\to 0} \frac{e^{x}-1}{x} = 1$]

$f'(x) = -e^{-x}$

2. e^{3x}

Solution:

We have to find the derivative of e^{3x} with the first principle method,

So, let $f(x) = e^{3x}$

By using the first principle formula, we get,

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$f'(x) = \lim_{h\to 0} \frac{e^{3(x+h)} - e^{3x}}{h}$

$f'(x) = \lim_{h\to 0} \frac{e^{3x}(e^{3h} - 1)}{h}$

$f'(x) = \lim_{h\to 0} \frac{e^{3x}(e^{3h} - 1)3}{3h}$

[By using $\lim_{x\to 0} \frac{e^{x} - 1}{x} = 1$]

$f'(x) = 3e^{3x}$

3. $e^{ax + b}$

Solution:

We have to find the derivative of e^{ax+b} with the first principle method,

So, let $f(x) = e^{ax+b}$

By using the first principle formula, we get,

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$f'(x) = \lim_{h\to 0} \frac{e^{a(x+h)+b} - e^{ax+b}}{h}$

$f'(x) = \lim_{h\to 0} \frac{e^{ax+b}(e^{ah} - 1)a}{ah}$

[By using $\lim_{x\to 0} \frac{e^{x} - 1}{x} = 1$]

$f'(x) = a\, e^{ax+b}$

4. $e^{\cos x}$

Solution:

We have to find the derivative of $e^{\cos x}$ with the first principle method, So, let f (x) = $e^{\cos x}$

By using the first principle formula, we get,

f ' (x) = $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

f ' (x) = $\lim_{h\to 0} \frac{e^{\cos(x+h)}-e^{\cos x}}{h}$

f ' (x) = $\lim_{h\to 0} \frac{e^{\cos x}(e^{\cos(x+h)-\cos x}-1)}{h}$

f ' (x) = $\lim_{h\to 0} \frac{e^{\cos x}(e^{\cos(x+h)-\cos x}-1)}{\cos(x+h)-\cos x} \frac{\cos(x+h)-\cos x}{h}$

[By using $\lim_{x\to 0} \frac{e^x-1}{x}$ = 1]

f ' (x) = $\lim_{h\to 0} e^{\cos x} \frac{\cos(x+h)-\cos x}{h}$

Now by using cos (x + h) = cos x cos h – sin x sin h

f ' (x) = $\lim_{h\to 0} e^{\cos x} \frac{\cos x \cos h-\sin x \sin h-\cos x}{h}$

f ' (x) = $\lim_{h\to 0} e^{\cos x} [\frac{\cos x(\cos h-1)}{h} - \frac{\sin x \sin h}{h}]$

[By using $\lim_{x\to 0} \frac{\sin x}{x}$ = 1 and cos 2x = 1–2sin^2 x]

f ' (x) = $\lim_{h\to 0} e^{\cos x} [\frac{\cos x\left(-2\sin^2\frac{h}{2}\right)(\frac{h}{4})}{h(\frac{h}{4})} - \sin x]$

f ' (x) = $\lim_{h\to 0} e^{\cos x} [\frac{\cos x\left(-2\sin^2\frac{h}{2}\right)(\frac{h}{4})}{\frac{h^2}{2^2}} - \sin x]$

f ' (x) = $-e^{\cos x} \sin x$

5. $e^{\sqrt{2x}}$

Solution:

We have to find the derivative of $e^{\sqrt{2x}}$ with the first principle method,

So, let $f(x) = e^{\sqrt{2x}}$

By using the first principle formula, we get,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{e^{\sqrt{2(x+h)}} - e^{\sqrt{2x}}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{e^{\sqrt{2x}}(e^{\sqrt{2(x+h)} - \sqrt{2x}} - 1)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{e^{\sqrt{2x}}(e^{\sqrt{2(x+h)} - \sqrt{2x}} - 1)}{h} \times \frac{\sqrt{2(x+h)} - \sqrt{2x}}{\sqrt{2(x+h)} - \sqrt{2x}}$$

[By using $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$]

$$f'(x) = \lim_{h \to 0} \frac{e^{\sqrt{2x}}}{h} \times (\sqrt{2(x+h)} - \sqrt{2x}) \times \frac{\sqrt{2(x+h)} + \sqrt{2x}}{\sqrt{2(x+h)} + \sqrt{2x}}$$

[By rationalizing]

$$f'(x) = \lim_{h \to 0} \frac{e^{\sqrt{2x}}}{h} \times \frac{(2(x+h) - 2x)}{\sqrt{2(x+h)} + \sqrt{2x}}$$

$$f'(x) = \frac{e^{\sqrt{2x}}}{\sqrt{2x}}$$

EXERCISE 11.2 PAGE NO: 11.37

Differentiate the following functions with respect to x:

1. Sin (3x + 5)

Solution:

Given Sin (3x + 5)

Let y = sin (3x + 5)

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\sin(3x+5)]$$

We know $\frac{d}{dx}(\sin x) = \cos x$

$\Rightarrow \frac{dy}{dx} = \cos(3x+5)\frac{d}{dx}(3x+5)$ [Using chain rule]

$$\Rightarrow \frac{dy}{dx} = \cos(3x+5)\left[\frac{d}{dx}(3x) + \frac{d}{dx}(5)\right]$$

$$\Rightarrow \frac{dy}{dx} = \cos(3x+5)\left[3\frac{d}{dx}(x) + \frac{d}{dx}(5)\right]$$

However, $\frac{d}{dx}(x) = 1$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \cos(3x+5)\,[3 \times 1 + 0]$$

$$\therefore \frac{dy}{dx} = 3\cos(3x+5)$$

Thus, $\frac{d}{dx}[\sin(3x+5)] = 3\cos(3x+5)$

2. tan^2 x

Solution:

Given $\tan^2 x$

Let y = $\tan^2 x$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^2 x)$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dy}{dx} = 2\tan^{2-1} x \frac{d}{dx}(\tan x)$ [Using chain rule]

$$\Rightarrow \frac{dy}{dx} = 2\tan x \frac{d}{dx}(\tan x)$$

However, $\frac{d}{dx}(\tan x) = \sec^2 x$

$$\Rightarrow \frac{dy}{dx} = 2\tan x(\sec^2 x)$$

$$\therefore \frac{dy}{dx} = 2\tan x \sec^2 x$$

Thus, $\frac{d}{dx}(\tan^2 x) = 2\tan x \sec^2 x$

3. tan (x° + 45°)

Solution:

Let y = tan (x° + 45°)

First, we will convert the angle from degrees to radians.

Let y = tan (x° + 45°)

First, we will convert the angle from degrees to radians.

We have $1° = \left(\frac{\pi}{180}\right)^c \Rightarrow (x+45)° = \left[\frac{(x+45)\pi}{180}\right]^c$

$$\Rightarrow y = \tan\left[\frac{(x+45)\pi}{180}\right]$$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\tan\left[\frac{(x+45)\pi}{180}\right]\right\}$$

We know $\frac{d}{dx}(\tan x) = \sec^2 x$

$$\Rightarrow \frac{dy}{dx} = \sec^2\left[\frac{(x+45)\pi}{180}\right]\frac{d}{dx}\left[\frac{(x+45)\pi}{180}\right]$$ [Using chain rule]

$$\Rightarrow \frac{dy}{dx} = \sec^2(x^\circ + 45^\circ)\frac{\pi}{180}\frac{d}{dx}(x+45)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\pi}{180}\sec^2(x^\circ + 45^\circ)\left[\frac{d}{dx}(x) + \frac{d}{dx}(45)\right]$$

However, $\frac{d}{dx}(x) = 1$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{\pi}{180}\sec^2(x^\circ + 45^\circ)\,[1+0]$$

$$\therefore \frac{dy}{dx} = \frac{\pi}{180}\sec^2(x^\circ + 45^\circ)$$

Thus, $\frac{d}{dx}[\tan(x^\circ + 45^\circ)] = \frac{\pi}{180}\sec^2(x^\circ + 45^\circ)$

4. Sin (log x)

Solution:

Given sin (log x)

Let y = sin (log x)

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\sin(\log x)]$$

We know $\frac{d}{dx}(\sin x) = \cos x$

$\Rightarrow \frac{dy}{dx} = \cos(\log x)\frac{d}{dx}(\log x)$ [Using chain rule]

However, $\frac{d}{dx}(\log x) = \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = \cos(\log x) \times \frac{1}{x}$

$\therefore \frac{dy}{dx} = \frac{1}{x}\cos(\log x)$

Thus, $\frac{d}{dx}[\sin(\log x)] = \frac{1}{x}\cos(\log x)$

5. $e^{sin\sqrt{x}}$

Solution:

Let $y = e^{\sin\sqrt{x}}$

On differentiating y with respect to x, we get

$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin\sqrt{x}}\right)$

We know $\frac{d}{dx}(e^x) = e^x$

$\Rightarrow \frac{dy}{dx} = e^{\sin\sqrt{x}}\frac{d}{dx}\left(\sin\sqrt{x}\right)$ [Using chain rule]

We have $\frac{d}{dx}(\sin x) = \cos x$

$\Rightarrow \frac{dy}{dx} = e^{\sin\sqrt{x}}\cos\sqrt{x}\frac{d}{dx}\left(\sqrt{x}\right)$ [Using chain rule]

$\Rightarrow \frac{dy}{dx} = e^{\sin\sqrt{x}}\cos\sqrt{x}\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$

However, $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dy}{dx} = e^{\sin\sqrt{x}} \cos\sqrt{x} \left[\frac{1}{2} x^{\left(\frac{1}{2}-1\right)}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} e^{\sin\sqrt{x}} \cos\sqrt{x}\, x^{-\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x}} e^{\sin\sqrt{x}} \cos\sqrt{x}$$

Thus, $\frac{d}{dx}\left(e^{\sin\sqrt{x}}\right) = \frac{1}{2\sqrt{x}} e^{\sin\sqrt{x}} \cos\sqrt{x}$

6. $e^{\tan x}$

Solution:

Let $y = e^{\tan x}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\tan x}\right)$$

We know $\frac{d}{dx}(e^x) = e^x$

$\Rightarrow \frac{dy}{dx} = e^{\tan x} \frac{d}{dx}(\tan x)$ [Using chain rule]

We have $\frac{d}{dx}(\tan x) = \sec^2 x$

$$\therefore \frac{dy}{dx} = e^{\tan x} \sec^2 x$$

Thus, $\frac{d}{dx}\left(e^{\tan x}\right) = e^{\tan x} \sec^2 x$

7. $\text{Sin}^2 (2x + 1)$

Solution:

Let $y = \sin^2 (2x + 1)$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\sin^2(2x+1)]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

Using chain rule we get.

$$\Rightarrow \frac{dy}{dx} = 2\sin^{2-1}(2x+1)\frac{d}{dx}[\sin(2x+1)]$$

$$\Rightarrow \frac{dy}{dx} = 2\sin(2x+1)\frac{d}{dx}[\sin(2x+1)]$$

We have $\frac{d}{dx}(\sin x) = \cos x$

Now by using chain rule we have

$$\Rightarrow \frac{dy}{dx} = 2\sin(2x+1)\cos(2x+1)\frac{d}{dx}(2x+1)$$

$$\Rightarrow \frac{dy}{dx} = \sin[2(2x+1)]\frac{d}{dx}(2x+1)$$ [∵ sin (2θ) = 2sinθcosθ]

$$\Rightarrow \frac{dy}{dx} = \sin(4x+2)\left[\frac{d}{dx}(2x) + \frac{d}{dx}(1)\right]$$

$$\Rightarrow \frac{dy}{dx} = \sin(4x+2)\left[2\frac{d}{dx}(x) + \frac{d}{dx}(1)\right]$$

However, $\frac{d}{dx}(x) = 1$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \sin(4x+2)\,[2 \times 1 + 0]$$

$$\therefore \frac{dy}{dx} = 2\sin(4x+2)$$

Thus, $\frac{d}{dx}[\sin^2(2x+1)] = 2\sin(4x+2)$

8. $\log_7(2x - 3)$

Solution:

Let $y = \log_7 (2x - 3)$

We know that $\log_a b = \frac{\log b}{\log a}$.

$$\Rightarrow \log_7(2x-3) = \frac{\log(2x-3)}{\log 7}$$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{\log(2x-3)}{\log 7}\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1}{\log 7}\right)\frac{d}{dx}[\log(2x-3)]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$

Now by using chain rule we get

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1}{\log 7}\right)\left(\frac{1}{2x-3}\right)\frac{d}{dx}(2x-3)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(2x-3)\log 7}\left[\frac{d}{dx}(2x) - \frac{d}{dx}(3)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(2x-3)\log 7}\left[2\frac{d}{dx}(x) - \frac{d}{dx}(3)\right]$$

However, $\frac{d}{dx}(x) = 1$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(2x-3)\log 7}[2 \times 1 - 0]$$

$$\therefore \frac{dy}{dx} = \frac{2}{(2x-3)\log 7}$$

Thus, $\frac{d}{dx}[\log_7(2x-3)] = \frac{2}{(2x-3)\log 7}$

9. tan 5x^o

Solution:

Let y = tan (5x°)

First, we will convert the angle from degrees to radians. We have

$$1^\circ = \left(\frac{\pi}{180}\right)^c \Rightarrow 5x^\circ = 5x \times \frac{\pi}{180}^c$$

$$\Rightarrow y = \tan\left(5x \times \frac{\pi}{180}\right)$$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\tan\left(5x \times \frac{\pi}{180}\right)\right]$$

We know $\frac{d}{dx}(\tan x) = \sec^2 x$

Now by using chain rule we have

$$\Rightarrow \frac{dy}{dx} = \sec^2\left(5x \times \frac{\pi}{180}\right)\frac{d}{dx}\left(5x \times \frac{\pi}{180}\right)$$

$$\Rightarrow \frac{dy}{dx} = \sec^2(5x^\circ)\frac{\pi}{180}\frac{d}{dx}(5x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\pi}{180}\sec^2(5x^\circ)\left[5\frac{d}{dx}(x)\right]$$

However, $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{\pi}{180}\sec^2(5x^\circ)\,[5]$$

$$\therefore \frac{dy}{dx} = \frac{5\pi}{180}\sec^2(5x^\circ)$$

Thus, $\frac{d}{dx}(\tan 5x^\circ) = \frac{5\pi}{180}\sec^2(5x^\circ)$

10. 2^{x^3}

Solution:

Let $y = 2^{x^3}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(2^{x^3}\right)$$

We know $\frac{d}{dx}(a^x) = a^x \log a$

Now by using chain rule,

$$\Rightarrow \frac{dy}{dx} = 2^{x^3} \log 2 \frac{d}{dx}(x^3)$$

We have $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dy}{dx} = 2^{x^3} \log 2 \times 3x^{3-1}$$

$$\Rightarrow \frac{dy}{dx} = 2^{x^3} \log 2 \times 3x^2$$

$$\therefore \frac{dy}{dx} = 2^{x^3} 3x^2 \log 2$$

Thus, $\frac{d}{dx}\left(2^{x^3}\right) = 2^{x^3} 3x^2 \log 2$

11. 3^{e^x}

Solution:

Let $y = 3^{e^x}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(3^{e^x}\right)$$

We know $\frac{d}{dx}(a^x) = a^x \log a$

Now by using chain rule,

$\Rightarrow \frac{dy}{dx} = 3^{e^x} \log 3 \frac{d}{dx}(e^x)$

We have $\frac{d}{dx}(e^x) = e^x$

$\Rightarrow \frac{dy}{dx} = 3^{e^x} \log 3 \times e^x$

$\therefore \frac{dy}{dx} = 3^{e^x} e^x \log 3$

Thus, $\frac{d}{dx}\left(3^{e^x}\right) = 3^{e^x} e^x \log 3$

12. $\log_x 3$

Solution:

Let y = $\log_x 3$

We know that $\log_a b = \frac{\log b}{\log a}$.

$\Rightarrow \log_x 3 = \frac{\log 3}{\log x}$

On differentiating y with respect to x, we get

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\log 3}{\log x}\right)$

$\Rightarrow \frac{dy}{dx} = \log 3 \frac{d}{dx}\left(\frac{1}{\log x}\right)$

$\Rightarrow \frac{dy}{dx} = \log 3 \frac{d}{dx}(\log x)^{-1}$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

Now by using chain rule,

$$\Rightarrow \frac{dy}{dx} = \log 3\,[-1 \times (\log x)^{-1-1}]\,\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{dy}{dx} = -\log 3\,(\log x)^{-2}\frac{d}{dx}(\log x)$$

We have $\frac{d}{dx}(\log x) = \frac{1}{x}$

$$\Rightarrow \frac{dy}{dx} = -\log 3\,(\log x)^{-2} \times \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{x}\frac{\log 3}{(\log x)^2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{x}\frac{\log 3}{(\log x)^2} \times \frac{\log 3}{\log 3}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{x\log 3}\frac{(\log 3)^2}{(\log x)^2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{x\log 3}\left(\frac{\log 3}{\log x}\right)^2$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{x\log 3 \times \left(\frac{\log x}{\log 3}\right)^2}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{x\log 3\,(\log_3 x)^2}$$

Thus, $\frac{d}{dx}(\log_x 3) = -\frac{1}{x\log 3(\log_3 x)^2}$

13. 3^{x^2+2x}

Solution:

Let $y = 3^{x^2+2x}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(3^{x^2+2x}\right)$$

We know $\frac{d}{dx}(a^x) = a^x \log a$

Now by using chain rule, we get

$$\Rightarrow \frac{dy}{dx} = 3^{x^2+2x} \log 3 \frac{d}{dx}(x^2 + 2x)$$

$$\Rightarrow \frac{dy}{dx} = 3^{x^2+2x} \log 3 \left[\frac{d}{dx}(x^2) + \frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{dy}{dx} = 3^{x^2+2x} \log 3 \left[\frac{d}{dx}(x^2) + 2\frac{d}{dx}(x)\right]$$

We have $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{dy}{dx} = 3^{x^2+2x} \log 3 \, [2x + 2 \times 1]$$

$$\Rightarrow \frac{dy}{dx} = 3^{x^2+2x} \log 3 \, (2x + 2)$$

$$\therefore \frac{dy}{dx} = (2x + 2) 3^{x^2+2x} \log 3$$

Thus, $\frac{d}{dx}\left(3^{x^2+2x}\right) = (2x + 2) 3^{x^2+2x} \log 3$

14. $\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$

Solution:

Let $y = \sqrt{\frac{a^2-x^2}{a^2+x^2}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[\left(\frac{a^2 - x^2}{a^2 + x^2}\right)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

Using chain rule

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{a^2-x^2}{a^2+x^2}\right)^{\frac{1}{2}-1} \frac{d}{dx}\left(\frac{a^2-x^2}{a^2+x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{a^2 - x^2}{a^2 + x^2}\right)^{-\frac{1}{2}} \frac{d}{dx}\left(\frac{a^2 - x^2}{a^2 + x^2}\right)$$

We know that $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{a^2 - x^2}{a^2 + x^2}\right)^{-\frac{1}{2}} \left[\frac{(a^2 + x^2)\frac{d}{dx}(a^2 - x^2) - (a^2 - x^2)\frac{d}{dx}(a^2 + x^2)}{(a^2 + x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{a^2 - x^2}{a^2 + x^2}\right)^{-\frac{1}{2}} \left[\frac{(a^2 + x^2)\left(\frac{d}{dx}(a^2) - \frac{d}{dx}(x^2)\right) - (a^2 - x^2)\left(\frac{d}{dx}(a^2) + \frac{d}{dx}(x^2)\right)}{(a^2 + x^2)^2}\right]$$

However, $\frac{d}{dx}(x^2) = 2x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{a^2 - x^2}{a^2 + x^2}\right)^{-\frac{1}{2}} \left[\frac{(a^2 + x^2)(0 - 2x) - (a^2 - x^2)(0 + 2x)}{(a^2 + x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{a^2 - x^2}{a^2 + x^2}\right)^{-\frac{1}{2}}\left[\frac{-2x(a^2 + x^2) - 2x(a^2 - x^2)}{(a^2 + x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{a^2 - x^2}{a^2 + x^2}\right)^{-\frac{1}{2}}\left[\frac{-2x(a^2 + x^2 + a^2 - x^2)}{(a^2 + x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{a^2 - x^2}{a^2 + x^2}\right)^{-\frac{1}{2}}\left[\frac{-2x(2a^2)}{(a^2 + x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{a^2 - x^2}{a^2 + x^2}\right)^{-\frac{1}{2}}\left[\frac{-2xa^2}{(a^2 + x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(a^2 - x^2)^{-\frac{1}{2}}}{(a^2 + x^2)^{-\frac{1}{2}}}\left[\frac{-2xa^2}{(a^2 + x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2xa^2(a^2 - x^2)^{-\frac{1}{2}}}{(a^2 + x^2)^{-\frac{1}{2}+2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2xa^2(a^2 - x^2)^{-\frac{1}{2}}}{(a^2 + x^2)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2xa^2}{(a^2 + x^2)^{\frac{3}{2}}(a^2 - x^2)^{\frac{1}{2}}}$$

$$\therefore \frac{dy}{dx} = \frac{-2xa^2}{(a^2 + x^2)^{\frac{3}{2}}\sqrt{a^2 - x^2}}$$

15. $3^{x \log x}$

Solution:

Let $y = 3^{x\log x}$

On differentiating y with respect to x, we get

$\frac{dy}{dx} = \frac{d}{dx}\left(3^{x\log x}\right)$

We know $\frac{d}{dx}(a^x) = a^x \log a$

Now by using chain rule

$\Rightarrow \frac{dy}{dx} = 3^{x\log x} \log 3 \frac{d}{dx}(x\log x)$

We know that by product rule (u v)' = vu' + u v'

$\Rightarrow \frac{dy}{dx} = 3^{x\log x} \log 3 \frac{d}{dx}(x \times \log x)$

$\Rightarrow \frac{dy}{dx} = 3^{x\log x} \log 3 \left[\log x \frac{d}{dx}(x) + x \frac{d}{dx}(\log x)\right]$

We have $\frac{d}{dx}(\log x) = \frac{1}{x}$ and $\frac{d}{dx}(x) = 1$

$\Rightarrow \frac{dy}{dx} = 3^{x\log x} \log 3 \left[\log x \times 1 + x \times \frac{1}{x}\right]$

$\Rightarrow \frac{dy}{dx} = 3^{x\log x} \log 3 \, [\log x + 1]$

$\therefore \frac{dy}{dx} = (1 + \log x) 3^{x\log x} \log 3$

Thus, $\frac{d}{dx}\left(3^{x\log x}\right) = (1 + \log x) 3^{x\log x} \log 3$

16. $\sqrt{\frac{1 + sinx}{1 - sinx}}$

Solution:

Let $y = \sqrt{\frac{1+\sin x}{1-\sin x}}$

On differentiating y with respect to x, we get

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[\left(\frac{1+\sin x}{1-\sin x}\right)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

Using chain rule, we get

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\sin x}{1-\sin x}\right)^{\frac{1}{2}-1} \frac{d}{dx}\left(\frac{1+\sin x}{1-\sin x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\sin x}{1-\sin x}\right)^{-\frac{1}{2}} \frac{d}{dx}\left(\frac{1+\sin x}{1-\sin x}\right)$$

We know that

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\sin x}{1-\sin x}\right)^{-\frac{1}{2}}\left[\frac{(1-\sin x)\frac{d}{dx}(1+\sin x) - (1+\sin x)\frac{d}{dx}(1-\sin x)}{(1-\sin x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx}$$

$$= \frac{1}{2}\left(\frac{1+\sin x}{1-\sin x}\right)^{-\frac{1}{2}}\left[\frac{(1-\sin x)\left(\frac{d}{dx}(1) + \frac{d}{dx}(\sin x)\right) - (1+\sin x)\left(\frac{d}{dx}(1) - \frac{d}{dx}(\sin x)\right)}{(1-\sin x)^2}\right]$$

We know $\frac{d}{dx}(\sin x) = \cos x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\sin x}{1-\sin x}\right)^{-\frac{1}{2}}\left[\frac{(1-\sin x)(0+\cos x) - (1+\sin x)(0-\cos x)}{(1-\sin x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\sin x}{1-\sin x}\right)^{-\frac{1}{2}}\left[\frac{(1-\sin x)\cos x + (1+\sin x)\cos x}{(1-\sin x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\sin x}{1-\sin x}\right)^{-\frac{1}{2}}\left[\frac{(1-\sin x + 1 + \sin x)\cos x}{(1-\sin x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\sin x}{1-\sin x}\right)^{-\frac{1}{2}}\left[\frac{2\cos x}{(1-\sin x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1+\sin x}{1-\sin x}\right)^{-\frac{1}{2}}\left[\frac{\cos x}{(1-\sin x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+\sin x)^{-\frac{1}{2}}}{(1-\sin x)^{-\frac{1}{2}}}\left[\frac{\cos x}{(1-\sin x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+\sin x)^{-\frac{1}{2}}\cos x}{(1-\sin x)^{-\frac{1}{2}+2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+\sin x)^{-\frac{1}{2}}\cos x}{(1-\sin x)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{(1-\sin x)^{1+\frac{1}{2}}(1+\sin x)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{(1-\sin x)(1-\sin x)^{\frac{1}{2}}(1+\sin x)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{(1-\sin x)\sqrt{(1-\sin x)(1+\sin x)}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{(1-\sin x)\sqrt{1-\sin^2 x}}$$

$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{(1-\sin x)\sqrt{\cos^2 x}}$ ($\because \sin^2\theta + \cos^2\theta = 1$)

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{(1-\sin x)\cos x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1-\sin x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1-\sin x} \times \frac{1+\sin x}{1+\sin x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1+\sin x}{1-\sin^2 x}$$

$\Rightarrow \frac{dy}{dx} = \frac{1+\sin x}{\cos^2 x}$ ($\because \sin^2\theta + \cos^2\theta = 1$)

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1}{\cos x}\right)^2 + \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \sec^2 x + \sec x \tan x$$

$$\therefore \frac{dy}{dx} = \sec x(\sec x + \tan x)$$

Thus, $\frac{d}{dx}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right) = \sec x(\sec x + \tan x)$

17. $\sqrt{\frac{1+x^2}{1-x^2}}$

Solution:

Let $y = \sqrt{\frac{1-x^2}{1+x^2}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{\frac{1-x^2}{1+x^2}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[\left(\frac{1-x^2}{1+x^2}\right)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

Now by using chain rule

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{\frac{1}{2}-1}\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{-\frac{1}{2}}\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

