EXERCISE 16.1 PAGE NO: 16.10

1. Find the Slopes of the tangent and the normal to the following curves at the indicated points:

(i) $y = \sqrt{x^3}$ at $x = 4$

Solution:

Given $y = \sqrt{x^3}$ at x = 4

First, we have to find $\frac{dy}{dx}$ of given function, f(x) that is to find the derivative of f (x)

$y = \sqrt{x^3}$

$\therefore \sqrt[n]{x} = x^{\frac{1}{n}}$

$\Rightarrow y = (x^3)^{\frac{1}{2}}$

$\Rightarrow y = (x)^{\frac{3}{2}}$

$\therefore \frac{dy}{dx}(x^n) = n\, x^{n-1}$

We know that the Slope of the tangent is $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{3}{2}(x)^{\frac{3}{2}-1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{2}(x)^{\frac{1}{2}}$$

Since, x = 4

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=4} = \frac{3}{2}(4)^{\frac{1}{2}}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=4} = \frac{3}{2} \times \sqrt{4}$$

$\Rightarrow \left(\frac{dy}{dx}\right)_{x=4} = \frac{3}{2} \times 2$

$\Rightarrow \left(\frac{dy}{dx}\right)_{x=4} = 3$

The Slope of the tangent at x = 4 is 3

$\Rightarrow$ The Slope of the normal = $\frac{-1}{\text{The Slope of the tangent}}$

$\Rightarrow$ The Slope of the normal = $\frac{-1}{\left(\frac{dy}{dx}\right)_{x=4}}$

$\Rightarrow$ The Slope of the normal = $\frac{-1}{3}$

(ii) y = √x at x = 9

Solution:

Given y = $\sqrt{x}$ at x = 9

First, we have to find $\frac{dy}{dx}$ of given function, f(x) that is to find the derivative of f(x)

$\Rightarrow y = \sqrt{x}$

$\therefore \sqrt[n]{x} = x^{\frac{1}{n}}$

$\Rightarrow y = (x)^{\frac{1}{2}}$

$\therefore \frac{dy}{dx}(x^n) = n\,x^{n-1}$

The Slope of the tangent is $\frac{dy}{dx}$

$\Rightarrow y = (x)^{\frac{1}{2}}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}(x)^{\frac{1}{2}-1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}(x)^{\frac{-1}{2}}$$

Since, x = 9

$$\left(\frac{dy}{dx}\right)_{x=9} = \frac{1}{2}(9)^{\frac{-1}{2}}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=9} = \frac{1}{2} \times \frac{1}{(9)^{\frac{1}{2}}}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=9} = \frac{1}{2} \times \frac{1}{\sqrt{9}}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=9} = \frac{1}{2} \times \frac{1}{3}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=9} = \frac{1}{6}$$

The Slope of the tangent at x = 9 is $\frac{1}{6}$

$\Rightarrow$ The Slope of the normal = $\frac{-1}{\text{The Slope of the tangent}}$

$\Rightarrow$ The Slope of the normal = $\frac{-1}{\left(\frac{dy}{dx}\right)x = 9}$

$\Rightarrow$ The Slope of the normal = $\frac{-1}{\frac{1}{6}}$

$\Rightarrow$ The Slope of the normal = – 6

(iii) y = x^3 – x at x = 2

Solution:

First, we have to find $\frac{dy}{dx}$ of given function f(x) that is to find the derivative of f(x)

$\therefore \frac{dy}{dx}(x^n) = n\,x^{n-1}$

The Slope of the tangent is $\frac{dy}{dx}$

$\Rightarrow y = x^3 - x$

$\Rightarrow \frac{dy}{dx} = \frac{dy}{dx}(x^3) + 3 \times \frac{dy}{dx}(x)$

$\Rightarrow \frac{dy}{dx} = 3.x^{3-1} - 1.x^{1-0}$

$\Rightarrow \frac{dy}{dx} = 3x^2 - 1$

Since, x = 2

$\Rightarrow \left(\frac{dy}{dx}\right)_{x=2} = 3 \times (2)^2 - 1$

$\Rightarrow \left(\frac{dy}{dx}\right)_{x=2} = (3 \times 4) - 1$

$\Rightarrow \left(\frac{dy}{dx}\right)_{x=2} = 12 - 1$

$\Rightarrow \left(\frac{dy}{dx}\right)_{x=2} = 11$

$\therefore$The Slope of the tangent at x = 2 is 11

$\Rightarrow$ The Slope of the normal = $\frac{-1}{\text{The Slope of the tangent}}$

$\Rightarrow$ The Slope of the normal = $\frac{-1}{\left(\frac{dy}{dx}\right)_{x=2}}$

$\Rightarrow$ The Slope of the normal = $\frac{-1}{11}$

(iv) y = 2x² + 3 sin x at x = 0

Solution:

Given $y = 2x^2 + 3\sin x$ at $x = 0$

First, we have to find $\frac{dy}{dx}$ of given function f(x) that is to find the derivative of f(x)

$\therefore \frac{dy}{dx}(x^n) = n\,x^{n-1}$

The Slope of the tangent is $\frac{dy}{dx}$

$\Rightarrow y = 2x^2 + 3\sin x$

$\Rightarrow \frac{dy}{dx} = 2\,dy/dx\,(x^2) + 3\,dy/dx\,(\sin x)$

$\Rightarrow \frac{dy}{dx} = 2\,(2x^{2-1}) + 3\,(\cos x)$

$\therefore \frac{d}{dx}(\text{Sin } x) = \cos x$

$\Rightarrow \frac{dy}{dx} = 4x + 3\cos x$

Since, x = 2

$\Rightarrow \left(\frac{dy}{dx}\right)_{x=0} = 4\,(0) + 3\cos(0)$

We know $\cos(0) = 1$

$\Rightarrow \left(\frac{dy}{dx}\right)_{x=0} = 0 + 3\,(1)$

$\Rightarrow \left(\frac{dy}{dx}\right)_{x=0} = 3$

$\therefore$The Slope of the tangent at x = 0 is 3

$\Rightarrow$ The Slope of the normal = $\frac{-1}{\text{The Slope of the tangent}}$

$\Rightarrow$ The Slope of the normal = $\frac{-1}{\left(\frac{dy}{dx}\right)_{x=0}}$

⇒ The Slope of the normal = $\frac{-1}{3}$

(v) x = a (θ - sin θ), y = a (1 + cos θ) at θ = -π /2

Solution:

Given x = a (θ - sin θ), y = a (1 + cos θ) at θ = -π /2

Here, to find $\frac{dy}{dx}$, we have to find $\frac{dy}{d\theta}$ & $\frac{dx}{d\theta}$ and divide $\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ and we get our desired $\frac{dy}{dx}$.

∴ $\frac{dy}{dx}(x^n) = n\, x^{n-1}$

⇒ x = a (θ – sin θ)

⇒ $\frac{dx}{d\theta} = a\,\{\frac{dx}{d\theta}(\theta) - \frac{dx}{d\theta}(\sin\theta)\}$

⇒ $\frac{dx}{d\theta} = a\,(1 - \cos\theta)$ ... (1)

∴ $\frac{d}{dx}(\text{Sin } x) = \cos x$

⇒ y = a (1 + cos θ)

⇒ $\frac{dy}{d\theta} = a\,[\frac{dx}{d\theta}(1) + \frac{dx}{d\theta}(\cos\theta)]$

∴ $\frac{d}{dx}(\text{Cos } x) = -\sin x$

∴ $\frac{d}{dx}(\text{Constant}) = 0$

⇒ $\frac{dy}{d\theta} = a\,(0 + (-\sin\theta))$

⇒ $\frac{dy}{d\theta} = a\,(-\sin\theta)$

$\Rightarrow \frac{dy}{d\theta} = -\, a \sin\theta$ ... (2)

$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a\sin\theta}{a(1-\cos\theta)}$

$\Rightarrow \frac{dy}{dx} = \frac{-\sin\theta}{(1-\cos\theta)}$

The Slope of the tangent is $\frac{-\sin\theta}{(1-\cos\theta)}$

Since, $\theta = \frac{-\pi}{2}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{\theta=\frac{-\pi}{2}} = \frac{-\sin\frac{-\pi}{2}}{(1-\cos\frac{-\pi}{2})}$

We know Cos (π /2) = 0 and sin (π /2) = 1

$\Rightarrow \left(\frac{dy}{dx}\right)_{\theta=\frac{-\pi}{2}} = \frac{-(-1)}{(1-(-0))}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{\theta=\frac{-\pi}{2}} = \frac{1}{(1-0)}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{\theta=\frac{-\pi}{2}} = 1$

∴The Slope of the tangent at x = $-\frac{\pi}{2}$ is 1

⇒ The Slope of the normal = $\frac{-1}{\text{The Slope of the tangent}}$

⇒ The Slope of the normal = $\frac{-1}{\left(\frac{dy}{dx}\right)_{\theta=\frac{-\pi}{2}}}$

⇒ The Slope of the normal = $\frac{-1}{1}$

⇒ The Slope of the normal = – 1

(vi) $x = a \cos^3 \theta$, $y = a \sin^3 \theta$ at $\theta = \pi/4$

Solution:

Given $x = a \cos^3 \theta$, $y = a \sin^3 \theta$ at $\theta = \pi/4$

Here, to find $\frac{dy}{dx}$, we have to find $\frac{dy}{d\theta}$ & $\frac{dx}{d\theta}$ and divide $\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ and we get $\frac{dy}{dx}$.

