

EXERCISE 9.1

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1. Test the continuity of the following function at the origin:

$$f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Solution:

Given

$$f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Consider LHL at $x = 0$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0 - h) = \lim_{h \rightarrow 0} f(-h)$$

$$\lim_{h \rightarrow 0} \frac{-h}{|-h|} = \lim_{h \rightarrow 0} \frac{-h}{h} = \lim_{h \rightarrow 0} -1 = -1$$

Consider RHL at $x = 0$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0 + h) = \lim_{h \rightarrow 0} f(h)$$

$$\lim_{h \rightarrow 0} \frac{h}{|h|} = \lim_{h \rightarrow 0} \frac{h}{h} = \lim_{h \rightarrow 0} 1 = 1$$

$$\therefore \lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$$

Hence LHL \neq RHL

Hence $f(x)$ is discontinuous at origin.

2. A function $f(x)$ is defined as

$$f(x) = \begin{cases} \frac{x^2 - x - 6}{x - 3}, & \text{if } x \neq 3 \\ 5, & \text{if } x = 3 \end{cases}$$

Show that $f(x)$ is continuous at $x = 3$.

Solution:

Given

$$f(x) = \begin{cases} \frac{x^2-x-6}{x-3}, & \text{if } x \neq 3 \\ 5, & \text{if } x = 3 \end{cases}$$

Consider LHL at $x = 3$

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{h \rightarrow 0} f(3-h)$$

$$\lim_{h \rightarrow 0} \frac{(3-h)^2 - (3-h) - 6}{(3-h) - 3} = \lim_{h \rightarrow 0} \frac{9 + h^2 - 6h - 3 + h - 6}{-h} = \lim_{h \rightarrow 0} \frac{h^2 - 5h}{-h} = \lim_{h \rightarrow 0} (5 - h) = 5$$

Consider RHL at $x = 3$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{h \rightarrow 0} f(3+h)$$

$$\lim_{h \rightarrow 0} \frac{(3+h)^2 - (3+h) - 6}{(3+h) - 3} = \lim_{h \rightarrow 0} \frac{9 + h^2 + 6h - 3 - h - 6}{h} = \lim_{h \rightarrow 0} \frac{h^2 + 5h}{h} = \lim_{h \rightarrow 0} (5 + h) = 5$$

Now, $f(3) = 5$

$$\therefore \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^-} f(x) = f(3)$$

Hence $f(x)$ is continuous at $x = 3$

3. A function $f(x)$ is defined as

$$f(x) = \begin{cases} \frac{x^2-9}{x-3}, & \text{if } x \neq 3 \\ 6, & \text{if } x = 3 \end{cases}$$

Show that $f(x)$ is continuous at $x = 3$.

Solution:

Given

$$f(x) = \begin{cases} \frac{x^2-9}{x-3}; & \text{if } x \neq 3 \\ 6; & \text{if } x = 3 \end{cases}$$

Consider LHL at $x = 3$

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{h \rightarrow 0} f(3 - h)$$

$$\lim_{h \rightarrow 0} \frac{(3 - h)^2 - 9}{(3 - h) - 3} = \lim_{h \rightarrow 0} \frac{3^2 + h^2 - 6h - 9}{3 - h - 3} = \lim_{h \rightarrow 0} \frac{h^2 - 6h}{-h} = \lim_{h \rightarrow 0} \frac{h(h - 6)}{-h} = \lim_{h \rightarrow 0} (6 - h) = 6$$

Consider RHL at $x = 3$

$$= \lim_{x \rightarrow 3^+} f(x) = \lim_{h \rightarrow 0} f(3 + h)$$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{h \rightarrow 0} f(3 + h)$$

$$\lim_{h \rightarrow 0} \frac{(3 + h)^2 - 9}{3 + h - 3} = \lim_{h \rightarrow 0} \frac{3^2 + h^2 + 6h - 9}{h} = \lim_{h \rightarrow 0} \frac{h^2 + 6h}{h} = \lim_{h \rightarrow 0} \frac{h(6 + h)}{h} = \lim_{h \rightarrow 0} (6 + h) = 6$$

We have $f(3) = 6$

$$\therefore \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = f(3)$$

Hence $f(x)$ is continuous at $x = 3$

$$4. f(x) = \begin{cases} \frac{x^2 - 1}{x - 1}; & \text{if } x \neq 1 \\ 2; & \text{if } x = 1 \end{cases}$$

Find whether $f(x)$ is continuous at $x = 1$.