We know that $\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{-\frac{1}{2}}\left[\frac{(1+x^2)\frac{d}{dx}(1-x^2)-(1-x^2)\frac{d}{dx}(1+x^2)}{(1+x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{-\frac{1}{2}}\left[\frac{(1+x^2)\left(\frac{d}{dx}(1)-\frac{d}{dx}(x^2)\right)-(1-x^2)\left(\frac{d}{dx}(1)+\frac{d}{dx}(x^2)\right)}{(1+x^2)^2}\right]$$

However, $\frac{d}{dx}(x^2) = 2x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{-\frac{1}{2}}\left[\frac{(1+x^2)(0-2x)-(1-x^2)(0+2x)}{(1+x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{-\frac{1}{2}}\left[\frac{-2x(1+x^2)-2x(1-x^2)}{(1+x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{-\frac{1}{2}}\left[\frac{-2x(1+x^2+1-x^2)}{(1+x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{-\frac{1}{2}}\left[\frac{-2x(2)}{(1+x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1-x^2}{1+x^2}\right)^{-\frac{1}{2}}\left[\frac{-2x}{(1+x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1-x^2)^{-\frac{1}{2}}}{(1+x^2)^{-\frac{1}{2}}}\left[\frac{-2x}{(1+x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x(1-x^2)^{-\frac{1}{2}}}{(1+x^2)^{-\frac{1}{2}+2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x(1-x^2)^{-\frac{1}{2}}}{(1+x^2)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x}{(1+x^2)^{\frac{3}{2}}(1-x^2)^{\frac{1}{2}}}$$

$$\therefore \frac{dy}{dx} = \frac{-2x}{(1+x^2)^{\frac{3}{2}}\sqrt{1-x^2}}$$

Thus, $\frac{d}{dx}\left(\sqrt{\frac{1-x^2}{1+x^2}}\right) = \frac{-2x}{(1+x^2)^{\frac{3}{2}}\sqrt{1-x^2}}$

18. $(\log \sin x)^2$

Solution:

Let $y = (\log \sin x)^2$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[(\log(\sin x))^2]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

Now by using chain rule,

$$\Rightarrow \frac{dy}{dx} = 2(\log(\sin x))^{2-1}\frac{d}{dx}[\log(\sin x)]$$

$$\Rightarrow \frac{dy}{dx} = 2\log(\sin x)\frac{d}{dx}[\log(\sin x)]$$

We have $\frac{d}{dx}(\log x) = \frac{1}{x}$

Now by using chain rule,

$$\Rightarrow \frac{dy}{dx} = 2\log(\sin x)\left[\frac{1}{\sin x}\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sin x}\log(\sin x)\frac{d}{dx}(\sin x)$$

However, $\frac{d}{dx}(\sin x) = \cos x$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sin x}\log(\sin x)\cos x$$

$$\Rightarrow \frac{dy}{dx} = 2\left(\frac{\cos x}{\sin x}\right)\log(\sin x)$$

$$\therefore \frac{dy}{dx} = 2\cot x\log(\sin x)$$

Thus, $\frac{d}{dx}[(\log(\sin x))^2] = 2\cot x\log(\sin x)$

19. $\sqrt{\frac{1+x}{1-x}}$

Solution:

Let $y = \sqrt{\frac{1+x}{1-x}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{\frac{1+x}{1-x}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

Now by using chain rule,

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}-1}\frac{d}{dx}\left(\frac{1+x}{1-x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\frac{d}{dx}\left(\frac{1+x}{1-x}\right)$$

We know that $\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\left[\frac{(1-x)\frac{d}{dx}(1+x)-(1+x)\frac{d}{dx}(1-x)}{(1-x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\left[\frac{(1-x)\left(\frac{d}{dx}(1)+\frac{d}{dx}(x)\right)-(1+x)\left(\frac{d}{dx}(1)-\frac{d}{dx}(x)\right)}{(1-x)^2}\right]$$

However, $\frac{d}{dx}(x) = 1$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\left[\frac{(1-x)(0+1)-(1+x)(0-1)}{(1-x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\left[\frac{(1-x)+(1+x)}{(1-x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\left[\frac{2}{(1-x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\left[\frac{1}{(1-x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+x)^{-\frac{1}{2}}}{(1-x)^{-\frac{1}{2}}}\left[\frac{1}{(1-x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+x)^{-\frac{1}{2}}}{(1-x)^{-\frac{1}{2}+2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+x)^{-\frac{1}{2}}}{(1-x)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(1-x)^{\frac{3}{2}}(1+x)^{\frac{1}{2}}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{(1-x)^{\frac{3}{2}}\sqrt{1+x}}$$

Thus, $\frac{d}{dx}\left(\sqrt{\frac{1+x}{1-x}}\right) = \frac{1}{(1-x)^{\frac{3}{2}}\sqrt{1+x}}$

20. $sin\left(\frac{1+x^2}{1-x^2}\right)$

Solution:

Let $y = \sin\left(\frac{1+x^2}{1-x^2}\right)$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin\left(\frac{1+x^2}{1-x^2}\right)\right]$$

We know $\frac{d}{dx}(\sin x) = \cos x$

Now by using chain rule

$$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{1+x^2}{1-x^2}\right)\frac{d}{dx}\left(\frac{1+x^2}{1-x^2}\right)$$

We know that $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{1+x^2}{1-x^2}\right)\left[\frac{(1-x^2)\frac{d}{dx}(1+x^2) - (1+x^2)\frac{d}{dx}(1-x^2)}{(1-x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx}$$
$$= \cos\left(\frac{1+x^2}{1-x^2}\right)\left[\frac{(1-x^2)\left(\frac{d}{dx}(1) + \frac{d}{dx}(x^2)\right) - (1+x^2)\left(\frac{d}{dx}(1) - \frac{d}{dx}(x^2)\right)}{(1-x^2)^2}\right]$$

However, $\frac{d}{dx}(x^2) = 2x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{1+x^2}{1-x^2}\right)\left[\frac{(1-x^2)(0+2x) - (1+x^2)(0-2x)}{(1-x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{1+x^2}{1-x^2}\right)\left[\frac{2x(1-x^2) + 2x(1+x^2)}{(1-x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{1+x^2}{1-x^2}\right)\left[\frac{2x(1-x^2+1+x^2)}{(1-x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{1+x^2}{1-x^2}\right)\left[\frac{2x(2)}{(1-x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{1+x^2}{1-x^2}\right)\left[\frac{4x}{(1-x^2)^2}\right]$$

$$\therefore \frac{dy}{dx} = \frac{4x}{(1-x^2)^2}\cos\left(\frac{1+x^2}{1-x^2}\right)$$

Thus, $\frac{d}{dx}\left[\sin\left(\frac{1+x^2}{1-x^2}\right)\right] = \frac{4x}{(1-x^2)^2}\cos\left(\frac{1+x^2}{1-x^2}\right)$

21. e^{3x} cos 2x

Solution:

Let y = e^{3x} cos (2x)

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{3x}\cos 2x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(e^{3x} \times \cos 2x)$$

We know that (u v)' = vu' + u v' (product rule)

$$\Rightarrow \frac{dy}{dx} = \cos 2x \frac{d}{dx}(e^{3x}) + e^{3x}\frac{d}{dx}(\cos 2x)$$

We know $\frac{d}{dx}(e^x) = e^x$ and $\frac{d}{dx}(\cos x) = -\sin x$

Now by using chain rule, we get

$$\Rightarrow \frac{dy}{dx} = \cos 2x\left[e^{3x}\frac{d}{dx}(3x)\right] + e^{3x}\left[-\sin 2x\frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{dy}{dx} = e^{3x}\cos 2x\left[\frac{d}{dx}(3x)\right] - e^{3x}\sin 2x\left[\frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{dy}{dx} = e^{3x}\cos 2x\left[3\frac{d}{dx}(x)\right] - e^{3x}\sin 2x\left[2\frac{d}{dx}(x)\right]$$

$$\Rightarrow \frac{dy}{dx} = 3e^{3x}\cos 2x\left[\frac{d}{dx}(x)\right] - 2e^{3x}\sin 2x\left[\frac{d}{dx}(x)\right]$$

We have $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{dy}{dx} = 3e^{3x}\cos 2x \times 1 - 2e^{3x}\sin 2x \times 1$$

$$\Rightarrow \frac{dy}{dx} = 3e^{3x}\cos 2x - 2e^{3x}\sin 2x$$

$$\therefore \frac{dy}{dx} = e^{3x}(3\cos 2x - 2\sin 2x)$$

Thus, $\frac{d}{dx}(e^{3x}\cos 2x) = e^{3x}(3\cos 2x - 2\sin 2x)$

22. Sin (log sin x)

Solution:

Let y = sin (log sin x)

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\sin(\log(\sin x))]$$

We know $\frac{d}{dx}(\sin x) = \cos x$

By using chain rule,

$$\Rightarrow \frac{dy}{dx} = \cos(\log(\sin x))\frac{d}{dx}[\log(\sin x)]$$

We have $\frac{d}{dx}(\log x) = \frac{1}{x}$

Now by using chain rule,

$$\Rightarrow \frac{dy}{dx} = \cos(\log(\sin x))\left[\frac{1}{\sin x}\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin x}\cos(\log(\sin x))\frac{d}{dx}(\sin x)$$

However, $\frac{d}{dx}(\sin x) = \cos x$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin x}\cos(\log(\sin x))\cos x$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{\cos x}{\sin x}\right)\cos(\log(\sin x))$$

$$\therefore \frac{dy}{dx} = \cot x\cos(\log(\sin x))$$

Thus, $\frac{d}{dx}[\sin(\log(\sin x))] = \cot x\cos(\log(\sin x))$

23. $e^{\tan 3x}$

Solution:

Let $y = e^{\tan 3x}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\tan 3x})$$

We know $\frac{d}{dx}(e^x) = e^x$

By using chain rule,

$$\Rightarrow \frac{dy}{dx} = e^{\tan 3x}\frac{d}{dx}(\tan 3x)$$

We have $\frac{d}{dx}(\tan x) = \sec^2 x$

Now by using chain rule, we get

$\Rightarrow \frac{dy}{dx} = e^{\tan 3x} \sec^2 3x \frac{d}{dx}(3x)$

$\Rightarrow \frac{dy}{dx} = 3e^{\tan 3x} \sec^2 3x \frac{d}{dx}(x)$

However, $\frac{d}{dx}(x) = 1$

$\Rightarrow \frac{dy}{dx} = 3e^{\tan 3x} \sec^2 3x \times 1$

$\therefore \frac{dy}{dx} = 3e^{\tan 3x} \sec^2 3x$

Thus, $\frac{d}{dx}(e^{\tan 3x}) = 3e^{\tan 3x} \sec^2 3x$

24. $e^{\sqrt{\cot x}}$

Solution:

Let $y = e^{\sqrt{\cot x}}$

On differentiating y with respect to x, we get

$\frac{dy}{dx} = \frac{d}{dx}(e^{\sqrt{\cot x}})$

We know $\frac{d}{dx}(e^x) = e^x$

By using chain rule, we get

$\Rightarrow \frac{dy}{dx} = e^{\sqrt{\cot x}} \frac{d}{dx}(\sqrt{\cot x})$

$\Rightarrow \frac{dy}{dx} = e^{\sqrt{\cot x}} \frac{d}{dx}\left[(\cot x)^{\frac{1}{2}}\right]$

We have $\frac{d}{dx}(x^n) = nx^{n-1}$

By using chain rule, we get

$$\Rightarrow \frac{dy}{dx} = e^{\sqrt{\cot x}}\left[\frac{1}{2}(\cot x)^{\frac{1}{2}-1}\frac{d}{dx}(\cot x)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}e^{\sqrt{\cot x}}(\cot x)^{-\frac{1}{2}}\frac{d}{dx}(\cot x)$$

However, $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2}e^{\sqrt{\cot x}}(\cot x)^{-\frac{1}{2}}\operatorname{cosec}^2 x$$

$$\Rightarrow \frac{dy}{dx} = -\frac{e^{\sqrt{\cot x}}\operatorname{cosec}^2 x}{2(\cot x)^{\frac{1}{2}}}$$

$$\therefore \frac{dy}{dx} = -\frac{e^{\sqrt{\cot x}}\operatorname{cosec}^2 x}{2\sqrt{\cot x}}$$

Thus, $\frac{d}{dx}\left(e^{\sqrt{\cot x}}\right) = -\frac{e^{\sqrt{\cot x}}\operatorname{cosec}^2 x}{2\sqrt{\cot x}}$

$$25.\ log\left(\frac{sinx}{1+cosx}\right)$$

Solution:

Let $y = \log\left(\frac{\sin x}{1+\cos x}\right)$

$$\Rightarrow y = \log\left(\frac{\sin 2\times\frac{x}{2}}{1+\cos 2\times\frac{x}{2}}\right)$$

We have sin2θ = 2sinθcosθ and 1 + cos2θ = 2cos²θ.

$$\Rightarrow y = \log\left(\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)$$

$$\Rightarrow y = \log\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)$$

$$\Rightarrow y = \log\left(\tan\frac{x}{2}\right)$$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(\tan\frac{x}{2}\right)\right]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$

Now by using chain rule we have,

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1}{\tan\frac{x}{2}}\right)\frac{d}{dx}\left(\tan\frac{x}{2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \cot\frac{x}{2}\frac{d}{dx}\left(\tan\frac{x}{2}\right)$$

We have $\frac{d}{dx}(\tan x) = \sec^2 x$

$$\Rightarrow \frac{dy}{dx} = \cot\frac{x}{2}\sec^2\frac{x}{2}\frac{d}{dx}\left(\frac{x}{2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\cot\frac{x}{2}\sec^2\frac{x}{2}\frac{d}{dx}(x)$$

However, $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\cot\frac{x}{2}\sec^2\frac{x}{2} \times 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \times \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} \times \frac{1}{\cos^2\frac{x}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin 2\times\frac{x}{2}}$ [∵ sin2θ = 2sinθcosθ]

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin x}$$

$$\therefore \frac{dy}{dx} = \operatorname{cosec} x$$

Thus, $\frac{d}{dx}\left[\log\left(\frac{\sin x}{1+\cos x}\right)\right] = \operatorname{cosec} x$

26. $\log\sqrt{\frac{1-\cos x}{1+\cos x}}$

Solution:

Let $y = \log\sqrt{\frac{1-\cos x}{1+\cos x}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\log\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[\log\left(\frac{1-\cos x}{1+\cos x}\right)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$

Now by using chain rule,

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\left(\frac{1-\cos x}{1+\cos x}\right)^{\frac{1}{2}}}\frac{d}{dx}\left[\left(\frac{1-\cos x}{1+\cos x}\right)^{\frac{1}{2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1-\cos x}{1+\cos x}\right)^{-\frac{1}{2}} \frac{d}{dx}\left[\left(\frac{1-\cos x}{1+\cos x}\right)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

Again by using chain rule, we get

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1-\cos x}{1+\cos x}\right)^{-\frac{1}{2}} \frac{1}{2}\left(\frac{1-\cos x}{1+\cos x}\right)^{\frac{1}{2}-1} \frac{d}{dx}\left(\frac{1-\cos x}{1+\cos x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-\cos x}{1+\cos x}\right)^{-\frac{1}{2}}\left(\frac{1-\cos x}{1+\cos x}\right)^{-\frac{1}{2}} \frac{d}{dx}\left(\frac{1-\cos x}{1+\cos x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-\cos x}{1+\cos x}\right)^{-1} \frac{d}{dx}\left(\frac{1-\cos x}{1+\cos x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\cos x}{1-\cos x}\right) \frac{d}{dx}\left(\frac{1-\cos x}{1+\cos x}\right)$$

We know that $\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\cos x}{1-\cos x}\right)\left[\frac{(1+\cos x)\frac{d}{dx}(1-\cos x) - (1-\cos x)\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\cos x}{1-\cos x}\right)\left[\frac{(1+\cos x)\left(\frac{d}{dx}(1) - \frac{d}{dx}(\cos x)\right) - (1-\cos x)\left(\frac{d}{dx}(1) + \frac{d}{dx}(\cos x)\right)}{(1+\cos x)^2}\right]$$

We know $\frac{d}{dx}(\cos x) = -\sin x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\cos x}{1-\cos x}\right)\left[\frac{(1+\cos x)(0+\sin x) - (1-\cos x)(0-\sin x)}{(1+\cos x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\cos x}{1-\cos x}\right)\left[\frac{(1+\cos x)\sin x + (1-\cos x)\sin x}{(1+\cos x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\cos x}{1-\cos x}\right)\left[\frac{(1+\cos x + 1 - \cos x)\sin x}{(1+\cos x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\cos x}{1-\cos x}\right)\left[\frac{2\sin x}{(1+\cos x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin x}{(1-\cos x)(1+\cos x)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin x}{1-\cos^2 x}$$

$\Rightarrow \frac{dy}{dx} = \frac{\sin x}{\sin^2 x}$ ($\because \sin^2\theta + \cos^2\theta = 1$)

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin x}$$

$$\therefore \frac{dy}{dx} = \operatorname{cosec} x$$

Thus, $\frac{d}{dx}\left(\log\sqrt{\frac{1-\cos x}{1+\cos x}}\right) = \operatorname{cosec} x$

27. tan ($e^{\sin x}$)

Solution:

Let y = tan ($e^{\sin x}$)

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\tan\left(e^{\sin x}\right)\right]$$

We know $\frac{d}{dx}(\tan x) = \sec^2 x$

Now by using chain rule,

$\Rightarrow \frac{dy}{dx} = \sec^2\left(e^{\sin x}\right)\frac{d}{dx}\left(e^{\sin x}\right)$

We have $\frac{d}{dx}(e^x) = e^x$

Again by using chain rule, we get

$\Rightarrow \frac{dy}{dx} = \sec^2\left(e^{\sin x}\right)e^{\sin x}\frac{d}{dx}(\sin x)$

However, $\frac{d}{dx}(\sin x) = \cos x$

$$\Rightarrow \frac{dy}{dx} = \sec^2\left(e^{\sin x}\right)e^{\sin x}\cos x$$

$$\therefore \frac{dy}{dx} = e^{\sin x}\cos x\sec^2\left(e^{\sin x}\right)$$

Thus, $\frac{d}{dx}\left[\tan\left(e^{\sin x}\right)\right] = e^{\sin x}\cos x\sec^2\left(e^{\sin x}\right)$

28. $\boldsymbol{log(x + \sqrt{x^2 + 1})}$

Solution:

Let $y = \log\left(x + \sqrt{x^2 + 1}\right)$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(x + \sqrt{x^2 + 1}\right)\right]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$

Using chain rule, we get

$\Rightarrow \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\frac{d}{dx}\left(x + \sqrt{x^2 + 1}\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + 1}}\left[\frac{d}{dx}(x) + \frac{d}{dx}\left(\sqrt{x^2 + 1}\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left[\frac{d}{dx}(x) + \frac{d}{dx}(x^2+1)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(x) = 1$ and $\frac{d}{dx}(x^n) = nx^{n-1}$

Again by using chain rule, we get

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left[1 + \frac{1}{2}(x^2+1)^{\frac{1}{2}-1}\frac{d}{dx}(x^2+1)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left[1 + \frac{1}{2}(x^2+1)^{-\frac{1}{2}}\left(\frac{d}{dx}(x^2) + \frac{d}{dx}(1)\right)\right]$$

However, $\frac{d}{dx}(x^2) = 2x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left[1 + \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x+0)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left[1 + \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \times 2x\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left[1 + x(x^2+1)^{-\frac{1}{2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left[1 + \frac{x}{\sqrt{x^2+1}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left[\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}\right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$$

Thus, $\frac{d}{dx}\left[\log\left(x+\sqrt{x^2+1}\right)\right] = \frac{1}{\sqrt{x^2+1}}$

29. $\frac{e^x \log x}{x^2}$

Solution:

Let $y = \frac{e^x \log x}{x^2}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x \log x}{x^2}\right)$$

We know that $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2)\frac{d}{dx}(e^x \log x) - (e^x \log x)\frac{d}{dx}(x^2)}{(x^2)^2}$$

We have (u v)' = vu' + u v' (product rule)

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2)\left[\log x \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(\log x)\right] - (e^x \log x)\frac{d}{dx}(x^2)}{x^4}$$

We know $\frac{d}{dx}(e^x) = e^x$, $\frac{d}{dx}(\log x) = \frac{1}{x}$ and $\frac{d}{dx}(x^2) = 2x$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2)\left[\log x \times e^x + e^x \times \frac{1}{x}\right] - (e^x \log x) \times 2x}{x^4}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2)\left[e^x \log x + \frac{e^x}{x}\right] - 2xe^x \log x}{x^4}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 e^x \log x + xe^x - 2xe^x \log x}{x^4}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 e^x \log x}{x^4} + \frac{xe^x}{x^4} - \frac{2xe^x \log x}{x^4}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^x \log x}{x^2} + \frac{e^x}{x^3} - \frac{2e^x \log x}{x^3}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^x}{x^2}\left(\log x + \frac{1}{x} - \frac{2 \log x}{x}\right)$$

$$\therefore \frac{dy}{dx} = e^x x^{-2}\left(\log x + \frac{1}{x} - \frac{2}{x}\log x\right)$$

Thus, $\frac{d}{dx}\left(\frac{e^x \log x}{x^2}\right) = e^x x^{-2}\left(\log x + \frac{1}{x} - \frac{2}{x}\log x\right)$

30. log (cosec x – cot x)

Solution:

Let y = log (cosec x – cot x)

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\log(\operatorname{cosec} x - \cot x)]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$

Now by using chain rule, we get

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{cosec} x - \cot x}\frac{d}{dx}(\operatorname{cosec} x - \cot x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{cosec} x - \cot x}\left[\frac{d}{dx}(\operatorname{cosec} x) - \frac{d}{dx}(\cot x)\right]$$

We know $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$ and $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{cosec} x - \cot x}[-\operatorname{cosec} x \cot x - (-\operatorname{cosec}^2 x)]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{cosec} x - \cot x}[-\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{cosec} x - \cot x}[\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{cosec} x - \cot x}[(\operatorname{cosec} x - \cot x)\operatorname{cosec} x]$$

$$\therefore \frac{dy}{dx} = \operatorname{cosec} x$$

Thus, $\frac{d}{dx}[\log(\operatorname{cosec}x - \cot x)] = \operatorname{cosec}x$

31. $\frac{e^{2x} + e^{-2x}}{e^{2x} - e^{-2x}}$

Solution:

Let $y = \frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}\right)$$

We know that $\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$ (quotient rule

$$\Rightarrow \frac{dy}{dx} = \frac{(e^{2x}-e^{-2x})\frac{d}{dx}(e^{2x}+e^{-2x}) - (e^{2x}+e^{-2x})\frac{d}{dx}(e^{2x}-e^{-2x})}{(e^{2x}-e^{-2x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(e^{2x}-e^{-2x})\left[\frac{d}{dx}(e^{2x}) + \frac{d}{dx}(e^{-2x})\right] - (e^{2x}+e^{-2x})\left[\frac{d}{dx}(e^{2x}) - \frac{d}{dx}(e^{-2x})\right]}{(e^{2x}-e^{-2x})^2}$$

We know $\frac{d}{dx}(e^x) = e^x$

$$\Rightarrow \frac{dy}{dx} = \frac{(e^{2x}-e^{-2x})\left[e^{2x}\frac{d}{dx}(2x) + e^{-2x}\frac{d}{dx}(-2x)\right] - (e^{2x}+e^{-2x})\left[e^{2x}\frac{d}{dx}(2x) - e^{-2x}\frac{d}{dx}(-2x)\right]}{(e^{2x}-e^{-2x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(e^{2x}-e^{-2x})\left[2e^{2x}\frac{d}{dx}(x) - 2e^{-2x}\frac{d}{dx}(x)\right] - (e^{2x}+e^{-2x})\left[2e^{2x}\frac{d}{dx}(x) + 2e^{-2x}\frac{d}{dx}(x)\right]}{(e^{2x}-e^{-2x})^2}$$

However, $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{(e^{2x} - e^{-2x})[2e^{2x} \times 1 - 2e^{-2x} \times 1] - (e^{2x} + e^{-2x})[2e^{2x} \times 1 + 2e^{-2x} \times 1]}{(e^{2x} - e^{-2x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(e^{2x} - e^{-2x})[2e^{2x} - 2e^{-2x}] - (e^{2x} + e^{-2x})[2e^{2x} + 2e^{-2x}]}{(e^{2x} - e^{-2x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(e^{2x} - e^{-2x})(e^{2x} - e^{-2x}) - 2(e^{2x} + e^{-2x})(e^{2x} + e^{-2x})}{(e^{2x} - e^{-2x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2[(e^{2x} - e^{-2x})^2 - (e^{2x} + e^{-2x})^2]}{(e^{2x} - e^{-2x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(e^{2x} - e^{-2x} + e^{2x} + e^{-2x})(e^{2x} - e^{-2x} - e^{2x} - e^{-2x})}{(e^{2x} - e^{-2x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(2e^{2x})(-2e^{-2x})}{(e^{2x} - e^{-2x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-8e^{2x+(-2x)}}{(e^{2x} - e^{-2x})^2}$$

$$\therefore \frac{dy}{dx} = \frac{-8}{(e^{2x} - e^{-2x})^2}$$

Thus, $\frac{d}{dx}\left(\frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}\right) = \frac{-8}{(e^{2x}-e^{-2x})^2}$

32. $\boldsymbol{log\left(\frac{x^2 + x + 1}{x^2 - x + 1}\right)}$

Solution:

Let $y = \log\left(\frac{x^2+x+1}{x^2-x+1}\right)$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(\frac{x^2 + x + 1}{x^2 - x + 1}\right)\right]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$

By using chain rule, we have

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\left(\frac{x^2+x+1}{x^2-x+1}\right)} \frac{d}{dx}\left(\frac{x^2+x+1}{x^2-x+1}\right)$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{x^2 - x + 1}{x^2 + x + 1}\right) \frac{d}{dx}\left(\frac{x^2 + x + 1}{x^2 - x + 1}\right)$$

We know that $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \left(\frac{x^2 - x + 1}{x^2 + x + 1}\right)\left[\frac{(x^2 - x + 1)\frac{d}{dx}(x^2 + x + 1) - (x^2 + x + 1)\frac{d}{dx}(x^2 - x + 1)}{(x^2 - x + 1)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{x^2 - x + 1}{x^2 + x + 1}\right)\left[\frac{(x^2 - x + 1)\left(\frac{d}{dx}(x^2) + \frac{d}{dx}(x) + \frac{d}{dx}(1)\right) - (x^2 + x + 1)\left(\frac{d}{dx}(x^2) - \frac{d}{dx}(x) + \frac{d}{dx}(1)\right)}{(x^2 - x + 1)^2}\right]$$