$\therefore \frac{dy}{dx}(x^n) = n\, x^{n-1}$

$\Rightarrow x = a\cos^3 \theta$

$\Rightarrow \frac{dx}{d\theta} = a\left(\frac{dx}{d\theta}(\cos^3 \theta)\right)$

$\therefore \frac{d}{dx}(\text{Cos } x) = -\sin x$

$\Rightarrow \frac{dx}{d\theta} = a\,(3\cos^{3-1} \theta \times -\sin \theta)$

$\Rightarrow \frac{dx}{d\theta} = a\,(3\cos^2 \theta \times -\sin \theta)$

$\Rightarrow \frac{dx}{d\theta} = -3a\cos^2 \theta \sin \theta$... (1)

$\Rightarrow y = a\sin^3 \theta$

$\Rightarrow \frac{dy}{d\theta} = a\left(\frac{dy}{d\theta}(\sin^3 \theta)\right)$

$\therefore \frac{d}{dx}(\text{Sin } x) = \cos x$

$\Rightarrow \frac{dy}{d\theta} = a\,(3\sin^{3-1} \theta \cos \theta)$

$\Rightarrow \frac{dy}{d\theta} = a\,(3\sin^2 \theta \cos \theta)$

$\Rightarrow \frac{dy}{d\theta} = 3\,a \sin^2 \theta \cos \theta$... (2)

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-3a\cos^2\theta\sin\theta}{3a\sin^2\theta\cos\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\cos\theta}{\sin\theta}$$

$$\Rightarrow \frac{dy}{dx} = -\tan\theta$$

The Slope of the tangent is – tan θ

Since, θ = π /4

$$\Rightarrow \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}} = -\tan(\pi/4)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}} = -1$$

We know tan (π /4) = 1

The Slope of the tangent at x = π /4 is – 1

$$\Rightarrow \text{The Slope of the normal} = \frac{-1}{\text{The Slope of the tangent}}$$

$$\Rightarrow \text{The Slope of the normal} = \frac{-1}{\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}}}$$

$$\Rightarrow \text{The Slope of the normal} = \frac{-1}{-1}$$

⇒ The Slope of the normal = 1

(vii) x = a (θ - sin θ), y = a (1 - cos θ) at θ = π /2

Solution:

Given x = a (θ - sin θ), y = a (1 - cos θ) at θ = π /2

Here, to find $\frac{dy}{dx}$, we have to find $\frac{dy}{d\theta}$ & $\frac{dx}{d\theta}$ and divide $\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ and we get $\frac{dy}{dx}$.

$\therefore \frac{dy}{dx}(x^n) = n\,x^{n-1}$

$\Rightarrow x = a\,(\theta - \sin\theta)$

$\Rightarrow \frac{dx}{d\theta} = a\,\{\frac{dx}{d\theta}(\theta) - \frac{dx}{d\theta}(\sin\theta)\}$

$\Rightarrow \frac{dx}{d\theta} = a\,(1 - \cos\theta)$ ... (1)

$\therefore \frac{d}{dx}(\text{Sin } x) = \cos x$

$\Rightarrow y = a\,(1 - \cos\theta)$

$\Rightarrow \frac{dy}{d\theta} = a\,(\frac{dx}{d\theta}(1) - \frac{dx}{d\theta}(\cos\theta))$

$\therefore \frac{d}{dx}(\text{Cos } x) = -\sin x$

$\therefore \frac{d}{dx}(\text{Constant}) = 0$

$\Rightarrow \frac{dy}{d\theta} = a\,(\theta - (-\sin\theta))$

$\Rightarrow \frac{dy}{d\theta} = a\sin\theta$ ... (2)

$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1-\cos\theta)}$

$\Rightarrow \frac{dy}{dx} = \frac{-\sin\theta}{(1-\cos\theta)}$

The Slope of the tangent is $\frac{-\sin\theta}{(1-\cos\theta)}$

Since, $\theta = \frac{\pi}{2}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{2}} = \frac{\sin\frac{\pi}{2}}{(1-\cos\frac{\pi}{2})}$

We know cos (π /2) = 0 and sin (π /2) = 1

$\Rightarrow \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{2}} = \frac{(1)}{(1-(-0))}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{2}} = \frac{1}{(1-0)}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{2}} = 1$

The Slope of the tangent at $x = \frac{\pi}{2}$ is 1

⇒ The Slope of the normal = $\frac{-1}{\text{The Slope of the tangent}}$

⇒ The Slope of the normal = $\frac{-1}{\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{2}}}$

⇒ The Slope of the normal = $\frac{-1}{1}$

⇒ The Slope of the normal = – 1

(viii) y = (sin 2x + cot x + 2)^2 at x = π /2

Solution:

Given y = (sin 2x + cot x + 2)^2 at x = π /2

First, we have to find $\frac{dy}{dx}$ of given function f(x) that is to find the derivative of f(x)

$\therefore \frac{dy}{dx}(x^n) = n\,x^{n-1}$

The Slope of the tangent is $\frac{dy}{dx}$

⇒ y = (sin2x + cot x + 2)^2

$\frac{dy}{dx} = 2 \times (\sin 2x + \cot x + 2)^{2-1}\{\frac{dy}{dx}(\sin 2x) + \frac{dy}{dx}(\cot x) + \frac{dy}{dx}(2)\}$

$\Rightarrow \frac{dy}{dx} = 2(\sin 2x + \cot x + 2)\ \{(\cos 2x) \times 2 + (-\operatorname{cosec}^2 x) + (0)\}$

$\therefore \frac{d}{dx}(\text{Sin } x) = \cos x$

$\therefore \frac{d}{dx}(\text{Cot } x) = -\operatorname{cosec}^2 x$

$\Rightarrow \frac{dy}{dx} = 2(\sin 2x + \cot x + 2)\ (2 \cos 2x - \operatorname{cosec}^2 x)$

Since, $x = \pi/2$

$\left(\frac{dy}{dx}\right)_{\theta = \frac{\pi}{2}} = 2 \times (\sin 2\ (\pi/2) + \cot (\pi/2) + 2)\ (2 \cos 2\ (\pi/2) - \operatorname{cosec}^2 (\pi/2))$

$\left(\frac{dy}{dx}\right)_{\theta = \frac{\pi}{2}} = 2 \times (\sin (\pi) + \cot (\pi/2) + 2) \times (2\cos (\pi) - \operatorname{cosec}^2 (\pi/2))$

$\Rightarrow \left(\frac{dy}{dx}\right)_{\theta = \frac{\pi}{2}} = 2 \times (0 + 0 + 2) \times (2(-1) - 1)$

We know $\sin (\pi) = 0$, $\cos (\pi) = -1$

$\text{Cot } (\pi/2) = 0$, $\operatorname{cosec} (\pi/2) = 1$

$\Rightarrow \left(\frac{dy}{dx}\right)_{\theta = \frac{\pi}{2}} = 2\ (2) \times (-2 - 1)$

$\Rightarrow \left(\frac{dy}{dx}\right)_{\theta = \frac{\pi}{2}} = 4 \times -3$

$\Rightarrow \left(\frac{dy}{dx}\right)_{\theta = \frac{\pi}{2}} = -12$

The Slope of the tangent at $x = \frac{\pi}{2}$ is -12

$\Rightarrow$ The Slope of the normal $= \frac{-1}{\text{The Slope of the tangent}}$

$\Rightarrow$ The Slope of the normal = $\frac{-1}{\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{2}}}$

$\Rightarrow$ The Slope of the normal = $\frac{-1}{-12}$

$\Rightarrow$ The Slope of the normal = $\frac{1}{12}$

(ix) $x^2 + 3y + y^2 = 5$ at (1, 1)

Solution:

Given $x^2 + 3y + y^2 = 5$ at (1, 1)

Here we have to differentiate the above equation with respect to x.

$\Rightarrow \frac{d}{dx}(x^2 + 3y + y^2) = \frac{d}{dx}(5)$

$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(3y) + \frac{d}{dx}(y^2) = \frac{d}{dx}(5)$

$\therefore \frac{dy}{dx}(x^n) = n\,x^{n-1}$

$\Rightarrow 2x + 3 \times \frac{dy}{dx} + 2y \times \frac{dy}{dx} = 0$

$\Rightarrow 2x + \frac{dy}{dx}(3 + 2y) = 0$

$\Rightarrow \frac{dy}{dx}(3 + 2y) = -2x$

$\Rightarrow \frac{dy}{dx} = \frac{-2x}{(3+2y)}$

The Slope of the tangent at (1, 1) is

$\Rightarrow \frac{dy}{dx} = \frac{-2 \times 1}{(3 + 2 \times 1)}$

$\Rightarrow \frac{dy}{dx} = \frac{-2}{(3+2)}$

$\Rightarrow \frac{dy}{dx} = \frac{-2}{5}$

The Slope of the tangent at (1, 1) is $\frac{-2}{5}$

$\Rightarrow$ The Slope of the normal = $\frac{-1}{\text{The Slope of the tangent}}$

$\Rightarrow$ The Slope of the normal = $\frac{-1}{\left(\frac{dy}{dx}\right)}$

$\Rightarrow$ The Slope of the normal = $\frac{-1}{\frac{-2}{5}}$

$\Rightarrow$ The Slope of the normal = $\frac{5}{2}$

(x) x y = 6 at (1, 6)

Solution:

Given xy = 6 at (1, 6)

Here we have to use the product rule for above equation, then we get

$\frac{d}{dx}(x\,y) = \frac{d}{dx}(6)$

$\Rightarrow x \times \frac{d}{dx}(y) + y\frac{d}{dx}(x) = \frac{d}{dx}(5)$

$\therefore \frac{d}{dx}$ (Constant) = 0

$\Rightarrow x\frac{dy}{dx} + y = 0$

$\Rightarrow x\frac{dy}{dx} = -y$

$\Rightarrow \frac{dy}{dx} = \frac{-y}{x}$

The Slope of the tangent at (1, 6) is

$\Rightarrow \frac{dy}{dx} = \frac{-6}{1}$

$\Rightarrow \frac{dy}{dx} = -6$

The Slope of the tangent at (1, 6) is – 6

$\Rightarrow$ The Slope of the normal = $\frac{-1}{\text{The Slope of the tangent}}$

$\Rightarrow$ The Slope of the normal = $\frac{-1}{\left(\frac{dy}{dx}\right)}$

$\Rightarrow$ The Slope of the normal = $\frac{-1}{-6}$

$\Rightarrow$ The Slope of the normal = $\frac{1}{6}$

2. Find the values of a and b if the Slope of the tangent to the curve x y + a x + by = 2 at (1, 1) is 2.

Solution:

Given the Slope of the tangent to the curve xy + ax + by = 2 at (1, 1) is 2

First, we will find The Slope of tangent by using product rule, we get

$\Rightarrow x\,y + ax + by = 2$

$\Rightarrow x\,\frac{d}{dx}(y) + y\frac{d}{dx}(x) + a\,\frac{d}{dx}(x) + b\,\frac{d}{dx}(y) + = \frac{d}{dx}(2)$

$\Rightarrow x\frac{dy}{dx} + y + a + b\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx}(x + b) + y + a = 0$

$\Rightarrow \frac{dy}{dx}(x + b) = -(a + y)$

$\Rightarrow \frac{dy}{dx} = \frac{-(a+y)}{x+b}$

Since, the Slope of the tangent to the curve xy + ax + by = 2 at (1, 1) is 2 that is,