Solution:

Given

$$f(x) = \begin{cases} \frac{x^2 - 1}{x - 1}; & \text{if } x \neq 1 \\ 2; & \text{if } x = 1 \end{cases}$$

Consider LHL at $x = 1$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{h \rightarrow 0} f(1 - h)$$

$$\lim_{h \rightarrow 0} \frac{(1 - h)^2 - 1}{(1 - h) - 1} = \lim_{h \rightarrow 0} \frac{1 + h^2 - 2h - 1}{1 - h - 1} = \lim_{h \rightarrow 0} \frac{h^2 - 2h}{-h} = \lim_{h \rightarrow 0} \frac{h(h - 2)}{-h} = \lim_{h \rightarrow 0} (2 - h) = 2$$

Consider RHL at $x = 1$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{h \rightarrow 0} f(1 + h)$$

$$\lim_{h \rightarrow 0} \frac{(1 + h)^2 - 1}{(1 + h) - 1} = \lim_{h \rightarrow 0} \frac{1 + h^2 + 2h - 1}{1 + h - 1} = \lim_{h \rightarrow 0} \frac{h^2 + 2h}{h} = \lim_{h \rightarrow 0} \frac{h(h + 2)}{h} = \lim_{h \rightarrow 0} (2 + h) = 2$$

Given $f(1) = 2$

$$\therefore \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = f(1)$$

Hence $f(x)$ is continuous at $x = 1$

$$5. \text{ If } f(x) = \begin{cases} \frac{\sin 3x}{x}; & \text{when } x \neq 0 \\ 1; & \text{when } x = 0 \end{cases}$$

Find whether $f(x)$ is continuous at $x = 0$.

Solution:

Given

$$f(x) = \begin{cases} \frac{\sin 3x}{x}; & \text{when } x \neq 0 \\ 1; & \text{when } x = 0 \end{cases}$$

Consider LHL at $x = 0$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0 - h) = \lim_{h \rightarrow 0} f(-h)$$

$$\lim_{h \rightarrow 0} \frac{\sin(-3h)}{-h} = \lim_{h \rightarrow 0} \frac{-\sin(3h)}{-h} = \lim_{h \rightarrow 0} \frac{3 \sin(3h)}{3h} = 3 \lim_{h \rightarrow 0} \frac{\sin(3h)}{3h} = 3 \cdot 1 = 3$$

Consider RHL at $x = 0$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(h)$$

$$\lim_{h \rightarrow 0} \frac{\sin 3h}{h} = \lim_{h \rightarrow 0} \frac{3 \sin 3h}{3h} = 3 \lim_{h \rightarrow 0} \frac{\sin(3h)}{3h} = 3 \cdot 1 = 3$$

Given $f(0) = 1$

$f(x)$ to be continuous at $x = a$

But here,

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) \neq f(0)$$

Hence $f(x)$ is discontinuous at $x = 0$

6. If $f(x) = \begin{cases} e^{\frac{1}{x}}; & \text{when } x \neq 0 \\ 1; & \text{when } x = 0 \end{cases}$

Find whether $f(x)$ is continuous at $x = 0$.

Solution:

Given

$$f(x) = \begin{cases} e^{\frac{1}{x}}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

Consider LHL at $x = 0$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0 - h) = \lim_{h \rightarrow 0} f(-h)$$

$$\lim_{h \rightarrow 0} e^{-\frac{1}{h}} = \lim_{h \rightarrow 0} \left(\frac{1}{e^{\frac{1}{h}}} \right) = \frac{1}{\lim_{h \rightarrow 0} e^{\frac{1}{h}}} = 0$$

Consider RHL at $x = 0$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(h)$$

$$\lim_{h \rightarrow 0} e^{\frac{1}{h}} = \infty$$

We have $f(0) = 1$

It is known that for a function $f(x)$ to be continuous at $x = a$

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$$

But

$$\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$$

Hence $f(x)$ is discontinuous at $x = 0$

7. Let $f(x) = \begin{cases} \frac{1-\cos x}{x^2}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0. \end{cases}$ Show that $f(x)$ is discontinuous at $x = 0$

Solution:

Given

$$f(x) = \begin{cases} \frac{1-\cos x}{x^2}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0. \end{cases}$$

Consider,

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \left(\frac{1 - \cos x}{x^2} \right)$$

$$\Rightarrow \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \left(\frac{2 \sin^2 \frac{x}{2}}{x^2} \right)$$

$$\Rightarrow \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \left(\frac{2 \sin^2 \frac{x}{2}}{4 \left(\frac{x^2}{4} \right)} \right)$$

$$\Rightarrow \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \left(\frac{2 \left(\sin \frac{x}{2} \right)^2}{4 \left(\frac{x}{2} \right)^2} \right)$$

$$\Rightarrow \lim_{x \rightarrow 0} f(x) = \frac{2}{4} \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2$$

$$\Rightarrow \lim_{x \rightarrow 0} f(x) = \frac{1}{2} \cdot 1^2 = \frac{1}{2}$$

We have $f(0) = 1$

$$\lim_{x \rightarrow 0} f(x) \neq f(0)$$

Thus $f(x)$ is discontinuous at $x = 0$

8. Show that $f(x) = \begin{cases} \frac{x-|x|}{2}, & \text{when } x \neq 0 \\ 2, & \text{when } x = 0. \end{cases}$ is discontinuous at $x = 0$

Solution:

Given

$$f(x) = \begin{cases} \frac{x-|x|}{2}, & \text{when } x \neq 0 \\ 2, & \text{when } x = 0. \end{cases}$$

The given function can be written as

$$f(x) = \begin{cases} \frac{x-x}{2}, & \text{when } x > 0 \\ \frac{x+x}{2}, & \text{when } x < 0 \\ 2, & \text{when } x = 0 \end{cases}$$

$$f(x) = \begin{cases} 0, & \text{when } x > 0 \\ x, & \text{when } x < 0 \\ 2, & \text{when } x = 0 \end{cases}$$

Consider LHL at $x = 0$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0 - h) = \lim_{h \rightarrow 0} f(-h)$$

$$= \lim_{h \rightarrow 0} (-h) = 0$$

Consider LHL at $x = 0$

$$\begin{aligned}\lim_{x \rightarrow 0^-} f(x) &= \lim_{h \rightarrow 0} f(0 - h) = \lim_{h \rightarrow 0} f(-h) \\ &= \lim_{h \rightarrow 0} (-h) = 0\end{aligned}$$

Consider RHL at $x = 0$

$$\begin{aligned}\lim_{x \rightarrow 0^+} f(x) &= \lim_{h \rightarrow 0} f(0 + h) = \lim_{h \rightarrow 0} f(h) \\ \lim_{h \rightarrow 0} 0 &= 0\end{aligned}$$

And we have $f(0) = 2$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) \neq f(0)$$

Hence, $f(x)$ is discontinuous at $x = 0$

9. Show that $f(x) = \begin{cases} \frac{|x-a|}{x-a}, & \text{when } x \neq a \\ 1, & \text{when } x = a. \end{cases}$ is discontinuous at $x = a$

Solution:

Given

$$f(x) = \begin{cases} \frac{|x-a|}{x-a}, & \text{when } x \neq a \\ 1, & \text{when } x = a. \end{cases}$$

The given function can be written as

$$f(x) = \begin{cases} \frac{x-a}{x-a}, & \text{when } x > a \\ \frac{a-x}{x-a}, & \text{when } x < a \\ 1, & \text{when } x = a \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} 1, & \text{when } x > a \\ -1, & \text{when } x < a \\ 1, & \text{when } x = a \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} 1, & \text{when } x \geq a \\ -1, & \text{when } x < a \end{cases}$$

Consider LHL at $x = a$

$$\begin{aligned} \lim_{x \rightarrow a^-} f(x) &= \lim_{h \rightarrow 0} f(a - h) \\ &= \lim_{h \rightarrow 0} (-1) = -1 \end{aligned}$$

Consider RHL at $x = a$

$$\begin{aligned} \lim_{x \rightarrow a^+} f(x) &= \lim_{h \rightarrow 0} f(a + h) \\ \lim_{h \rightarrow 0} (1) &= 1 \end{aligned}$$

$$\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$$

Thus $f(x)$ is discontinuous at $x = a$

10. Discuss the continuity of the following functions at the indicated point(s):

$$(i) f(x) = \begin{cases} |x| \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases} \text{ at } x = 0$$

Solution:

Given

$$f(x) = \begin{cases} |x| \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Consider,

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} |x| \cos\left(\frac{1}{x}\right)$$

$$\Rightarrow \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} |x| \lim_{x \rightarrow 0} \cos\left(\frac{1}{x}\right)$$

$$\Rightarrow \lim_{x \rightarrow 0} f(x) = 0 \times \lim_{x \rightarrow 0} \cos\left(\frac{1}{x}\right) = 0$$

$$\Rightarrow \lim_{x \rightarrow 0} f(x) = f(0)$$

Hence $f(x)$ is continuous at $x = 0$

$$(ii) f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases} \text{ at } x = 0$$

Solution:

Given

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Consider,

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right) = \lim_{x \rightarrow 0} x^2 \lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right) = 0 \times \lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right) = 0$$

$$\Rightarrow \lim_{x \rightarrow 0} f(x) = f(0)$$

Hence $f(x)$ is continuous at $x = 0$

$$(iii) f(x) = \begin{cases} (x - a) \sin\left(\frac{1}{x-a}\right), & x \neq a \\ 0, & x = a \end{cases} \text{ at } x = a$$

Solution:

Given

$$f(x) = \begin{cases} (x - a) \sin\left(\frac{1}{x-a}\right), & x \neq a \\ 0, & x = a \end{cases}$$

Now substitute $x - a = y$ in above equation then we get,

$$\begin{aligned} \lim_{x \rightarrow a} (x - a) \sin\left(\frac{1}{x - a}\right) &= \lim_{y \rightarrow 0} y \sin\left(\frac{1}{y}\right) \\ &= \lim_{y \rightarrow 0} y \lim_{y \rightarrow 0} \sin\left(\frac{1}{y}\right) = 0 \times \lim_{y \rightarrow 0} \sin\left(\frac{1}{y}\right) = 0 \\ \Rightarrow \lim_{x \rightarrow a} f(x) &= f(a) = 0 \end{aligned}$$