We know $\frac{d}{dx}(x^2) = 2x$, $\frac{d}{dx}(x) = 1$ and derivative of constant is 0.

$$\Rightarrow \frac{dy}{dx} = \left(\frac{x^2 - x + 1}{x^2 + x + 1}\right)\left[\frac{(x^2 - x + 1)(2x + 1 + 0) - (x^2 + x + 1)(2x - 1 + 0)}{(x^2 - x + 1)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{x^2 - x + 1}{x^2 + x + 1}\right)\left[\frac{(2x + 1)(x^2 - x + 1) - (2x - 1)(x^2 + x + 1)}{(x^2 - x + 1)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{x^2-x+1}{x^2+x+1}\right)\left[\frac{2x(x^2-x+1)+(x^2-x+1)-2x(x^2+x+1)+(x^2+x+1)}{(x^2-x+1)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{x^2-x+1}{x^2+x+1}\right)\left[\frac{2x(x^2-x+1-x^2-x-1)+(x^2-x+1+x^2+x+1)}{(x^2-x+1)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{x^2-x+1}{x^2+x+1}\right)\left[\frac{2x(-2x)+(2x^2+2)}{(x^2-x+1)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{x^2-x+1}{x^2+x+1}\right)\left[\frac{-4x^2+2x^2+2}{(x^2-x+1)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{x^2-x+1}{x^2+x+1}\right)\left[\frac{2-2x^2}{(x^2-x+1)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{2-2x^2}{(x^2+x+1)(x^2-x+1)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(1-x^2)}{(x^2+1)^2-x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(1-x^2)}{(x^2+1)^2-x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(1-x^2)}{x^4+2x^2+1-x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(1-x^2)}{(x^2+1)^2-x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(1-x^2)}{x^4+2x^2+1-x^2}$$

33. $\tan^{-1}(e^x)$

Solution:

Let $y = \tan^{-1}(e^x)$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1} e^x)$$

We know $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

Now by using chain rule, we get

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+(e^x)^2}\frac{d}{dx}(e^x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+e^{2x}}\frac{d}{dx}(e^x)$$

However, $\frac{d}{dx}(e^x) = e^x$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+e^{2x}} \times e^x$$

$$\therefore \frac{dy}{dx} = \frac{e^x}{1+e^{2x}}$$

Thus, $\frac{d}{dx}(\tan^{-1} e^x) = \frac{e^x}{1+e^{2x}}$

34. $e^{\sin^{-1} 2x}$

Solution:

Let $y = e^{\sin^{-1} 2x}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\sin^{-1} 2x})$$

We know $\frac{d}{dx}(e^x) = e^x$

Using chain rule, we can write as

$\Rightarrow \frac{dy}{dx} = e^{\sin^{-1} 2x} \frac{d}{dx}(\sin^{-1} 2x)$

We have $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

Using chain rule we get

$\Rightarrow \frac{dy}{dx} = e^{\sin^{-1} 2x} \frac{1}{\sqrt{1-(2x)^2}} \frac{d}{dx}(2x)$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sin^{-1} 2x}}{\sqrt{1-4x^2}} \times 2 \frac{d}{dx}(x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2e^{\sin^{-1} 2x}}{\sqrt{1-4x^2}} \times \frac{d}{dx}(x)$$

However, $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{2e^{\sin^{-1} 2x}}{\sqrt{1-4x^2}} \times 1$$

$$\therefore \frac{dy}{dx} = \frac{2e^{\sin^{-1} 2x}}{\sqrt{1-4x^2}}$$

Thus, $\frac{d}{dx}\left(e^{\sin^{-1} 2x}\right) = \frac{2e^{\sin^{-1} 2x}}{\sqrt{1-4x^2}}$

35. sin (2 sin^{-1} x)

Solution:

Let y = sin (2sin^{-1}x)

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\sin(2\sin^{-1} x)]$$

We know $\frac{d}{dx}(\sin x) = \cos x$

By using chain rule we get,

$\Rightarrow \frac{dy}{dx} = \cos(2\sin^{-1}x)\frac{d}{dx}(2\sin^{-1}x)$

$$\Rightarrow \frac{dy}{dx} = \cos(2\sin^{-1}x) \times 2\frac{d}{dx}(\sin^{-1}x)$$

$$\Rightarrow \frac{dy}{dx} = 2\cos(2\sin^{-1}x)\frac{d}{dx}(\sin^{-1}x)$$

We have $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$$\Rightarrow \frac{dy}{dx} = 2\cos(2\sin^{-1}x) \times \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = \frac{2\cos(2\sin^{-1}x)}{\sqrt{1-x^2}}$$

Thus, $\frac{d}{dx}[\sin(2\sin^{-1}x)] = \frac{2\cos(2\sin^{-1}x)}{\sqrt{1-x^2}}$

36. $e^{\tan^{-1}\sqrt{x}}$

Solution:

Let $y = e^{\tan^{-1}\sqrt{x}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\tan^{-1}\sqrt{x}}\right)$$

We know $\frac{d}{dx}(e^x) = e^x$

Now by using chain rule, we can write as

$\Rightarrow \frac{dy}{dx} = e^{\tan^{-1}\sqrt{x}}\frac{d}{dx}(\tan^{-1}\sqrt{x})$

We have $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

Again by using chain rule we get,

$$\Rightarrow \frac{dy}{dx} = e^{\tan^{-1}\sqrt{x}} \frac{1}{1+(\sqrt{x})^2} \frac{d}{dx}(\sqrt{x})$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\tan^{-1}\sqrt{x}}}{1+x} \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

However, $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\tan^{-1}\sqrt{x}}}{1+x}\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\tan^{-1}\sqrt{x}}}{1+x}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\tan^{-1}\sqrt{x}}}{1+x}\left(\frac{1}{2\sqrt{x}}\right)$$

$$\therefore \frac{dy}{dx} = \frac{e^{\tan^{-1}\sqrt{x}}}{2\sqrt{x}(1+x)}$$

Thus, $\frac{d}{dx}\left(e^{\tan^{-1}\sqrt{x}}\right) = \frac{e^{\tan^{-1}\sqrt{x}}}{2\sqrt{x}(1+x)}$

37. $\sqrt{\tan^{-1}\left(\frac{x}{2}\right)}$

Solution:

Let $y = \sqrt{\tan^{-1}\frac{x}{2}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{\tan^{-1}\frac{x}{2}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[\left(\tan^{-1}\frac{x}{2}\right)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

Now by using chain rule, we get

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\tan^{-1}\frac{x}{2}\right)^{\frac{1}{2}-1}\frac{d}{dx}\left(\tan^{-1}\frac{x}{2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\tan^{-1}\frac{x}{2}\right)^{-\frac{1}{2}}\frac{d}{dx}\left(\tan^{-1}\frac{x}{2}\right)$$

We have $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

Again by using chain rule, we can write as

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\tan^{-1}\frac{x}{2}\right)^{-\frac{1}{2}}\frac{1}{1+\left(\frac{x}{2}\right)^2}\frac{d}{dx}\left(\frac{x}{2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\tan^{-1}\frac{x}{2}\right)^{-\frac{1}{2}}\frac{1}{1+\frac{x^2}{4}} \times \frac{1}{2}\frac{d}{dx}(x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\tan^{-1}\frac{x}{2}\right)^{-\frac{1}{2}}\frac{4}{4+x^2} \times \frac{1}{2}\frac{d}{dx}(x)$$

$$\Rightarrow \frac{dy}{dx} = \left(\tan^{-1}\frac{x}{2}\right)^{-\frac{1}{2}}\frac{1}{4+x^2} \times \frac{d}{dx}(x)$$

However, $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{dy}{dx} = \left(\tan^{-1}\frac{x}{2}\right)^{-\frac{1}{2}}\frac{1}{4+x^2} \times 1$$

$$\Rightarrow \frac{dy}{dx} = \left(\tan^{-1}\frac{x}{2}\right)^{-\frac{1}{2}}\frac{1}{4+x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(4+x^2)\left(\tan^{-1}\frac{x}{2}\right)^{\frac{1}{2}}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{(4+x^2)\sqrt{\tan^{-1}\frac{x}{2}}}$$

EXERCISE 11.3 PAGE NO: 11.62

Differentiate the following functions with respect to x:

1. $\cos^{-1}\left\{2x\sqrt{1-x^2}\right\},\ \frac{1}{\sqrt{2}} < x < 1$

Solution:

Let $y = \cos^{-1}\{2x\sqrt{1-x^2}\}$

let $x = \cos\theta$

Now

$$y = \cos^{-1}\{2\cos\theta\sqrt{1-\cos^2\theta}\}$$

$$= \cos^{-1}\{2\cos\theta\sqrt{\sin^2\theta}\}$$

Using $\sin^2\theta + \cos^2\theta = 1$ and $2\sin\theta\cos\theta = \sin 2\theta$

$$= \cos^{-1}(2\cos\theta\sin\theta)$$

$$= \cos^{-1}(\sin 2\theta)$$

$$y = \cos^{-1}\left(\cos\left(\frac{\pi}{2} - 2\theta\right)\right)$$

Now by considering the limits,

$$\frac{1}{\sqrt{2}} < x < 1$$

$$\Rightarrow \frac{1}{\sqrt{2}} < \cos\theta < 1$$

$$\Rightarrow 0 < \theta < \frac{\pi}{4}$$

$$\Rightarrow 0 < 2\theta < \frac{\pi}{2}$$

$$\Rightarrow 0 > -2\theta > -\frac{\pi}{2}$$

$$\Rightarrow \frac{\pi}{2} > \frac{\pi}{2} - 2\theta > \frac{\pi}{2} - \frac{\pi}{2}$$

$$\Rightarrow 0 < \frac{\pi}{2} - 2\theta < \frac{\pi}{2}$$

Therefore,

$$y = \cos^{-1}\left(\cos\left(\frac{\pi}{2} - 2\theta\right)\right)$$

$$y = \cos^{-1}\left(\cos\left(\frac{\pi}{2} - 2\theta\right)\right)$$

$$y = \left(\frac{\pi}{2} - 2\theta\right)$$

$$y = \frac{\pi}{2} - 2\cos^{-1}x$$

Differentiating with respect to x, we get

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - 2\cos^{-1}x\right)$$

$$\Rightarrow \frac{dy}{dx} = 0 - 2\left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

2. $\cos^{-1}\left\{\sqrt{\frac{1+x}{2}}\right\}, \ -1 < x < 1$

Solution:

Let

$$y = \cos^{-1}\left\{\sqrt{\frac{1+x}{2}}\right\}$$

$\text{let } x = \cos 2\theta$

Now

$$y = \cos^{-1}\left\{\sqrt{\frac{1+\cos 2\theta}{2}}\right\}$$

$$y = \cos^{-1}\left\{\sqrt{\frac{2\cos^2\theta}{2}}\right\}$$

Now by using $\cos 2\theta = 2\cos^2\theta - 1$

$y = \cos^{-1}(\cos\theta)$

Considering the limits,

$-1 < x < 1$

$-1 < \cos 2\theta < 1$

$0 < 2\theta < \pi$

$0 < \theta < \frac{\pi}{2}$

Now, $y = \cos^{-1}(\cos\theta)$

$y = \theta$

$y = \frac{1}{2}\cos^{-1}x$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{1}{2}\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

3. $\sin^{-1}\left\{\sqrt{\frac{1-x}{2}}\right\}, \; 0 < x < 1$

Solution:

Let, $y = \sin^{-1}\left\{\sqrt{\frac{1-x}{2}}\right\}$

$\text{let } x = \cos 2\theta$

Now

$$y = \sin^{-1}\left\{\sqrt{\frac{1-\cos 2\theta}{2}}\right\}$$

$$y = \sin^{-1}\left\{\sqrt{\frac{2\sin^2\theta}{2}}\right\}$$

Using $\cos 2\theta = 1 - 2\sin^2\theta$

$y = \sin^{-1}(\sin\theta)$

Considering the limits,

$0 < x < 1$

$0 < \cos 2\theta < 1$

$$0 < 2\theta < \frac{\pi}{2}$$

$$0 < \theta < \frac{\pi}{4}$$

Now, $y = \sin^{-1}(\sin\theta)$

$y = \theta$

$$y = \frac{1}{2}\cos^{-1}x$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{1}{2}\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

4. $\sin^{-1}\left\{\sqrt{1-x^2}\right\},\ 0 < x < 1$

Solution:

Let,

$$y = \sin^{-1}\left\{\sqrt{1 - x^2}\right\}$$

let $x = \cos\theta$

Now

$$y = \sin^{-1}\left\{\sqrt{1 - \cos^2\theta}\right\}$$

Using $\sin^2\theta + \cos^2\theta = 1$

$y = \sin^{-1}(\sin\theta)$

Considering the limits,

$0 < x < 1$

$0 < \cos\theta < 1$

$$0 < \theta < \frac{\pi}{2}$$

Now, $y = \sin^{-1}(\sin\theta)$

$y = \theta$

$y = \cos^{-1}x$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$$

5. $\tan^{-1}\left\{\frac{x}{\sqrt{a^2 - x^2}}\right\}, \ -a < x < a$

Solution:

$$y = \tan^{-1}\left\{\frac{x}{\sqrt{a^2 - x^2}}\right\}$$

Let $x = a\sin\theta$

Now

$$y = \tan^{-1}\left\{\frac{a\sin\theta}{\sqrt{a^2 - a^2\sin^2\theta}}\right\}$$

Using $\sin^2\theta + \cos^2\theta = 1$

$$y = \tan^{-1}\left\{\frac{a\sin\theta}{a\sqrt{1 - \sin^2\theta}}\right\}$$

$$y = \tan^{-1}\left\{\frac{\sin\theta}{\cos\theta}\right\}$$

$y = \tan^{-1}(\tan\theta)$

Considering the limits,

$-a < x < a$

$-a < a\sin\theta < a$

$-1 < \sin\theta < 1$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Now, $y = \tan^{-1}(\tan\theta)$

$y = \theta$

$$y = \sin^{-1}\left(\frac{x}{a}\right)$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right)$$

$$\frac{dy}{dx} = \frac{a}{\sqrt{a^2 - x^2}} \times \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$$

6. $\sin^{-1}\left\{\frac{x}{\sqrt{x^2 + a^2}}\right\}$

Solution:

Let,

$$y = \sin^{-1}\left\{\frac{x}{\sqrt{x^2 + a^2}}\right\}$$

Let x = a tan θ

Now

$$y = \sin^{-1}\left\{\frac{a\tan\theta}{\sqrt{a^2\tan^2\theta + a^2}}\right\}$$

Using $1 + \tan^2\theta = \sec^2\theta$

$$y = \sin^{-1}\left\{\frac{a\tan\theta}{a\sqrt{\tan^2\theta + 1}}\right\}$$

$$y = \sin^{-1}\left\{\frac{a\tan\theta}{a\sqrt{\sec^2\theta}}\right\}$$

$$y = \sin^{-1}\left\{\frac{\tan\theta}{\sec\theta}\right\}$$

$y = \sin^{-1}(\sin\theta)$

$y = \theta$

$$y = \tan^{-1}\left(\frac{x}{a}\right)$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right)$$

$$\frac{dy}{dx} = \frac{a^2}{a^2 + x^2} \times \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{a}{a^2 + x^2}$$

7. $\sin^{-1}(2x^2 - 1)$, $0 < x < 1$

Solution:

Let,

$y = \sin^{-1}\{2x^2 - 1\}$

let $x = \cos\theta$

Now

$y = \sin^{-1}\left\{\sqrt{2\cos^2\theta - 1}\right\}$

Using $2\cos^2\theta - 1 = \cos 2\theta$

$y = \sin^{-1}(\cos 2\theta)$

$y = \sin^{-1}\left\{\sin\left(\frac{\pi}{2} - 2\theta\right)\right\}$

Considering the limits,

$0 < x < 1$

$0 < \cos\theta < 1$

$0 < \theta < \frac{\pi}{2}$

$0 < 2\theta < \pi$

$0 > -2\theta > -\pi$

$\frac{\pi}{2} > \frac{\pi}{2} - 2\theta > -\frac{\pi}{2}$

Now,

$y = \sin^{-1}\left\{\sin\left(\frac{\pi}{2} - 2\theta\right)\right\}$

$y = \frac{\pi}{2} - 2\theta$

$y = \frac{\pi}{2} - 2\cos^{-1}x$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - 2\cos^{-1}x\right)$$

$$\frac{dy}{dx} = 0 - 2\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

8. $\sin^{-1}(1 - 2x^2)$, $0 < x < 1$

Solution:

Let,

$y = \sin^{-1}\{1 - 2x^2\}$

let $x = \sin\theta$

Now

$y = \sin^{-1}\left\{\sqrt{1 - 2\sin^2\theta}\right\}$

Using $1 - 2\sin^2\theta = \cos 2\theta$

$y = \sin^{-1}(\cos 2\theta)$

$y = \sin^{-1}\left\{\sin\left(\frac{\pi}{2} - 2\theta\right)\right\}$

Considering the limits,

$0 < x < 1$

$0 < \sin\theta < 1$

$0 < \theta < \frac{\pi}{2}$

$0 < 2\theta < \pi$

$0 > -2\theta > -\pi$

$$\frac{\pi}{2} > \frac{\pi}{2} - 2\theta > -\frac{\pi}{2}$$

Now,

$$y = \sin^{-1}\left\{\sin\left(\frac{\pi}{2} - 2\theta\right)\right\}$$

$$y = \frac{\pi}{2} - 2\theta$$

$$y = \frac{\pi}{2} - 2\sin^{-1} x$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - 2\cos^{-1} x\right)$$

$$\frac{dy}{dx} = 0 - 2\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$\frac{dy}{dx} = \frac{-2}{\sqrt{1-x^2}}$$

9. $\cos^{-1}\left\{\frac{x}{\sqrt{x^2+a^2}}\right\}$

Solution:

Let,

$$y = \cos^{-1}\left\{\frac{x}{\sqrt{x^2+a^2}}\right\}$$

Let x = a cot θ

Now

$$y = \cos^{-1}\left\{\frac{a\cot\theta}{\sqrt{a^2\cot^2\theta + a^2}}\right\}$$

Using $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$$y = \cos^{-1}\left\{\frac{a\cot\theta}{a\sqrt{\cot^2\theta + 1}}\right\}$$

$$y = \cos^{-1}\left\{\frac{a\cot\theta}{a\sqrt{\operatorname{cosec}^2\theta}}\right\}$$

$$y = \cos^{-1}\left\{\frac{\cot\theta}{\operatorname{cosec}\theta}\right\}$$

$y = \cos^{-1}(\cos\theta)$

$y = \theta$

$$y = \cot^{-1}\left(\frac{x}{a}\right)$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cot^{-1}\left(\frac{x}{a}\right)\right)$$

$$\frac{dy}{dx} = \frac{-a^2}{a^2 + x^2} \times \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{-a}{a^2 + x^2}$$

10. $\sin^{-1}\left\{\frac{\sin x + \cos x}{\sqrt{2}}\right\}, -\frac{3\pi}{4} < x < \frac{\pi}{4}$

Solution:

Let,

$$y = \sin^{-1}\left\{\frac{\sin x + \cos x}{\sqrt{2}}\right\}$$

Now

$$y = \sin^{-1}\left\{\sin x \frac{1}{\sqrt{2}} + \cos x \frac{1}{\sqrt{2}}\right\}$$

$$y = \sin^{-1}\left\{\sin x \cos\left(\frac{\pi}{4}\right) + \cos x \sin\left(\frac{\pi}{4}\right)\right\}$$

Using sin (A + B) = sin A cos B + cos A sin B

$$y = \sin^{-1}\left\{\sin\left(x + \frac{\pi}{4}\right)\right\}$$

Considering the limits,

$$-\frac{3\pi}{4} < x < \frac{\pi}{4}$$

Differentiating it with respect to x,

$$y = x + \frac{\pi}{4}$$

$$\frac{dy}{dx} = 1$$

11. $\cos^{-1}\left\{\frac{\cos x + \sin x}{\sqrt{2}}\right\}, \ -\frac{\pi}{4} < x < \frac{\pi}{4}$

Solution:

Let,

$$y = \cos^{-1}\left\{\frac{\cos x + \sin x}{\sqrt{2}}\right\}$$

Now

$$y = \cos^{-1}\left\{\cos x \frac{1}{\sqrt{2}} + \sin x \frac{1}{\sqrt{2}}\right\}$$

$$y = \cos^{-1}\left\{\cos x \cos\left(\frac{\pi}{4}\right) + \sin x \sin\left(\frac{\pi}{4}\right)\right\}$$

Using cos (A – B) = cos A cos B + sin A sin B

$$y = \cos^{-1}\left\{\cos\left(x - \frac{\pi}{4}\right)\right\}$$

Considering the limits,

$$-\frac{\pi}{4} < x < \frac{\pi}{4}$$

$$-\frac{\pi}{2} < x - \frac{\pi}{4} < 0$$

Now,

$$y = -x + \frac{\pi}{4}$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = -1$$

12. $\tan^{-1}\left\{\frac{x}{1+\sqrt{1-x^2}}\right\}, \ -1 < x < 1$

Solution:

Let,

$$y = \tan^{-1}\left\{\frac{x}{1+\sqrt{1-x^2}}\right\}$$

Let x = sin θ

Now

$$y = \tan^{-1}\left\{\frac{\sin\theta}{1+\sqrt{1-\sin^2\theta}}\right\}$$

Using $\sin^2\theta + \cos^2\theta = 1$

$$y = \tan^{-1}\left\{\frac{\sin\theta}{1+\sqrt{\cos^2\theta}}\right\}$$

$$y = \tan^{-1}\left\{\frac{\sin\theta}{1+\cos\theta}\right\}$$

Using $2\cos^2\theta = 1 + \cos 2\theta$ and $2 \sin\theta \cos\theta = \sin 2\theta$

$$y = \tan^{-1}\left\{\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right\}$$

$$y = \tan^{-1}\left\{\tan\frac{\theta}{2}\right\}$$

Considering the limits,

$-1 < x < 1$

$-1 < \sin\theta < 1$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$-\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{4}$$

Now,

$$y = \tan^{-1}\left\{\tan\frac{\theta}{2}\right\}$$

$$y = \frac{\theta}{2}$$

$$y = \frac{1}{2}\sin^{-1}x$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}\sin^{-1}x\right)$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}}$$

13. $\tan^{-1}\left\{\frac{x}{a+\sqrt{a^2-x^2}}\right\},\ -a < x < a$

Solution:

Let,

$$y = \tan^{-1}\left\{\frac{x}{a+\sqrt{a^2-x^2}}\right\}$$

Let $x = a\sin\theta$

Now

$$y = \tan^{-1}\left\{\frac{a\sin\theta}{a+\sqrt{a^2-a^2\sin^2\theta}}\right\}$$

Using $\sin^2\theta + \cos^2\theta = 1$

$$y = \tan^{-1}\left\{\frac{a\sin\theta}{a+a\sqrt{\cos^2\theta}}\right\}$$

$$y = \tan^{-1}\left\{\frac{\sin\theta}{1+\cos\theta}\right\}$$

Using $2\cos^2\theta = 1 + \cos\theta$ and $2\sin\theta\cos\theta = \sin 2\theta$

$$y = \tan^{-1}\left\{\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right\}$$

$$y = \tan^{-1}\left\{\tan\frac{\theta}{2}\right\}$$

Considering the limits,

$-a < x < a$

$-1 < \sin\theta < 1$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$-\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{4}$$

Now,

$$y = \tan^{-1}\left\{\tan\frac{\theta}{2}\right\}$$

$$y = \frac{\theta}{2}$$

$$y = \frac{1}{2}\sin^{-1}\frac{x}{a}$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}\sin^{-1}\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{a}{2\sqrt{a^2 - x^2}} \times \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{a^2 - x^2}}$$

14. $\sin^{-1}\left\{\frac{x + \sqrt{1 - x^2}}{\sqrt{2}}\right\}, \ -1 < x < 1$

Solution:

Let,

$$y = \sin^{-1}\left\{\frac{x + \sqrt{1 - x^2}}{\sqrt{2}}\right\}$$

Let x = sin θ

Now

$$y = \sin^{-1}\left\{\frac{\sin\theta + \sqrt{1 - \sin^2\theta}}{\sqrt{2}}\right\}$$

Using $\sin^2\theta + \cos^2\theta = 1$

$$y = \sin^{-1}\left\{\frac{\sin\theta + \cos\theta}{\sqrt{2}}\right\}$$

Now

$$y = \sin^{-1}\left\{\sin\theta\frac{1}{\sqrt{2}} + \cos\theta\frac{1}{\sqrt{2}}\right\}$$

$$y = \sin^{-1}\left\{\sin\theta\cos\left(\frac{\pi}{4}\right) + \cos\theta\sin\left(\frac{\pi}{4}\right)\right\}$$

Using sin (A + B) = sin A cos B + cos A sin B

$$y = \sin^{-1}\left\{\sin\left(\theta + \frac{\pi}{4}\right)\right\}$$

Considering the limits,

$-1 < x < 1$

$-1 < \sin\theta < 1$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$-\frac{\pi}{2} + \frac{\pi}{4} < \theta + \frac{\pi}{4} < \frac{\pi}{2} + \frac{\pi}{4}$$

$$-\frac{\pi}{4} < \theta + \frac{\pi}{4} < \frac{3\pi}{4}$$

Now,

$$y = \sin^{-1}\left\{\sin\left(\theta + \frac{\pi}{4}\right)\right\}$$

$$y = \theta + \frac{\pi}{4}$$

$$y = \sin^{-1}x + \frac{\pi}{4}$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1}x + \frac{\pi}{4}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

15. $\cos^{-1}\left\{\frac{x+\sqrt{1-x^2}}{\sqrt{2}}\right\}, \ -1 < x < 1$

Solution:

Let,

$$y = \cos^{-1}\left\{\frac{x+\sqrt{1-x^2}}{\sqrt{2}}\right\}$$

Let $x = \sin\theta$

Now

$$y = \cos^{-1}\left\{\frac{\sin\theta + \sqrt{1-\sin^2\theta}}{\sqrt{2}}\right\}$$

