EXERCISE 14.1 PAGE NO: 14.9

1. If y = sin x and x changes from π/2 to 22/14, what is the approximate change in y?

Solution:

Given y = sin x and x changes from $\frac{\pi}{2}$ to $\frac{22}{14}$.

Let $x = \frac{\pi}{2}$ so that $x + \Delta x = \frac{22}{14}$

$$\Rightarrow \frac{\pi}{2} + \Delta x = \frac{22}{14}$$

$$\therefore \Delta x = \frac{22}{14} - \frac{\pi}{2}$$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x)$$

We know that $\frac{d}{dx}(\sin x) = \cos x$

$$\therefore \frac{dy}{dx} = \cos x$$

When $x = \frac{\pi}{2}$, we have $\frac{dy}{dx} = \cos\left(\frac{\pi}{2}\right)$.

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{2}} = 0$$

We know that if y = f(x) and Δ x is a small increment in x, then the corresponding increment in y, Δ y = f(x + Δ x) – f(x), is approximately given as

$$\Delta y = \left(\frac{dy}{dx}\right)\Delta x$$

Here, $\frac{dy}{dx} = 0$ and $\Delta x = \frac{22}{14} - \frac{\pi}{2}$

$$\Rightarrow \Delta y = (0)\left(\frac{22}{14} - \frac{\pi}{2}\right)$$

$\therefore \Delta y = 0$

Thus, there is approximately no change in y.

2. The radius of a sphere shrinks from 10 to 9.8 cm. Find approximately the decrease in its volume.

Solution:

Given the radius of a sphere changes from 10 cm to 9.8 cm.
Let x be the radius of the sphere and Δx be the change in the value of x.
Hence, we have x = 10 and $x + \Delta x = 9.8$
$\Rightarrow 10 + \Delta x = 9.8$
$\Rightarrow \Delta x = 9.8 - 10$
$\therefore \Delta x = -0.2$
The volume of a sphere of radius x is given by

$$V = \frac{4}{3}\pi x^3$$

On differentiating V with respect to x, we get

$$\frac{dV}{dx} = \frac{d}{dx}\left(\frac{4}{3}\pi x^3\right)$$

$$\Rightarrow \frac{dV}{dx} = \frac{4\pi}{3}\frac{d}{dx}(x^3)$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dV}{dx} = \frac{4\pi}{3}(3x^2)$$

$$\therefore \frac{dV}{dx} = 4\pi x^2$$

When x = 10, we have $\frac{dV}{dx} = 4\pi(10)^2$.

$$\Rightarrow \left(\frac{dV}{dx}\right)_{x=10} = 4\pi \times 100$$

$$\Rightarrow \left(\frac{dV}{dx}\right)_{x=10} = 400\pi$$

We know that if y = f(x) and Δ x is a small increment in x, then the corresponding increment in y, Δ y = f(x + Δ x) – f(x), is approximately given as

$$\Delta y = \left(\frac{dy}{dx}\right)\Delta x$$

Here, $\frac{dV}{dx} = 400\pi$ and Δ x = –0.2

⇒ ΔV = (400π) (–0.2)

∴ ΔV = –80π

Thus, the approximate decrease in the volume of the sphere is 80π cm^3.

3. A circular metal plate expands under heating so that its radius increases by k%. Find the approximate increase in the area of the plate, if the radius of the plate before heating is 10 cm.

Solution:

Given the radius of a circular plate initially is 10 cm and it increases by k%.

Let x be the radius of the circular plate, and Δ x is the change in the value of x.

Hence, we have x = 10 and $\Delta x = \frac{k}{100} \times 10$

∴ Δ x = 0.1k

The area of a circular plate of radius x is given by

A = πx^2

On differentiating A with respect to x, we get

$$\frac{dA}{dx} = \frac{d}{dx}(\pi x^2)$$

$\Rightarrow \frac{dA}{dx} = \pi \frac{d}{dx}(x^2)$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dA}{dx} = \pi(2x)$

$\therefore \frac{dA}{dx} = 2\pi x$

When x = 10, we have $\frac{dA}{dx} = 2\pi(10)$.

$\Rightarrow \left(\frac{dA}{dx}\right)_{x=10} = 20\pi$

We know that if y = f(x) and Δx is a small increment in x, then the corresponding increment in y, $\Delta y = f(x + \Delta x) - f(x)$, is approximately given as

$\Delta y = \left(\frac{dy}{dx}\right)\Delta x$

Here, $\frac{dA}{dx} = 20\pi$ and $\Delta x = 0.1k$

$\Rightarrow \Delta A = (20\pi)(0.1k)$

$\therefore \Delta A = 2k\pi$

Thus, the approximate increase in the area of the circular plate is $2k\pi$ cm^2.