Hence $f(x)$ is continuous at $x = a$

$$(iv) f(x) = \begin{cases} \frac{e^x - 1}{\log(1 + 2x)}, & \text{if } x \neq 0 \\ 7, & \text{if } x = 0 \end{cases}$$

Solution:

Given

$$f(x) = \begin{cases} \frac{e^x - 1}{\log(1 + 2x)}, & \text{if } x \neq 0 \\ 7, & \text{if } x = 0 \end{cases}$$

Consider,

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{e^x - 1}{\log(1 + 2x)}$$

$$\Rightarrow \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{e^x - 1}{\frac{2x \log(1 + 2x)}{2x}}$$

$$\Rightarrow \lim_{x \rightarrow 0} f(x) = \frac{1}{2} \lim_{x \rightarrow 0} \frac{\left(\frac{e^x - 1}{x}\right)}{\left(\frac{\log(1 + 2x)}{2x}\right)}$$

$$\Rightarrow \lim_{x \rightarrow 0} f(x) = \frac{1}{2} \times \frac{\left(\lim_{x \rightarrow 0} \frac{e^x - 1}{x}\right)}{\left(\lim_{x \rightarrow 0} \frac{\log(1+2x)}{2x}\right)} = \frac{1}{2} \times \frac{1}{1} = \frac{1}{2}$$

And we have $f(0) = 7$

$$\Rightarrow \lim_{x \rightarrow 0} f(x) \neq f(0)$$

Hence $f(x)$ is discontinuous at $x = 0$

$$(v) f(x) = \begin{cases} \frac{1-x^n}{1-x}, & x \neq 1 \\ n-1, & x = 1 \end{cases} \quad n \in N \text{ at } x = 1$$

Solution:

Given

$$f(x) = \begin{cases} \frac{1-x^n}{1-x}, & x \neq 1 \\ n-1, & x = 1 \end{cases} \quad n \in N$$

Clearly, $f(1) = n - 1$

$$\text{LHL} = \lim_{h \rightarrow 0} f(1-h) = \lim_{h \rightarrow 0} \frac{1-(1-h)^n}{1-(1-h)} = \lim_{h \rightarrow 0} \frac{1-(1-h)^n}{h}$$

Using binomial theorem we get

$$(1-h)^n = \sum_{k=0}^n \binom{n}{k} (-h)^k 1^{n-k}$$

$$(1-h)^n = 1 - nh + \binom{n}{2}h^2 - \dots$$

LHL =

$$\lim_{h \rightarrow 0} \frac{1-1+nh-\binom{n}{2}h^2+\dots\text{higher deg terms}}{h} = \lim_{h \rightarrow 0} \{n - \binom{n}{2}h + \binom{n}{3}h^2 - \dots\text{higher deg terms}\}$$

Putting $h=0$ we get,

$$\text{LHL} = n$$

$$\text{RHL} = \lim_{h \rightarrow 0} f(1+h) = \lim_{h \rightarrow 0} \frac{1-(1+h)^n}{1-(1+h)} = \lim_{h \rightarrow 0} \frac{1-(1+h)^n}{-h}$$

Using binomial expansion as used above we get the following expression

Similarly,

$$\text{RHL} =$$

$$\lim_{h \rightarrow 0} \frac{1-1-nh-\binom{n}{2}h^2-\dots-\text{higher deg terms}}{-h} = \lim_{h \rightarrow 0} \{n + \binom{n}{2}h + \binom{n}{3}h^2 - \dots \text{higher deg terms}\}$$

Putting $h=0$ we get,

$$\text{RHL} = n$$

$$\text{Thus RHL} = \text{LHL} \neq f(1)$$

Hence $f(x)$ is discontinuous at $x=1$

$$(vi) f(x) = \begin{cases} \frac{|x^2-1|}{x-1}, & \text{for } x \neq 1 \\ 2, & \text{for } x = 1 \end{cases} \text{ at } x = 1$$

Solution:

Given

$$f(x) = \begin{cases} \frac{|x^2-1|}{x-1}, & \text{for } x \neq 1 \\ 2, & \text{for } x = 1 \end{cases}$$

Clearly, $f(1) = 2$

$$\text{LHL} = \lim_{h \rightarrow 0} f(1-h) = \lim_{h \rightarrow 0} \frac{|(1-h)^2-1|}{1-h-1} = \lim_{h \rightarrow 0} \frac{|1+h^2-2h-1|}{-h} = \lim_{h \rightarrow 0} \frac{|h(h-2)|}{-h}$$

Since h is positive no which is very close to 0

$\therefore (h-2)$ is negative and hence $h(h-2)$ is also negative.

$$|h(h-2)| = -h(h-2)$$

$$\text{LHL} = \lim_{h \rightarrow 0} \frac{-h(h-2)}{-h} = \lim_{h \rightarrow 0} (h-2) = -2$$

$$\text{RHL} = \lim_{h \rightarrow 0} f(1+h) = \lim_{h \rightarrow 0} \frac{|(1+h)^2-1|}{1+h-1} = \lim_{h \rightarrow 0} \frac{|1+h^2+2h-1|}{h} = \lim_{h \rightarrow 0} \frac{|h(h+2)|}{h}$$