Using $\sin^2\theta + \cos^2\theta = 1$

$$y = \cos^{-1}\left\{\frac{\sin\theta + \cos\theta}{\sqrt{2}}\right\}$$

Now

$$y = \cos^{-1}\left\{\sin\theta\frac{1}{\sqrt{2}} + \cos\theta\frac{1}{\sqrt{2}}\right\}$$

$$y = \cos^{-1}\left\{\sin\theta\sin\left(\frac{\pi}{4}\right) + \cos\theta\cos\left(\frac{\pi}{4}\right)\right\}$$

Using cos (A – B) = cos A cos B + sin A sin B

$$y = \cos^{-1}\left\{\cos\left(\theta - \frac{\pi}{4}\right)\right\}$$

Considering the limits,

$-1 < x < 1$

$-1 < \sin\theta < 1$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$-\frac{\pi}{2} - \frac{\pi}{4} < \theta - \frac{\pi}{4} < \frac{\pi}{2} - \frac{\pi}{4}$$

$$-\frac{3\pi}{4} < \theta - \frac{\pi}{4} < \frac{\pi}{4}$$

Now,

$$y = \cos^{-1}\left\{\cos\left(\theta - \frac{\pi}{4}\right)\right\}$$

$$y = -\left(\theta - \frac{\pi}{4}\right)$$

$$y = -\sin^{-1} x + \frac{\pi}{4}$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(-\sin^{-1} x + \frac{\pi}{4}\right)$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

16. $\tan^{-1}\left\{\frac{4x}{1-4x^2}\right\}, \ -\frac{1}{2} < x < \frac{1}{2}$

Solution:

Let,

$$y = \tan^{-1}\left\{\frac{4x}{1-4x^2}\right\}$$

Let $2x = \tan\theta$

$$y = \tan^{-1}\left\{\frac{2\tan\theta}{1-\tan^2\theta}\right\}$$

Using $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$$y = \tan^{-1}(\tan 2\theta)$$

Considering the limits,

$$-\frac{1}{2} < x < \frac{1}{2}$$

$$-1 < 2x < 1$$

$$-1 < \tan\theta < 1$$

$$-\frac{\pi}{4} < \theta < \frac{\pi}{4}$$

$$-\frac{\pi}{2} < 2\theta < \frac{\pi}{2}$$

Now,

$y = \tan^{-1}(\tan 2\theta)$

$y = 2\theta$

$y = 2\tan^{-1}(2x)$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1} 2x)$$

$$\frac{dy}{dx} = 2 \times \frac{2}{1 + (2x)^2}$$

$$\frac{dy}{dx} = \frac{4}{1 + 4x^2}$$

17. $\tan^{-1}\left\{\frac{2^{x+1}}{1 - 4^x}\right\}, \ -\infty < x < 0$

Solution:

Let,

$$y = \tan^{-1}\left\{\frac{2^{x+1}}{1 - 4^x}\right\}$$

Let $2^x = \tan\theta$

$$y = \tan^{-1}\left\{\frac{2 \times 2^x}{1 - (2^x)^2}\right\}$$

$$y = \tan^{-1}\left\{\frac{2\tan\theta}{1 - \tan^2\theta}\right\}$$

Using $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$

$$y = \tan^{-1}(\tan 2\theta)$$

Considering the limits,

$-\infty < x < 0$

$2^{-\infty} < 2^x < 2^0$

$0 < \tan\theta < 1$

$0 < \theta < \frac{\pi}{4}$

$0 < 2\theta < \frac{\pi}{2}$

Now,

$y = \tan^{-1}(\tan 2\theta)$

$y = 2\theta$

$y = 2\tan^{-1}(2^x)$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1} 2^x)$$

$$\frac{dy}{dx} = 2 \times \frac{2^x \log 2}{1 + (2^x)^2}$$

$$\frac{dy}{dx} = \frac{2^{x+1} \log 2}{1 + 4^x}$$

18. $\tan^{-1}\left\{\frac{2a^x}{1 - a^{2x}}\right\}$, $a > 1$, $-\infty < x < 0$

Solution:

Let,

$$y = \tan^{-1}\left\{\frac{2a^x}{1 - a^{2x}}\right\}$$

Let $a^x = \tan\theta$

$$y = \tan^{-1}\left\{\frac{2\tan\theta}{1 - \tan^2\theta}\right\}$$

Using $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$y = \tan^{-1}(\tan 2\theta)$

Considering the limits,

$-\infty < x < 0$

$a^{-\infty} < a^x < a^0$

$0 < \tan\theta < 1$

$0 < \theta < \frac{\pi}{4}$

$0 < 2\theta < \frac{\pi}{2}$

Now, $y = \tan^{-1}(\tan 2\theta)$

$y = 2\theta$

$y = 2\tan^{-1}(a^x)$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1} a^x)$$

$$\frac{dy}{dx} = 2 \times \frac{a^x \log a}{1+(a^x)^2}$$

$$\frac{dy}{dx} = \frac{2a^x \log a}{1+a^{2x}}$$

19. $\sin^{-1}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right\},\ 0 < x < 1$

Solution:

Let, $y = \sin^{-1}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right\}$

Let x = cos2θ

Now

$$y = \sin^{-1}\left\{\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{2}\right\}$$

Using $1 - 2\sin^2\theta = \cos 2\theta$ and $2\cos^2\theta - 1 = \cos 2\theta$

$$y = \sin^{-1}\left\{\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{2}\right\}$$

Now

$$y = \sin^{-1}\left\{\sin\theta\frac{1}{\sqrt{2}}+\cos\theta\frac{1}{\sqrt{2}}\right\}$$

$$y = \sin^{-1}\left\{\sin\theta\cos\left(\frac{\pi}{4}\right)+\cos\theta\sin\left(\frac{\pi}{4}\right)\right\}$$

Using sin (A + B) = sin A cos B + cos A sin B

$$y = \sin^{-1}\left\{\sin\left(\theta+\frac{\pi}{4}\right)\right\}$$

Considering the limits,

$0 < x < 1$

$0 < \cos 2\theta < 1$

$$0 < 2\theta < \frac{\pi}{2}$$

$$0 < \theta < \frac{\pi}{4}$$

Now,

$$y = \sin^{-1}\left\{\sin\left(\theta+\frac{\pi}{4}\right)\right\}$$

$$y = \theta + \frac{\pi}{4}$$

$$y = \frac{1}{2}\cos^{-1}x + \frac{\pi}{4}$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}\cos^{-1}x + \frac{\pi}{4}\right)$$

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{-1}{2\sqrt{1-x^2}}$$

20. $\tan^{-1}\left\{\frac{\sqrt{1+a^2x^2}-1}{ax}\right\},\ x \neq 0$

Solution:

Let,

$$y = \tan^{-1}\left\{\frac{\sqrt{1+a^2x^2}-1}{ax}\right\}$$

Let ax = tan θ

Now

$$y = \tan^{-1}\left\{\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right\}$$

Using $\sec^2\theta = 1 + \tan^2\theta$

$$y = \tan^{-1}\left\{\frac{\sqrt{\sec^2\theta}-1}{\tan\theta}\right\}$$

$$y = \tan^{-1}\left\{\frac{\sec\theta - 1}{\tan\theta}\right\}$$

$$y = \tan^{-1}\left\{\frac{1-\cos\theta}{\sin\theta}\right\}$$

Using 2 $\sin^2\theta = 1 - \cos 2\theta$ and $2 \sin \theta \cos \theta = \sin 2\theta$

$$y = \tan^{-1}\left\{\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right\}$$

$$y = \tan^{-1}\left\{\tan\frac{\theta}{2}\right\}$$

$$y = \frac{\theta}{2}$$

$$y = \frac{1}{2}\tan^{-1} ax$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}\tan^{-1} ax\right)$$

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{a}{1 + (ax)^2}$$

$$\frac{dy}{dx} = \frac{a}{2(1 + a^2x^2)}$$

21. $\tan^{-1}\left\{\frac{\sin x}{1+\cos x}\right\}, -\pi < x < \pi$

Solution:

Let,

$$y = \tan^{-1}\left\{\frac{\sin x}{1 + \cos x}\right\}$$

Function y is defined for all real numbers where $\cos x \neq -1$

Using $2 \cos^2\theta = 1 + \cos 2\theta$ and $2 \sin \theta \cos \theta = \sin 2\theta$

$$y = \tan^{-1}\left\{\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right\}$$

$$y = \tan^{-1}\left\{\tan\frac{x}{2}\right\}$$

$$y = \frac{x}{2}$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right)$$

$$\frac{dy}{dx} = \frac{1}{2}$$

22. $\sin^{-1}\left\{\frac{1}{\sqrt{1+x^2}}\right\}$

Solution:

Let,

$$y = \sin^{-1}\left\{\frac{1}{\sqrt{1+x^2}}\right\}$$

Let $x = \cot\theta$

Now

$$y = \sin^{-1}\left\{\frac{1}{\sqrt{1+\cot^2\theta}}\right\}$$

Using, $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

Now

$$y = \sin^{-1}\left\{\frac{1}{\sqrt{\operatorname{cosec}^2\theta}}\right\}$$

$$y = \sin^{-1}\left\{\frac{1}{\operatorname{cosec}\theta}\right\}$$

$y = \sin^{-1}(\sin\theta)$

$y = \theta$

$y = \cot^{-1}x$

Differentiating with respect to x we get

$$\frac{dy}{dx} = \frac{d}{dx}(\cot^{-1}x)$$

$$\frac{dy}{dx} = -\frac{1}{1+x^2}$$

23. $\cos^{-1}\left\{\frac{1-x^{2n}}{1+x^{2n}}\right\},\ 0 < x < \infty$

Solution:

Let,

$$y = \cos^{-1}\left\{\frac{1-x^{2n}}{1+x^{2n}}\right\}$$

Let $x^n = \tan\theta$

Now

$$y = \cos^{-1}\left\{\frac{1-\tan^2\theta}{1+\tan^2\theta}\right\}$$

Using $\frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$

$$y = \cos^{-1}\{\cos 2\theta\}$$

Considering the limits,

$0 < x < \infty$

$0 < x^n < \infty$

$0 < \theta < \frac{\pi}{2}$

Now, $y = \cos^{-1}(\cos 2\theta)$

$y = 2\theta$

$y = \tan^{-1}(x^n)$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}(x^n))$$

$$\frac{dy}{dx} = \frac{2nx^{n-1}}{1 + (x^n)^2}$$

$$\frac{dy}{dx} = \frac{2nx^{n-1}}{1 + x^{2n}}$$

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Find dy/dx in each of the following:

1. $xy = c^2$

Solution:

Given $xy = c^2$;

Now we have to find $\frac{dy}{dx}$ of given equation, so by differentiating the equation on both sides with respect to x, we get,

By using the product rule on the left hand side,

$$\frac{d(xy)}{dx} = \frac{dc^2}{dx}$$

x (dy/dx) + y (1) = 0

$$\frac{dy}{dx} = \frac{-y}{x}$$

We can further solve it by putting the value of y,

$$\frac{dy}{dx} = \frac{-c^2}{x^2}$$

2. $y^3 - 3xy^2 = x^3 + 3x^2y$

Solution:

Given $y^3 - 3xy^2 = x^3 + 3x^2y$,

Now we have to find dy/dx of given equation, so by differentiating the equation on both sides with respect to x, we get,

$$\Rightarrow \frac{d}{dx}(y^3) - \frac{d}{dx}(3xy^2) = \frac{d}{dx}(x^3) + \frac{d}{dx}(3x^2y)$$

Now by using product rule we get,

$$\Rightarrow 3y^2\frac{dy}{dx} - 3\left[x\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x)\right] = 3x^2 + 3\left[x^2\frac{d}{dx}(y) + y\frac{d}{dx}(x^2)\right]$$

$$\Rightarrow 3y^2\frac{dy}{dx} - 3\left[x(2y)\frac{dy}{dx} + y^2\right] = 3x^2 + 3\left[x^2\frac{dy}{dx} + y(2x)\right]$$

$$\Rightarrow 3y^2\frac{dy}{dx} - 6xy\frac{dy}{dx} - 3y^2 = 3x^2 + 3x^2\frac{dy}{dx} + 6xy$$

$$\Rightarrow 3y^2\frac{dy}{dx} - 6xy\frac{dy}{dx} - 3x^2\frac{dy}{dx} = 3x^2 + 6xy + 3y^2$$

$$\Rightarrow 3\frac{dy}{dx}\left(y^2 - 2xy - x^2\right) = 3\left(x^2 + 2xy + y^2\right)$$

Now by taking 3 as common we get,

$$\Rightarrow \frac{dy}{dx} = \frac{3(x+y)^2}{3\left(y^2 - 2xy - x^2\right)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x+y)^2}{y^2 - 2xy - x^2}$$

3. $x^{2/3} + y^{2/3} = a^{2/3}$

Solution:

Given $x^{2/3} + y^{2/3} = a^{2/3}$,

Now we have to find dy/dx of given equation, so by differentiating the equation on both sides with respect to x, we get,

$$\frac{2}{3}\frac{1}{x^{1/3}} + \frac{2}{3}\frac{1}{y^{1/3}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-y^{1/3}}{x^{1/3}}$$

Now by substituting the value, we get

$$\frac{dy}{dx} = \frac{-\sqrt{a^{2/3} - x^{2/3}}}{x^{1/3}}$$

4. 4x + 3y = log (4x – 3y)

Solution:

Given 4x + 3y = log (4x – 3y),

Now we have to find dy/dx of it, so by differentiating the equation on both sides with respect to x, we get,

$$\frac{d}{dx}(4x)+\frac{d}{dx}(3y)=\frac{d}{dx}\{\log(4x-3y)\}$$

$$\Rightarrow 4+3\frac{dy}{dx}=\frac{1}{(4x-3y)}\frac{d}{dx}(4x-3y)$$

$$\Rightarrow 4+3\frac{dy}{dx}=\frac{1}{(4x-3y)}\left(4-3\frac{dy}{dx}\right)$$

$$\Rightarrow 3\frac{dy}{dx}+\frac{3}{(4x-3y)}\frac{dy}{dx}=\frac{4}{(4x-3y)}-4$$

$$\Rightarrow 3\frac{dy}{dx}\left\{1+\frac{1}{(4x-3y)}\right\}=4\left\{\frac{1}{(4x-3y)}-1\right\}$$

$$\Rightarrow 3\frac{dy}{dx}\left\{\frac{4x-3y+1}{(4x-3y)}\right\}=4\left\{\frac{1-4x+3y}{(4x-3y)}\right\}$$

$$\Rightarrow \frac{dy}{dx}=\frac{4}{3}\left\{\frac{1-4x+3y}{(4x-3y)}\right\}\left(\frac{4x-3y}{4x-3y+1}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{4}{3}\left(\frac{1-4x+3y}{4x-3y+1}\right)$$

5. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Solution:

Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, Now we have to find dy/dx of given equation, so by differentiating the equation on both sides with respect to x, we get,

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-xb^2}{ya^2}$$

6. $x^5 + y^5 = 5xy$

Solution:

Given $x^5 + y^5 = 5xy$

Now we have to find dy/dx of given equation, so by differentiating the equation on both sides with respect to x, we get,

$$\frac{d}{dx}(x^5) + \frac{d}{dx}(y^5) = \frac{d}{dx}(5xy)$$

Now by using product rule, we get

$$\Rightarrow 5x^4 + 5y^4\frac{dy}{dx} = 5\left[x\frac{dy}{dx} + y\frac{d}{dx}(x)\right]$$

$$\Rightarrow 5x^4 + 5y^4\frac{dy}{dx} = 5\left[x\frac{dy}{dx} + y(1)\right]$$

$$\Rightarrow 5x^4 + 5y^4\frac{dy}{dx} = 5x\frac{dy}{dx} + 5y$$

$$\Rightarrow 5y^4\frac{dy}{dx} - 5x\frac{dy}{dx} = 5y - 5x^4$$

$$\Rightarrow 5\frac{dy}{dx}(y^4 - x) = 5(y - x^4)$$

$$\Rightarrow \frac{dy}{dx} = \frac{5(y - x^4)}{5(y^4 - x)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y - x^4}{y^4 - x}$$

7. $(x + y)^2 = 2axy$

Solution:

Given $(x + y)^2 = 2axy$

Now we have to find dy/dx of given equation, so by differentiating the equation on both sides with respect to x, we get,

$$\Rightarrow \frac{d}{dx}(x+y)^2 = \frac{d}{dx}(2axy)$$

Now by using product rule, we get

$$\Rightarrow 2(x+y)\frac{d}{dx}(x+y) = 2a\left[x\frac{dy}{dx} + y\frac{d}{dx}(x)\right]$$

$$\Rightarrow 2(x+y)\left[1 + \frac{dy}{dx}\right] = 2a\left[x\frac{dy}{dx} + y(1)\right]$$

$$\Rightarrow 2(x+y) + 2(x+y)\frac{dy}{dx} = 2ax\frac{dy}{dx} + 2ay$$

$$\Rightarrow \frac{dy}{dx}[2(x+y) - 2ax] = 2ay - 2(x+y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2[ay - x - y]}{2[x + y - ax]}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{ay - x - y}{x + y - ax}\right)$$

8. $(x^2 + y^2)^2 = xy$

Solution:

Given $(x + y)^2 = 2axy$

Now we have to find dy/dx of given equation, so by differentiating the equation on both sides with respect to x, we get,

$$\Rightarrow \frac{d}{dx}\left[\left(x^2+y^2\right)^2\right]=\frac{d}{dx}(xy)$$

Now by applying product rule we get,

$$\Rightarrow 2\left(x^2+y^2\right)\frac{d}{dx}\left(x^2+y^2\right)=x\frac{dy}{dx}+y\frac{d}{dx}(x)$$

$$\Rightarrow 2\left(x^2+y^2\right)\left(2x+2y\frac{dy}{dx}\right)=x\frac{dy}{dx}+y(1)$$

$$\Rightarrow 4x\left(x^2+y^2\right)+4y\left(x^2+y^2\right)\frac{dy}{dx}=x\frac{dy}{dx}+y$$

$$\Rightarrow 4y\left(x^2+y^2\right)\frac{dy}{dx}-x\frac{dy}{dx}=y-4x\left(x^2+y^2\right)$$

$$\Rightarrow \frac{dy}{dx}\left[4y\left(x^2+y^2\right)-x\right]=y-4x\left(x^2+y^2\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{y-4x\left(x^2+y^2\right)}{4y\left(x^2+y^2\right)-x}$$

$$\Rightarrow \frac{dy}{dx}=\frac{4x\left(x^2+y^2\right)-y}{x-4y\left(x^2+y^2\right)}$$

9. Tan^{-1} (x^2 + y^2)

Solution:

Given tan $^{-1}(x^2 + y^2) = a$,

Now we have to find dy/dx of given function, so by differentiating the equation on both sides with respect to x, we get,

$$\frac{1}{x^2+y^2}\left(2x+2y\frac{dy}{dx}\right)=0$$

$$\frac{dy}{dx}=\frac{-x}{y}$$

10. $e^{x-y} = log\left(\frac{x}{y}\right)$

Solution:

Given $e^{x-y} = log\left(\frac{x}{y}\right)$

Now we have to find dy/dx of given function, so by differentiating the equation on both sides with respect to x, we get,

$$\frac{d}{dx}\left(e^{x-y}\right) = \frac{d}{dx}\left\{\log\left(\frac{x}{y}\right)\right\}$$

$$\Rightarrow e^{(x-y)}\frac{d}{dx}(x-y) = \frac{1}{\left(\frac{x}{y}\right)} \times \frac{d}{dx}\left(\frac{x}{y}\right)$$

Now by applying quotient rule we get

$$\Rightarrow e^{(x-y)}\left(1-\frac{dy}{dx}\right) = \frac{y}{x}\left[\frac{y\frac{d}{dx}(x) - x\frac{dy}{dx}}{y^2}\right]$$

$$\Rightarrow e^{(x-y)} - e^{(x-y)}\frac{dy}{dx} = \frac{1}{xy}\left[y(1) - x\frac{dy}{dx}\right]$$

$$\Rightarrow e^{(x-y)} - e^{(x-y)}\frac{dy}{dx} = \frac{1}{x} - \frac{1}{y}\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} - e^{(x-y)}\frac{dy}{dx} = \frac{1}{x} - e^{(x-y)}$$

$$\Rightarrow \frac{dy}{dx}\left[\frac{1}{y} - \frac{e^{(x-y)}}{1}\right] = \frac{1}{x} - \frac{e^{(x-y)}}{1}$$

$$\Rightarrow \frac{dy}{dx}\left[\frac{1 - ye^{(x-y)}}{y}\right] = \frac{1 - xe^{(x-y)}}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}\left[\frac{1 - xe^{(x-y)}}{1 - ye^{(x-y)}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{-x}\left[\frac{xe^{(x-y)} - 1}{ye^{(x-y)} - 1}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}\left[\frac{xe^{(x-y)} - 1}{ye^{(x-y)} - 1}\right]$$

11. Sin xy + cos (x + y) = 1

Solution:

Given Sin x y + cos (x + y) = 1

Now we have to find dy/dx of given function, so by differentiating the equation on both sides with respect to x, we get,

$$\frac{d}{dx}(\sin xy) + \frac{d}{dx}\cos(x+y) = \frac{d}{dx}(1)$$

$$\Rightarrow \cos xy \frac{d}{dx}(xy) - \sin(x+y)\frac{d}{dx}(x+y) = 0$$

$$\Rightarrow \cos xy\left[x\frac{dy}{dx} + y\frac{d}{dx}(x)\right] - \sin(x+y)\left[1 + \frac{dy}{dx}\right] = 0$$

$$\Rightarrow \cos xy\left[x\frac{dy}{dx} + y(1)\right] - \sin(x+y) - \sin(x+y)\frac{dy}{dx} = 0$$

$$\Rightarrow x\cos xy\frac{dy}{dx} + y\cos xy - \sin(x+y) - \sin(x+y)\frac{dy}{dx} = 0$$

$$\Rightarrow [x\cos xy - \sin(x+y)]\frac{dy}{dx} = [\sin(x+y) - y\cos xy]$$

$$\Rightarrow \frac{dy}{dx} = \left[\frac{\sin(x+y) - y\cos xy}{x\cos xy - \sin(x+y)}\right]$$

12. $If\ \sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y),\ prove\ that\ \frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}.$

Solution:

Given $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$

Let x = sin A and y = sin B

Then given equation becomes,

$$\Rightarrow \sqrt{1-\sin^2 A}+\sqrt{1-\sin^2 B}=a(\sin A-\sin B)$$

$$\Rightarrow \cos A+\cos B=a(\sin A-\sin B)$$

$$\Rightarrow a=\frac{\cos A+\cos B}{\sin A-\sin B}$$

Now by applying the formula we get,

$$\Rightarrow a=\frac{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}$$

$$\Rightarrow a=\cot\left(\frac{A-B}{2}\right)$$

$$\Rightarrow \cot^{-1}a=\frac{A-B}{2}$$

$$\Rightarrow 2\cot^{-1}a=A-B$$

$$\Rightarrow 2\cot^{-1}a=\sin^{-1}x-\sin^{-1}y.$$

Now by differentiating with respect to x we get,

$$\frac{d}{dx}(2cot^{-1}a)=\frac{d}{dx}(\sin^{-1}x)-\frac{d}{dx}(\sin^{-1}y)$$

$$\Rightarrow 0 = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$$

13. $\boldsymbol{If\ y = \sqrt{1-x^2} + x\sqrt{1-y^2} = 1,\ prove\ that\ \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}.}$

Solution:

Given, $y = \sqrt{1-x^2} + x\sqrt{1-y^2} = 1$

Let x = sin A and y = sin B

Then given equation becomes,

$$\Rightarrow \sin B\sqrt{1-\sin^2 A} + \sin A\sqrt{1-\sin^2 B} = 1$$

Now by applying the identity, we get

$$\Rightarrow \sin B\cos A + \sin A\cos B = 1$$

$$\Rightarrow \sin(A+B) = 1$$

$$\Rightarrow A + B = \sin^{-1}(1)$$

Now by substituting the values of A and B, we get

$$\Rightarrow \sin^{-1}x + \sin^{-1}y = \frac{\pi}{2}$$

Now by differentiating with respect to x, we get

$$\Rightarrow \frac{d}{dx}(\sin^{-1} x) + \frac{d}{dx}(\sin^{-1} y) = \frac{d}{dx}\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$$

$$14.\ If\ xy = 1,\ prove\ that\ \frac{dy}{dx} + y^2 = 0.$$

Solution:

Given x y = 1

Differentiating with respect to x, we get

$$\frac{d}{dx}(xy) = \frac{d}{dx}(1)$$

By using product rule,

$$\Rightarrow x\frac{dy}{dx} + y\frac{d}{dx}(x) = 0.$$

$$\Rightarrow x\frac{dy}{dx} + y(1) = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

We have x y = 1, therefore x = 1/y

$$\Rightarrow \frac{dy}{dx} = -\frac{y}{\frac{1}{y}}$$

$$\Rightarrow \frac{dy}{dx} = -y^2$$

$$\Rightarrow \frac{dy}{dx} + y^2 = 0$$

15. $If\ xy^2 = 1,\ prove\ that\ 2\frac{dy}{dx} + y^3 = 0.$

Solution:

Given $xy^2 = 1$

Now differentiating given equation with respect to x, we get

$$\frac{d}{dx}(xy^2) = \frac{d}{dx}(1)$$

$$\Rightarrow x\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x) = 0$$

$$\Rightarrow x(2y)\frac{dy}{dx} + y^2(1) = 0$$

$$\Rightarrow 2xy\frac{dy}{dx} = -y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y^2}{2xy}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{2x}$$

Now by substituting $x = 1/y^2$ in above equation we get

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{2\left(\frac{1}{y^2}\right)}$$

$$\Rightarrow 2\frac{dy}{dx} = -y^3$$

$$\Rightarrow 2\frac{dy}{dx} + y^3 = 0$$

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Differentiate the following functions with respect to x:

1. $x^{1/x}$

Solution:

Let $y = x^{\frac{1}{x}}$

Taking log both the sides:

$$\Rightarrow \log y = \log x^{\frac{1}{x}}$$

We know that log x^a = a log x, substituting this in above equation we get

$$\Rightarrow \log y = \frac{1}{x}\log x$$

Differentiating with respect to x, we get

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d\left(\frac{1}{x}\log x\right)}{dx}$$

Now by using the product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{1}{x} \times \frac{d(\log x)}{dx} + \log x \times \frac{d(x^{-1})}{dx}$$

We have $\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}; \frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}\right\}$, by using this we get,

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x} \times \frac{1}{x}\frac{dx}{dx} + \log x\left(\frac{-1}{x^2}\right)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2} - \frac{1}{x^2}\log x$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1 - \log x}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1 - \log x}{x^2}\right)$$