4. Find the percentage error in calculating the surface area of a cubical box if an error of 1% is made in measuring the lengths of the edges of the cube.

Solution:

Given the error in the measurement of the edge of a cubical box is 1%.

Let x be the edge of the cubical box, and Δx is the error in the value of x.

Hence, we have $\Delta x = \frac{1}{100} \times x$

∴ $\Delta x = 0.01x$

The surface area of a cubical box of radius x is given by

$S = 6x^2$

On differentiating A with respect to x, we get

$$\frac{dS}{dx} = \frac{d}{dx}(6x^2)$$

$$\Rightarrow \frac{dS}{dx} = 6\frac{d}{dx}(x^2)$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dS}{dx} = 6(2x)$$

$$\therefore \frac{dS}{dx} = 12x$$

We know that if $y = f(x)$ and Δx is a small increment in x, then the corresponding increment in y, $\Delta y = f(x + \Delta x) - f(x)$, is approximately given as

$$\Delta y = \left(\frac{dy}{dx}\right)\Delta x$$

Here, $\frac{dS}{dx} = 12x$ and $\Delta x = 0.01x$

$\Rightarrow \Delta S = (12x)(0.01x)$

$\therefore \Delta S = 0.12x^2$

The percentage error is,

$$\text{Error} = \frac{0.12x^2}{6x^2} \times 100\%$$

$\Rightarrow$ Error = $0.02 \times 100\%$

∴ Error = 2%

Thus, the error in calculating the surface area of the cubical box is 2%.

5. If there is an error of 0.1% in the measurement of the radius of a sphere, find approximately the percentage error in the calculation of the volume of the sphere.

Solution:

Given the error in the measurement of the radius of a sphere is 0.1%. Let x be the radius of the sphere and Δ x be the error in the value of x.

Hence, we have $\Delta x = \frac{0.1}{100} \times x$

$\therefore \Delta x = 0.001x$

The volume of a sphere of radius x is given by

$$V = \frac{4}{3}\pi x^3$$

On differentiating V with respect to x, we get

$$\frac{dV}{dx} = \frac{d}{dx}\left(\frac{4}{3}\pi x^3\right)$$

$$\Rightarrow \frac{dV}{dx} = \frac{4\pi}{3}\frac{d}{dx}(x^3)$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dV}{dx} = \frac{4\pi}{3}(3x^2)$$

$$\therefore \frac{dV}{dx} = 4\pi x^2$$

We know that if y = f(x) and Δ x is a small increment in x, then the corresponding increment in y, Δ y = f(x + Δ x) – f(x), is approximately given as

$$\Delta y = \left(\frac{dy}{dx}\right)\Delta x$$

Here, $\frac{dV}{dx} = 4\pi x^2$ and Δ x = 0.001x

$\therefore \Delta V = 0.004\pi x^3$

The percentage error is,

$$\text{Error} = \frac{0.004\pi x^3}{\frac{4}{3}\pi x^3} \times 100\%$$

$$\Rightarrow \text{Error} = \frac{0.004 \times 3}{4} \times 100\%$$

$\Rightarrow$ Error = 0.003 × 100%

$\therefore$ Error = 0.3%

Thus, the error in calculating the volume of the sphere is 0.3%.

6. The pressure p and the volume v of a gas are connected by the relation $pv^{1.4}$ = const. Find the percentage error in p corresponding to a decrease of ½ % in v.

Solution:

Given $pv^{1.4}$ = constant and the decrease in v is ½ %.

Hence, we have $\Delta v = -\frac{\frac{1}{2}}{100} \times v$

$\therefore \Delta v = -0.005v$

We have $pv^{1.4}$ = constant

Taking log on both sides, we get

$\text{Log}(pv^{1.4}) = \log(\text{constant})$

$\Rightarrow \text{Log } p + \log v^{1.4} = 0$ [$\because \log(ab) = \log a + \log b$]

$\Rightarrow \text{Log } p + 1.4 \log v = 0$ [$\because \log(a^m) = m \log a$]

