EXERCISE 10J

1. Find the second derivative of:

(i) x^{11}

(ii) 5^x

(iii) tan x

(iv) $\cos^{-1} x$

Solution:

(i) By differentiating w.r.t. x

$f'(x) = 11x^{11-1} = 11x^{10}$

Again by differentiating w.r.t. x

$f''(x) = 110x^{10-1} = 110x^9$

(ii) By differentiating w.r.t. x

$f'(x) = 5^x \log_e 5$

Again by differentiating w.r.t. x

$f''(x) = \log_e 5 \, . \, 5^x \log_e 5 = 5^x(\log_e 5)^2$

(iii) By differentiating w.r.t. x

$f'(x) = \sec^2 x$

Again by differentiating w.r.t. x

$f''(x) = 2\sec x \, . \, \sec x \tan x = 2\sec^2 x \tan x$

(iv) By differentiating w.r.t. x

$$f'(x) = \frac{-1}{\sqrt{1-x^2}}$$

Again by differentiating w.r.t. x

$$f''(x) = \frac{-1}{2} \times \frac{-1}{(1-x^2)^{\frac{3}{2}}} \times -2x = \frac{-x}{(1-x^2)^{\frac{3}{2}}}$$

2. Find the second derivative of:

(i) x sin x

(ii) $e^{2x} \cos 3x$

(iii) $x^3 \log x$

Solution:

(i) By differentiating w.r.t. x

$f'(x) = \sin x + x\cos x$

Again by differentiating w.r.t. x

$f''(x) = \cos x + \cos x - x\sin x = -\sin x + 2\cos x$

(ii) By differentiating w.r.t. x

$f'(x) = 2e^{2x}\cos 3x + e^{2x}(-\sin 3x).3$

So we get

$= 2e^{2x}\cos 3x - 3e^{2x}\sin 3x$

Again by differentiating w.r.t. x

$f''(x) = 2.2e^{2x}\cos 3x + 2e^{2x}(-\sin 3x).3 - 3.2e^{2x}\sin 3x - 3e^{2x}\cos 3x.3$

On further calculation

$= 4e^{2x}\cos 3x - 6e^{2x}\sin 3x - 6e^{2x}\sin 3x - 9e^{2x}\cos 3x$

So we get

$= -12e^{2x}\sin 3x - 5e^{2x}\cos 3x$

(iii) By differentiating w.r.t. x

$$f'(x) = 3x^2 \log x + \frac{x^3}{x}$$

So we get

$f'(x) = 3x^2\log x + x^2$

Again by differentiating w.r.t. x

$$f'(x) = 6x \log x + \frac{3x^2}{x} + 2x$$

On further calculation

$= 6x \log x + 3x + 2x$

So we get

$= 6x \log x + 5x$

3. If y = x + tan x, show that $\cos^2 x \frac{d^2y}{dx^2} - 2y + 2x = 0$.

Solution:

y = x + tan x

It can be written as

tan x = y – x

By differentiating w.r.t.x

$$\frac{dy}{dx} = 1 + \sec^2 x$$

Again by differentiating w.r.t. x

$$\frac{d^2y}{dx^2} = 2\sec x\ .\sec x \tan x$$

It can be written as

$$\frac{d^2y}{dx^2} = 2\sec^2 x\ \tan x$$

So we get

$$\frac{d^2y}{dx^2} = \frac{2\tan x}{\cos^2 x}$$

By substituting the value of $\frac{d^2y}{dx^2}$

$$\cos^2 x \frac{d^2y}{dx^2} = 2\tan x$$

Substituting the value of tan x

$$\cos^2 x \frac{d^2y}{dx^2} = 2y - 2x$$

So we get

$$\cos^2 x \frac{d^2y}{dx^2} - 2y + 2x = 0$$

4. If $y = 2\sin x + 3\cos x$, **show that** $y + \frac{d^2y}{dx^2} = 0$.

Solution:

We know that

y = 2 sin x + 3 cos x

By differentiating w.r.t. x

$$\frac{dy}{dx} = 2\cos x - 3\sin x$$

Again by differentiating w.r.t. x

$$\frac{d^2y}{dx^2} = -2\sin x - 3\cos x$$

So we get

$$\frac{d^2y}{dx^2} = -y$$

It can be written as

$$\frac{d^2y}{dx^2} + y = 0$$

Therefore, it is proved.

5. If y = 3 cos (log x) + 4 sin (log x), prove that $x^2y_2 + xy_1 + y = 0$.
Solution:

We know that

y = 3 cos (log x) + 4 sin (log x)

By differentiating w.r.t. x

$$y_1 = -3\sin(\log x)\frac{1}{x} + 4\cos(\log x)\frac{1}{x}$$

It can be written as

$$y_1 = \frac{-3\sin(\log x) + 4\cos(\log x)}{x}$$

Again by differentiating w.r.t. x

$$y_2 = \frac{x\left(-3\cos(\log x)\frac{1}{x} - 4\sin(\log x)\frac{1}{x}\right) - (-3\sin(\log x) + 4\cos(\log x))}{x^2}$$

By cross multiplication

$$x^2 y_2 = \frac{-x}{x}(3\cos(\log x) - 4\sin(\log x)) - (y_1 x)$$

It can be written as

$$x^2 y_2 = -y - xy_1$$

So we get

$$x^2y_2 + xy_1 + y = 0$$

Therefore, it is proved.