Put the value of $y = x^{\frac{1}{x}}$

$$\Rightarrow \frac{dy}{dx} = x^{\frac{1}{x}}\left(\frac{1 - \log x}{x^2}\right)$$

2. $x^{\sin x}$

Solution:

Let $y = x^{\sin x}$

Taking log both the sides

$\log y = \log (x^{\sin x})$

$\log y = \sin x \log x$ $\{\log x^a = a \log x\}$

Differentiating with respect to x, we get

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\sin x \log x)}{dx}$$

Now by using product rule, we can write as

$$\Rightarrow \frac{d(\log y)}{dx} = \sin x \times \frac{d(\log x)}{dx} + \log x \times \frac{d(\sin x)}{dx}$$

Again we have, $\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx} \;\&\; \frac{d(\sin x)}{dx} = \cos x\right\}$, by using this we can write as

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \sin x \times \frac{1}{x}\frac{dx}{dx} + \log x(\cos x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\sin x}{x} + \log x \cos x$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{\sin x}{x} + \log x \cos x\right)$$

Put the value of $y = x^{\sin x}$

$$\Rightarrow \frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \log x \cos x\right)$$

3. $(1 + \cos x)^x$

Solution:

Let $y = (1 + \cos x)^x$

Taking log on both the sides

$\Rightarrow \log y = \log (1 + \cos x)^x$

$\Rightarrow \log y = x \log (1+ \cos x)$ $\{\log x^a = a \log x\}$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d[x \log (1+ \cos x)]}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = x \times \frac{d[\log(1+ \cos x)]}{dx} + \log(1+ \cos x) \times \frac{dx}{dx}$$

Again we have, $\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{(1+ \cos x)}\frac{d(1+ \cos x)}{dx} + \log(1+ \cos x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{(1+ \cos x)}(-\sin x) + \log(1+ \cos x)$$

$$\left\{\frac{d(1+\cos x)}{dx} = \frac{d(1)}{dx} + \frac{d(\cos x)}{dx} = 0 + (-\sin x)\frac{dx}{dx} = -\sin x\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{-x\sin x}{1+\cos x} + \log(1+ \cos x)$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{-x\sin x}{1+\cos x} + \log(1+ \cos x)\right\}$$

Put the value of $y = (1 + \cos x)^x$

$$\Rightarrow \frac{dy}{dx} = (1+ \cos x)^x\left\{\frac{-x\sin x}{1+\cos x} + \log(1+ \cos x)\right\}$$

4. $x^{\cos^{-1} x}$

Solution:

Let $y = x^{\cos^{-1} x}$

Taking log both the sides

$\Rightarrow \log y = \log x^{\cos^{-1} x}$

$\Rightarrow \log y = \cos^{-1} x \log x$ {log x^a = a log x}

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\cos^{-1} x \log x)}{dx}$$

By using product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = \cos^{-1} x \times \frac{d(\log x)}{dx} + \log x \times \frac{d(\cos^{-1} x)}{dx}$$

Again we have, $\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx} \;\&\; \frac{d(\cos^{-1} x)}{dx} = \frac{-1}{\sqrt{1-x^2}}\right\}$, from this we can write as

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\cos^{-1} x}{x} + \log x\left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\cos^{-1} x}{x} - \frac{\log x}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{\cos^{-1} x}{x} - \frac{\log x}{\sqrt{1-x^2}}\right\}$$

Put the value of $y = x^{\cos^{-1} x}$

$$\Rightarrow \frac{dy}{dx} = x^{\cos^{-1} x}\left\{\frac{\cos^{-1} x}{x} - \frac{\log x}{\sqrt{1-x^2}}\right\}$$

5. (log x)^x

Solution:

Let y = $(\log x)^x$

Taking log both the sides

$\Rightarrow \log y = \log (\log x)^x$

$\Rightarrow \log y = x \log (\log x)$ $\{\log x^a = a \log x\}$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(x \log\log x)}{dx}$$

By product rule, we have

$$\Rightarrow \frac{d(\log y)}{dx} = x \times \frac{d(\log\log x)}{dx} + \log\log x \times \frac{dx}{dx}$$

We know that $\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\right\}$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{\log x}\frac{d(\log x)}{dx} + \log\log x$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{x}{\log x} \times \frac{1}{x} + \log\log x$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{1}{\log x} + \log\log x\right\}$$

Put the value of $y = (\log x)^x$

$$\Rightarrow \frac{dy}{dx} = (\log x)^x\left\{\frac{1}{\log x} + \log\log x\right\}$$

6. $(\log x)^{\cos x}$

Solution:

Let y = $(\log x)^{\cos x}$

Taking log both the sides, we get

$\Rightarrow$ Log y = log (log x)$^{\cos x}$

$\Rightarrow$ Log y = cos x log (log x) {log x^a = a log x}

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\cos x \log \log x)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = \cos x \times \frac{d(\log \log x)}{dx} + \log \log x \times \frac{d(\cos x)}{dx}$$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}$ & $\frac{d(\cos x)}{dx} = -\sin x$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cos x \times \frac{1}{\log x}\frac{d(\log x)}{dx} + \log \log x \ (-\sin x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\cos x}{\log x} \times \frac{1}{x} - \sin x \log \log x$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{\cos x}{x \log x} - \sin x \log \log x\right\}$$

Put the value of y = (log x)$^{\cos x}$

$$\Rightarrow \frac{dy}{dx} = (\log x)^{\cos x}\left\{\frac{\cos x}{x \log x} - \sin x \log \log x\right\}$$

7. (Sin x)$^{\cos x}$

Solution:

Let y = (sin x)$^{\cos x}$

Taking log both the sides

$\Rightarrow$ Log y = log (sin x)$^{\cos x}$

$\Rightarrow$ Log y = cos x log sin x {log x^a = a log x}

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\cos x \log \sin x)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = \cos x \times \frac{d(\log \sin x)}{dx} + \log \sin x \times \frac{d(\cos x)}{dx}$$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}$; $\frac{d(\cos x)}{dx} = -\sin x$; $\frac{d(\sin x)}{dx} = \cos x$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cos x \times \frac{1}{\sin x}\frac{d(\sin x)}{dx} + \log \sin x\,(-\sin x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cot x\,(\cos x) - \sin x \log \sin x$$

$$\Rightarrow \frac{dy}{dx} = y\{\cos x \cot x - \sin x \log \sin x\}$$

Put the value of $y = (\sin x)^{\cos x}$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\cos x}\{\cos x \cot x - \sin x \log \sin x\}$$

8. $e^{x \log x}$

Solution:

Let $y = e^{x \log x}$

Taking log both the sides, we get

$\Rightarrow$ Log y = log $(e)^{x \log x}$

$\Rightarrow$ Log y = x log x log e {log x^a = a log x}

$\Rightarrow$ Log y = x log x {log e = 1}

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(x \log x)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = x \times \frac{d(\log x)}{dx} + \log x \times \frac{dx}{dx}$$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{x}\frac{dx}{dx} + \log x$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{x}{x} + \log x$$

$$\Rightarrow \frac{dy}{dx} = y\{1 + \log x\}$$

Put the value of $y = e^{x \log x}$

$$\Rightarrow \frac{dy}{dx} = e^{x\log x}\{1 + \log x\}$$

$$\Rightarrow \frac{dy}{dx} = e^{\log x^x}\{1 + \log x\}$$ $\{e^{\log a} = a;\ a \log x = x^a\}$

$$\Rightarrow \frac{dy}{dx} = x^x\{1 + \log x\}$$

9. $(\text{Sin } x)^{\log x}$

Solution:

Let $y = (\sin x)^{\log x}$

Taking log both the sides

$\Rightarrow \text{Log } y = \log (\sin x)^{\log x}$

$\Rightarrow \text{Log } y = \log x \log \sin x$ $\{\log x^a = a \log x\}$

Differentiating with respect to x, then we get

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\log x \log \sin x)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = \log x \times \frac{d(\log \sin x)}{dx} + \log \sin x \times \frac{d(\log x)}{dx}$$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}$; $\frac{d(\sin x)}{dx} = \cos x$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log x \times \frac{1}{\sin x}\frac{d(\sin x)}{dx} + \log \sin x \left(\frac{1}{x}\frac{dx}{dx}\right)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\log x}{\sin x}(\cos x) + \frac{\log \sin x}{x}$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\log x \cot x + \frac{\log \sin x}{x}\right\}$$

Put the value of $y = (\sin x)^{\log x}$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\log x}\left\{\log x \cot x + \frac{\log \sin x}{x}\right\}$$

10. $10^{\log \sin x}$

Solution:

Let $y = 10^{\log \sin x}$

Taking log both the sides

$\Rightarrow$ Log y = log $10^{\log \sin x}$

$\Rightarrow$ Log y = log sin x log 10 {log x^a = a log x}

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\log 10 \log \sin x)}{dx}$$

Now by using chain rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = \log 10 \times \frac{d(\log \sin x)}{dx}$$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}; \frac{d(\sin x)}{dx} = \cos x$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log 10 \times \frac{1}{\sin x}\frac{d(\sin x)}{dx}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\log 10}{\sin x}(\cos x)$$

$$\Rightarrow \frac{dy}{dx} = y\{\log 10 \cot x\}$$

Put the value of $y = 10^{\log \sin x}$

$$\Rightarrow \frac{dy}{dx} = 10^{\log \sin x}\{\log 10 \cot x\}$$

11. $(\log x)^{\log x}$

Solution:

Let $y = (\log x)^{\log x}$

Taking log both the sides

$\Rightarrow$ Log $y = \log (\log x)^{\log x}$

$\Rightarrow$ Log $y = \log x \log (\log x)$ $\{\log x^a = a \log x\}$

Differentiating with respect to x, then we get

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\log x \log(\log x))}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = \log x \times \frac{d(\log(\log x))}{dx} + \log(\log x) \times \frac{d(\log x)}{dx}$$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log x \times \frac{1}{\log x}\frac{d(\log x)}{dx} + \log\log x\left(\frac{1}{x}\frac{dx}{dx}\right)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\log x}{\log x}\left(\frac{1}{x}\frac{dx}{dx}\right) + \frac{\log(\log x)}{x}$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{1}{x} + \frac{\log(\log x)}{x}\right\}$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{1+\log(\log x)}{x}\right\}$$

Put the value of $y = (\log x)^{\log x}$

$$\Rightarrow \frac{dy}{dx} = (\log x)^{\log x}\left\{\frac{1+\log(\log x)}{x}\right\}$$

12. $10^{(10^x)}$

Solution:

Let $y = 10^{(10^x)}$

Taking log both the sides

$\Rightarrow$ Log y = log $10^{(10^x)}$

$\Rightarrow$ Log y = 10 x log 10 {log x^a = a log x}

$\Rightarrow$ Log y = (10 log 10) x

Differentiating with respect to x,

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d\{(10\log 10)x\}}{dx}$$

Here 10 log 10 is a constant term, therefore by using chain rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = 10 \times \log(10) \times \frac{d(x)}{dx}$$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}$; $\frac{d(\sin x)}{dx} = \cos x$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 10\log(10)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 10\log(10)$$

$$\Rightarrow \frac{dy}{dx} = y\{10\log(10)\}$$

Put the value of y = $10^{(10^x)}$

$$\Rightarrow \frac{dy}{dx} = 10^{10x}\{10\log(10)\}$$

13. Sin (x^x)

Solution:

Let y = sin (x^x)

Take sin inverse both sides

⇒ $\sin^{-1} y = \sin^{-1} (\sin x^x)$

⇒ $\sin^{-1} y = x^x$

Taking log both the sides

⇒ Log ($\sin^{-1}$ y) = log x^x

⇒ Log ($\sin^{-1}$ y) = x log x {log x^a = a log x}

Differentiating with respect to x

$$\Rightarrow \frac{d(\log(\sin^{-1} y))}{dx} = \frac{d(x\log x)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log(\sin^{-1} y))}{dx} = x \times \frac{d(\log x)}{dx} + \log x \times \frac{dx}{dx}$$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}$

$$\Rightarrow \frac{1}{\sin^{-1} y}\frac{d(\sin^{-1} y)}{dx} = x \times \frac{1}{x}\frac{dx}{dx} + \log x$$

Again we have, $\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$ by using this result we get

$$\Rightarrow \frac{1}{\sin^{-1} y} \times \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = \frac{x}{x} + \log x$$

$$\Rightarrow \frac{1}{\sin^{-1} y(\sqrt{1-y^2})}\frac{dy}{dx} = 1 + \log x$$

$$\Rightarrow \frac{dy}{dx} = \sin^{-1} y\left(\sqrt{1-y^2}\right)(1 + \log x)$$

Put the value of $y = \sin(x^x)$

$$\Rightarrow \frac{dy}{dx} = \sin^{-1}(\sin x^x)\left(\sqrt{1-\sin^2(x^x)}\right)(1 + \log x)$$

From $\sin^2 x + \cos^2 x = 1$, we can write as

$$\Rightarrow \frac{dy}{dx} = x^x\left(\sqrt{\cos^2(x^x)}\right)(1 + \log x)$$

$$\Rightarrow \frac{dy}{dx} = x^x \cos x^x (1 + \log x)$$

14. $(\mathrm{Sin}^{-1} x)^x$

Solution:

Let $y = (\sin^{-1} x)^x$

Taking log both the sides

$\Rightarrow \text{Log } y = \log(\sin^{-1} x)^x$

$\Rightarrow \text{Log } y = x \log(\sin^{-1} x)$ $\{\log x^a = a \log x\}$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(x \log(\sin^{-1}x))}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = x \times \frac{d(\log(\sin^{-1}x))}{dx} + \log(\sin^{-1}x) \times \frac{dx}{dx}$$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{\sin^{-1}x}\frac{d(\sin^{-1}x)}{dx} + \log(\sin^{-1}x)$$

Again we have, $\frac{d(\sin^{-1}u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$ by using this result we get

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{x}{\sin^{-1}x} \times \frac{1}{\sqrt{1-x^2}}\frac{dx}{dx} + \log(\sin^{-1}x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{x}{\sin^{-1}x\sqrt{1-x^2}} + \log(\sin^{-1}x)$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{x}{\sin^{-1}x\sqrt{1-x^2}} + \log(\sin^{-1}x)\right\}$$

Put the value of $y = (\sin^{-1}x)^x$

$$\Rightarrow \frac{dy}{dx} = (\sin^{-1}x)^x\left\{\frac{x}{\sin^{-1}x\sqrt{1-x^2}} + \log(\sin^{-1}x)\right\}$$

15. $x^{\sin^{-1}x}$

Solution:

Let $y = x^{\sin^{-1}x}$

Taking log both the sides

$$\Rightarrow \log y = \log x^{\sin^{-1}x}$$

$\Rightarrow$ Log y = $\sin^{-1}x \log x$ {$\log x^a = a \log x$}

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\sin^{-1} x \ \log x)}{dx}$$

By using product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = \sin^{-1} x \times \frac{d(\log x)}{dx} + \log x \times \frac{d(\sin^{-1} x)}{dx}$$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}$; $\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \sin^{-1} x \times \frac{1}{x}\frac{dx}{dx} + \log x \times \frac{1}{\sqrt{1-x^2}}\frac{dx}{dx}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\sin^{-1} x}{x} + \frac{\log x}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{\sin^{-1} x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right\}$$

Put the value of $y = x^{\sin^{-1} x}$:

$$\Rightarrow \frac{dy}{dx} = x^{\sin^{-1} x}\left\{\frac{\sin^{-1} x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right\}$$

16. $(\tan x)^{1/x}$

Solution:

Let $y = (\tan x)^{\frac{1}{x}}$

Taking log both the sides, we get

$$\Rightarrow \log y = \log(\tan x)^{\frac{1}{x}}$$

$$\Rightarrow \log y = \frac{1}{x}\log\tan x$$ {Log x^a = a log x}

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d\left(\frac{1}{x}\log\tan x\right)}{dx}$$

By using product rule, we can write as

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{1}{x} \times \frac{d(\log\tan x)}{dx} + \log\tan x \times \frac{d(x^{-1})}{dx}$$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}; \frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x} \times \frac{1}{\tan x}\frac{d(\tan x)}{dx} + \log\tan x\,(-x^{-2})$$

Again we have $\frac{d(\tan x)}{dx} = \sec^2 x$ by using this result in the above expression

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x\tan x}(\sec^2 x) - \frac{\log\tan x}{x^2}$$

$$\frac{dy}{dx} = y\left\{\frac{\sec^2 x}{x\tan x} - \frac{\log\tan x}{x^2}\right\}$$

Put the value of $y = (\tan x)^{\frac{1}{x}}$

$$\frac{dy}{dx} = (\tan x)^{\frac{1}{x}}\left\{\frac{\sec^2 x}{x\tan x} - \frac{\log\tan x}{x^2}\right\}$$

17. $x^{\tan^{-1} x}$

Solution:

Let $y = x^{\tan^{-1} x}$

Taking log both the sides

$\Rightarrow \log y = \log x^{\tan^{-1} x}$

$\Rightarrow$ Log y = $\tan^{-1} x \log x$ {$\log x^a = a \log x$}

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\tan^{-1} x \ \log x)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = \tan^{-1} x \times \frac{d(\log x)}{dx} + \log x \times \frac{d(\tan^{-1} x)}{dx}$$

Again we know that $\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}; \frac{d(\tan^{-1} u)}{dx} = \frac{1}{u^2+1}\frac{du}{dx}$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \tan^{-1} x \times \frac{1}{x}\frac{dx}{dx} + \log x \times \frac{1}{x^2+1}\frac{dx}{dx}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\tan^{-1} x}{x} + \frac{\log x}{x^2+1}$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{\tan^{-1} x}{x} + \frac{\log x}{x^2+1}\right\}$$

Put the value of $y = x^{\tan^{-1} x}$

$$\Rightarrow \frac{dy}{dx} = x^{\tan^{-1} x}\left\{\frac{\tan^{-1} x}{x} + \frac{\log x}{x^2+1}\right\}$$

18. (i) $(x^x)\sqrt{x}$

Solution:

Let $y = (x)^x \sqrt{x}$

Taking log both the sides

$$\Rightarrow \log y = \log(x)^x \sqrt{x}$$

$$\Rightarrow \log y = \log(x)^x + \log\sqrt{x}$$ {Log (ab) = log a +log b}

$$\Rightarrow \log y = \log(x)^x + \log x^{\frac{1}{2}}$$

$$\Rightarrow \log y = x\log x + \frac{1}{2}\log x$$ {Log x^a = a log x}

$$\Rightarrow \log y = \left(x+\frac{1}{2}\right)\log x$$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d\left(\left(x+\frac{1}{2}\right)\log x\right)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log y)}{dx} = \left(x+\frac{1}{2}\right)\times\frac{d(\log x)}{dx} + \log x \times \frac{d\left(x+\frac{1}{2}\right)}{dx}$$

Again we have to use chain rule for the above expression,

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \left(x+\frac{1}{2}\right)\times\frac{1}{x}\frac{dx}{dx} + \log x\frac{dx}{dx}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{(2x+1)}{2}\times\frac{1}{x} + \log x$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{(2x+1)}{2x} + \log x\right\}$$

Put the value of $y = (x)^x\sqrt{x}$

$$\Rightarrow \frac{dy}{dx} = (x)^x\sqrt{x}\left\{\frac{(2x+1)}{2x} + \log x\right\}$$

$$\Rightarrow \frac{dy}{dx} = (x)^x\sqrt{x}\left\{\frac{2x}{2x} + \frac{1}{2x} + \log x\right\}$$

$$\Rightarrow \frac{dy}{dx} = (x)^x\sqrt{x}\left\{1 + \frac{1}{2x} + \log x\right\}$$

18. (ii) $x^{(\sin x - \cos x)} + \dfrac{x^2-1}{x^2+1}$

Solution:

Let $y = x^{(\sin x - \cos x)} + \dfrac{x^2-1}{x^2+1}$

$\Rightarrow y = a + b$

where $a = x^{(\sin x - \cos x)}; b = \frac{x^2 - 1}{x^2 + 1}$

Now we have to differentiate y = a + b with respect to x

By using chain rule, we can write as

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$a = x^{(\sin x - \cos x)}$

Taking log both the sides to the above expressions we get

$$\Rightarrow \log a = \log x^{(\sin x - \cos x)}$$

$\Rightarrow \log a = (\sin x - \cos x) \log x$ {Log x^a = a log x}

Differentiating with respect to x

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d((\sin x - \cos x)\log x)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log a)}{dx} = (\sin x - \cos x) \times \frac{d(\log x)}{dx} + \log x \times \frac{d(\sin x - \cos x)}{dx}$$

To the above expression we have to use chain rule,

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = (\sin x - \cos x) \times \frac{1}{x}\frac{dx}{dx} + \log x\left(\frac{d(\sin x)}{dx} - \frac{d(\cos x)}{dx}\right)$$

We know that $\frac{d(\cos x)}{dx} = -\sin x; \frac{d(\sin x)}{dx} = \cos x$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{(\sin x - \cos x)}{x} + \log x(\cos x - (-\sin x))$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{(\sin x - \cos x)}{x} + \log x(\cos x + \sin x)$$

$$\Rightarrow \frac{da}{dx} = a\left\{\frac{\sin x - \cos x}{x} + \log x\,(\cos x + \sin x)\right\}$$

Put the value of a = $x^{(\sin x - \cos x)}$

$$\Rightarrow \frac{da}{dx} = x^{(\sin x - \cos x)}\left\{\frac{\sin x - \cos x}{x} + \log x\,(\cos x + \sin x)\right\}$$

$$b = \frac{x^2 - 1}{x^2 + 1}$$

To differentiate above expression with respect to x we have to use quotient rule,

$$\Rightarrow \frac{db}{dx} = \frac{(x^2+1)\frac{d(x^2-1)}{dx} - (x^2-1)\ \frac{d(x^2+1)}{dx}}{(x^2+1)^2}$$

Now by using chain rule, we get

$$\Rightarrow \frac{db}{dx} = \frac{(x^2+1)(2x) - (x^2-1)\ (2x)}{(x^2+1)^2}$$

$$\Rightarrow \frac{db}{dx} = \frac{(2x^3+2x) - (2x^3-2x)}{(x^2+1)^2}$$

$$\Rightarrow \frac{db}{dx} = \frac{(2x^3+2x-2x^3+2x)}{(x^2+1)^2}$$

$$\Rightarrow \frac{db}{dx} = \frac{4x}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

Now by substituting all the values in above expressions we get

$$\Rightarrow \frac{dy}{dx} = x^{(\sin x - \cos x)}\left\{\frac{\sin x - \cos x}{x} + \log x\,(\cos x + \sin x)\right\} + \frac{4x}{(x^2+1)^2}$$

$$18.(iii)\ x^{x\cos x} + \frac{x^2+1}{x^2-1}$$

Solution:

$$\text{Let } y = x^{x\cos x} + \frac{x^2+1}{x^2-1}$$

$\Rightarrow y = a + b$

$$\text{where } a = x^{x\cos x};\ b = \frac{x^2+1}{x^2-1}$$

Now we have to differentiate y = a + b with respect to x

By using chain rule, we can write as

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$a = x^{x\cos x}$$

Taking log both the sides to the above equation we get

$$\Rightarrow \log a = \log x^{x\cos x}$$

$$\Rightarrow \log a = x\cos x \log x$$

$\{\text{Log } x^a = a \log x\}$

Differentiating with respect to x,

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d(x\cos x\log x)}{dx}$$

Now by using product rule, we can write as

$$\Rightarrow \frac{d(\log a)}{dx} = x\cos x \times \frac{d(\log x)}{dx} + \log x \times \frac{d(x\cos x)}{dx}$$

$$\Rightarrow \frac{d(\log a)}{dx} = x\cos x \times \frac{d(\log x)}{dx} + \log x\left\{x\frac{d(\cos x)}{dx} + \cos x\right\}$$

Again we have, $\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}$ by using this result in the above expressions we get

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = x\cos x \times \frac{1}{x}\frac{dx}{dx} + \log x\{x(-\sin x) + \cos x\}$$

We know that $\frac{d(\cos x)}{dx} = -\sin x;\ \frac{d(\sin x)}{dx} = \cos x$

$$\Rightarrow \frac{1}{a}\ \frac{da}{dx} = \frac{x\cos x}{x} + \log x(\cos x - x\sin x)$$

$$\Rightarrow \frac{da}{dx} = a\{\cos x + \log x(\cos x - x\sin x)\}$$

Put the value of $a = x^{x\cos x}$:

$$\Rightarrow \frac{da}{dx} = x^{x\cos x}\{\cos x + \log x(\cos x - x\sin x)\}$$

$$\Rightarrow \frac{da}{dx} = x^{x\cos x}\{\cos x + \log x\cos x - x\sin x\log x\}$$

$$\Rightarrow \frac{da}{dx} = x^{x\cos x}\{\cos x(1 + \log x) - x\sin x\log x\}$$

$$b = \frac{x^2+1}{x^2-1}$$

Now we have to differentiate above expression using quotient rule, then we get

$$\Rightarrow \frac{db}{dx} = \frac{(x^2-1)\frac{d(x^2+1)}{dx} - (x^2+1)\frac{d(x^2-1)}{dx}}{(x^2-1)^2}$$

Now apply chain rule for the above equation,

$$\Rightarrow \frac{db}{dx} = \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2+1)^2}$$

$$\Rightarrow \frac{db}{dx} = \frac{(2x^3-2x) - (2x^3+2x)}{(x^2+1)^2}$$

$$\Rightarrow \frac{db}{dx} = \frac{(2x^3-2x-2x^3-2x)}{(x^2+1)^2}$$

$$\Rightarrow \frac{db}{dx} = \frac{-4x}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

By substituting all values in the above expression we get

$$\Rightarrow \frac{dy}{dx} = x^{x\cos x}\{\cos x\,(1+\log x) - x\sin x\log x\} - \frac{4x}{(x^2+1)^2}$$

$18.(iv)\ (x\cos x)^x + (x\sin x)^{\frac{1}{x}}$

Solution:

Let $y = (x\cos x)^x + (x\sin x)^{\frac{1}{x}}$

$\Rightarrow y = a + b$

where $a = (x\cos x)^x; b = (x\sin x)^{\frac{1}{x}}$

Now we have to differentiate y = a + b with respect to x

By using chain rule, we can write as

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$a = (x\cos x)^x$

Taking log both the sides, we get

$\Rightarrow \log a = \log(x\cos x)^x$

$\Rightarrow \log a = x\log(x\cos x)$

$\{\text{Log } x^a = a \log x\}$

Differentiating with respect to x

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d(x\log(x\cos x))}{dx}$$

By using product rule, we get

$$\Rightarrow \frac{d(\log a)}{dx} = x \times \frac{d(\log(x\cos x))}{dx} + \log(x\cos x) \times \frac{dx}{dx}$$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = x \times \frac{1}{x\cos x}\frac{d(x\cos x)}{dx} + \log(x\cos x)$$