Since h is a positive no which is very close to 0

$(h+2)$ is positive and hence $h(h+2)$ is also positive.

$$\therefore |h(h+2)| = h(h+2)$$

$$\therefore \text{RHL} = \lim_{h \rightarrow 0} \frac{h(h+2)}{h} = \lim_{h \rightarrow 0} (h+2) = 2$$

Clearly, $\text{LHL} \neq \text{RHL}$

Hence $f(x)$ is discontinuous at $x=1$

$$\text{(vii) } f(x) = \begin{cases} \frac{2|x|+x^2}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \text{ at } x = 0$$

Solution:

Given

$$f(x) = \begin{cases} \frac{2|x|+x^2}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Clearly, $f(0) = 0$

$$\begin{aligned} \text{LHL} &= \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} f(-h) = \lim_{h \rightarrow 0} \frac{2|-h|+(-h)^2}{-h} \\ &= \lim_{h \rightarrow 0} \frac{2h+h^2}{-h} = \lim_{h \rightarrow 0} (-2-h) = -2 \end{aligned}$$

$$\begin{aligned} \text{RHL} &= \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} f(h) = \lim_{h \rightarrow 0} \frac{2|h|+(h)^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{2h+h^2}{h} = \lim_{h \rightarrow 0} (2+h) = 2 \end{aligned}$$

Clearly, $LHL \neq RHL \neq f(0)$

Hence $f(x)$ is discontinuous at $x=0$

$$(viii) f(x) = \begin{cases} |x - a| \sin \frac{1}{x-a}, & \text{when } x \neq a \\ 0, & \text{when } x = a. \end{cases} \quad \text{at } x = a$$

Solution:

Given

$$f(x) = \begin{cases} |x - a| \sin \frac{1}{x-a}, & \text{when } x \neq a \\ 0, & \text{when } x = a. \end{cases}$$

Clearly, $f(a) = 0$

$$LHL = \lim_{h \rightarrow 0} f(a - h) = \lim_{h \rightarrow 0} |(a - h - a)| \sin \left(\frac{1}{a-h-a} \right)$$

$$= \lim_{h \rightarrow 0} |-h| \sin \left(\frac{1}{-h} \right) = \lim_{h \rightarrow 0} h \sin \left(\frac{1}{h} \right) = 0$$

$$RHL = \lim_{h \rightarrow 0} f(a + h) = \lim_{h \rightarrow 0} |a + h - a| \sin \left(\frac{1}{a+h-a} \right) = \lim_{h \rightarrow 0} |h| \sin \left(\frac{1}{h} \right)$$

$$= \lim_{h \rightarrow 0} h \sin \left(\frac{1}{h} \right) = 0$$

Since whatever is value of h , $\sin(1/h)$ is going to range from -1 to 1

As $h \rightarrow 0$, i.e. approximately 0

Clearly, $LHL = RHL = f(a)$

Hence $f(x)$ is continuous at $x = 0$

11. Show that $f(x) = \begin{cases} 1 + x^2, & \text{if } 0 \leq x \leq 1 \\ 2 - x, & \text{if } x > 1. \end{cases}$ is discontinuous at $x = 1$

Solution:

Given

$$f(x) = \begin{cases} 1 + x^2, & \text{if } 0 \leq x \leq 1 \\ 2 - x, & \text{if } x > 1 \end{cases}$$

Consider LHL at $x = 1$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{h \rightarrow 0} f(1 - h)$$

$$= \lim_{h \rightarrow 0} (1 + (1 - h)^2) = \lim_{h \rightarrow 0} (2 + h^2 - 2h) = 2$$

Now again consider RHL at $x = 1$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{h \rightarrow 0} f(1 + h)$$

$$= \lim_{h \rightarrow 0} (2 - (1 + h)) = \lim_{h \rightarrow 0} (1 - h) = 1$$

$$\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$$

Hence $f(x)$ is discontinuous at $x = 1$

12. Show that $f(x) = \begin{cases} \frac{\sin 3x}{\tan 2x}, & \text{if } x < 0 \\ \frac{3}{2}, & \text{if } x = 0 \\ \frac{\log(1+3x)}{e^{2x}-1}, & \text{if } x > 0 \end{cases}$ is continuous at $x = 0$

Solution:

Given

$$f(x) = \begin{cases} \frac{\sin 3x}{\tan 2x}, & \text{if } x < 0 \\ \frac{3}{2}, & \text{if } x = 0 \\ \frac{\log(1+3x)}{e^{2x}-1}, & \text{if } x > 0 \end{cases}$$