On differentiating both sides with respect to v, we get

$$\frac{d}{dp}(\log p) \times \frac{dp}{dv} + \frac{d}{dv}(1.4 \log v) = 0$$

$$\Rightarrow \frac{d}{dp}(\log p) \times \frac{dp}{dv} + 1.4\frac{d}{dv}(\log v) = 0$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$

$$\Rightarrow \frac{1}{p} \times \frac{dp}{dv} + 1.4 \times \frac{1}{v} = 0$$

$$\Rightarrow \frac{1}{p}\frac{dp}{dv} + \frac{1.4}{v} = 0$$

$$\Rightarrow \frac{1}{p}\frac{dp}{dv} = -\frac{1.4}{v}$$

$$\therefore \frac{dp}{dv} = -\frac{1.4}{v}p$$

We know that if y = f(x) and Δ x is a small increment in x, then the corresponding increment in y, Δ y = f(x + Δ x) – f(x), is approximately given as

$$\Delta y = \left(\frac{dy}{dx}\right)\Delta x$$

Here, $\frac{dp}{dv} = -\frac{1.4}{v}p$ and Δ v = –0.005v

$$\Rightarrow \Delta p = \left(-\frac{1.4}{v}p\right)(-0.005v)$$

⇒ Δ p = (–1.4p) (–0.005)

∴ Δ p = 0.007p

The percentage error is,

$$\text{Error} = \frac{0.007p}{p} \times 100\%$$

⇒ Error = 0.007 × 100%

∴ Error = 0.7%

Thus, the error in p corresponding to the decrease in v is 0.7%.

7. The height of a cone increases by k%, its semi-vertical angle remaining the same. What is the approximate percentage increase in (i) in total surface area, and (ii) in the volume, assuming that k is small.

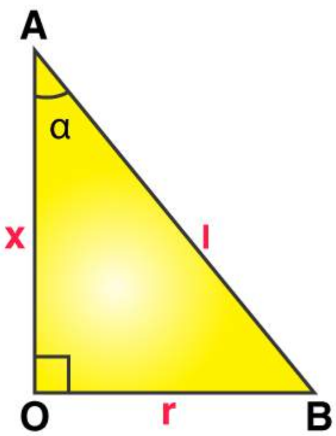
Solution:

Given the height of a cone increases by k%. Let x be the height of the cone and Δx be the change in the value of x.

Hence, we have $\Delta x = \frac{k}{100} \times x$

$\therefore \Delta x = 0.01kx$

Let us assume the radius, the slant height and the semi-vertical angle of the cone to be r, l and α respectively as shown in the figure below.



From the above figure, using trigonometry, we have

$$\tan \alpha = \frac{OB}{OA}$$

$$\Rightarrow \tan \alpha = \frac{r}{x}$$

$\therefore r = x \tan(\alpha)$

We also have

$$\cos \alpha = \frac{OA}{AB}$$

$$\Rightarrow \cos \alpha = \frac{x}{l}$$

$\Rightarrow l = \frac{x}{\cos\alpha}$

∴ $l = x \sec(\alpha)$

(i) The total surface area of the cone is given by

$S = \pi r^2 + \pi r l$

From above, we have $r = x \tan(\alpha)$ and $l = x \sec(\alpha)$.

$\Rightarrow S = \pi (x \tan(\alpha))^2 + \pi (x \tan(\alpha))(x \sec(\alpha))$

$\Rightarrow S = \pi x^2 \tan^2\alpha + \pi x^2 \tan(\alpha) \sec(\alpha)$

$\Rightarrow S = \pi x^2 \tan(\alpha) [\tan(\alpha) + \sec(\alpha)]$

On differentiating S with respect to x, we get

$$\frac{dS}{dx} = \frac{d}{dx}[\pi x^2 \tan\alpha (\tan\alpha + \sec\alpha)]$$

$$\Rightarrow \frac{dS}{dx} = \pi \tan\alpha (\tan\alpha + \sec\alpha) \frac{d}{dx}(x^2)$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dS}{dx} = \pi \tan\alpha (\tan\alpha + \sec\alpha)(2x)$$

$$\therefore \frac{dS}{dx} = 2\pi x \tan\alpha (\tan\alpha + \sec\alpha)$$

We know that if $y = f(x)$ and Δx is a small increment in x, then the corresponding increment in y, $\Delta y = f(x + \Delta x) - f(x)$, is approximately given as

$$\Delta y = \left(\frac{dy}{dx}\right) \Delta x$$

Here, $\frac{dS}{dx} = 2\pi x \tan\alpha (\tan\alpha + \sec\alpha)$ and $\Delta x = 0.01kx$

$\Rightarrow \Delta S = (2\pi x \tan(\alpha) [\tan(\alpha) + \sec(\alpha)]) (0.01kx)$