Again by using product rule, we can write as

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{x}{x\cos x}\left\{x\frac{d(\cos x)}{dx} + \cos x\right\} + \log(x\cos x)$$

We have $\frac{d(\cos x)}{dx} = -\sin x$ using this result we can write as

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{1}{\cos x}\{x(-\sin x) + \cos x\} + \log(x\cos x)$$

$$\Rightarrow \frac{da}{dx} = a\left\{\frac{\cos x - x\sin x}{\cos x} + \log(x\cos x)\right\}$$

Put the value of a = $(x\cos x)^x$:

$$\Rightarrow \frac{da}{dx} = (x\cos x)^x\left\{\frac{\cos x - x\sin x}{\cos x} + \log(x\cos x)\right\}$$

$$\Rightarrow \frac{da}{dx} = (x\cos x)^x\{1 - x\tan x + \log(x\cos x)\}$$

$$b = (x\sin x)^{\frac{1}{x}}$$

Taking log both the sides

$$\Rightarrow \log b = \log(x\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log b = \frac{1}{x}\log(x\sin x)$$ {Log x^a = a log x}

Differentiating with respect to x

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{d\left(\frac{1}{x}\log(x\sin x)\right)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{1}{x} \times \frac{d(\log(x\sin x))}{dx} + \log(x\sin x) \times \frac{d(x^{-1})}{dx}$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \frac{1}{x} \times \frac{1}{x\sin x}\frac{d(x\sin x)}{dx} + \log(x\sin x)\,(-x^{-2})$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \frac{1}{x^2\sin x}\left(x\frac{d(\sin x)}{dx} + \sin x\frac{dx}{dx}\right) - \frac{\log(x\sin x)}{x^2}$$

We know that $\frac{d(\sin x)}{dx} = \cos x$

$$\Rightarrow \frac{db}{dx} = b\left\{\frac{x\cos x + \sin x}{x^2\sin x} - \frac{\log(x\sin x)}{x^2}\right\}$$

Put the value of $b = (x\sin x)^{\frac{1}{x}}$:

$$\Rightarrow \frac{db}{dx} = (x\sin x)^{\frac{1}{x}}\left\{\frac{x\cos x + \sin x}{x^2\sin x} - \frac{\log(x\sin x)}{x^2}\right\}$$

$$\Rightarrow \frac{db}{dx} = (x\sin x)^{\frac{1}{x}}\left\{\frac{x\cot x + 1}{x^2} - \frac{\log(x\sin x)}{x^2}\right\}$$

$$\Rightarrow \frac{db}{dx} = (x\sin x)^{\frac{1}{x}}\left\{\frac{x\cot x + 1 - \log(x\sin x)}{x^2}\right\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

Now by substituting all the values in above expression we get

$$\Rightarrow \frac{dy}{dx} = (x\cos x)^x\{1 - x\tan x + \log(x\cos x)\} + (x\sin x)^{\frac{1}{x}}\left\{\frac{x\cot x + 1 - \log(x\sin x)}{x^2}\right\}$$

18. (v) $\left(x + \frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

Solution:

Let $y = \left(x+\frac{1}{x}\right)^{x} + x^{\left(1+\frac{1}{x}\right)}$

$\Rightarrow y = a + b$

where $a = \left(x+\frac{1}{x}\right)^{x}; b = x^{\left(1+\frac{1}{x}\right)}$

Now we have to differentiate y = a + b with respect to x

By using chain rule, we can write as

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$a = \left(x+\frac{1}{x}\right)^{x}$$

Taking log both the sides, we get

$$\Rightarrow \log a = \log\left(x+\frac{1}{x}\right)^{x}$$

$\Rightarrow \log a = x\log\left(x+\frac{1}{x}\right)$ {Log x^a = a log x}

Differentiating with respect to x

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d\left(x\log\left(x+\frac{1}{x}\right)\right)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log a)}{dx} = x \times \frac{d\left(\log\left(x+\frac{1}{x}\right)\right)}{dx} + \log\left(x+\frac{1}{x}\right) \times \frac{dx}{dx}$$

Again we know that $\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}$.

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = x \times \frac{1}{x+\frac{1}{x}}\frac{d\left(x+\frac{1}{x}\right)}{dx} + \log\left(x+\frac{1}{x}\right)$$

Again by using chain rule in the above expression we get

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{x}{\frac{x^2+1}{x}}\left\{\frac{dx}{dx} + \frac{d\left(\frac{1}{x}\right)}{dx}\right\} + \log\left(x + \frac{1}{x}\right)$$

By using $\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{x^2}{x^2+1}\left\{1 + \left(-\frac{1}{x^2}\right)\right\} + \log\left(x + \frac{1}{x}\right)$$

$$\Rightarrow \frac{da}{dx} = a\left\{\frac{x^2}{x^2+1}\left\{1 - \frac{1}{x^2}\right\} + \log\left(x + \frac{1}{x}\right)\right\}$$

Put the value of $a = \left(x + \frac{1}{x}\right)^x$:

$$\Rightarrow \frac{da}{dx} = \left(x + \frac{1}{x}\right)^x\left\{\frac{x^2}{x^2+1}\left\{1 - \frac{1}{x^2}\right\} + \log\left(x + \frac{1}{x}\right)\right\}$$

$$\Rightarrow \frac{da}{dx} = \left(x + \frac{1}{x}\right)^x\left\{\frac{x^2}{x^2+1} - \frac{1}{x^2+1} + \log\left(x + \frac{1}{x}\right)\right\}$$

$$\Rightarrow \frac{da}{dx} = \left(x + \frac{1}{x}\right)^x\left\{\frac{x^2-1}{x^2+1} + \log\left(x + \frac{1}{x}\right)\right\}$$

$b = x^{\left(1+\frac{1}{x}\right)}$

Taking log both the sides

$$\Rightarrow \log b = \log x^{\left(1+\frac{1}{x}\right)}$$

$$\Rightarrow \log b = \left(1 + \frac{1}{x}\right)\log x$$ {Log x^a = a log x}

Differentiating with respect to x

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{d\left(\left(1 + \frac{1}{x}\right)\log x\right)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log b)}{dx} = \left(1+\frac{1}{x}\right) \times \frac{d(\log x)}{dx} + \log x \times \frac{d\left(1+\frac{1}{x}\right)}{dx}$$

Again for the above expression we have to apply chain rule,

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \frac{x+1}{x} \times \frac{1}{x}\frac{dx}{dx} + \log x \left(\frac{d(1)}{dx} + \frac{d\left(\frac{1}{x}\right)}{dx}\right)$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \frac{x+1}{x^2} + \log x \left(-\frac{1}{x^2}\right)$$

$$\Rightarrow \frac{db}{dx} = b\left\{\frac{x+1}{x^2} - \frac{\log x}{x^2}\right\}$$

$$\Rightarrow \frac{db}{dx} = b\left\{\frac{x+1-\log x}{x^2}\right\}$$

Put the value of $b = x^{\left(1+\frac{1}{x}\right)}$:

$$\Rightarrow \frac{db}{dx} = x^{\left(1+\frac{1}{x}\right)}\left\{\frac{x+1-\log x}{x^2}\right\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

Now by substituting the all the values in above expression we get

$$\Rightarrow \frac{dy}{dx} = \left(x+\frac{1}{x}\right)^x\left\{\frac{x^2-1}{x^2+1} + \log\left(x+\frac{1}{x}\right)\right\} + x^{\left(1+\frac{1}{x}\right)}\left\{\frac{x+1-\log x}{x^2}\right\}$$

18. (vi) $e^{\sin x} + (\tan x)^x$

Solution:

Let $y = e^{\sin x} + (\tan x)^x$

$\Rightarrow y = a + b$

Where $a = e^{\sin x}$; $b = (\tan x)^x$

Now we have to differentiate y = a + b with respect to x

By using chain rule, we can write as

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

a= $e^{\sin x}$

Taking log both the sides, we get

⇒ Log a= log $e^{\sin x}$

⇒ Log a= sin x log e {Log x^a = a log x}

⇒ Log a= sin x {log e =1}

Differentiating with respect to x

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d(\sin x)}{dx}$$

Again we have $\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}$; $\frac{d(\sin x)}{dx} = \cos x$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \cos x$$

$$\Rightarrow \frac{da}{dx} = a\,(\cos x)$$

Put the value of a = $e^{\sin x}$

$$\Rightarrow \frac{da}{dx} = e^{\sin x}\cos x$$

b = $(\tan x)^x$

Taking log both the sides:

⇒ Log b= log $(\tan x)^x$

⇒ Log b= x log (tan x) {Log x^a = a log x}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{d(x \log(\tan x))}{dx}$$

Again by using product rule,

$$\Rightarrow \frac{d(\log b)}{dx} = x \times \frac{d(\log(\tan x))}{dx} + \log(\tan x) \times \frac{dx}{dx}$$

We know that $\frac{d(\tan x)}{dx} = \sec^2 x$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = x \times \frac{1}{\tan x}\frac{d(\tan x)}{dx} + \log(\tan x)$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \frac{x}{\tan x}(\sec^2 x) + \log(\tan x)$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \frac{x\cos x}{\sin x}\left(\frac{1}{\cos^2 x}\right) + \log(\tan x)$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \frac{x}{\sin x}\left(\frac{1}{\cos x}\right) + \log(\tan x)$$

$$\Rightarrow \frac{db}{dx} = b\left\{\frac{x}{\sin x\cos x} + \log(\tan x)\right\}$$

Put the value of b = $(\tan x)^x$

$$\Rightarrow \frac{db}{dx} = (\tan x)^x\left\{\frac{x}{\sin x\cos x} + \log(\tan x)\right\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\Rightarrow \frac{dy}{dx} = e^{\sin x}\cos x + (\tan x)^x\left\{\frac{x}{\sin x\cos x} + \log(\tan x)\right\}$$

18. (vii) $(\cos x)^x + (\sin x)^{1/x}$

Solution:

Let $y = (\cos x)^x + (\sin x)^{\frac{1}{x}}$

$\Rightarrow y = a + b$

where $a = (\cos x)^x; b = (\sin x)^{\frac{1}{x}}$

Now we have to differentiate y = a + b with respect to x

By using chain rule, we can write as

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$a = (\cos x)^x$$

Taking log both the sides

$$\Rightarrow \log a = \log(\cos x)^x$$

$$\Rightarrow \log a = x\log(\cos x)$$ {Log x^a = a log x}

Differentiating with respect to x

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d(x\log(\cos x))}{dx}$$

Now by using product rule, we have

$$\Rightarrow \frac{d(\log a)}{dx} = x \times \frac{d(\log(\cos x))}{dx} + \log(\cos x) \times \frac{dx}{dx}$$

Again we have $\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = x \times \frac{1}{\cos x}\frac{d(\cos x)}{dx} + \log(\cos x)$$

We know that $\frac{d(\cos x)}{dx} = -\sin x$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{x}{\cos x}(-\sin x) + \log(\cos x)$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{-x\sin x}{\cos x} + \log(\cos x)$$

$$\Rightarrow \frac{da}{dx} = a\{-x\tan x + \log(\cos x)\}$$

Put the value of $a = (\cos x)^x$:

$$\Rightarrow \frac{da}{dx} = (\cos x)^x\{-x\tan x + \log(\cos x)\}$$

$b = (\sin x)^{\frac{1}{x}}$

Taking log both the sides

$$\Rightarrow \log b = \log(\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log b = \frac{1}{x}\log(\sin x)$$ {Log x^a = a log x}

Differentiating with respect to x

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{d\left(\frac{1}{x}\log(\sin x)\right)}{dx}$$

Again by product rule we have

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{1}{x} \times \frac{d(\log(\sin x))}{dx} + \log(\sin x) \times \frac{d(x^{-1})}{dx}$$

We know that $\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}$.

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \frac{1}{x} \times \frac{1}{\sin x}\frac{d(\sin x)}{dx} + \log(\sin x)\,(-x^{-2})$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \frac{1}{x\sin x}(\cos x) - \frac{\log(\sin x)}{x^2}$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \frac{\cos x}{x\sin x} - \frac{\log(\sin x)}{x^2}$$

$$\Rightarrow \frac{db}{dx} = b\left\{\frac{\cot x}{x} - \frac{\log(\sin x)}{x^2}\right\}$$

Put the value of $b = (\sin x)^{\frac{1}{x}}$:

$\Rightarrow \frac{db}{dx} = (\sin x)^{\frac{1}{x}} \left\{\frac{\cot x}{x} - \frac{\log(\sin x)}{x^2}\right\}$

$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$

$\Rightarrow \frac{dy}{dx} = (\cos x)^x\{-x\tan x + \log(\cos x)\} + (\sin x)^{\frac{1}{x}} \left\{\frac{\cot x}{x} - \frac{\log(\sin x)}{x^2}\right\}$

$18.(viii)\ x^{x^2-3} + (x-3)^{x^2}$

Solution:

Let $y = x^{x^2-3} + (x-3)^{x^2}$

$\Rightarrow y = a + b$

where $a = x^{x^2-3}; b = (x-3)^{x^2}$

Now we have to differentiate y = a + b with respect to x

By using chain rule, we can write as

$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$

$a = x^{x^2-3}$

Taking log both the sides

$\Rightarrow \log a = \log x^{x^2-3}$

$\Rightarrow \log a = (x^2-3)\log x$ {Log x^a = a log x}

Differentiating with respect to x

$\Rightarrow \frac{d(\log a)}{dx} = \frac{d((x^2-3)\log x)}{dx}$

Now by using product rule,

$\Rightarrow \frac{d(\log a)}{dx} = (x^2-3) \times \frac{d(\log x)}{dx} + \log x \times \frac{d(x^2-3)}{dx}$

Again by using chain rule we get

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = (x^2 - 3) \times \frac{1}{x}\frac{dx}{dx} + \log x \times (2x)$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{(x^2-3)}{x} + 2x\log x$$

$$\Rightarrow \frac{da}{dx} = a\left\{\frac{(x^2-3)}{x} + 2x\log x\right\}$$

Put the value of $a = x^{x^2-3}$:

$$\Rightarrow \frac{da}{dx} = x^{x^2-3}\left\{\frac{(x^2-3)}{x} + 2x\log x\right\}$$

$b = (x-3)^{x^2}$

Taking log both the sides:

$$\Rightarrow \log b = (x-3)^{x^2}$$

$$\Rightarrow \log b = x^2\log(x-3) \ \{\text{Log } x^a = a \log x\}$$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{d(x^2\log(x-3))}{dx}$$

Again by using product rule, we get

$$\Rightarrow \frac{d(\log b)}{dx} = x^2 \times \frac{d(\log\ (x-3))}{dx} + \log(x-3) \times \frac{d(x^2)}{dx}$$

For the above expression now we have to use chain rule,

$$\Rightarrow \frac{1}{b}\ \frac{db}{dx} = x^2 \times \frac{1}{(x-3)}\ \frac{d(x-3)}{dx} + \log(x-3) \times (2x)$$

$$\Rightarrow \frac{1}{b}\ \frac{db}{dx} = \frac{x^2}{(x-3)}\left(\frac{dx}{dx} - \frac{d(3)}{dx}\right) + 2x\log(x-3)$$

$$\Rightarrow \frac{1}{b}\ \frac{db}{dx} = \frac{x^2}{(x-3)}(1) + 2x\log\ (x-3)$$

$$\Rightarrow \frac{db}{dx} = b\left\{\frac{x^2}{(x-3)} + 2x\log(x-3)\right\}$$

Put the value of b = $(x-3)^{x^2}$:

$$\Rightarrow \frac{db}{dx} = (x-3)^{x^2}\left\{\frac{x^2}{(x-3)} + 2x\log(x-3)\right\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\Rightarrow \frac{dy}{dx} = x^{x^2-3}\left\{\frac{(x^2-3)}{x} + 2x\log x\right\} + (x-3)^{x^2}\left\{\frac{x^2}{(x-3)} + 2x\log(x-3)\right\}$$

19. $y = e^x + 10^x + x^x$

Solution:

Let $y = e^x + 10^x + x^x$

⇒ y = a + b + c

Where $a = e^x$; $b = 10^x$; $c = x^x$

Now we have to differentiate y = a + b + c with respect to x

By using chain rule, we can write as

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx} + \frac{dc}{dx}$$

$a = e^x$

Taking log both the sides

⇒ Log a= Log e^x

⇒ Log a= x log e

{Log x^a = a log x}

⇒ Log a= x {log e =1}

Differentiating with respect to x

$\Rightarrow \frac{d(\log a)}{dx} = \frac{dx}{dx}$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}$

$\Rightarrow \frac{1}{a}\frac{da}{dx} = 1$

$\Rightarrow \frac{da}{dx} = a$

Put the value of a = e^x

$\Rightarrow \frac{da}{dx} = e^x$

b = 10^x

Taking log both the sides:

⇒ Log b= log 10^x

⇒ Log b= x log 10

{Log x^a = a log x}

Differentiating with respect to x

$\Rightarrow \frac{d(\log b)}{dx} = \frac{d(x \log 10)}{dx}$

Now by using chain rule,

$\Rightarrow \frac{d(\log b)}{dx} = \log 10 \times \frac{dx}{dx}$

$\Rightarrow \frac{1}{b}\frac{db}{dx} = b(\log 10)$

$\Rightarrow \frac{db}{dx} = b(\log 10)$

Put the value of b = 10^x

$$\Rightarrow \frac{db}{dx} = 10^x(\log 10)$$

c = x^x

Taking log both the sides

⇒ Log c= log x^x

⇒ Log c= x log x

{Log x^a = a log x}

Differentiating with respect to x

$$\Rightarrow \frac{d(\log c)}{dx} = \frac{d(x \log x)}{dx}$$

By using product rule, we get

$$\Rightarrow \frac{d(\log c)}{dx} = x \times \frac{d(\log x)}{dx} + \log x \times \frac{dx}{dx}$$

$$\Rightarrow \frac{1}{c}\frac{dc}{dx} = x \times \frac{1}{x}\frac{dx}{dx} + \log x$$

$$\Rightarrow \frac{1}{c}\frac{dc}{dx} = 1 + \log x$$

$$\Rightarrow \frac{dc}{dx} = c\{1 + \log x\}$$

Put the value of c = x^x

$$\Rightarrow \frac{dc}{dx} = x^x\{1 + \log x\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx} + \frac{dc}{dx}$$

$$\Rightarrow \frac{dy}{dx} = e^x + 10^x(\log 10) + x^x\{1 + \log x\}$$

20. y = $x^n + n^x + x^x + n^n$

Solution:

Let $y = x^n + n^x + x^x + n^n$

$\Rightarrow y = a + b + c + m$

Where $a= x^n$; $b = n^x$; $c = x^x$; $m= n^n$

Now we have to differentiate $y = a + b + c + m$ with respect to x

By using chain rule, we can write as

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx} + \frac{dc}{dx} + \frac{dm}{dx}$$

$a= x^n$

Taking log both the sides

$\Rightarrow$ Log a= log x^n

$\Rightarrow$ Log a= n log x

{Log x^a = a log x}

$\Rightarrow$ Log a= n log x {log e =1}

Differentiating with respect to x

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d(n \log x)}{dx}$$

Again by chain rule, we can write as

$$\Rightarrow \frac{d(\log a)}{dx} = n\frac{d(\log x)}{dx}$$

We know that $\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = n \times \frac{1}{x}\frac{dx}{dx}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{n}{x}$$

$$\Rightarrow \frac{da}{dx} = \frac{an}{x}$$

Put the value of a = x^n

$$\frac{da}{dx} = \frac{nx^n}{x}$$

$$\frac{da}{dx} = nx^{n-1}$$

b = n^x

Taking log both the sides

⇒ Log b= log n^x

⇒ Log b= x log n {Log x^a = a log x}

Differentiating with respect to x using chain rule, we get

$$\Rightarrow \frac{d(\log b)}{dx} = \log n \times \frac{dx}{dx}$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = b(\log n)$$

$$\Rightarrow \frac{db}{dx} = b(\log n)$$

Put the value of b = n^x

$$\Rightarrow \frac{db}{dx} = n^x(\log n)$$

c = x^x

Taking log both the sides

⇒ Log c= log x^x

⇒ Log c= x log x

{Log x^a = a log x}

Differentiating with respect to x

$$\Rightarrow \frac{d(\log c)}{dx} = \frac{d(x \log x)}{dx}$$

Now by using product rule, we get

$$\Rightarrow \frac{d(\log c)}{dx} = x \times \frac{d(\log x)}{dx} + \log x \times \frac{dx}{dx}$$

$$\Rightarrow \frac{1}{c}\frac{dc}{dx} = x \times \frac{1}{x}\frac{dx}{dx} + \log x$$

$$\Rightarrow \frac{1}{c}\frac{dc}{dx} = 1 + \log x$$

$$\Rightarrow \frac{dc}{dx} = c\{1 + \log x\}$$

Put the value of $c = x^x$

$$\Rightarrow \frac{dc}{dx} = x^x\{1 + \log x\}$$

$m = n^n$

$$\Rightarrow \frac{dm}{dx} = \frac{d(n^n)}{dx}$$

$$\Rightarrow \frac{dm}{dx} = 0$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx} + \frac{dc}{dx} + \frac{dm}{dx}$$

$$\Rightarrow \frac{dy}{dx} = nx^{n-1} + n^x(\log n) + x^x\{1 + \log x\} + 0$$

$$\Rightarrow \frac{dy}{dx} = nx^{n-1} + n^x(\log n) + x^x\{1 + \log x\}$$

EXERCISE 11.6 PAGE NO: 11.98

1. $If\ y = \sqrt{x + \sqrt{x + \sqrt{x + ... to\ \infty}}},\ prove\ that\ \frac{dy}{dx} = \frac{1}{2y-1}.$

Solution:

Given,

$$y = \sqrt{x + \sqrt{x + \sqrt{x + \cdots \text{to } \infty}}}$$

$$y = \sqrt{x + y}$$

Where $y = \sqrt{x + \sqrt{x + \cdots \text{to } \infty}}$

On squaring both sides,

$y^2 = x + y$

Differentiating both sides with respect to x,

$$2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx}(2y - 1) = 1$$

$$\frac{dy}{dx} = \frac{1}{2y - 1}$$

Hence proved.

2. $If\ y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + ... to\ \infty}}},\ prove\ that\ \frac{dy}{dx} = \frac{\sin x}{1-2y}.$

Solution:

Given,

$$y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \cdots \text{to } \infty}}}$$

$y = \sqrt{\cos x + y}$

Where y = $\sqrt{\cos x + \sqrt{\cos x + \cdots \text{to } \infty}}$

Squaring on both sides,

$y^2 = \cos x + y$

Differentiating both sides with respect to x,

$$2y\frac{dy}{dx} = -\sin x + \frac{dy}{dx}$$

$$\frac{dy}{dx}(2y-1) = -\sin x$$

$$\frac{dy}{dx} = -\frac{\sin x}{2y-1}$$

$$\frac{dy}{dx} = \frac{\sin x}{1-2y}$$

Hence proved.

3. $If\ y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots.. \ to\ \infty}}},\ prove\ that\ (2y-1)\frac{dy}{dx} = \frac{1}{x}.$

Solution:

Given

$$y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \cdots \text{to } \infty}}}$$

$y = \sqrt{\log x + y}$

Where y = $\sqrt{\log x + \sqrt{\log x + \cdots \text{to } \infty}}$

Squaring on both sides,

$y^2 = \text{Log } x + y$

Differentiating both sides with respect to x,

$$2y\frac{dy}{dx} = \frac{1}{x} + \frac{dy}{dx}$$

$$\frac{dy}{dx}(2y - 1) = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{x(2y - 1)}$$

Hence proved.

4. $If\ y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots\ldots\ to\ \infty}}},\ prove\ that\ \frac{dy}{dx} = \frac{sec^2 x}{2y - 1}.$

Solution:

Given,

$$y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \cdots \text{to } \infty}}}$$

$$y = \sqrt{\tan x + y}$$

On squaring both sides,

$y^2 = \tan x + y$

Differentiating both sides with respect to x,

$$2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\frac{dy}{dx}(2y - 1) = \sec^2 x$$

$$\frac{dy}{dx} = \frac{\sec^2 x}{(2y - 1)}$$

Hence proved.