$\frac{dy}{dx} = 2$

$\Rightarrow \left\{ \frac{-(a+y)}{x+b} \right\}_{(x=1,\, y=1)} = 2$

$\Rightarrow \frac{-(a+1)}{1+b} = 2$

⇒ – a – 1 = 2(1 + b)
⇒ – a – 1 = 2 + 2b
⇒ a + 2b = – 3 ... (1)
Also, the point (1, 1) lies on the curve xy + ax + by = 2, we have
1 × 1 + a × 1 + b × 1 = 2
⇒ 1 + a + b = 2
⇒ a + b = 1 ... (2)
From (1) & (2), we get b = -4
Substitute b = – 4 in a + b = 1
a – 4 = 1
⇒ a = 5
So the value of a = 5 & b = – 4

3. If the tangent to the curve y = x^3 + a x + b at (1, – 6) is parallel to the line x – y + 5 = 0, find a and b

Solution:
Given the Slope of the tangent to the curve y = x^3 + ax + b at (1, – 6)

First, we will find the slope of tangent

$y = x^3 + ax + b$

$\frac{dy}{dx} = \frac{d}{dx}(x^3) + \frac{d}{dx}(ax) + \frac{d}{dx}(b)$

$\Rightarrow \frac{dy}{dx} = 3x^{3-1} + a\left(\frac{dx}{dx}\right) + 0$

$\Rightarrow \frac{dy}{dx} = 3x^2 + a$

The Slope of the tangent to the curve $y = x^3 + ax + b$ at $(1, -6)$ is

$\Rightarrow \frac{dy}{dx}_{(x=1, y=-6)} = 3(1)^2 + a$

$\Rightarrow \frac{dy}{dx}_{(x=1, y=-6)} = 3 + a$ ... (1)

The given line is $x - y + 5 = 0$

$y = x + 5$ is the form of equation of a straight line $y = mx + c$, where m is the Slope of the line.

So the slope of the line is $y = 1 \times x + 5$

So the Slope is 1. ... (2)

Also the point $(1, -6)$ lie on the tangent, so

$x = 1$ & $y = -6$ satisfies the equation, $y = x^3 + ax + b$

$-6 = 1^3 + a \times 1 + b$

$\Rightarrow -6 = 1 + a + b$

$\Rightarrow a + b = -7$ ... (3)

Since, the tangent is parallel to the line, from (1) & (2)

Hence, $3 + a = 1$

$\Rightarrow a = -2$

From (3)

$a + b = -7$

$\Rightarrow -2 + b = -7$

$\Rightarrow b = -5$

So the value is $a = -2$ & $b = -5$

4. Find a point on the curve $y = x^3 - 3x$ where the tangent is parallel to the chord joining $(1, -2)$ and $(2, 2)$.

Solution:

Given curve $y = x^3 - 3x$

First, we will find the Slope of the tangent

$y = x^3 - 3x$

$\frac{dy}{dx} = \frac{d}{dx}(x^3) - \frac{d}{dx}(3x)$

$\Rightarrow \frac{dy}{dx} = 3x^{3-1} - 3(\frac{dx}{dx})$

$\Rightarrow \frac{dy}{dx} = 3x^2 - 3$ ... (1)

The equation of line passing through (x_0, y_0) and The Slope m is $y - y_0 = m(x - x_0)$.

So the Slope, m = $\frac{y - y_0}{x - x_0}$

The Slope of the chord joining (1, – 2) & (2, 2)

$\Rightarrow \frac{dy}{dx} = \frac{2-(-2)}{2-1}$

$\Rightarrow \frac{dy}{dx} = \frac{4}{1}$

$\Rightarrow \frac{dy}{dx} = 4$ ... (2)

From (1) & (2)

$3x^2 - 3 = 4$

$\Rightarrow 3x^2 = 7$

$\Rightarrow x^2 = \frac{7}{3}$

$\Rightarrow x = \pm\sqrt{\frac{7}{3}}$

$y = x^3 - 3x$

$\Rightarrow y = x(x^2 - 3)$

$\Rightarrow y = \pm\sqrt{\frac{7}{3}}((\pm\sqrt{\frac{7}{3}})^2 - 3)$

$\Rightarrow y = \pm\sqrt{\frac{7}{3}}\left(\left(\frac{7}{3} - 3\right)\right)$

$\Rightarrow y = \pm\sqrt{\frac{7}{3}}\left(\frac{-2}{3}\right)$

$\Rightarrow y = \mp\left(\frac{-2}{3}\right)\sqrt{\frac{7}{3}}$

Thus, the required point is $x = \pm\sqrt{\frac{7}{3}}$ & $y = \mp\left(\frac{-2}{3}\right)\sqrt{\frac{7}{3}}$

5. Find a point on the curve $y = x^3 - 2x^2 - 2x$ at which the tangent lines are parallel to the line $y = 2x - 3$.

Solution:

Given the curve $y = x^3 - 2x^2 - 2x$ and a line $y = 2x - 3$

First, we will find the slope of tangent

$y = x^3 - 2x^2 - 2x$

$\frac{dy}{dx} = \frac{d}{dx}(x^3) - \frac{d}{dx}(2x^2) - \frac{d}{dx}(2x)$

$\Rightarrow \frac{dy}{dx} = 3x^{3-1} - 2 \times 2\,(x^{2-1}) - 2 \times x^{1-1}$

$\Rightarrow \frac{dy}{dx} = 3x^2 - 4x - 2$ ... (1)

$y = 2x - 3$ is the form of equation of a straight line $y = mx + c$, where m is the Slope of the line.

So the slope of the line is $y = 2 \times (x) - 3$

Thus, the Slope = 2. ... (2)

From (1) & (2)

$\Rightarrow 3x^2 - 4x - 2 = 2$

$\Rightarrow 3x^2 - 4x = 4$

$\Rightarrow 3x^2 - 4x - 4 = 0$

We will use factorization method to solve the above Quadratic equation.

$\Rightarrow 3x^2 - 6x + 2x - 4 = 0$

$\Rightarrow 3x(x - 2) + 2(x - 2) = 0$

⇒ $(x - 2)(3x + 2) = 0$
⇒ $(x - 2) = 0$ & $(3x + 2) = 0$
⇒ $x = 2$ or
$x = -2/3$
Substitute $x = 2$ & $x = -2/3$ in $y = x^3 - 2x^2 - 2x$
When $x = 2$
⇒ $y = (2)^3 - 2 \times (2)^2 - 2 \times (2)$
⇒ $y = 8 - (2 \times 4) - 4$
⇒ $y = 8 - 8 - 4$
⇒ $y = -4$
When $x = \frac{-2}{3}$

⇒ $y = \left(\frac{-2}{3}\right)^3 - 2 \times \left(\frac{-2}{3}\right)^2 - 2 \times \left(\frac{-2}{3}\right)$

⇒ $y = \left(\frac{-8}{27}\right) - 2 \times \left(\frac{4}{9}\right) + \left(\frac{4}{3}\right)$

⇒ $y = \left(\frac{-8}{27}\right) - \left(\frac{8}{9}\right) + \left(\frac{4}{3}\right)$

Taking LCM

⇒ $y = \frac{(-8\times1)-(8\times3) + (4\times9)}{27}$

⇒ $y = \frac{-8-24+36}{27}$

⇒ $y = \frac{4}{27}$

Thus, the points are $(2, -4)$ & $\left(\frac{-2}{3}, \frac{4}{27}\right)$

6. Find a point on the curve $y^2 = 2x^3$ at which the Slope of the tangent is 3

Solution:
Given the curve $y^2 = 2x^3$ and the Slope of tangent is 3
$y^2 = 2x^3$
Differentiating the above with respect to x

$\Rightarrow 2y^{2-1}\frac{dy}{dx} = 2 \times 3x^{3-1}$

$\Rightarrow y\frac{dy}{dx} = 3x^2$

$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{y}$

Since, The Slope of tangent is 3

$\frac{3x^2}{y} = 3$

$\Rightarrow \frac{x^2}{y} = 1$

$\Rightarrow x^2 = y$

Substituting $x^2 = y$ in $y^2 = 2x^3$,

$(x^2)^2 = 2x^3$

$x^4 - 2x^3 = 0$

$x^3(x - 2) = 0$

$x^3 = 0$ or $(x - 2) = 0$

$x = 0$ or $x = 2$

If $x = 0$

$\Rightarrow \frac{dy}{dx} = \frac{3(0)^2}{y}$

dy/dx = 0 which is not possible.
So we take x = 2 and substitute it in $y^2 = 2x^3$, we get
$y^2 = 2(2)^3$
$y^2 = 2 \times 8$
$y^2 = 16$
$y = 4$
Thus, the required point is (2, 4)

7. Find a point on the curve x y + 4 = 0 at which the tangents are inclined at an angle of

45° with the x–axis.

Solution:

Given the curve is xy + 4 = 0

If a tangent line to the curve y = f(x) makes an angle θ with x – axis in the positive direction, then

$\frac{dy}{dx}$ = The Slope of the tangent = tan θ

xy + 4 = 0

Differentiating the above with respect to x

$\Rightarrow x\frac{d}{dx}(y) + y\frac{d}{dx}(x) + \frac{d}{dx}(4) = 0$

$\Rightarrow x\frac{dy}{dx} + y = 0$

$\Rightarrow x\frac{dy}{dx} = -y$

$\Rightarrow \frac{dy}{dx} = \frac{-y}{x}$ ... (1)

Also, $\frac{dy}{dx} = \tan 45° = 1$ ... (2)

From (1) & (2), we get,

$\Rightarrow \frac{-y}{x} = 1$

$\Rightarrow x = -y$

Substitute in xy + 4 = 0, we get

$\Rightarrow x(-x) + 4 = 0$

$\Rightarrow -x^2 + 4 = 0$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

So when x = 2, y = – 2

And when x = – 2, y = 2

Thus, the points are (2, – 2) & (– 2, 2)

8. Find a point on the curve $y = x^2$ where the Slope of the tangent is equal to the x – coordinate of the point.

Solution:

Given the curve is $y = x^2$

$y = x^2$

Differentiating the above with respect to x

$\Rightarrow \frac{dy}{dx} = 2x^{2-1}$

$\Rightarrow \frac{dy}{dx} = 2x$ ... (1)

Also given the Slope of the tangent is equal to the x – coordinate,

$\frac{dy}{dx} = x$ ... (2)

From (1) & (2), we get,

$2x = x$

$\Rightarrow x = 0.$

Substituting this in $y = x^2$, we get,

$y = 0^2$

$\Rightarrow y = 0$

Thus, the required point is (0, 0)

9. At what point on the circle $x^2 + y^2 – 2x – 4y + 1 = 0$, the tangent is parallel to x – axis.

Solution:

Given the curve is $x^2 + y^2 – 2x – 4y + 1 = 0$

Differentiating the above with respect to x

$\Rightarrow x^2 + y^2 – 2x – 4y + 1 = 0$

$\Rightarrow 2x^{2-1} + 2y^{2-1} \times \frac{dy}{dx} – 2 – 4 \times \frac{dy}{dx} + 0 = 0$

$\Rightarrow 2x + 2y\frac{dy}{dx} – 2 – 4\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx}(2y - 4) = -2x + 2$

$\Rightarrow \frac{dy}{dx} = \frac{-2(x-1)}{2(y-2)}$

$\Rightarrow \frac{dy}{dx} = \frac{-(x-1)}{(y-2)}$ ... (1)

$\therefore \frac{dy}{dx}$ = The Slope of the tangent = tan θ

Since, the tangent is parallel to x – axis

$\Rightarrow \frac{dy}{dx} = \tan(0) = 0$ ... (2)

Because tan (0) = 0

From (1) & (2), we get,

$\Rightarrow \frac{-(x-1)}{(y-2)} = 0$

$\Rightarrow -(x - 1) = 0$

$\Rightarrow x = 1$

Substituting x = 1 in $x^2 + y^2 - 2x - 4y + 1 = 0$, we get,

$\Rightarrow 1^2 + y^2 - 2(1) - 4y + 1 = 0$

$\Rightarrow 1 - y^2 - 2 - 4y + 1 = 0$

$\Rightarrow y^2 - 4y = 0$

$\Rightarrow y(y - 4) = 0$

$\Rightarrow y = 0$ and $y = 4$

Thus, the required point is (1, 0) and (1, 4)

10. At what point of the curve $y = x^2$ does the tangent make an angle of 45° with the x–axis?

Solution:

Given the curve is $y = x^2$

Differentiating the above with respect to x

$\Rightarrow y = x^2$

$\Rightarrow \frac{dy}{dx} = 2x^{2-1}$

$\Rightarrow \frac{dy}{dx} = 2x$ ... (1)

$\therefore \frac{dy}{dx}$ = The Slope of the tangent = tan θ

Since, the tangent make an angle of 45° with x – axis

$\Rightarrow \frac{dy}{dx} = \tan(45°) = 1$ ... (2)

Because tan (45°) = 1

From (1) & (2), we get,

$\Rightarrow 2x = 1$

$\Rightarrow x = \frac{1}{2}$

Substituting $x = \frac{1}{2}$ in $y = x^2$, we get,

$\Rightarrow y = (\frac{1}{2})^2$

$\Rightarrow y = \frac{1}{4}$

Thus, the required point is $(\frac{1}{2}, \frac{1}{4})$

EXERCISE 16.2 PAGE NO: 15.27

1. Find the equation of the tangent to the curve √x + √y = a, at the point ($a^2/4$, $a^2/4$).

Solution:

Given √x + √y = a

To find the slope of the tangent of the given curve we have to differentiate the given equation

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\left(\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{x}}{\sqrt{y}}$$

At $\left(\frac{a^2}{4}, \frac{a^2}{4}\right)$ slope m, is – 1

The equation of the tangent is given by $y - y_1 = m(x - x_1)$

$$y - \frac{a^2}{4} = -1\left(x - \frac{a^2}{4}\right)$$

$$x + y = \frac{a^2}{2}$$

2. Find the equation of the normal to y = $2x^3 - x^2 + 3$ at (1, 4).

Solution:

Given $y = 2x^3 - x^2 + 3$

By differentiating the given curve, we get the slope of the tangent

$$m = \frac{dy}{dx} = 6x^2 - 2x$$

m = 4 at (1, 4)

Normal is perpendicular to tangent so, $m_1 m_2 = -1$

$$m(\text{normal}) = -\frac{1}{4}$$

Equation of normal is given by $y - y_1 = m\ (\text{normal})\ (x - x_1)$

$$y - 4 = \left(-\frac{1}{4}\right)(x - 1)$$

$x + 4y = 17$

3. Find the equation of the tangent and the normal to the following curves at the indicated points:
(i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0, 5)

Solution:
Given $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0, 5)

By differentiating the given curve, we get the slope of the tangent

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

m (tangent) at (0, 5) = – 10

$$m(\text{normal}) \text{ at } (0,5) = \frac{1}{10}$$

Equation of tangent is given by $y - y_1 = m\ (\text{tangent})\ (x - x_1)$

$y - 5 = -10x$

$y + 10x = 5$

Equation of normal is given by $y - y_1 = m\ (\text{normal})\ (x - x_1)$

$$y - 5 = \frac{1}{10}x$$

(ii) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at x = 1 y = 3

Solution:

Given $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $x = 1$ $y = 3$

By differentiating the given curve, we get the slope of the tangent

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

m (tangent) at (x = 1) = 2

Normal is perpendicular to tangent so, $m_1m_2 = -1$

$$m(\text{normal}) \text{ at } (x = 1) = -\frac{1}{2}$$

Equation of tangent is given by $y - y_1 = m\text{ (tangent) } (x - x_1)$

$y - 3 = 2(x - 1)$

$y = 2x + 1$

Equation of normal is given by $y - y_1 = m\text{ (normal) } (x - x_1)$

$$y - 3 = -\frac{1}{2}(x - 1)$$

$$2y = 7 - x$$

(iii) $y = x^2$ at (0, 0)

Solution:

Given $y = x^2$ at (0, 0)

By differentiating the given curve, we get the slope of the tangent

$$\frac{dy}{dx} = 2x$$

m (tangent) at (x = 0) = 0

Normal is perpendicular to tangent so, $m_1m_2 = -1$

$$m(\text{normal}) \text{ at } (x = 0) = \frac{1}{0}$$

We can see that the slope of normal is not defined

Equation of tangent is given by $y - y_1 = m\text{ (tangent) } (x - x_1)$

$y = 0$

Equation of normal is given by $y - y_1 = m\text{ (normal) } (x - x_1)$

x = 0

(iv) $y = 2x^2 - 3x - 1$ at (1, – 2)

Solution:

Given $y = 2x^2 - 3x - 1$ at (1, – 2)

By differentiating the given curve, we get the slope of the tangent

$$\frac{dy}{dx} = 4x - 3$$

m (tangent) at (1, – 2) = 1

Normal is perpendicular to tangent so, $m_1m_2 = -1$

m (normal) at (1, – 2) = – 1

Equation of tangent is given by $y - y_1 = m \text{ (tangent) } (x - x_1)$

y + 2 = 1(x – 1)

y = x – 3

Equation of normal is given by $y - y_1 = m \text{ (normal) } (x - x_1)$

y + 2 = – 1(x – 1)

y + x + 1 = 0

$$(v)\ y^2 = \frac{x^3}{4 - x}$$

Solution:

By differentiating the given curve, we get the slope of the tangent

$$2y\frac{dy}{dx} = \frac{(4 - x)3x^2 + x^4}{(4 - x)^2}$$

$$\frac{dy}{dx} = \frac{(4 - x)3x^2 + x^4}{2y(4 - x)^2}$$

m (tangent) at (2, – 2) = – 2

$$\text{m(normal) at } (2, -2) = \frac{1}{2}$$

Equation of tangent is given by $y - y_1 = m \text{ (tangent) } (x - x_1)$

y + 2 = – 2(x – 2)

y + 2x = 2

Equation of normal is given by $y - y_1 = m$ (normal) $(x - x_1)$

$$y + 2 = \frac{1}{2}(x - 2)$$

$2y + 4 = x - 2$

$2y - x + 6 = 0$

4. Find the equation of the tangent to the curve x = θ + sin θ, y = 1 + cos θ at θ = π/4.

Solution:

Given x = θ + sin θ, y = 1 + cos θ at θ = π/4

By differentiating the given equation with respect to θ, we get the slope of the tangent

$$\frac{dx}{d\theta} = 1 + \cos\theta$$

$$\frac{dy}{d\theta} = -\sin\theta$$

Dividing both the above equations

$$\frac{dy}{dx} = -\frac{\sin\theta}{1 + \cos\theta}$$

m at θ = ($\pi/4$) = $-1 + \frac{1}{\sqrt{2}}$

Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$

$$y - 1 - \frac{1}{\sqrt{2}} = \left(-1 + \frac{1}{\sqrt{2}}\right)\left(x - \frac{\pi}{4} - \frac{1}{\sqrt{2}}\right)$$

5. Find the equation of the tangent and the normal to the following curves at the indicated points:

(i) x = θ + sin θ, y = 1 + cos θ at θ = π/2

Solution:

Given x = θ + sin θ, y = 1 + cos θ at θ = π/2

By differentiating the given equation with respect to θ, we get the slope of the tangent

$$\frac{dx}{d\theta} = 1 + \cos\theta$$

$$\frac{dy}{d\theta} = -\sin\theta$$

Dividing both the above equations

$$\frac{dy}{dx} = -\frac{\sin\theta}{1 + \cos\theta}$$

m (tangent) at $\theta = (\pi/2) = -1$

Normal is perpendicular to tangent so, $m_1m_2 = -1$

m (normal) at $\theta = (\pi/2) = 1$

Equation of tangent is given by $y - y_1 = m \text{ (tangent) } (x - x_1)$

$$y - 1 = -1\left(x - \frac{\pi}{2} - 1\right)$$

Equation of normal is given by $y - y_1 = m \text{ (normal) } (x - x_1)$

$$y - 1 = 1\left(x - \frac{\pi}{2} - 1\right)$$

(ii) $x = \frac{2at^2}{1+t^2},\ y = \frac{2at^3}{1+t^2}$ at $t = \frac{1}{2}$

Solution:

By differentiating the given equation with respect to t, we get the slope of the tangent

$$\frac{dx}{dt} = \frac{(1 + t^2)4at - 2at^2(2t)}{(1 + t^2)^2}$$

$$\frac{dx}{dt} = \frac{4at}{(1 + t^2)^2}$$

$$\frac{dy}{dt} = \frac{(1 + t^2)6at^2 - 2at^3(2t)}{(1 + t^2)^2}$$

$$\frac{dy}{dt} = \frac{6at^2 + 2at^4}{(1 + t^2)^2}$$

Now dividing $\frac{dy}{dt}$ and $\frac{dx}{dt}$ to obtain the slope of tangent

$$\frac{dy}{dx} = \frac{6at^2 + 2at^4}{4at}$$

m (tangent) at t = $\frac{1}{2}$ is $\frac{13}{16}$

Normal is perpendicular to tangent so, $m_1 m_2 = -1$

m (normal) at t = $\frac{1}{2}$ is $-\frac{16}{13}$

Equation of tangent is given by $y - y_1$ = m (tangent) $(x - x_1)$

$$y - \frac{a}{5} = \frac{13}{16}\left(x - \frac{2a}{5}\right)$$

Equation of normal is given by $y - y_1$ = m (normal) $(x - x_1)$

$$y - \frac{a}{5} = -\frac{16}{13}\left(x - \frac{2a}{5}\right)$$

(iii) $x = at^2$, $y = 2at$ at $t = 1$.