Consider LHL at $x = 0$

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \lim_{h \rightarrow 0} f(0 - h) = \lim_{h \rightarrow 0} f(-h) \\ &= \lim_{h \rightarrow 0} \left(\frac{\sin 3(-h)}{\tan 2(-h)} \right) = \lim_{h \rightarrow 0} \left(\frac{\sin 3h}{\tan 2h} \right) = \lim_{h \rightarrow 0} \left(\frac{\frac{3 \sin 3h}{3h}}{\frac{2 \tan 2h}{2h}} \right) \\ &= \frac{\lim_{h \rightarrow 0} \left(\frac{3 \sin 3h}{3h} \right)}{\lim_{h \rightarrow 0} \left(\frac{2 \tan 2h}{2h} \right)} = \frac{3 \lim_{h \rightarrow 0} \left(\frac{\sin 3h}{3h} \right)}{2 \lim_{h \rightarrow 0} \left(\frac{\tan 2h}{2h} \right)} = \frac{3 \times 1}{2 \times 1} = \frac{3}{2} \end{aligned}$$

Consider RHL at $x = 0$

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{h \rightarrow 0} f(0 + h) = \lim_{h \rightarrow 0} f(h) \\ &= \lim_{h \rightarrow 0} \left(\frac{\log(1 + 3h)}{e^{2h} - 1} \right) = \lim_{h \rightarrow 0} \left(\frac{3h \frac{\log(1+3h)}{3h}}{\frac{2h(e^{2h}-1)}{2h}} \right) \\ &= \frac{3}{2} \lim_{h \rightarrow 0} \left(\frac{\frac{\log(1+3h)}{3h}}{\frac{(e^{2h}-1)}{2h}} \right) = \frac{3}{2} \frac{\lim_{h \rightarrow 0} \left(\frac{\log(1+3h)}{3h} \right)}{\lim_{h \rightarrow 0} \left(\frac{(e^{2h}-1)}{2h} \right)} = \frac{3 \times 1}{2 \times 1} = \frac{3}{2} \end{aligned}$$

We have $f(0) = 2/3$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0)$$

Thus $f(x)$ is continuous at $x = 0$

13. Find the value of a for which the function f defined by

$$f(x) = \begin{cases} a \sin \frac{\pi}{2}(x + 1), & x \leq 0 \\ \frac{\tan x - \sin x}{x^3}, & x > 0 \end{cases} \text{ is continuous at } x = 0$$

Solution:

Given

$$f(x) = \begin{cases} a \sin \frac{\pi}{2}(x + 1), & x \leq 0 \\ \frac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$$

Consider LHL at $x = 0$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0 - h) = \lim_{h \rightarrow 0} f(-h) = \lim_{h \rightarrow 0} a \sin \frac{\pi}{2}(-h + 1) = a \sin \frac{\pi}{2} = a$$

Now again consider RHL at $x = 0$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0 + h) = \lim_{h \rightarrow 0} f(h) = \lim_{h \rightarrow 0} \frac{\tan h - \sin h}{h^3}$$

$$\Rightarrow \lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} \frac{\frac{\sin h}{\cos h} - \sin h}{h^3}$$

$$\Rightarrow \lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} \frac{\frac{\sin h}{\cos h}(1 - \cos h)}{h^3}$$

$$\Rightarrow \lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} \frac{(1 - \cos h) \tan h}{h^3}$$

$$\Rightarrow \lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} \frac{2 \sin^2 \frac{h}{2} \tan h}{4 \frac{h^2}{4} \times h}$$

$$\Rightarrow \lim_{x \rightarrow 0^+} f(x) = \frac{2}{4} \lim_{h \rightarrow 0} \frac{\sin^2 \frac{h}{2} \tan h}{\frac{h^2}{4} \times h}$$

$$\Rightarrow \lim_{x \rightarrow 0^+} f(x) = \frac{1}{2} \lim_{h \rightarrow 0} \left(\frac{\sin \frac{h}{2}}{\frac{h}{2}} \right)^2 \lim_{h \rightarrow 0} \frac{\tan h}{h}$$