EXERCISE 11.7 PAGE NO: 11.103

Find dy/dx, when
1. $x = at^2$ and y = 2 at

Solution:

Given that $x = at^2$, $y = 2at$

Now by differentiating $x = at^2$ with respect to t we get

$$\frac{dx}{dt} = \frac{d(at^2)}{dt} = 2at$$

Again by differentiating y = 2at with respect to t we get

$$\frac{dy}{dt} = \frac{d(2at)}{dt} = 2a$$

Therefore,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

2. x = a (θ + sin θ) and y = a (1 – cos θ)

Solution:

Given that $x = at^2$, $y = 2at$

Now by differentiating $x = at^2$ with respect to t we get

$$\frac{dx}{dt} = \frac{d(at^2)}{dt} = 2at$$

Again by differentiating y = 2at with respect to t we get

$$\frac{dy}{dt} = \frac{d(2at)}{dt} = 2a$$

Therefore,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

x = a (θ + sin θ)

Differentiating it with respect to θ,

$\frac{dx}{d\theta} = a(1 + \cos\theta)$ ...... (1)

And,

y = a (1- cos θ)

Differentiating it with respect to θ,

$$\frac{dy}{d\theta} = a(0 + \sin\theta)$$

$\frac{dy}{d\theta} = a\sin\theta$ ...... (2)

Using equation (1) and (2),

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{a\sin\theta}{a(1 - \cos\theta)}$$

$$= \frac{\frac{2\sin\theta}{2}\frac{(\cos\theta)}{2}}{\frac{2\sin^2\theta}{2}},$$

{Since, $1 - \cos\theta = \frac{2\sin^2\theta}{2}$ }

$$= \frac{dy}{dx} = \frac{\tan\theta}{2}$$

3. x = a cos θ and y = b sin θ

Solution:

Given x = a cos θ and y = b sin θ

Now by differentiating x with respect to θ we get,

$$\frac{dx}{d\theta} = \frac{d(a\cos\theta)}{d\theta} = -a\sin\theta$$

Again by differentiating y with respect to θ we get,

$$\frac{dy}{d\theta} = \frac{d(b\sin\theta)}{d\theta} = b\cos\theta$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$$

4. x = a e^θ (sin θ – cos θ), y = a e^θ (sin θ + cos θ)

Solution:

Given that x = a e^θ (sin θ – cos θ)

Differentiating it with respect to θ

$$\frac{dx}{d\theta} = a[e^\theta \frac{d(\sin\theta - \cos\theta)}{d\theta} + (\sin\theta - \cos\theta)\frac{d(e^\theta)}{d\theta}]$$

= a [e^θ (cos θ + sin θ) + (sin θ-cos θ) e^θ]

$\frac{dx}{d\theta} = a[2e^\theta \sin\theta]$ ...... (1)

And also given that, y = a e^θ (sin θ + cos θ)

Differentiating it with respect to θ,

$$\frac{dy}{d\theta} = a[e^\theta \frac{d(\sin\theta + \cos\theta)}{d\theta} + (\sin\theta + \cos\theta)\frac{d(e^\theta)}{d\theta}]$$

= a [e^θ (cos θ - sin θ) + (sin θ + cos θ) e^θ]

$\frac{dy}{d\theta} = a[2e^\theta \cos\theta]$ ...... (2)

Dividing equation (2) by equation (1),

$$\frac{dy}{dx} = \frac{a(2e^{\theta}\cos\theta)}{a(2e^{\theta}\sin\theta)}$$

$$\frac{dy}{dx} = \cot\theta$$

5. x = b sin² θ and y = a cos² θ

Solution:

Given that $x = b\sin^2\theta$

Now by differentiating above equation with respect to θ, we get

$$\frac{dx}{d\theta} = \frac{d(b\sin^2\theta)}{d\theta} = 2b\sin\theta\cos\theta$$

And also given that $y = a\cos^2\theta$

Now by differentiating above equation with respect to θ, we get

$$\frac{dy}{d\theta} = d(a\cos^2\theta) = -2a\cos\theta\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{2a\cos\theta\sin\theta}{2b\sin\theta\cos\theta} = -\frac{a}{b}$$

6. x = a (1 – cos θ) and y = a (θ + sin θ) at θ = π/2

Solution:

Given $x = a(1 - \cos\theta)$

Differentiate x with respect to θ, we get

$$\frac{dx}{d\theta} = \frac{d[a(1-\cos\theta)]}{d\theta} = a(\sin\theta)$$

And also given that $y = a(\theta + \sin\theta)$

Differentiate x with respect to θ, we get

$$\frac{dy}{d\theta} = \frac{d(\theta + \sin\theta)}{d\theta} = a(1 + \cos\theta)$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a(1 + \cos\theta)}{a(\sin\theta)}\bigg|\left(\theta = \frac{\pi}{2}\right)$$

$$= \frac{a(1 + 0)}{a} = 1$$

7. $x = \dfrac{e^t + e^{-t}}{2}$ and $y = \dfrac{e^t - e^{-t}}{2}$

Solution:

Given $x = \dfrac{e^t + e^{-t}}{2}$

Differentiating above equation with respect to t

$$\frac{dx}{dt} = \frac{1}{2}\left[\frac{d(e^t)}{dt} + \frac{d(e^{-t})}{dt}\right]$$

$$= \frac{1}{2}\left[e^t + e^{-t}\frac{d(-t)}{dt}\right]$$

$\frac{dx}{dt} = \frac{1}{2}(e^t - e^{-t}) = y$ ...... (1)

And also given that $y = \dfrac{e^t - e^{-t}}{2}$

Differentiating above equation with respect to t,

$$\frac{dy}{dt} = \frac{1}{2}\left[\frac{d(e^t)}{dt} - \frac{d(e^{-t})}{dt}\right]$$

$$= \frac{1}{2}\left[e^{t-} - e^{-t}\frac{d(-t)}{dt}\right]$$

$$= \frac{1}{2}\left(e^{t} - e^{-t}(-1)\right)$$

$\frac{dy}{dt} = \frac{e^{\theta}+e^{\theta}}{2} = x$ ...... (2)

Dividing equation (2) by (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{x}{y}$$

$$\frac{dy}{dx} = \frac{x}{y}$$

$$8.\ x = \frac{3at}{1+t^2}\ and\ y = \frac{3at^2}{1+t^2}$$

Solution:

Given $x = \frac{3at}{1+t^2}$

Differentiating above equation with respect to t using quotient rule,

$$\frac{dx}{dt} = \left[\frac{\left((1+t^2)\frac{d(3at)}{dt} - 3at\frac{d(1+t^2)}{dt}\right)}{(1+t^2)^2}\right]$$

$$= \left[\frac{(1+t^2)(3a) - 3at(2t)}{(1+t^2)^2}\right]$$

$$= \left[\frac{(3a) + 3at^2 - 6at^2}{(1+t^2)^2}\right]$$

$$= \left[\frac{3a - 3at^2}{(1+t^2)^2}\right]$$

$\frac{dx}{dt} = \frac{3a(1-t^2)}{(1+t^2)^2}$ ...... (1)

And also given that $y = \frac{3at^2}{1+t^2}$

Differentiating above equation with respect to t using quotient rule

$$\frac{dy}{dx} = \left[\frac{(1+t^2)\frac{d(3at^2)}{dt} - 3at^2\frac{d(1+t^2)}{dt}}{(1+t^2)^2}\right]$$

$$\frac{dy}{dt} = \left[\frac{(1+t^2)(6at) - (3at^2)(2t)}{(1+t^2)^2}\right]$$

$$= \left[\frac{6at + 6at^3 - 6at^3}{(1+t^2)^2}\right]$$

$\frac{dy}{dt} = \frac{6at}{(1+t^2)^2}$ ..... (2)

Dividing equation (2) by (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6at}{(1+t^2)^2} \times \frac{3a(1-t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dx} = \frac{2t}{1-t^2}$$

9. x = a (cos θ + θ sin θ) and y = a (sin θ – θ cos θ)

Solution:

Given x = a (cos θ + θ sin θ)

Now differentiating x with respect to θ

$\frac{dx}{d\theta} = a\left[\frac{d}{d\theta}\cos\theta + \frac{d}{d\theta}(\theta\sin\theta)\right]$

$$= a\left[-\sin\theta + \frac{\theta d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(\theta)\right]$$

$$= a[-\sin\theta + \theta\cos\theta + \sin\theta] = a\theta\cos\theta$$

And also given y = a (sin θ – cos θ),

Now differentiating x with respect to θ

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\theta\cos\theta)\right]$$

$$= a\left[\cos\theta - \left\{\frac{\theta d}{d\theta}(\cos\theta) + \cos\theta\frac{d}{d\theta}(\theta)\right\}\right]$$

$$= a[\cos\theta + \theta\sin\theta - \cos\theta]$$

= a θ sin θ

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

10. $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right)$ *and* $y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$

Solution:

Given $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right)$

Differentiating x with respect to θ using the product rule,

$$\frac{dx}{d\theta} = e^{\theta}\frac{d}{d\theta}\left(\theta + \frac{1}{\theta}\right) + \left(\theta + \frac{1}{\theta}\right)\frac{d}{d\theta}(e^{\theta})$$

$$= e^{\theta}\left(1 - \frac{1}{\theta^2}\right) + \frac{\theta^2 + 1}{\theta}(e^{\theta})$$

$$= e^{\theta}\left(1 - \frac{1}{\theta^2} + \frac{\theta^2 + 1}{\theta}\right)$$

$$= e^{\theta}\left(\frac{\theta^2 - 1 + \theta^3 + \theta}{\theta^2}\right)$$

$\frac{dx}{d\theta} = e^{\theta}\left(\frac{\theta^3+\theta^2+\theta-1}{\theta^2}\right)$ ...... (1)

And also given that, $y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$

Differentiating y with respect to θ using the product rule,

$$\frac{dy}{d\theta} = e^{-\theta}\frac{d}{d\theta}\left(\theta - \frac{1}{\theta}\right) + \left(\theta - \frac{1}{\theta}\right)\frac{d}{d\theta}(e^{-\theta})$$

$$= e^{-\theta}\left(1 + \frac{1}{\theta^2}\right) + \left(\theta - \frac{1}{\theta}\right)e^{-\theta}\frac{d}{d\theta}(-\theta)$$

$$= e^{-\theta}\left(1 + \frac{1}{\theta^2}\right) + \left(\theta - \frac{1}{\theta}\right)e^{-\theta}(-1)$$

$$\frac{dy}{d\theta} = e^{-\theta}\left(1 + \frac{1}{\theta^2} - \theta + \frac{1}{\theta}\right)$$

$$= e^{-\theta}\left(\frac{\theta^2 + 1 - \theta^3 + \theta}{\theta^2}\right)$$

$\frac{dy}{d\theta} = e^{-\theta}\left(\frac{-\theta^3+\theta^2+\theta+1}{\theta^2}\right)$ ...... (2)

Divide equation (2) by (1)

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = e^{-\theta}\left(\frac{-\theta^3 + \theta^2 + \theta + 1}{\theta^2}\right) \times \frac{1}{e^{\theta}\left(\frac{\theta^3 + \theta^2 + \theta - 1}{\theta^2}\right)}$$

$$= e^{-2\theta}\left(\frac{-\theta^3 + \theta^2 + \theta + 1}{\theta^3 + \theta^2 + \theta - 1}\right)$$

$$11.\ x = \frac{2t}{1+t^2}\ and\ y = \frac{1-t^2}{1+t^2}$$

Solution:

Given, $x = \frac{2t}{1+t^2}$

Differentiating x with respect to t using quotient rule,

$$\frac{dx}{dt} = \left[\frac{(1+t^2)\frac{d}{dt}(2t) - 2t\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right]$$

$$= \left[\frac{(1+t^2)(2) - 2t(2t)}{(1+t^2)^2}\right]$$

$$= \left[\frac{2+2t^2-4t^2}{(1+t^2)^2}\right]$$

$$= \left[\frac{2-2t^2}{(1+t^2)^2}\right]$$

$\frac{dx}{dt} = \left[\frac{2-2t^2}{(1+t^2)^2}\right]$ ...... (1)

And also given that, $y = \frac{1-t^2}{1+t^2}$

Differentiating y with respect to t using quotient rule,

$$\frac{dy}{dt} = \left[\frac{(1+t^2)\frac{d}{dt}(1-t^2) - (1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right]$$

$$= \left[\frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2}\right]$$

$$= \left[\frac{-2t-2t^3-2t+2t^3}{(1+t^2)^2}\right]$$

$\frac{dy}{dt} = \left[\frac{-4t}{(1+t^2)^2}\right]$ ...... (2)

Dividing equation (2) by (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \left[\frac{-4t}{(1+t^2)^2}\right] \times \frac{1}{\left[\frac{2-2t^2}{(1+t^2)^2}\right]}$$

$$= -\frac{2t}{1-t^2}$$

$$\frac{dy}{dx} = -\frac{x}{y} \text{ [since, } \frac{x}{y} = \frac{2t}{1+t^2} \times \frac{1+t^2}{1-t^2} = \frac{2t}{1-t^2} \text{]}$$

12. $x = cos^{-1}\frac{1}{\sqrt{1+t^2}}$ *and* $y = sin^{-1}\frac{1}{\sqrt{1+t^2}},\ t\epsilon R$

Solution:

Given $x = \cos^{-1}\frac{1}{\sqrt{1+t^2}}$

Differentiating x with respect to t using chain rule,

$$\frac{dx}{dt} = -\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1+t^2}}\right)^2}}\frac{d}{dt}\left(\frac{1}{\sqrt{1+t^2}}\right)$$

$$= -\frac{1}{\sqrt{1-\frac{1}{1+t^2}}}\left\{-\frac{1}{2(1+t^2)^{\frac{3}{2}}}\right\}\frac{d}{dt}(1+t^2)$$

$$= -\frac{(1+t^2)^{\frac{1}{2}}}{\sqrt{(1+t^2-1)}}\left\{-\frac{1}{2(1+t^2)^{\frac{3}{2}}}\right\}(2t)$$

$$= -\frac{t}{\sqrt{t^2} \times (1+t^2)}$$

$\frac{dx}{dt} = -\frac{1}{1+t^2}$ ...... (1)

Also given that, $y = \sin^{-1}\frac{1}{\sqrt{1+t^2}}$

Differentiating y with respect to t using chain rule,

$$\frac{dy}{dt} = \frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1+t^2}}\right)^2}}\frac{d}{dt}\left(\frac{1}{\sqrt{1+t^2}}\right)$$

$$= \frac{1}{\sqrt{1-\frac{1}{1+t^2}}}\left\{-\frac{1}{2(1+t^2)^{\frac{3}{2}}}\right\}\frac{d}{dt}(1+t^2)$$

$$= \frac{(1+t^2)^{\frac{1}{2}}}{\sqrt{(1+t^2-1)}}\left\{-\frac{1}{2(1+t^2)^{\frac{3}{2}}}\right\}(2t)$$

$$= \frac{t}{\sqrt{t^2}\times(1+t^2)}$$

$\frac{dy}{dt} = -\frac{1}{1+t^2}$ ...... (2)

Dividing equation (2) by (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{1}{1+t^2}\times -\frac{1+t^2}{1}$$

$$\frac{dy}{dx} = 1$$

13. $x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$

Solution:

Given $x = \frac{1-t^2}{1+t^2}$

Differentiating x with respect to t using quotient rule,

$$\frac{dx}{dt} = \left[\frac{(1+t^2)\frac{d}{dt}(1-t^2)-(1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right]$$

$$= \left[\frac{(1+t^2)(-2t)-(1-t^2)(2t)}{(1+t^2)^2}\right]$$

$$= \left[\frac{-2t - 2t^3 - 2t + 2t^3}{(1+t^2)^2}\right]$$

$\frac{dx}{dt} = \left[\frac{-4t}{(1+t^2)^2}\right]$ ...... (1)

And also given that, $y = \frac{2t}{1+t^2}$

Differentiating y with respect to t using quotient rule,

$$\frac{dy}{dt} = \left[\frac{(1+t^2)\frac{d}{dt}(2t) - (2t)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right]$$

$$= \left[\frac{(1+t^2)(2) - (2t)(2t)}{(1+t^2)^2}\right]$$

$$= \left[\frac{2 + 2t^2 - 4t^2}{(1+t^2)^2}\right]$$

$\frac{dy}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}$ ...... (2)

Divide equation (2) by (1) so,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{1}{\frac{-4t}{(1+t^2)^2}}$$

$$\frac{dy}{dx} = \frac{2(1-t^2)}{-4t}$$

14. $If\ x = 2\cos\theta - \cos 2\theta\ and\ y = 2\sin\theta - sin2\theta,\ prove\ that\ \frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right).$

Solution:
Given x = 2cos θ – cos 2θ

Differentiating x with respect to θ using chain rule,

$$\frac{dx}{d\theta} = 2(-\sin\theta) - (-\sin 2\theta)\frac{d}{d\theta}(2\theta)$$

$$= -2\sin\theta + 2\sin 2\theta$$

$\frac{dx}{d\theta} = 2(\sin 2\theta - \sin\theta)$ ...... (1)

And also given that, y = 2sin θ – sin 2θ

Differentiating y with respect to θ using chain rule,

$$\frac{dy}{d\theta} = 2\cos\theta - \cos 2\theta\frac{d}{d\theta}(2\theta)$$

$$= 2\cos\theta - \cos 2\theta(2)$$

$$= 2\cos\theta - 2\cos 2\theta$$

$\frac{dy}{d\theta} = 2(\cos\theta - \cos 2\theta)$ ...... (2)

Dividing equation (2) by equation (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2(\cos\theta - \cos 2\theta)}{2(\sin 2\theta - \sin\theta)}$$

$$= \frac{(\cos\theta - \cos 2\theta)}{(\sin 2\theta - \sin\theta)}$$

$$\frac{dy}{dx} = \frac{-2\sin\left(\frac{\theta + 2\theta}{2}\right)\sin\left(\frac{\theta - 2\theta}{2}\right)}{2\cos\left(\frac{\theta + 2\theta}{2}\right)\sin\left(\frac{2\theta - \theta}{2}\right)}$$

$$\left[\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)\right]$$

$$= -\frac{\sin\left(\frac{3\theta}{2}\right)\left(\sin\left(-\frac{\theta}{2}\right)\right)}{\cos\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}$$

$$= -\frac{\sin\left(\frac{3\theta}{2}\right)\left(-\sin\frac{\theta}{2}\right)}{\cos\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}$$

$$= \frac{\sin\left(\frac{3\theta}{2}\right)}{\cos\left(\frac{3\theta}{2}\right)}$$

$$\frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right)$$

EXERCISE 11.8 PAGE NO: 11.112

1. Differentiate x^2 with respect to x^3.

Solution:

Let $u = x^2$ and $v = x^3$.

We have to differentiate u with respect to v that is find $\frac{du}{dv}$.

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}(x^2)$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{du}{dx} = 2x^{2-1}$$

$$\therefore \frac{du}{dx} = 2x$$

Now, on differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(x^3)$$

$$\Rightarrow \frac{dv}{dx} = 3x^{3-1}$$

$$\therefore \frac{dv}{dx} = 3x^2$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{2x}{3x^2}$$

$$\therefore \frac{du}{dv} = \frac{2}{3x}$$

Thus, $\frac{du}{dv} = \frac{2}{3x}$

2. Differentiate log (1 +x^2) with respect to $\tan^{-1}$ x.

Solution:

Let u = log (1 + x^2) and v = $\tan^{-1}x$.

We have to differentiate u with respect to v that is find $\frac{du}{dv}$.

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}[\log(1 + x^2)]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$

$$\Rightarrow \frac{du}{dx} = \frac{1}{1+x^2}\frac{d}{dx}(1 + x^2)$$

Now by using chain rule, we get

$$\Rightarrow \frac{du}{dx} = \frac{1}{1 + x^2}\left[\frac{d}{dx}(1) + \frac{d}{dx}(x^2)\right]$$

However, $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = \frac{1}{1 + x^2}[0 + 2x^{2-1}]$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{1 + x^2}[2x]$$

$$\therefore \frac{du}{dx} = \frac{2x}{1 + x^2}$$

Now, on differentiating v with respect to x, we get

$$\therefore \frac{dv}{dx} = \frac{1}{1 + x^2}$$

We have $\frac{du}{dv}=\frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv}=\frac{\frac{2x}{1+x^2}}{\frac{1}{1+x^2}}$$

$$\Rightarrow \frac{du}{dv}=\frac{2x}{1+x^2}\times(1+x^2)$$

$$\therefore \frac{du}{dv}=2x$$

Thus, $\frac{du}{dv}=2x$

3. Differentiate $(\log x)^x$ with respect to log x.

Solution:

Let u = $(\log x)^x$ and v = log x.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have u = $(\log x)^x$

Taking log on both sides, we get

Log u = log $(\log x)^x$

⇒ Log u = x × log (log x) [∵ log a^m = m × log a]

On differentiating both the sides with respect to x, we get

$$\frac{d}{du}(\log u)\times\frac{du}{dx}=\frac{d}{dx}[x\times\log(\log x)]$$

We know that (u v)' = vu' + uv'

$$\Rightarrow \frac{d}{du}(\log u)\times\frac{du}{dx}=\log(\log x)\frac{d}{dx}(x)+x\frac{d}{dx}[\log(\log x)]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$ and $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{1}{u} \times \frac{du}{dx} = \log(\log x) \times 1 + x\left[\frac{1}{\log x}\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \log(\log x) + \frac{x}{\log x}\frac{d}{dx}(\log x)$$

But, u = (log x)^x and $\frac{d}{dx}(\log x) = \frac{1}{x}$

$$\Rightarrow \frac{1}{(\log x)^x}\frac{du}{dx} = \log(\log x) + \frac{x}{\log x} \times \frac{1}{x}$$

$$\Rightarrow \frac{1}{(\log x)^x}\frac{du}{dx} = \log(\log x) + \frac{1}{\log x}$$

$$\therefore \frac{du}{dx} = (\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right]$$

Now, on differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(\log x)$$

$$\therefore \frac{dv}{dx} = \frac{1}{x}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{(\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right]}{\frac{1}{x}}$$

$$\Rightarrow \frac{du}{dv} = x(\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right]$$

$$\Rightarrow \frac{du}{dv} = x(\log x)^x\left[\frac{\log(\log x)\log x + 1}{\log x}\right]$$

$\Rightarrow \frac{du}{dv} = \frac{x(\log x)^x}{\log x}[\log(\log x)\log x + 1]$

$\therefore \frac{du}{dv} = x(\log x)^{x-1}[1 + \log x \log(\log x)]$

Thus, $\frac{du}{dv} = x(\log x)^{x-1}[1 + \log x \log(\log x)]$

4. Differentiate $\sin^{-1}\sqrt{(1-x^2)}$ with respect to $\cos^{-1}x$, if
(i) $x \in (0, 1)$
(ii) $x \in (-1, 0)$

Solution:
(i) Given $\sin^{-1}\sqrt{(1-x^2)}$

Let $u = \sin^{-1}\sqrt{1-x^2}$ and $v = \cos^{-1}x$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\sqrt{1-x^2}$

By substituting $x = \cos\theta$, we have

$u = \sin^{-1}\sqrt{1-(\cos\theta)^2}$

$\Rightarrow u = \sin^{-1}\sqrt{1-\cos^2\theta}$

$\Rightarrow u = \sin^{-1}\sqrt{\sin^2\theta}$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow u = \sin^{-1}(\sin\theta)$

Given $x \in (0, 1)$

However, $x = \cos\theta$.

$\Rightarrow \cos\theta \in (0, 1)$

$\Rightarrow \theta \in \left(0, \frac{\pi}{2}\right)$

Hence, $u = \sin^{-1}(\sin\theta) = \theta$.

$\Rightarrow u = \cos^{-1}x$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}(\cos^{-1}x)$$

We know $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$

$$\therefore \frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Now, on differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(\cos^{-1}x)$$

$$\therefore \frac{dv}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

We have, $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{-\frac{1}{\sqrt{1-x^2}}}{-\frac{1}{\sqrt{1-x^2}}}$$

$$\Rightarrow \frac{du}{dv} = -\frac{1}{\sqrt{1-x^2}} \times \left(-\sqrt{1-x^2}\right)$$

$$\therefore \frac{du}{dv} = 1$$

Thus, $\frac{du}{dv} = 1$

(ii) Given $\sin^{-1}\sqrt{(1-x^2)}$

Let $u = \sin^{-1}\sqrt{1-x^2}$ and $v = \cos^{-1}x$.

Now we have to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\sqrt{1 - x^2}$

By substituting x = cos θ, we get

$u = \sin^{-1}\sqrt{1 - (\cos\theta)^2}$

$\Rightarrow u = \sin^{-1}\sqrt{1 - \cos^2\theta}$

$\Rightarrow u = \sin^{-1}\sqrt{\sin^2\theta}$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow u = \sin^{-1}(\sin\theta)$

Given $x \in (-1, 0)$

However, x = cos θ.

$\Rightarrow \cos\theta \in (-1, 0)$

$\Rightarrow \theta \in \left(\frac{\pi}{2}, \pi\right)$

Hence, $u = \sin^{-1}(\sin\theta) = \pi - \theta$.

$\Rightarrow u = \pi - \cos^{-1}x$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}(\pi - \cos^{-1}x)$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(\pi) - \frac{d}{dx}(\cos^{-1}x)$$

We know $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 0 - \left(-\frac{1}{\sqrt{1 - x^2}}\right)$$

$$\therefore \frac{du}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

Now, on differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(\cos^{-1} x)$$

$$\therefore \frac{dv}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{1}{\sqrt{1-x^2}}}{-\frac{1}{\sqrt{1-x^2}}}$$

$$\Rightarrow \frac{du}{dv} = \frac{1}{\sqrt{1-x^2}} \times \left(-\sqrt{1-x^2}\right)$$

$$\therefore \frac{du}{dv} = -1$$

Thus, $\frac{du}{dv} = -1$

5. *Differentiate* $\sin^{-1}\left(4x\sqrt{1-4x^2}\right)$ *with respect to* $\sqrt{1-4x^2}$ *if,*

(i) $x \epsilon \left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$

(ii) $x \epsilon \left(\frac{1}{2\sqrt{2}}, \frac{1}{2}\right)$

(iii) $x \epsilon \left(-\frac{1}{2}, -\frac{1}{2\sqrt{2}}\right)$

Solution:

(i) Let

$u = \sin^{-1}\left(4x\sqrt{1-4x^2}\right)$ And $v = \sqrt{1-4x^2}$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\left(4x\sqrt{1-4x^2}\right)$

$\Rightarrow u = \sin^{-1}\left(4x\sqrt{1-(2x)^2}\right)$

By substituting 2x = cos θ, we have

$u = \sin^{-1}\left(2\cos\theta\sqrt{1-(\cos\theta)^2}\right)$

$\Rightarrow u = \sin^{-1}\left(2\cos\theta\sqrt{1-(\cos\theta)^2}\right)$

$\Rightarrow u = \sin^{-1}\left(2\cos\theta\sqrt{\sin^2\theta}\right)$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow u = \sin^{-1}(2\cos\theta\sin\theta)$

$\Rightarrow u = \sin^{-1}(\sin 2\theta)$

Given $x \in \left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$

However, 2x = cos θ ⇒ $x = \frac{\cos\theta}{2}$

$\Rightarrow \frac{\cos\theta}{2} \in \left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$

$\Rightarrow \cos\theta \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\Rightarrow \theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

$\Rightarrow 2\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Hence, $u = \sin^{-1}(\sin 2\theta) = \pi - 2\theta$.