$\therefore \Delta S = 0.02 k \pi x^2 \tan(\alpha) [\tan(\alpha) + \sec(\alpha)]$

The percentage increase in S is,

$$\text{Increase} = \frac{\Delta S}{S} \times 100\%$$

$$\Rightarrow \text{Increase} = \frac{0.02k\pi x^2 \tan\alpha\,(\tan\alpha + \sec\alpha)}{\pi x^2 \tan\alpha\,(\tan\alpha + \sec\alpha)} \times 100\%$$

$\Rightarrow$ Increase = 0.02k × 100%

∴ Increase = 2k%

Thus, the approximate increase in the total surface area of the cone is 2k%.

(ii) The volume of the cone is given by

$$V = \frac{1}{3}\pi r^2 x$$

From above, we have r = x tan (α).

$$\Rightarrow V = \frac{1}{3}\pi(x\tan\alpha)^2 x$$

$$\Rightarrow V = \frac{1}{3}\pi(x^2\tan^2\alpha)x$$

$$\Rightarrow V = \frac{1}{3}\pi x^3\tan^2\alpha$$

On differentiating V with respect to x, we get

$$\frac{dV}{dx} = \frac{d}{dx}\left(\frac{1}{3}\pi x^3 \tan^2\alpha\right)$$

$$\Rightarrow \frac{dV}{dx} = \frac{1}{3}\pi\tan^2\alpha\frac{d}{dx}(x^3)$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dV}{dx} = \frac{1}{3}\pi\tan^2\alpha\,(3x^2)$$

$$\therefore \frac{dV}{dx} = \pi x^2\tan^2\alpha$$

We know that if y = f(x) and Δ x is a small increment in x, then the corresponding increment in y, Δ y = f(x + Δ x) – f(x), is approximately given as

$$\Delta y = \left(\frac{dy}{dx}\right)\Delta x$$

Here, $\frac{dV}{dx} = \pi x^2 \tan^2 \alpha$ and Δ x = 0.01kx

$\Rightarrow \Delta V = (\pi x^2 \tan^2\alpha)(0.01kx)$

$\therefore \Delta V = 0.01k\pi x^3 \tan^2\alpha$

The percentage increase in V is,

$$\text{Increase} = \frac{\Delta V}{V} \times 100\%$$

$$\Rightarrow \text{Increase} = \frac{0.01k\pi x^3 \tan^2 \alpha}{\frac{1}{3}\pi x^3 \tan^2 \alpha} \times 100\%$$

$$\Rightarrow \text{Increase} = \frac{0.01k}{\frac{1}{3}} \times 100\%$$

⇒ Increase = 0.03k × 100%

∴ Increase = 3k%

Thus, the approximate increase in the volume of the cone is 3k%.

8. Show that the relative error in computing the volume of a sphere, due to an error in measuring the radius, is approximately equal to three times the relative error in the radius.

Solution:

Let the error in measuring the radius of a sphere be k%.

Let x be the radius of the sphere and Δ x be the error in the value of x

Hence, we have $\Delta x = \frac{k}{100} \times x$

$\therefore \Delta x = 0.01kx$

The volume of a sphere of radius x is given by

$$V = \frac{4}{3}\pi x^3$$

On differentiating V with respect to x, we get

$$\frac{dV}{dx} = \frac{d}{dx}\left(\frac{4}{3}\pi x^3\right)$$

$$\Rightarrow \frac{dV}{dx} = \frac{4\pi}{3}\frac{d}{dx}(x^3)$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dV}{dx} = \frac{4\pi}{3}(3x^2)$$

$$\therefore \frac{dV}{dx} = 4\pi x^2$$

We know that if y = f(x) and Δx is a small increment in x, then the corresponding increment in y, $\Delta y = f(x + \Delta x) - f(x)$, is approximately given as

$$\Delta y = \left(\frac{dy}{dx}\right)\Delta x$$

Here, $\frac{dV}{dx} = 4\pi x^2$ and $\Delta x = 0.01kx$

$\Rightarrow \Delta V = (4\pi x^2)(0.01kx)$

$\therefore \Delta V = 0.04k\pi x^3$

The percentage error is,

$$\text{Error} = \frac{0.04k\pi x^3}{\frac{4}{3}\pi x^3} \times 100\%$$

$$\Rightarrow \text{Error} = \frac{0.04k \times 3}{4} \times 100\%$$

$\Rightarrow$ Error = 0.03k × 100%

$\therefore$ Error = 3k%