$$\Rightarrow \lim_{x \rightarrow 0^+} f(x) = \frac{1}{2} \times 1 \times 1$$

$$\Rightarrow \lim_{x \rightarrow 0^+} f(x) = \frac{1}{2}$$

If $f(x)$ is continuous at $x = 0$, then

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x)$$

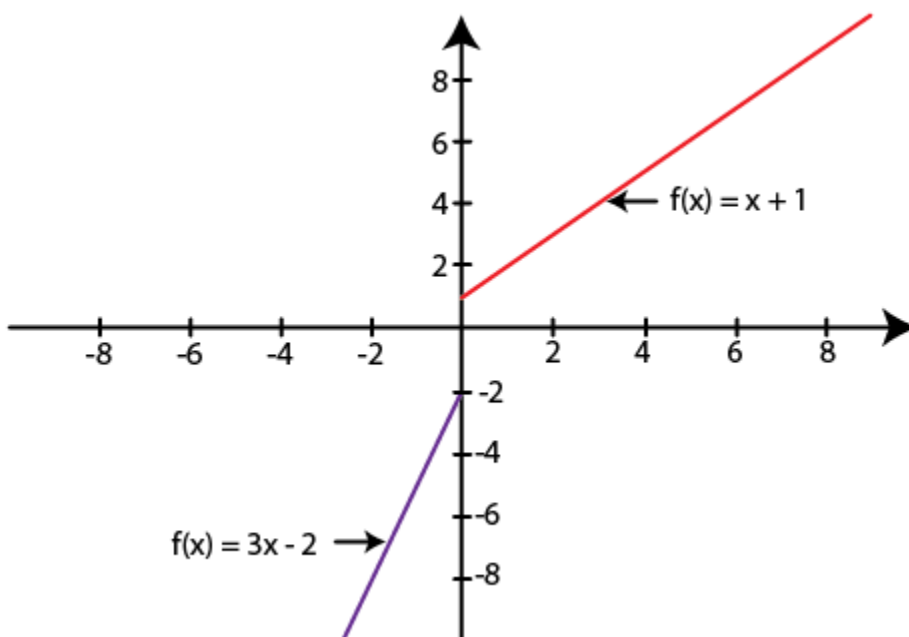
$$\Rightarrow a = \frac{1}{2}$$

14. Examine the continuity of the function

$$f(x) = \begin{cases} 3x - 2, & x \leq 0 \\ x + 1, & x > 0 \end{cases} \text{ at } x = 0$$

Also sketch the graph of this function.

Solution:



Given

$$f(x) = \begin{cases} 3x - 2, & x \leq 0 \\ x + 1, & x > 0 \end{cases} \text{ at } x = 0$$

The given function can be written as

$$f(x) = \begin{cases} 3x - 2, & x < 0 \\ 3(0) - 2, & x = 0 \\ x + 1, & x > 0 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} 3x - 2, & x < 0 \\ -2, & x = 0 \\ x + 1, & x > 0 \end{cases}$$

Consider LHL at $x = 0$

$$= \lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0 - h) = \lim_{h \rightarrow 0} f(-h)$$

$$\lim_{h \rightarrow 0} 3(-h) - 2 = -2$$

Now again consider RHL at $x = 0$

$$= \lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0 + h) = \lim_{h \rightarrow 0} f(h)$$

$$\lim_{h \rightarrow 0} (h + 1) = 1$$

$$\therefore \lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$$

Hence $f(x)$ is discontinuous at $x = 0$

15. Discuss the continuity of the function

$$f(x) = \begin{cases} x, & x > 0 \\ 1, & x = 0 \\ -x, & x < 0 \end{cases} \text{ at the point } x = 0$$

Solution:

Given

$$f(x) = \begin{cases} x, & x > 0 \\ 1, & x = 0 \\ -x, & x < 0 \end{cases} \text{ at the point } x = 0$$

Consider LHL at $x = 0$

$$= \lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0 - h) = \lim_{h \rightarrow 0} f(-h)$$

$$\lim_{h \rightarrow 0} -(-h) = 0$$

Consider RHL at $x = 0$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0 + h) = \lim_{h \rightarrow 0} f(h)$$

$$\lim_{h \rightarrow 0} (h) = 0$$

And we have $f(0) = 1$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) \neq f(0)$$

Hence $f(x)$ is discontinuous at $x = 0$

16. Discuss the continuity of the function

$$f(x) = \begin{cases} x, & 0 \leq x < \frac{1}{2} \\ 12, & x = \frac{1}{2} \\ 1 - x, & \frac{1}{2} < x \leq 1 \end{cases} \text{ at the point } x = \frac{1}{2}$$

Solution:

Given

$$f(x) = \begin{cases} x, & 0 \leq x < \frac{1}{2} \\ 12, & x = \frac{1}{2} \\ 1 - x, & \frac{1}{2} < x \leq 1 \end{cases}$$

Consider LHL at $x = \frac{1}{2}$

$$\lim_{x \rightarrow \frac{1}{2}^-} f(x) = \lim_{h \rightarrow 0} f\left(\frac{1}{2} - h\right)$$

$$\lim_{h \rightarrow 0} \left(\frac{1}{2} - h\right) = \frac{1}{2}$$

Again consider RHL at $x = \frac{1}{2}$

$$\lim_{x \rightarrow \frac{1}{2}^+} f(x) = \lim_{h \rightarrow 0} f\left(\frac{1}{2} + h\right)$$

$$\lim_{h \rightarrow 0} \left(1 - \left(\frac{1}{2} + h\right)\right) = \frac{1}{2}$$

We have $f\left(\frac{1}{2}\right) = \frac{1}{2}$

$$\therefore \lim_{x \rightarrow \frac{1}{2}^-} f(x) = \lim_{x \rightarrow \frac{1}{2}^+} f(x) = f\left(\frac{1}{2}\right)$$

Hence $f(x)$ is continuous at $x = \frac{1}{2}$

17. Discuss the continuity of

$$f(x) = \begin{cases} 2x - 1, & x < 0 \\ 2x + 1, & x \geq 0 \end{cases} \text{ at } x = 0$$