$\Rightarrow u = \pi - 2\cos^{-1}(2x)$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}[\pi - 2\cos^{-1}(2x)]$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(\pi) - \frac{d}{dx}[2\cos^{-1}(2x)]$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(\pi) - 2\frac{d}{dx}[\cos^{-1}(2x)]$$

We know $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 0 - 2\left[-\frac{1}{\sqrt{1-(2x)^2}}\frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{2}{\sqrt{1-4x^2}}\left[\frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{2}{\sqrt{1-4x^2}}\left[2\frac{d}{dx}(x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}}\frac{d}{dx}(x)$$

However, $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}} \times 1$$

$$\therefore \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}}$$

Now, we have $v = \sqrt{1-4x^2}$

On differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}\left(\sqrt{1-4x^2}\right)$$

$$\Rightarrow \frac{dv}{dx} = \frac{d}{dx}(1-4x^2)^{\frac{1}{2}}$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2}(1-4x^2)^{\frac{1}{2}-1}\frac{d}{dx}(1-4x^2)$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2}(1-4x^2)^{-\frac{1}{2}}\left[\frac{d}{dx}(1) - \frac{d}{dx}(4x^2)\right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1-4x^2}}\left[\frac{d}{dx}(1) - 4\frac{d}{dx}(x^2)\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1-4x^2}}[0 - 4(2x^{2-1})]$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1-4x^2}}[-8x]$$

$$\therefore \frac{dv}{dx} = -\frac{4x}{\sqrt{1-4x^2}}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{4}{\sqrt{1-4x^2}}}{-\frac{4x}{\sqrt{1-4x^2}}}$$

$$\Rightarrow \frac{du}{dv} = \frac{4}{\sqrt{1-4x^2}} \times \left(-\frac{\sqrt{1-x^2}}{4x}\right)$$

$$\therefore \frac{du}{dv} = -\frac{1}{x}$$

Thus, $\frac{du}{dv} = -\frac{1}{x}$

(ii) Let

$u = \sin^{-1}\left(4x\sqrt{1-4x^2}\right)$ and $v = \sqrt{1-4x^2}$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\left(4x\sqrt{1-4x^2}\right)$

$\Rightarrow u = \sin^{-1}\left(4x\sqrt{1-(2x)^2}\right)$

By substituting 2x = cos θ, we have

$u = \sin^{-1}\left(2\cos\theta\sqrt{1-(\cos\theta)^2}\right)$

$\Rightarrow u = \sin^{-1}\left(2\cos\theta\sqrt{1-(\cos\theta)^2}\right)$

$\Rightarrow u = \sin^{-1}\left(2\cos\theta\sqrt{\sin^2\theta}\right)$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow u = \sin^{-1}(2\cos\theta\sin\theta)$

$\Rightarrow u = \sin^{-1}(\sin 2\theta)$

Given $x \in \left(\frac{1}{2\sqrt{2}}, \frac{1}{2}\right)$

However, 2x = cos θ ⇒ $x = \frac{\cos\theta}{2}$

$\Rightarrow \frac{\cos\theta}{2} \in \left(\frac{1}{2\sqrt{2}}, \frac{1}{2}\right)$

$\Rightarrow \cos\theta \in \left(\frac{1}{\sqrt{2}}, 1\right)$

$\Rightarrow \theta \in \left(0, \frac{\pi}{4}\right)$

$\Rightarrow 2\theta \in \left(0, \frac{\pi}{2}\right)$

Hence, $u = \sin^{-1}(\sin 2\theta) = 2\theta$.

$\Rightarrow u = 2\cos^{-1}(2x)$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}[2\cos^{-1}(2x)]$$

$$\Rightarrow \frac{du}{dx} = 2\frac{d}{dx}[\cos^{-1}(2x)]$$

We know $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 2\left[-\frac{1}{\sqrt{1-(2x)^2}}\frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{du}{dx} = -\frac{2}{\sqrt{1-4x^2}}\left[\frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{du}{dx} = -\frac{2}{\sqrt{1-4x^2}}\left[2\frac{d}{dx}(x)\right]$$

$$\Rightarrow \frac{du}{dx} = -\frac{4}{\sqrt{1-4x^2}}\frac{d}{dx}(x)$$

However, $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{du}{dx} = -\frac{4}{\sqrt{1-4x^2}} \times 1$$

$$\therefore \frac{du}{dx} = -\frac{4}{\sqrt{1-4x^2}}$$

We have $\frac{dv}{dx} = -\frac{4x}{\sqrt{1-4x^2}}$

We know that $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{-\frac{4}{\sqrt{1-4x^2}}}{-\frac{4x}{\sqrt{1-4x^2}}}$$

$$\Rightarrow \frac{du}{dv} = -\frac{4}{\sqrt{1-4x^2}} \times \left(-\frac{\sqrt{1-x^2}}{4x}\right)$$

$$\therefore \frac{du}{dv} = \frac{1}{x}$$

Thus, $\frac{du}{dv} = \frac{1}{x}$

(iii) Let

$u = \sin^{-1}\left(4x\sqrt{1-4x^2}\right)$ And $v = \sqrt{1-4x^2}$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\left(4x\sqrt{1-4x^2}\right)$

$$\Rightarrow u = \sin^{-1}\left(4x\sqrt{1-(2x)^2}\right)$$

By substituting 2x = cos θ, we have

$$u = \sin^{-1}\left(2\cos\theta\sqrt{1-(\cos\theta)^2}\right)$$

$$\Rightarrow u = \sin^{-1}\left(2\cos\theta\sqrt{1-(\cos\theta)^2}\right)$$

$\Rightarrow u = \sin^{-1}\left(2\cos\theta\sqrt{\sin^2\theta}\right)$ [∵ $\sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow u = \sin^{-1}(2\cos\theta\sin\theta)$

$\Rightarrow u = \sin^{-1}(\sin 2\theta)$

Given $x \in \left(-\frac{1}{2}, -\frac{1}{2\sqrt{2}}\right)$

However, 2x = cos θ ⇒ $x = \frac{\cos\theta}{2}$

$$\Rightarrow \frac{\cos\theta}{2} \in \left(-\frac{1}{2}, -\frac{1}{2\sqrt{2}}\right)$$

$\Rightarrow \cos\theta \in \left(-1, -\frac{1}{\sqrt{2}}\right)$

$\Rightarrow \theta \in \left(\frac{3\pi}{4}, \pi\right)$

$\Rightarrow 2\theta \in \left(\frac{3\pi}{2}, 2\pi\right)$

Hence, $u = \sin^{-1}(\sin 2\theta) = 2\pi - 2\theta$.

$\Rightarrow u = 2\pi - 2\cos^{-1}(2x)$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}[2\pi - 2\cos^{-1}(2x)]$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(2\pi) - \frac{d}{dx}[2\cos^{-1}(2x)]$$

$$\Rightarrow \frac{du}{dx} = 2\frac{d}{dx}(\pi) - 2\frac{d}{dx}[\cos^{-1}(2x)]$$

We know $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 0 - 2\left[-\frac{1}{\sqrt{1-(2x)^2}}\frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{2}{\sqrt{1-4x^2}}\left[\frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{2}{\sqrt{1-4x^2}}\left[2\frac{d}{dx}(x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}}\frac{d}{dx}(x)$$

However, $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}} \times 1$$

$$\therefore \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}}$$

We have $\frac{dv}{dx} = -\frac{4x}{\sqrt{1-4x^2}}$

We know that $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{4}{\sqrt{1-4x^2}}}{-\frac{4x}{\sqrt{1-4x^2}}}$$

$$\Rightarrow \frac{du}{dv} = \frac{4}{\sqrt{1-4x^2}} \times \left(-\frac{\sqrt{1-x^2}}{4x}\right)$$

$$\therefore \frac{du}{dv} = -\frac{1}{x}$$

Thus, $\frac{du}{dv} = -\frac{1}{x}$

6. *Differentiate* $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ *with respect to* $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, *if* $-1 < x < 1,\ x \neq 0.$

Solution:

Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

By substituting x = tan θ, we have

$$u = \tan^{-1}\left(\frac{\sqrt{1+(\tan\theta)^2}-1}{\tan\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\sqrt{\sec^2\theta-1}}{\tan\theta}\right)$$ [∵ $\sec^2\theta - \tan^2\theta = 1$]

$$\Rightarrow u = \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{1-\cos\left(2\times\frac{\theta}{2}\right)}{\sin\left(2\times\frac{\theta}{2}\right)}\right)$$

But, $\cos 2\theta = 1 - 2\sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.

$$\Rightarrow u = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

Given $-1 < x < 1 \Rightarrow x \in (-1, 1)$

However, $x = \tan\theta$

$\Rightarrow$ Tan $\theta \in (-1, 1)$

$$\Rightarrow \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$$

$\Rightarrow \frac{\theta}{2} \epsilon \left(-\frac{\pi}{8}, \frac{\pi}{8}\right)$

Hence, $u = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$

$\Rightarrow u = \frac{1}{2}\tan^{-1} x$

On differentiating u with respect to x, we get

$\frac{du}{dx} = \frac{d}{dx}\left(\frac{1}{2}\tan^{-1} x\right)$

$\Rightarrow \frac{du}{dx} = \frac{1}{2}\frac{d}{dx}(\tan^{-1} x)$

We know $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\Rightarrow \frac{du}{dx} = \frac{1}{2} \times \frac{1}{1+x^2}$

$\therefore \frac{du}{dx} = \frac{1}{2(1+x^2)}$

Now, we have $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

By substituting x = tan θ, we have

$v = \sin^{-1}\left(\frac{2\tan\theta}{1+(\tan\theta)^2}\right)$

$\Rightarrow v = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$\Rightarrow v = \sin^{-1}\left(\frac{2\tan\theta}{\sec^2\theta}\right)$ [$\because \sec^2\theta - \tan^2\theta = 1$]

$\Rightarrow v = \sin^{-1}\left(\frac{2 \times \frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos^2\theta}}\right)$

$\Rightarrow v = \sin^{-1}\left(2 \times \frac{\sin\theta}{\cos\theta} \times \cos^2\theta\right)$

$\Rightarrow v = \sin^{-1}(2\sin\theta\cos\theta)$

But, $\sin2\theta = 2\sin\theta\cos\theta$

$\Rightarrow v = \sin^{-1}(\sin2\theta)$

However, $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \Rightarrow 2\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Hence, $v = \sin^{-1}(\sin2\theta) = 2\theta$

$\Rightarrow v = 2\tan^{-1}x$

On differentiating v with respect to x, we get

$\frac{dv}{dx} = \frac{d}{dx}(2\tan^{-1}x)$

$\Rightarrow \frac{dv}{dx} = 2\frac{d}{dx}(\tan^{-1}x)$

We know $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$\Rightarrow \frac{dv}{dx} = 2 \times \frac{1}{1+x^2}$

$\therefore \frac{dv}{dx} = \frac{2}{1+x^2}$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$\Rightarrow \frac{du}{dv} = \frac{\frac{1}{2(1+x^2)}}{\frac{2}{1+x^2}}$

$\Rightarrow \frac{du}{dv} = \frac{1}{2(1+x^2)} \times \frac{1+x^2}{2}$

$\therefore \frac{du}{dv} = \frac{1}{4}$

Thus, $\frac{du}{dv} = \frac{1}{4}$

7. $Differentiate\, \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ *with respect to* $\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$, *if,*

(i) x ∈ (0, 1/ √2)
(ii) x ∈ (1/√2, 1)

Solution:

(i) Let

$u = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ And $v = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$.

We have to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

By substituting x = sin θ, we have

$u = \sin^{-1}\left(2\sin\theta\sqrt{1-(\sin\theta)^2}\right)$

$\Rightarrow u = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$\Rightarrow u = \sin^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow u = \sin^{-1}(2\sin\theta\cos\theta)$

$\Rightarrow u = \sin^{-1}(\sin 2\theta)$

Now, we have $v = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

By substituting x = sin θ, we have

$v = \sec^{-1}\left(\frac{1}{\sqrt{1-(\sin\theta)^2}}\right)$

$\Rightarrow v = \sec^{-1}\left(\frac{1}{\sqrt{1-\sin^2\theta}}\right)$

$\Rightarrow v = \sec^{-1}\left(\frac{1}{\sqrt{\cos^2\theta}}\right)$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow v = \sec^{-1}\left(\frac{1}{\cos\theta}\right)$

$\Rightarrow v = \sec^{-1}(\sec\theta)$

Given $x \in \left(0, \frac{1}{\sqrt{2}}\right)$

However, x = sin θ

$\Rightarrow \sin\theta \in \left(0, \frac{1}{\sqrt{2}}\right)$

$\Rightarrow \theta \in \left(0, \frac{\pi}{4}\right)$

$\Rightarrow 2\theta \in \left(0, \frac{\pi}{2}\right)$

Hence, $u = \sin^{-1}(\sin 2\theta) = 2\theta$.

$\Rightarrow u = 2\sin^{-1}(x)$

On differentiating u with respect to x, we get

$\frac{du}{dx} = \frac{d}{dx}(2\sin^{-1}x)$

$\Rightarrow \frac{du}{dx} = 2\frac{d}{dx}(\sin^{-1}x)$

We know $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \frac{du}{dx} = 2 \times \frac{1}{\sqrt{1-x^2}}$

$\therefore \frac{du}{dx} = \frac{2}{\sqrt{1-x^2}}$

We have $\theta \in \left(0, \frac{\pi}{4}\right)$

Hence, v = $\sec^{-1}(\sec \theta) = \theta$

$\Rightarrow v = \sin^{-1}x$

On differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(\sin^{-1} x)$$

We know $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$$\therefore \frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{2}{\sqrt{1-x^2}}}{\frac{1}{\sqrt{1-x^2}}}$$

$$\Rightarrow \frac{du}{dv} = \frac{2}{\sqrt{1-x^2}} \times \sqrt{1-x^2}$$

$$\therefore \frac{du}{dv} = 2$$

Thus, $\frac{du}{dv} = 2$

(ii) Let

$u = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ And $v = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$.

We have to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

By substituting x = sin θ, we have

$$u = \sin^{-1}\left(2\sin\theta\sqrt{1-(\sin\theta)^2}\right)$$

$$\Rightarrow u = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$$

$$\Rightarrow u = \sin^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$$ [∵ $\sin^2\theta + \cos^2\theta = 1$]

$$\Rightarrow u = \sin^{-1}(2\sin\theta\cos\theta)$$

$$\Rightarrow u = \sin^{-1}(\sin 2\theta)$$

Now, we have $v = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

By substituting x = sin θ, we have

$$v = \sec^{-1}\left(\frac{1}{\sqrt{1-(\sin\theta)^2}}\right)$$

$$\Rightarrow v = \sec^{-1}\left(\frac{1}{\sqrt{1-\sin^2\theta}}\right)$$

$$\Rightarrow v = \sec^{-1}\left(\frac{1}{\sqrt{\cos^2\theta}}\right)$$ [∵ $\sin^2\theta + \cos^2\theta = 1$]

$$\Rightarrow v = \sec^{-1}\left(\frac{1}{\cos\theta}\right)$$

$$\Rightarrow v = \sec^{-1}(\sec\theta)$$

Given $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$

However, x = sin θ

$$\Rightarrow \sin\theta \in \left(\frac{1}{\sqrt{2}}, 1\right)$$

$$\Rightarrow \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$$

$$\Rightarrow 2\theta \in \left(\frac{\pi}{2}, \pi\right)$$

Hence, $u = \sin^{-1}(\sin 2\theta) = \pi - 2\theta$.

$\Rightarrow u = \pi - 2\sin^{-1}(x)$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}(\pi - 2\sin^{-1}x)$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(\pi) - \frac{d}{dx}(2\sin^{-1}x)$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(\pi) - 2\frac{d}{dx}(\sin^{-1}x)$$

We know $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 0 - 2 \times \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{du}{dx} = \frac{-2}{\sqrt{1-x^2}}$$

We have $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Hence, $v = \sec^{-1}(\sec\theta) = \theta$

$\Rightarrow v = \sin^{-1}x$

On differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(\sin^{-1}x)$$

We know $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$$\therefore \frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{-\frac{2}{\sqrt{1-x^2}}}{\frac{1}{\sqrt{1-x^2}}}$$

$$\Rightarrow \frac{du}{dv} = -\frac{2}{\sqrt{1-x^2}} \times \sqrt{1-x^2}$$

$$\therefore \frac{du}{dv} = -2$$

Thus, $\frac{du}{dv} = -2$

8. Differentiate (cos x)$^{\sin x}$ with respect to (sin x)$^{\cos x}$.

Solution:

Let u = (cos x)$^{\sin x}$ and v = (sin x)$^{\cos x}$.

We have to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have u = (cos x)$^{\sin x}$

Taking log on both sides, we get

Log u = log (cos x)$^{\sin x}$

⇒ Log u = (sin x) × log (cos x) [∵ log a^m = m × log a]

On differentiating both the sides with respect to x, we get

$$\frac{d}{du}(\log u) \times \frac{du}{dx} = \frac{d}{dx}[\sin x \times \log(\cos x)]$$

We know that (uv)' = vu' + uv'

$$\Rightarrow \frac{d}{du}(\log u) \times \frac{du}{dx} = \log(\cos x)\frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}[\log(\cos x)]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$ and $\frac{d}{dx}(\sin x) = \cos x$

$$\Rightarrow \frac{1}{u} \times \frac{du}{dx} = \log(\cos x) \times \cos x + \sin x \left[\frac{1}{\cos x} \frac{d}{dx}(\cos x)\right]$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \cos x \log(\cos x) + \frac{\sin x}{\cos x}\frac{d}{dx}(\cos x)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \cos x \log(\cos x) + \tan x \frac{d}{dx}(\cos x)$$

We know $\frac{d}{dx}(\cos x) = -\sin x$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \cos x \log(\cos x) + \tan x (-\sin x)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \cos x \log(\cos x) - \tan x \sin x$$

But, $u = (\cos x)^{\sin x}$

$$\Rightarrow \frac{1}{(\cos x)^{\sin x}}\frac{du}{dx} = \cos x \log(\cos x) - \tan x \sin x$$

$$\therefore \frac{du}{dx} = (\cos x)^{\sin x}[\cos x \log(\cos x) - \tan x \sin x]$$

Now, we have $v = (\sin x)^{\cos x}$

Taking log on both sides, we get

$\text{Log } v = \log (\sin x)^{\cos x}$

$\Rightarrow \text{Log } v = (\cos x) \times \log (\sin x)$ [$\because \log a^m = m \times \log a$]

On differentiating both the sides with respect to x, we get

$$\frac{d}{dv}(\log v) \times \frac{dv}{dx} = \frac{d}{dx}[\cos x \times \log(\sin x)]$$

We know that (uv)’ = vu’ + uv’ (product rule)

$$\Rightarrow \frac{d}{du}(\log u) \times \frac{dv}{dx} = \log(\sin x)\frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}[\log(\sin x)]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$ and $\frac{d}{dx}(\cos x) = -\sin x$

$$\Rightarrow \frac{1}{v} \times \frac{dv}{dx} = \log(\sin x) \times (-\sin x) + \cos x \left[\frac{1}{\sin x} \frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = -\sin x \log(\sin x) + \frac{\cos x}{\sin x}\frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = -\sin x \log(\sin x) + \cot x \frac{d}{dx}(\sin x)$$

We know $\frac{d}{dx}(\sin x) = \cos x$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = -\sin x \log(\sin x) + \cot x \times (\cos x)$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = -\sin x \log(\sin x) + \cot x \cos x$$

But, $v = (\sin x)^{\cos x}$

$$\Rightarrow \frac{1}{(\sin x)^{\cos x}}\frac{dv}{dx} = -\sin x \log(\sin x) + \cot x \cos x$$

$$\therefore \frac{dv}{dx} = (\sin x)^{\cos x}[-\sin x \log(\sin x) + \cot x \cos x]$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{(\cos x)^{\sin x}[\cos x \log(\cos x) - \tan x \sin x]}{(\sin x)^{\cos x}[-\sin x \log(\sin x) + \cot x \cos x]}$$

$$\therefore \frac{du}{dv} = \frac{(\cos x)^{\sin x}[\cos x \log(\cos x) - \tan x \sin x]}{(\sin x)^{\cos x}[\cot x \cos x - \sin x \log(\sin x)]}$$

Thus, $\frac{du}{dv} = \frac{(\cos x)^{\sin x}[\cos x \log(\cos x) - \tan x \sin x]}{(\sin x)^{\cos x}[\cot x \cos x - \sin x \log(\sin x)]}$

9. $Differentiate \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ *with respect to* $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, *if* $0 < x < 1.$

Solution:

Let $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

By substituting x = tan θ, we have

$$u = \sin^{-1}\left(\frac{2\tan\theta}{1+(\tan\theta)^2}\right)$$

$$\Rightarrow u = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$\Rightarrow u = \sin^{-1}\left(\frac{2\tan\theta}{\sec^2\theta}\right) \text{ [}\because \sec^2\theta - \tan^2\theta = 1\text{]}$$

$$\Rightarrow u = \sin^{-1}\left(\frac{2\times\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos^2\theta}}\right)$$

$$\Rightarrow u = \sin^{-1}\left(2\times\frac{\sin\theta}{\cos\theta}\times\cos^2\theta\right)$$

$\Rightarrow u = \sin^{-1}(2\sin\theta\cos\theta)$

But, $\sin 2\theta = 2\sin\theta\cos\theta$

$\Rightarrow u = \sin^{-1}(\sin 2\theta)$

Given $0 < x < 1 \Rightarrow x \in (0, 1)$

However, $x = \tan\theta$

$\Rightarrow \tan\theta \in (0, 1)$

$$\Rightarrow \theta \in \left(0, \frac{\pi}{4}\right)$$

$$\Rightarrow 2\theta \in \left(0, \frac{\pi}{2}\right)$$

Hence, $u = \sin^{-1}(\sin 2\theta) = 2\theta$

$\Rightarrow u = 2\tan^{-1}x$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}(2\tan^{-1}x)$$

$$\Rightarrow \frac{du}{dx} = 2\frac{d}{dx}(\tan^{-1}x)$$

We know $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$$\Rightarrow \frac{du}{dx} = 2 \times \frac{1}{1+x^2}$$

$$\therefore \frac{du}{dx} = \frac{2}{1+x^2}$$

Now, we have $v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

By substituting $x = \tan\theta$, we have

$$v = \cos^{-1}\left(\frac{1-(\tan\theta)^2}{1+(\tan\theta)^2}\right)$$

$$\Rightarrow v = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$\Rightarrow v = \cos^{-1}\left(\frac{1-\tan^2\theta}{\sec^2\theta}\right)$ [$\because \sec^2\theta - \tan^2\theta = 1$]

$$\Rightarrow v = \cos^{-1}\left(\frac{1}{\sec^2\theta} - \frac{\tan^2\theta}{\sec^2\theta}\right)$$

$$\Rightarrow v = \cos^{-1}\left(\frac{1}{\frac{1}{\cos^2\theta}} - \frac{\frac{\sin^2\theta}{\cos^2\theta}}{\frac{1}{\cos^2\theta}}\right)$$

$\Rightarrow v = \cos^{-1}(\cos^2\theta - \sin^2\theta)$

But, $\cos2\theta = \cos^2\theta - \sin^2\theta$

$\Rightarrow v = \cos^{-1}(\cos 2\theta)$

However, $\theta \in \left(0, \frac{\pi}{4}\right) \Rightarrow 2\theta \in \left(0, \frac{\pi}{2}\right)$

Hence, $v = \cos^{-1}(\cos 2\theta) = 2\theta$

$\Rightarrow v = 2\tan^{-1}x$

On differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(2\tan^{-1}x)$$

$$\Rightarrow \frac{dv}{dx} = 2\frac{d}{dx}(\tan^{-1}x)$$

We know $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$$\Rightarrow \frac{dv}{dx} = 2 \times \frac{1}{1+x^2}$$

$$\therefore \frac{dv}{dx} = \frac{2}{1+x^2}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{2}{1+x^2}}{\frac{2}{1+x^2}}$$

$$\Rightarrow \frac{du}{dv} = \frac{2}{1+x^2} \times \frac{1+x^2}{2}$$

$$\therefore \frac{du}{dv} = 1$$

Thus, $\frac{du}{dv} = 1$

10. $Differentiate \tan^{-1}\left(\frac{1+ax}{1-ax}\right)$ $with\ respect\ to \sqrt{1+a^2x^2}.$

Solution:

Let $u=\tan^{-1}\left(\frac{1+ax}{1-ax}\right)$ and $v=\sqrt{1+a^2x^2}$.

We have to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u=\tan^{-1}\left(\frac{1+ax}{1-ax}\right)$

By substituting ax = tan θ, we have

$$u=\tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$\Rightarrow u=\tan^{-1}\left(\frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\tan\theta}\right)$$

$$\Rightarrow u=\tan^{-1}\left(\tan\left(\frac{\pi}{4}+\theta\right)\right)\ \left[\because \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}\right]$$

$$\Rightarrow u=\frac{\pi}{4}+\theta$$

$$\Rightarrow u=\frac{\pi}{4}+\tan^{-1}(ax)$$

On differentiating u with respect to x, we get

$$\frac{du}{dx}=\frac{d}{dx}\left[\frac{\pi}{4}+\tan^{-1}(ax)\right]$$

$$\Rightarrow \frac{du}{dx}=\frac{d}{dx}\left(\frac{\pi}{4}\right)+\frac{d}{dx}[\tan^{-1}(ax)]$$

We know $\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx}=0+\frac{1}{1+(ax)^2}\frac{d}{dx}(ax)$$

$$\Rightarrow \frac{du}{dx} = 0 + \frac{1}{1+(ax)^2}\frac{d}{dx}(ax)$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{1+a^2x^2}\left[a\frac{d}{dx}(x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{a}{1+a^2x^2}\frac{d}{dx}(x)$$

We know $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{du}{dx} = \frac{a}{1+a^2x^2} \times 1$$

$$\therefore \frac{du}{dx} = \frac{a}{1+a^2x^2}$$

Now, we have $v = \sqrt{1+a^2x^2}$

On differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}\left(\sqrt{1+a^2x^2}\right)$$

$$\Rightarrow \frac{dv}{dx} = \frac{d}{dx}(1+a^2x^2)^{\frac{1}{2}}$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2}(1+a^2x^2)^{\frac{1}{2}-1}\frac{d}{dx}(1+a^2x^2)$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2}(1+a^2x^2)^{-\frac{1}{2}}\left[\frac{d}{dx}(1) + \frac{d}{dx}(a^2x^2)\right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1+a^2x^2}}\left[\frac{d}{dx}(1) + a^2\frac{d}{dx}(x^2)\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1+a^2x^2}}\left[0 + a^2\left(2x^{2-1}\right)\right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1+a^2x^2}}\left[2a^2x\right]$$

$$\therefore \frac{dv}{dx} = \frac{a^2x}{\sqrt{1+a^2x^2}}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{a}{1+a^2x^2}}{\frac{a^2x}{\sqrt{1+a^2x^2}}}$$

$$\Rightarrow \frac{du}{dv} = \frac{a}{1+a^2x^2} \times \frac{\sqrt{1+a^2x^2}}{a^2x}$$

$$\therefore \frac{du}{dv} = \frac{1}{ax\sqrt{1+a^2x^2}}$$

Thus, $\frac{du}{dv} = \frac{1}{ax\sqrt{1+a^2x^2}}$