Solution:

Given $x = at^2$, $y = 2at$ at $t = 1$.

By differentiating the given equation with respect to t, we get the slope of the tangent

$$\frac{dx}{dt} = 2at$$

$$\frac{dy}{dt} = 2a$$

Now dividing $\frac{dy}{dt}$ and $\frac{dx}{dt}$ to obtain the slope of tangent

$$\frac{dy}{dx} = \frac{1}{t}$$

m (tangent) at t = 1 is 1

Normal is perpendicular to tangent so, $m_1m_2 = -1$

m (normal) at t = 1 is – 1

Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$

$y - 2a = 1(x - a)$

Equation of normal is given by $y - y_1 = m$ (normal) $(x - x_1)$

$y - 2a = -1(x - a)$

(iv) x = a sec t, y = b tan t at t.

Solution:

Given x = a sec t, y = b tan t at t.

By differentiating the given equation with respect to t, we get the slope of the tangent

$$\frac{dx}{dt} = a\sec t \tan t$$

$$\frac{dy}{dt} = b\sec^2 t$$

Now dividing $\frac{dy}{dt}$ and $\frac{dx}{dt}$ to obtain the slope of tangent

$$\frac{dy}{dx} = \frac{b\operatorname{cosec} t}{a}$$

m (tangent) at t = $\frac{b\operatorname{cosec} t}{a}$

Normal is perpendicular to tangent so, $m_1m_2 = -1$

m (normal) at t = $-\frac{a}{b}\sin t$

Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$

$$y - b\tan t = \frac{b\operatorname{cosec} t}{a}(x - a\sec t)$$

Equation of normal is given by $y - y_1 = m$ (normal) $(x - x_1)$

$$y - b\tan t = -\frac{a\sin t}{b}(x - a\sec t)$$

(v) x = a (θ + sin θ), y = a (1 – cos θ) at θ

Solution:

Given $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at θ

By differentiating the given equation with respect to θ, we get the slope of the tangent

$$\frac{dx}{d\theta} = a(1 + \cos\theta)$$

$$\frac{dy}{d\theta} = a(\sin\theta)$$

Now dividing $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ to obtain the slope of tangent

$$\frac{dy}{dx} = \frac{\sin\theta}{1 + \cos\theta}$$

m (tangent) at theta is $\frac{\sin\theta}{1 + \cos\theta}$

Normal is perpendicular to tangent so, $m_1 m_2 = -1$

m (normal) at theta is $-\frac{\sin\theta}{1 + \cos\theta}$

Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$

$$y - a(1 - \cos\theta) = \frac{\sin\theta}{1 + \cos\theta}(x - a(\theta + \sin\theta))$$

Equation of normal is given by $y - y_1 = m$ (normal) $(x - x_1)$

$$y - a(1-\cos\theta) = \frac{1+\cos\theta}{-\sin\theta}(x - a(\theta + \sin\theta))$$

(vi) $x = 3\cos\theta - \cos^3\theta$, $y = 3\sin\theta - \sin^3\theta$

Solution:

Given $x = 3\cos\theta - \cos^3\theta$, $y = 3\sin\theta - \sin^3\theta$

By differentiating the given equation with respect to θ, we get the slope of the tangent

$$\frac{dx}{d\theta} = -3\sin\theta + 3\cos^2\theta\sin\theta$$

$$\frac{dy}{d\theta} = 3\cos\theta - 3\sin^2\theta\cos\theta$$

Now dividing $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ to obtain the slope of tangent

$$\frac{dy}{dx} = \frac{3\cos\theta - 3\sin^2\theta\cos\theta}{-3\sin\theta + 3\cos^2\theta\sin\theta} = -\tan^3\theta$$

m (tangent) at theta is $-\tan^3\theta$

Normal is perpendicular to tangent so, $m_1m_2 = -1$

m (normal) at theta is $\cot^3\theta$

Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$

$$y - 3\sin\theta + \sin^3\theta = -\tan^3\theta(x - 3\cos\theta + 3\cos^3\theta)$$

Equation of normal is given by $y - y_1 = m$ (normal) $(x - x_1)$

$$y - 3\sin\theta + \sin^3\theta = \cot^3\theta(x - 3\cos\theta + 3\cos^3\theta)$$

6. Find the equation of the normal to the curve $x^2 + 2y^2 - 4x - 6y + 8 = 0$ at the point whose abscissa is 2.

Solution:

Given $x^2 + 2y^2 - 4x - 6y + 8 = 0$

By differentiating the given curve, we get the slope of the tangent

$$2x + 4y\frac{dy}{dx} - 4 - 6\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{4 - 2x}{4y - 6}$$

Finding y co – ordinate by substituting x in the given curve

$2y^2 - 6y + 4 = 0$

$y^2 - 3y + 2 = 0$

$y = 2$ or $y = 1$

m (tangent) at $x = 2$ is 0

Normal is perpendicular to tangent so, $m_1m_2 = -1$

m (normal) at $x = 2$ is 1/0, which is undefined

Equation of normal is given by $y - y_1 = m$ (normal) $(x - x_1)$

$x = 2$

7. Find the equation of the normal to the curve $ay^2 = x^3$ at the point (am^2, am^3).

Solution:

Given $ay^2 = x^3$

By differentiating the given curve, we get the slope of the tangent

$$2ay\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{2ay}$$

m (tangent) at (am^2, am^3) is $\frac{3m}{2}$

Normal is perpendicular to tangent so, $m_1m_2 = -1$

m (normal) at (am^2, am^3) is $-\frac{2}{3m}$

Equation of normal is given by $y - y_1 = m$ (normal) $(x - x_1)$

$$y - am^3 = -\frac{2}{3m}(x - am^2)$$

8. The equation of the tangent at (2, 3) on the curve $y^2 = ax^3 + b$ is $y = 4x – 5$. Find the values of a and b.

Solution:

Given $y^2 = ax^3 + b$ is $y = 4x – 5$

By differentiating the given curve, we get the slope of the tangent

$$2y\frac{dy}{dx} = 3ax^2$$

$$\frac{dy}{dx} = \frac{3ax^2}{2y}$$

m (tangent) at (2, 3) = 2a

Equation of tangent is given by $y – y_1 = m$ (tangent) $(x – x_1)$

Now comparing the slope of a tangent with the given equation

2a = 4

a = 2

Now (2, 3) lies on the curve, these points must satisfy

$3^2 = 2 \times 2^3 + b$

b = – 7

9. Find the equation of the tangent line to the curve $y = x^2 + 4x – 16$ which is parallel to the line $3x – y + 1 = 0$.

Solution:

Given $y = x^2 + 4x – 16$

By differentiating the given curve, we get the slope of the tangent

$$\frac{dy}{dx} = 2x + 4$$

m (tangent) = 2x + 4

Equation of tangent is given by $y – y_1 = m$ (tangent) $(x – x_1)$

Now comparing the slope of a tangent with the given equation

$2x + 4 = 3$

$$x = -\frac{1}{2}$$

Now substituting the value of x in the curve to find y

$$y = \frac{1}{4} - 2 - 16 = -\frac{71}{4}$$

Therefore, the equation of tangent parallel to the given line is

$$y + \frac{71}{4} = 3\left(x + \frac{1}{2}\right)$$

10. Find the equation of normal line to the curve $y = x^3 + 2x + 6$ which is parallel to the line $x + 14y + 4 = 0$.

Solution:

Given $y = x^3 + 2x + 6$

By differentiating the given curve, we get the slope of the tangent

$$\frac{dy}{dx} = 3x^2 + 2$$

m (tangent) $= 3x^2 + 2$

Normal is perpendicular to tangent so, $m_1m_2 = -1$

m (normal) $= \frac{-1}{3x^2 + 2}$

Equation of normal is given by $y - y_1 = $ m (normal) $(x - x_1)$

Now comparing the slope of normal with the given equation

m (normal) $= -\frac{1}{14}$

$$-\frac{1}{14} = -\frac{1}{3x^2 + 2}$$

$x = 2$ or -2

Hence the corresponding value of y is 18 or – 6

So, equations of normal are

$$y - 18 = -\frac{1}{14}(x - 2)$$

Or

$$y + 6 = -\frac{1}{14}(x + 2)$$

EXERCISE 16.3 PAGE NO: 16.40

1. Find the angle to intersection of the following curves:
(i) $y^2 = x$ and $x^2 = y$

Solution:

Given curves $y^2 = x$ ... (1)

And $x^2 = y$ ... (2)

First curve is $y^2 = x$

Differentiating above with respect to x,

$\Rightarrow 2y.\frac{dy}{dx} = 1$

$\Rightarrow m_1 = \frac{dy}{dx} = \frac{1}{2x}$ ... (3)

The second curve is $x^2 = y$

$\Rightarrow 2x = \frac{dy}{dx}$

$\Rightarrow m_2 = \frac{dy}{dx} = 2x$ ... (4)

Substituting (1) in (2), we get

$\Rightarrow x^2 = y$

$\Rightarrow (y^2)^2 = y$

$\Rightarrow y^4 - y = 0$

$\Rightarrow y (y^3 - 1) = 0$

$\Rightarrow y = 0$ or $y = 1$

Substituting y = 0 & y = 1 in (1) in (2),

$x = y^2$

When y = 0, x = 0

When y = 1, x = 1

Substituting above values for m_1 & m_2, we get,

When x = 0,

$m_1 = \frac{dy}{dx} = \frac{1}{2\times0} = \infty$

When x = 1,

$m_1 = \frac{dy}{dx} = \frac{1}{2\times1} = \frac{1}{2}$

Values of m_1 is ∞ & $\frac{1}{2}$

When y = 0,

$m_2 = \frac{dy}{dx} = 2x = 2 \times 0 = 0$

When x = 1,

$m_2 = \frac{dy}{dx} = 3x = 2\times1 = 2$

Values of m_2 is 0 & 2

When $m_1 = \infty$ & $m_2 = 0$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\tan\theta = \left|\frac{0-\infty}{1 + \infty\times0}\right|$