Solution:

Given

$$f(x) = \begin{cases} 2x - 1, & x < 0 \\ 2x + 1, & x \geq 0 \end{cases} \text{ at } x = 0$$

Consider LHL at $x = 0$

$$\lim_{x \rightarrow 0^-} f(x) = 2(0) - 1 = -1$$

Again consider RHL at $x = 0$

$$\lim_{x \rightarrow 0^+} f(x) = 2(0) + 1 = 1$$

$$\Rightarrow \lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$$

Hence $f(x)$ is discontinuous at $x = 0$

18. For what value of k is the function

$$f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ k, & x = 1 \end{cases} \text{ continuous at } x = 1?$$

Solution:

Given

$$f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ k, & x = 1 \end{cases}$$

If $f(x)$ is continuous at $x = 1$, then

$$\lim_{x \rightarrow 1} f(x) = f(1)$$

$$\lim_{x \rightarrow 1} \frac{x^2-1}{x-1} = k$$

$$\lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{x-1} = k$$

$$\lim_{x \rightarrow 1} (x+1) = k$$

$$k = 2$$

19. Determine the value of the constant k so that the function

$$f(x) = \begin{cases} \frac{x^2-3x+2}{x-1}, & x \neq 1 \\ k, & x = 1 \end{cases} \text{ continuous at } x = 1$$

Solution:

Given

$$f(x) = \begin{cases} \frac{x^2-3x+2}{x-1}, & x \neq 1 \\ k, & x = 1 \end{cases}$$

If $f(x)$ is continuous at $x = 1$, then

$$\lim_{x \rightarrow 1} f(x) = f(1)$$

$$\lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{x - 1} = k$$

$$\lim_{x \rightarrow 1} \frac{(x - 2)(x - 1)}{x - 1} = k$$

$$\lim_{x \rightarrow 1} (x - 2) = k$$

$$k = -1$$

20. For what value of k is the function

$$f(x) = \begin{cases} \frac{\sin 5x}{3x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases} \text{ continuous at } x = 0?$$

Solution:

Given

$$f(x) = \begin{cases} \frac{\sin 5x}{3x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$

If $f(x)$ is continuous at $x = 0$, then we have

$$\lim_{x \rightarrow 0} f(x) = f(0)$$

$$\lim_{x \rightarrow 0} \frac{\sin 5x}{3x} = k$$

$$\lim_{x \rightarrow 0} \frac{5 \sin 5x}{3 \times 5x} = k$$

$$\frac{5}{3} \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} = k$$

$$\frac{5}{3} \times 1 = k$$

$$k = \frac{5}{3}$$

21. Determine the value of the constant k so that the function

$$f(x) = \begin{cases} kx^2, & \text{if } x \leq 2 \\ 3, & \text{if } x > 2 \end{cases} \text{ is continuous at } x = 2$$

Solution:

Given

$$f(x) = \begin{cases} kx^2, & \text{if } x \leq 2 \\ 3, & \text{if } x > 2 \end{cases}$$

If $f(x)$ is continuous at $x = 2$, then we have

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$$

Now,

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{h \rightarrow 0} f(2 - h) = \lim_{h \rightarrow 0} k(2 - h)^2 = 4k$$

$$f(2) = 3$$

From the above equation we can write as

$$4k = 3$$

$$\Rightarrow k = \frac{3}{4}$$

22. Determine the value of the constant k so that the function

$$f(x) = \begin{cases} \frac{\sin 2x}{5x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases} \text{ is continuous at } x = 0$$

Solution:

Given

$$f(x) = \begin{cases} \frac{\sin 2x}{5x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$

If $f(x)$ is continuous at $x = 0$

$$\lim_{x \rightarrow 0} f(x) = f(0)$$

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{5x} = k$$

$$\lim_{x \rightarrow 0} \frac{2 \sin 2x}{5 \times 2x} = k$$

$$\frac{2}{5} \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} = k$$

$$\frac{2}{5} \times 1 = k$$

$$k = \frac{2}{5}$$

23. Find the values of a so that the function

$$f(x) = \begin{cases} ax + 5, & \text{if } x \leq 2 \\ x - 1, & \text{if } x > 2 \end{cases} \text{ is continuous at } x = 2$$

Solution:

Given

$$f(x) = \begin{cases} ax + 5, & \text{if } x \leq 2 \\ x - 1, & \text{if } x > 2 \end{cases}$$

Consider LHL at $x = 2$

$$\lim_{h \rightarrow 0} a(2 - h) + 5 = 2a + 5$$

Now again consider

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{h \rightarrow 0} f(2 + h)$$

$$\lim_{h \rightarrow 0} (2 + h - 1)$$

$$= 1$$

$$f(2) = a(2) + 5 = 2a + 5$$

Since $f(x)$ is continuous at $x = 2$ we have

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$$

$$2a + 5 = 1$$

$$2a = -4$$

$$a = -2$$