$\tan\theta = \infty$

$\theta = \tan^{-1}(\infty)$

$\therefore \tan^{-1}(\infty) = \frac{\pi}{2}$

$\theta = \frac{\pi}{2}$

When $m_1 = \frac{1}{2}$ & $m_2 = 2$

Angle of intersection of two curves is given by $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\tan\theta = \left|\frac{2-\frac{1}{2}}{1+\frac{1}{2}\times 2}\right|$

$\tan\theta = \left|\frac{\frac{3}{2}}{2}\right|$

$\tan\theta = \left|\frac{3}{4}\right|$

$\theta = \tan^{-1}\left(\frac{3}{4}\right)$

(ii) $y = x^2$ and $x^2 + y^2 = 20$

Solution:

Given curves $y = x^2$ ... (1) and $x^2 + y^2 = 20$ ... (2)

Now consider first curve $y = x^2$

$\Rightarrow m_1 = \frac{dy}{dx} = 2x$ ... (3)

Consider second curve is $x^2 + y^2 = 20$

Differentiating above with respect to x,

$\Rightarrow 2x + 2y.\frac{dy}{dx} = 0$

$\Rightarrow y.\frac{dy}{dx} = -x$

$\Rightarrow m_2 = \frac{dy}{dx} = \frac{-x}{y}$ ... (4)

Substituting (1) in (2), we get

$\Rightarrow y + y^2 = 20$

$\Rightarrow y^2 + y - 20 = 0$

We will use factorization method to solve the above Quadratic equation

$\Rightarrow y^2 + 5y - 4y - 20 = 0$

$\Rightarrow y(y + 5) - 4(y + 5) = 0$

$\Rightarrow (y + 5)(y - 4) = 0$

$\Rightarrow y = -5$ & $y = 4$

Substituting $y = -5$ & $y = 4$ in (1) in (2),

$y = x^2$

When $y = -5$,

$\Rightarrow -5 = x^2$

$\Rightarrow x = \sqrt{-5}$

When $y = 4$,

$\Rightarrow 4 = x^2$

$\Rightarrow x = \pm 2$

Substituting above values for m_1 & m_2, we get,

When $x = 2$,

$m_1 = \frac{dy}{dx} = 2 \times 2$

$= 4$

When $x = 1$,

$m_1 = \frac{dy}{dx} = 2 \times -2$

$= -4$

Values of m_1 is 4 & -4

When $y = 4$ & $x = 2$

$m_2 = \frac{dy}{dx} = \frac{-x}{y} = \frac{-2}{4} = \frac{-1}{2}$

When $y = 4$ & $x = -2$

$m_2 = \frac{dy}{dx} = \frac{-x}{y} = \frac{2}{4} = \frac{1}{2}$

Values of m_2 is $\frac{-1}{2}$ & $\frac{1}{2}$

When $m_1 = \infty$ & $m_2 = 0$

Angle of intersection of two curves is given by $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\text{Tan}\,\theta = \left|\frac{\frac{-1}{2} - 4}{1 + 2\times 4}\right|$

$\text{Tan}\,\theta = \left|\frac{\frac{-9}{2}}{1-2}\right|$

$\text{Tan}\,\theta = \left|\frac{9}{2}\right|$

$\theta = \tan^{-1}(\frac{9}{2})$

(iii) $2y^2 = x^3$ and $y^2 = 32x$

Solution:

Given curves $2y^2 = x^3$ ... (1) and $y^2 = 32x$ ... (2)

First curve is $2y^2 = x^3$

Differentiating above with respect to x,

$\Rightarrow 4y.\frac{dy}{dx} = 3x^2$

$\Rightarrow m_1 = \frac{dy}{dx} = \frac{3x^2}{4y}$ ... (3)

Second curve is $y^2 = 32x$

$\Rightarrow 2y.\frac{dy}{dx} = 32$

$\Rightarrow y.\frac{dy}{dx} = 16$

$\Rightarrow m_2 = \frac{dy}{dx} = \frac{16}{y}$ ... (4)

Substituting (2) in (1), we get

$\Rightarrow 2y^2 = x^3$

$\Rightarrow 2(32x) = x^3$

$\Rightarrow 64\,x = x^3$

$\Rightarrow x^3 - 64x = 0$

$\Rightarrow x\,(x^2 - 64) = 0$

$\Rightarrow x = 0\ \&\ (x^2 - 64) = 0$

$\Rightarrow x = 0\ \&\ \pm 8$

Substituting $x = 0$ & $x = \pm 8$ in (1) in (2),

$y^2 = 32x$

When $x = 0$, $y = 0$

When $x = 8$

$\Rightarrow y^2 = 32 \times 8$

$\Rightarrow y^2 = 256$

$\Rightarrow y = \pm 16$

Substituting above values for m_1 & m_2, we get,

When $x = 0$, $y = 16$

$m_1 = \frac{dy}{dx}$

$\Rightarrow \frac{3\times 0^2}{4\times 8}$

$= 0$

When $x = 8$, $y = 16$

$m_1 = \frac{dy}{dx}$

$\Rightarrow \frac{3\times 8^2}{4\times 16}$

$= 3$

Values of m_1 is 0 & 3

When $x = 0$, $y = 0$,

$m_2 = \frac{dy}{dx}$

$\Rightarrow \frac{16}{y} = \frac{16}{0} = \infty$

When y = 16,

$m_2 = \frac{dy}{dx}$

$\Rightarrow \frac{16}{y} = \frac{16}{16}$

= 1

Values of m_2 is ∞ & 1

When $m_1 = 0$ & $m_2 = \infty$

$\Rightarrow \text{Tan}\ \theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\Rightarrow \text{Tan}\ \theta = \left|\frac{\infty - 0}{1 + \infty \times 0}\right|$

$\Rightarrow \text{Tan}\ \theta = \infty$

$\Rightarrow \theta = \tan^{-1}(\infty)$

$\therefore \text{Tan}^{-1}(\infty) = \frac{\pi}{2}$

$\Rightarrow \theta = \frac{\pi}{2}$

When $m_1 = \frac{1}{2}$ & $m_2 = 2$

Angle of intersection of two curves is given by $\tan \theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\Rightarrow \text{Tan}\ \theta = \left|\frac{3-1}{1 + 3 \times 1}\right|$

$\Rightarrow \text{Tan}\ \theta = \left|\frac{2}{4}\right|$

$\Rightarrow \text{Tan}\ \theta = \left|\frac{1}{2}\right|$

$\Rightarrow \theta = \tan^{-1}(\frac{1}{2})$

(iv) $x^2 + y^2 - 4x - 1 = 0$ and $x^2 + y^2 - 2y - 9 = 0$

Solution:

Given curves $x^2 + y^2 - 4x - 1 = 0$ ... (1) and $x^2 + y^2 - 2y - 9 = 0$ ... (2)
First curve is $x^2 + y^2 - 4x - 1 = 0$
$\Rightarrow x^2 - 4x + 4 + y^2 - 4 - 1 = 0$
$\Rightarrow (x - 2)^2 + y^2 - 5 = 0$
Now, Subtracting (2) from (1), we get
$\Rightarrow x^2 + y^2 - 4x - 1 - (x^2 + y^2 - 2y - 9) = 0$
$\Rightarrow x^2 + y^2 - 4x - 1 - x^2 - y^2 + 2y + 9 = 0$
$\Rightarrow - 4x - 1 + 2y + 9 = 0$
$\Rightarrow - 4x + 2y + 8 = 0$
$\Rightarrow 2y = 4x - 8$
$\Rightarrow y = 2x - 4$
Substituting $y = 2x - 4$ in (3), we get,
$\Rightarrow (x - 2)^2 + (2x - 4)^2 - 5 = 0$
$\Rightarrow (x - 2)^2 + 4(x - 2)^2 - 5 = 0$
$\Rightarrow (x - 2)^2(1 + 4) - 5 = 0$
$\Rightarrow 5(x - 2)^2 - 5 = 0$
$\Rightarrow (x - 2)^2 - 1 = 0$
$\Rightarrow (x - 2)^2 = 1$
$\Rightarrow (x - 2) = \pm1$
$\Rightarrow x = 1 + 2$ or $x = - 1 + 2$
$\Rightarrow x = 3$ or $x = 1$
So, when $x = 3$
$y = 2\times3 - 4$
$\Rightarrow y = 6 - 4 = 2$
So, when $x = 3$
$y = 2 \times 1 - 4$
$\Rightarrow y = 2 - 4 = - 2$
The point of intersection of two curves are (3, 2) & (1, – 2)
Now, differentiating curves (1) & (2) with respect to x, we get

$\Rightarrow x^2 + y^2 - 4x - 1 = 0$

$\Rightarrow 2x + 2y\frac{dy}{dx} - 4 - 0 = 0$

$\Rightarrow x + y\frac{dy}{dx} - 2 = 0$

$\Rightarrow y\frac{dy}{dx} = 2 - x$

$\Rightarrow \frac{dy}{dx} = \frac{2-x}{y}$ ... (3)

$\Rightarrow x^2 + y^2 - 2y - 9 = 0$

$\Rightarrow 2x + 2y\frac{dy}{dx} - 2\frac{dy}{dx} - 0 = 0$

$\Rightarrow x + y\frac{dy}{dx} - \frac{dy}{dx} = 0$

$\Rightarrow x + (y - 1)\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-x}{y-1}$ ... (4)

At (3, 2) in equation (3), we get

$\Rightarrow \frac{dy}{dx} = \frac{2-3}{2}$

$\Rightarrow m_1 = \frac{dy}{dx} = \frac{-1}{2}$

At (3, 2) in equation (4), we get

$\Rightarrow \frac{dy}{dx} = \frac{-3}{2-1}$

$\Rightarrow \frac{dy}{dx} = -3$

$\Rightarrow m_2 = \frac{dy}{dx} = -3$

When $m_1 = \frac{-1}{2}$ & $m_2 = 0$

Angle of intersection of two curves is given by $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\Rightarrow$ Tan θ $= \left|\frac{\frac{-1}{2}+3}{1+\frac{3}{2}}\right| = 1$

$\Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$

$(v)\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ and\ x^2 + y^2 = ab$

Solution:

Given curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ... (1) and $x^2 + y^2 = ab$ ... (2)

Second curve is $x^2 + y^2 = ab$

$y^2 = ab - x^2$

Substituting this in equation (1),

$\Rightarrow \frac{x^2}{a^2} + \frac{ab-x^2}{b^2} = 1$

$\Rightarrow \frac{x^2b^2 + a^2(ab-x^2)}{a^2b^2} = 1$

$\Rightarrow x^2b^2 + a^3b - a^2x^2 = a^2b^2$

$\Rightarrow x^2b^2 - a^2x^2 = a^2b^2 - a^3b$

$\Rightarrow x^2(b^2 - a^2) = a^2b(b - a)$

$\Rightarrow x^2 = \frac{a^2b(b-a)}{x^2\ (b^2-a^2)}$

$\Rightarrow x^2 = \frac{a^2b(b-a)}{x^2\ (b-a)(b+a)}$

$\Rightarrow x^2 = \frac{a^2b}{(b+a)}$

$\therefore a^2 - b^2 = (a + b)(a - b)$

$\Rightarrow x = \pm\sqrt{\frac{a^2b}{(b+a)}}$ ... (3)

Since, $y^2 = ab - x^2$

$\Rightarrow y^2 = ab - (\frac{a^2b}{(b+a)})$

$\Rightarrow y^2 = \frac{ab^2 + a^2b - a^2b}{(b+a)}$

$\Rightarrow y^2 = \frac{ab^2}{(b+a)}$

$\Rightarrow y = \pm\sqrt{\frac{ab^2}{(b+a)}}$ ... (4)

Since, curves are $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ & $x^2 + y^2 = ab$

Differentiating above with respect to x

$\Rightarrow \frac{2x}{a^2} + \frac{2y}{b^2}.\frac{dy}{dx} = 0$

$\Rightarrow \frac{y}{b^2}.\frac{dy}{dx} = -\frac{x}{a^2}$

$\Rightarrow \frac{dy}{dx} = \frac{-\frac{x}{a^2}}{\frac{y}{b^2}}$

$\Rightarrow \frac{dy}{dx} = \frac{-b^2x}{a^2y}$

$\Rightarrow m_1 = \frac{dy}{dx} = \frac{-b^2x}{a^2y}$ ... (5)

Second curve is $x^2 + y^2 = ab$

$\Rightarrow 2x + 2y.\frac{dy}{dx} = 0$

$\Rightarrow m_2 = \frac{dy}{dx} = \frac{-x}{y}$ ... (6)

Substituting (3) in (4), above values for m_1 & m_2, we get, At $\left(\sqrt{\frac{a^2b}{(b+a)}}, \sqrt{\frac{ab^2}{(b+a)}}\right)$ in equation (5), we get

$$\Rightarrow \frac{dy}{dx} = \frac{-b^2 \times \sqrt{\frac{a^2b}{(b\ +\ a)}}}{a^2 \times \sqrt{\frac{ab^2}{(b\ +\ a)}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-b^2 \times a\sqrt{\frac{b}{(b\ +\ a)}}}{a^2 \times b\sqrt{\frac{a}{(b\ +\ a)}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-b^2 a\sqrt{b}}{a^2 b\sqrt{a}}$$

$$\Rightarrow m_1 = \frac{dy}{dx} = \frac{-b\sqrt{b}}{a\sqrt{a}}$$

At $\left(\sqrt{\frac{a^2b}{(b+a)}}, \sqrt{\frac{ab^2}{(b+a)}}\right)$ in equation (6), we get

$$\Rightarrow \frac{dy}{dx} = \frac{-\sqrt{\frac{a^2b}{(b\ +\ a)}}}{\sqrt{\frac{ab^2}{(b\ +\ a)}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-a\sqrt{\frac{b}{(b\ +\ a)}}}{b\sqrt{\frac{a}{(b\ +\ a)}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-a\sqrt{b}}{b\sqrt{a}}$$

$$\Rightarrow m_2 = \frac{dy}{dx} = -\sqrt{\frac{a}{b}}$$

When $m_1 = \frac{-b\sqrt{b}}{a\sqrt{a}}$ & $m_2 = -\sqrt{\frac{a}{b}}$

Angle of intersection of two curves is given by $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$\Rightarrow \text{Tan } \theta = \left|\frac{\frac{-b\sqrt{b}}{a\sqrt{a}} - -\sqrt{\frac{a}{b}}}{1 + \frac{-b\sqrt{b}}{a\sqrt{a}} \times -\sqrt{\frac{a}{b}}}\right|$$

$$\Rightarrow \text{Tan } \theta = \left|\frac{\frac{-b\sqrt{b}}{a\sqrt{a}} + \sqrt{\frac{a}{b}}}{1 + \frac{b}{a}}\right|$$

$$\Rightarrow \text{Tan } \theta = \left|\frac{\frac{-b\sqrt{b}\times\sqrt{b} + a\sqrt{a}\times\sqrt{a}}{a\sqrt{a}\times\sqrt{b}}}{1 + \frac{b}{a}}\right|$$

$$\Rightarrow \text{Tan } \theta = \left|\frac{\frac{-b\times b + a\times a}{a\sqrt{ab}}}{1 + \frac{b}{a}}\right|$$

$$\Rightarrow \text{Tan } \theta = \left|\frac{\frac{a^2 - b^2}{a\sqrt{ab}}}{\frac{a+b}{a}}\right|$$

$$\Rightarrow \text{Tan } \theta = \left|\frac{\frac{(a+b)(a-b)}{\sqrt{ab}}}{a+b}\right|$$

$$\Rightarrow \text{Tan } \theta = \left|\frac{(a-b)}{\sqrt{ab}}\right|$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{(a-b)}{\sqrt{ab}}\right)$$

2. Show that the following set of curves intersect orthogonally:
(i) $y = x^3$ and $6y = 7 - x^2$

Solution:
Given curves $y = x^3$ ... (1) and $6y = 7 - x^2$ ... (2)
Solving (1) & (2), we get
$\Rightarrow 6y = 7 - x^2$
$\Rightarrow 6(x^3) = 7 - x^2$
$\Rightarrow 6x^3 + x^2 - 7 = 0$

Since $f(x) = 6x^3 + x^2 - 7$,

We have to find $f(x) = 0$, so that x is a factor of f(x).

When x = 1

$f(1) = 6(1)^3 + (1)^2 - 7$

$f(1) = 6 + 1 - 7$

$f(1) = 0$

Hence, x = 1 is a factor of f(x).

Substituting x = 1 in $y = x^3$, we get

$y = 1^3$

$y = 1$

The point of intersection of two curves is (1, 1)

First curve $y = x^3$

Differentiating above with respect to x,

$\Rightarrow 6\frac{dy}{dx} = 0 - 2x$

$\Rightarrow m_2 = \frac{-2x}{6}$

$\Rightarrow m_2 = \frac{-x}{3}$ ... (4)

At (1, 1), we have,

$m_1 = 3x^2$

$\Rightarrow 3 \times (1)^2$

$m_1 = 3$

At (1, 1), we have,

$\Rightarrow m_2 = \frac{-x}{3}$

$\Rightarrow \frac{-1}{3}$

$\Rightarrow m_2 = \frac{-1}{3}$

When $m_1 = 3$ & $m_2 = \frac{-1}{3}$

Two curves intersect orthogonally if $m_1m_2 = -1$

$\Rightarrow 3x^{\frac{-1}{3}} = -1$

∴ Two curves $y = x^3$ & $6y = 7 - x^2$ intersect orthogonally.

(ii) $x^3 - 3xy^2 = -2$ and $3x^2 y - y^3 = 2$

Solution:

Given curves $x^3 - 3xy^2 = -2$ ... (1) and $3x^2y - y^3 = 2$ ... (2)
Adding (1) & (2), we get
$\Rightarrow x^3 - 3xy^2 + 3x^2y - y^3 = -2 + 2$
$\Rightarrow x^3 - 3xy^2 + 3x^2y - y^3 = -0$
$\Rightarrow (x - y)^3 = 0$
$\Rightarrow (x - y) = 0$
$\Rightarrow x = y$
Substituting x = y on $x^3 - 3xy^2 = -2$
$\Rightarrow x^3 - 3 \times x \times x^2 = -2$
$\Rightarrow x^3 - 3x^3 = -2$
$\Rightarrow -2x^3 = -2$
$\Rightarrow x^3 = 1$
$\Rightarrow x = 1$
Since x = y
y = 1
The point of intersection of two curves is (1, 1)
First curve $x^3 - 3xy^2 = -2$
Differentiating above with respect to x,

$\Rightarrow 3x^2 - 3(1 \times y^2 + x \times 2y\frac{dy}{dx}) = 0$

$\Rightarrow 3x^2 - 3y^2 - 6xy\frac{dy}{dx} = 0$

$\Rightarrow 3x^2 - 3y^2 = 6xy\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{3x^2 - 3y^2}{6xy}$

$\Rightarrow \frac{dy}{dx} = \frac{3(x^2 - y^2)}{6xy}$

$\Rightarrow m_1 = \frac{(x^2-y^2)}{2xy}$ ... (3)

Second curve $3x^2y - y^3 = 2$

Differentiating above with respect to x

$\Rightarrow 3(2x \times y + x^2 \times \frac{dy}{dx}) - 3y^2\frac{dy}{dx} = 0$

$\Rightarrow 6xy + 3x^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$

$\Rightarrow 6xy + (3x^2 - 3y^2)\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-6xy}{3x^2-3y^2}$

$\Rightarrow \frac{dy}{dx} = \frac{-2xy}{x^2-y^2}$

$\Rightarrow m_2 = \frac{-2xy}{x^2-y^2}$ ... (4)

When $m_1 = \frac{(x^2-y^2)}{2xy}$ & $m_2 = \frac{-2xy}{x^2-y^2}$

Two curves intersect orthogonally if $m_1m_2 = -1$

$\Rightarrow \frac{(x^2-y^2)}{2xy} \times \frac{-2xy}{x^2-y^2} = -1$

∴ Two curves $x^3 - 3xy^2 = -2$ & $3x^2y - y^3 = 2$ intersect orthogonally.

(iii) $x^2 + 4y^2 = 8$ and $x^2 - 2y^2 = 4$.

Solution:

Given curves $x^2 + 4y^2 = 8$ ... (1) and $x^2 - 2y^2 = 4$ ... (2)

Solving (1) & (2), we get,

From 2nd curve,

$x^2 = 4 + 2y^2$

Substituting on $x^2 + 4y^2 = 8$,

$\Rightarrow 4 + 2y^2 + 4y^2 = 8$

$\Rightarrow 6y^2 = 4$

$\Rightarrow y^2 = \frac{4}{6}$

$\Rightarrow y = \pm\sqrt{\frac{2}{3}}$

Substituting on $y = \pm\sqrt{\frac{2}{3}}$, we get,

$\Rightarrow x^2 = 4 + 2(\pm\sqrt{\frac{2}{3}})^2$

$\Rightarrow x^2 = 4 + 2(\frac{2}{3})$

$\Rightarrow x^2 = 4 + \frac{4}{3}$

$\Rightarrow x^2 = \frac{16}{3}$

$\Rightarrow x = \pm\sqrt{\frac{16}{3}}$

$\Rightarrow x = \pm\frac{4}{\sqrt{3}}$

$\therefore$ The point of intersection of two curves $(\frac{4}{\sqrt{3}}, \sqrt{\frac{2}{3}})$ & $(-\frac{4}{\sqrt{3}}, -\sqrt{\frac{2}{3}})$

Now, differentiating curves (1) & (2) with respect to x, we get

$\Rightarrow x^2 + 4y^2 = 8$

$\Rightarrow 2x + 8y.\frac{dy}{dx} = 0$

$\Rightarrow 8y.\frac{dy}{dx} = -2x$

$\Rightarrow \frac{dy}{dx} = \frac{-x}{4y}$ ... (3)

$\Rightarrow x^2 - 2y^2 = 4$

$\Rightarrow 2x - 4y.\frac{dy}{dx} = 0$

$\Rightarrow x - 2y.\frac{dy}{dx} = 0$

$\Rightarrow 4y\frac{dy}{dx} = x$

$\Rightarrow \frac{dy}{dx} = \frac{x}{2y}$ ... (4)

At $(\frac{4}{\sqrt{3}}, \sqrt{\frac{2}{3}})$ in equation (3), we get

$$\Rightarrow \frac{dy}{dx} = \frac{-\frac{4}{\sqrt{3}}}{4 \times \sqrt{\frac{2}{3}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\frac{1}{\sqrt{3}}}{\sqrt{\frac{2}{3}}}$$

$\Rightarrow m_1 = \frac{-1}{\sqrt{2}}$

At $(\frac{4}{\sqrt{3}}, \sqrt{\frac{2}{3}})$ in equation (4), we get

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{4}{\sqrt{3}}}{2 \times \sqrt{\frac{2}{3}})}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{2}{\sqrt{3}}}{\sqrt{\frac{2}{3}})}$$

$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{2}}$

$\Rightarrow \frac{dy}{dx} = \sqrt{2}$

$\Rightarrow m_2 = 1$

When $m_1 = \frac{-1}{\sqrt{2}}$ & $m_2 = \sqrt{2}$

Two curves intersect orthogonally if $m_1m_2 = -1$

$\Rightarrow \frac{-1}{\sqrt{2}} \times \sqrt{2} = -1$

∴ Two curves $x^2 + 4y^2 = 8$ & $x^2 - 2y^2 = 4$ intersect orthogonally.

3. $x^2 = 4y$ and $4y + x^2 = 8$ at (2, 1)

Solution:

Given curves $x^2 = 4y$ ... (1) and $4y + x^2 = 8$ ... (2)

The point of intersection of two curves (2, 1)

Solving (1) & (2), we get,

First curve is $x^2 = 4y$

Differentiating above with respect to x,

$\Rightarrow 2x = 4.\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{2x}{4}$

$\Rightarrow m_1 = \frac{x}{2}$ ... (3)

Second curve is $4y + x^2 = 8$

$\Rightarrow 4.\frac{dy}{dx} + 2x = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-2x}{4}$

$\Rightarrow m_2 = \frac{-x}{2}$ ... (4)

Substituting (2, 1) for m_1 & m_2, we get,

$m_1 = \frac{x}{2}$

$\Rightarrow \frac{2}{2}$

$m_1 = 1$ ... (5)

$m_2 = \frac{-x}{2}$

$\Rightarrow \frac{-2}{2}$

$m_2 = -1$ ... (6)

When $m_1 = 1$ & $m_2 = -1$

Two curves intersect orthogonally if $m_1 m_2 = -1$

$\Rightarrow 1 \times -1 = -1$

$\therefore$ Two curves $x^2 = 4y$ & $4y + x^2 = 8$ intersect orthogonally.

(ii) $x^2 = y$ and $x^3 + 6y = 7$ at (1, 1)

Solution:

Given curves $x^2 = y$ ... (1) and $x^3 + 6y = 7$ ... (2)
The point of intersection of two curves (1, 1)
Solving (1) & (2), we get,
First curve is $x^2 = y$
Differentiating above with respect to x,

$\Rightarrow 2x = \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = 2x$

$\Rightarrow m_1 = 2x$ ... (3)

Second curve is $x^3 + 6y = 7$

Differentiating above with respect to x,

$\Rightarrow 3x^2 + 6.\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-3x^2}{6}$

$\Rightarrow \frac{dy}{dx} = \frac{-x^2}{2}$

$\Rightarrow m_2 = \frac{-x^2}{2}$ ... (4)

Substituting (1, 1) for m_1 & m_2, we get,

$m_1 = 2x$

$\Rightarrow 2 \times 1$

$m_1 = 2$ ... (5)

$m_2 = \frac{-x^2}{2}$

$\Rightarrow \frac{-1^2}{2}$

$m_2 = -\frac{-1}{2}$ ... (6)

When $m_1 = 2$ & $m_2 = -\frac{-1}{2}$

Two curves intersect orthogonally if $m_1m_2 = -1$

$\Rightarrow 2\times\frac{-1}{2} = -1$

∴ Two curves $x^2 = y$ & $x^3 + 6y = 7$ intersect orthogonally.

(iii) $y^2 = 8x$ and $2x^2 + y^2 = 10$ at $(1, 2\sqrt{2})$

Solution:

Given curves $y^2 = 8x$ ... (1) and $2x^2 + y^2 = 10$ ... (2)

The point of intersection of two curves are (0, 0) & (1, 2√

Now, differentiating curves (1) & (2) w.r.t x, we get

$\Rightarrow y^2 = 8x$

$\Rightarrow 2y.\frac{dy}{dx} = 8$

$\Rightarrow \frac{dy}{dx} = \frac{8}{2y}$

$\Rightarrow \frac{dy}{dx} = \frac{4}{y}$ ... (3)

$\Rightarrow 2x^2 + y^2 = 10$

Differentiating above with respect to x,

$\Rightarrow 4x + 2y.\frac{dy}{dx} = 0$

$\Rightarrow 2x + y.\frac{dy}{dx} = 0$

$\Rightarrow y.\frac{dy}{dx} = -2x$

$\Rightarrow \frac{dy}{dx} = \frac{-2x}{y}$ ... (4)

Substituting (1, 2√2) for m_1 & m_2, we get,

$m_1 = \frac{4}{y}$

$\Rightarrow \frac{4}{2\sqrt{2}}$

$m_1 = \sqrt{2}$ ... (5)

$m_2 = \frac{-2x}{y}$

$\Rightarrow \frac{-2\times1}{2\sqrt{2}}$

$m_2 = -\frac{-1}{\sqrt{2}}$ ... (6)

When $m_1 = \sqrt{2}$ & $m_2 = \frac{-1}{\sqrt{2}}$

Two curves intersect orthogonally if $m_1m_2 = -1$

$\Rightarrow \sqrt{2} \times \frac{-1}{\sqrt{2}} = -1$

$\therefore$ Two curves $y^2 = 8x$ & $2x^2 + y^2 = 10$ intersect orthogonally.

4. Show that the curves $4x = y^2$ and $4xy = k$ cut at right angles, if $k^2 = 512$.

Solution:

Given curves $4x = y^2$ ... (1) and $4xy = k$ ... (2)

We have to prove that two curves cut at right angles if $k^2 = 512$

Now, differentiating curves (1) & (2) w.r.t x, we get

$\Rightarrow 4x = y^2$

$\Rightarrow 4 = 2y.\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{2}{y}$

$m_1 = \frac{2}{y}$ ... (3)

$\Rightarrow 4xy = k$

Differentiating above with respect to x,

$\Rightarrow 4(y + x\frac{dy}{dx}) = 0$

$\Rightarrow y + x\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-y}{x}$

$\Rightarrow m_2 = \frac{-y}{x}$ ... (4)

Two curves intersect orthogonally if $m_1m_2 = -1$

Since m_1 and m_2 cuts orthogonally,

$\Rightarrow \frac{2}{y} \times \frac{-y}{x} = -1$

$\Rightarrow \frac{-2}{x} = -1$

$\Rightarrow x = 2$

Now, Solving (1) & (2), we get,

$4xy = k$ & $4x = y^2$

$\Rightarrow (y^2)\, y = k$

$\Rightarrow y^3 = k$

$\Rightarrow y = k^{\frac{1}{3}}$

Substituting $y = k^{\frac{1}{3}}$ in $4x = y^2$, we get,

$\Rightarrow 4x = (k^{\frac{1}{3}})^2$

$\Rightarrow 4 \times 2 = k^{\frac{2}{3}}$

$\Rightarrow k^{\frac{2}{3}} = 8$

$\Rightarrow k^2 = 8^3$

$\Rightarrow k^2 = 512$

5. Show that the curves $2x = y^2$ and $2xy = k$ cut at right angles, if $k^2 = 8$.

Solution:

Given curves $2x = y^2$ ... (1) and $2xy = k$ ... (2)

We have to prove that two curves cut at right angles if $k^2 = 8$

Now, differentiating curves (1) & (2) with respect to x, we get

$\Rightarrow 2x = y^2$

$\Rightarrow 2 = 2y.\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{y}$

$m_1 = \frac{1}{y}$ ... (3)

$\Rightarrow 2xy = k$

Differentiating above with respect to x,

$\Rightarrow 2(y + x\frac{dy}{dx}) = 0$

$\Rightarrow y + x\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-y}{x}$

$\Rightarrow m_2 = \frac{-y}{x}$ ... (4)

Two curves intersect orthogonally if $m_1m_2 = -1$

Since m_1 and m_2 cuts orthogonally,

$\Rightarrow \frac{1}{y} \times \frac{-y}{x} = -1$

$\Rightarrow \frac{-1}{x} = -1$

$\Rightarrow x = 1$

Now, solving (1) & (2), we get,

$2xy = k$ & $2x = y^2$

$\Rightarrow (y^2)\, y = k$

$\Rightarrow y^3 = k$

$\Rightarrow y = k^{\frac{1}{3}}$

Substituting y = $k^{\frac{1}{3}}$ in 2x = y^2, we get,

$\Rightarrow 2x = (k^{\frac{1}{3}})^2$

$\Rightarrow 2\times1 = k^{\frac{2}{3}}$

$\Rightarrow k^{\frac{2}{3}} = 2$

$\Rightarrow k^2 = 2^3$

$\Rightarrow k^2 = 8$