6. If $y = e^{-x}\cos x$, show that $\frac{d^2y}{dx^2} = 2e^{-x}\sin x$.

Solution:

We know that

$y = e^{-x} \cos x$

By differentiating w.r.t. x

$$\frac{dy}{dx} = -e^{-x}\cos x + e^{-x}(-\sin x)$$

It can be written as

$$\frac{dy}{dx} = -e^{-x}\cos x - e^{-x}\sin x$$

So we get

$$\frac{dy}{dx} = -e^{-x}(\cos x + \sin x)$$

Again by differentiating w.r.t. x

$$\frac{d^2y}{dx^2} = e^{-x}(\cos x + \sin x) - e^{-x}(-\sin x + \cos x)$$

By multiplication

$$\frac{d^2y}{dx^2} = e^{-x}(\cos x + \sin x - (-\sin x) - \cos x)$$

On further calculation

$$\frac{d^2y}{dx^2} = e^{-x}(\sin x + \sin x)$$

So we get

$$\frac{d^2y}{dx^2} = 2e^{-x}\sin x$$

7.If $y = \sec x - \tan x$, show that $\cos x\frac{d^2y}{dx^2} = y^2$.

Solution:

We know that

y = sec x - tanx

By differentiating w.r.t. x

$$\frac{dy}{dx} = \sec x \tan x - \sec^2 x$$

Again by differentiating w.r.t. x

$$\frac{d^2y}{dx^2} = \sec x \tan x \times \tan x + \sec x \times \sec^2 x - 2\sec x \times \sec x \tan x$$

On further calculation

$$\frac{d^2y}{dx^2} = \sec x \tan^2 x + \sec^3 x - 2\sec^2 x \tan x$$

Taking sec x as common

$$\frac{d^2y}{dx^2} = \sec x\,(\tan^2 x + \sec^2 x - 2\sec x \tan x)$$

It can be written as

$$\frac{1}{\sec x}\frac{d^2y}{dx^2} = (\sec x - \tan x)^2$$

So we get

$$\cos x\frac{d^2y}{dx^2} = y^2$$

Therefore, it is proved.

8. $\mathbf{If\ y = (cosecx + cotx),\ prove\ that\ sinx\frac{d^2y}{dx^2} - y^2 = 0.}$

Solution:

We know that

y = (cosec x + cot x)

By differentiating w.r.t. x

$$\frac{dy}{dx} = -\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x$$

Again by differentiating w.r.t. x

$$\frac{d^2y}{dx^2} = \operatorname{cosec} x \cot^2 x + \operatorname{cosec}^3 x + 2\operatorname{cosec} x \times \operatorname{cosec} x \cot x$$

Taking cosec x as common

$$\frac{d^2y}{dx^2} = \operatorname{cosec} x\,(\cot^2 x + \operatorname{cosec}^2 x + 2\operatorname{cosec} x \cot x)$$

It can be written as

$$\frac{1}{\operatorname{cosec} x}\frac{d^2y}{dx^2} = (\cot x + \operatorname{cosec} x)^2$$

So we get

$$\sin x\frac{d^2y}{dx^2} = y^2$$

Here

$$\sin x\frac{d^2y}{dx^2} - y^2 = 0$$

Therefore, it is proved.

9. $\mathbf{If\ y = tan^{-1}x,\ show\ that\ (1 + x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0.}$

Solution:

We know that

$y = \tan^{-1} x$

By differentiating w.r.t. x

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

So we get

$$(1+x^2)\frac{dy}{dx} = 1$$

Again by differentiating w.r.t. x

$$(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$$

Therefore, it is proved.

10. If $y = \sin(\sin x)$, prove that $\frac{d^2y}{dx^2} + (\tan x)\frac{dy}{dx} + y\cos^2 x = 0$.

Solution:

We know that

$y = \sin(\sin x)$

By differentiating w.r.t. x

$$\frac{dy}{dx} = \cos(\sin x)\ \cos x$$

Again by differentiating w.r.t. x

$$\frac{d^2y}{dx^2} = -\sin(\sin x)\cos x\ \cos x - \sin x\cos(\sin x)$$

It can be written as

$$\frac{d^2y}{dx^2} = -y\cos^2 x - \sin x\,\frac{\frac{dy}{dx}}{\cos x}$$

So we get

$$\frac{d^2y}{dx^2} = -y\cos^2 x - \tan x\frac{dy}{dx}$$

Here

$$\frac{d^2y}{dx^2} + y\cos^2 x + \tan x\frac{dy}{dx} = 0$$

11. If y = a cos (log x) + b sin (log x), prove that $x^2 y_2 + xy_1 + y = 0$
Solution:

We know that

y = a cos (log x) + b sin (log x)

By differentiating w.r.t. x

$$y_1 = -a\sin(\log x)\frac{1}{x}$$

Again by differentiating w.r.t. x

$$y_2 = \frac{-x\,a\cos(\log x)\frac{1}{x} + a\sin(\log x)}{x^2}$$

It can be written as

$$x^2 y_2 = -y - xy_1$$

So we get

$$x^2 y_2 + xy_1 + y = 0$$

Therefore, it is proved.

12. Find the second derivative of e^{3x} sin 4x.
Solution:

We know that

$y = e^{3x}\sin 4x$

By differentiating w.r.t. x

$$\frac{dy}{dx} = 3e^{3x}\sin 4x + 4e^{3x}\cos 4x$$

Again by differentiating w.r.t. x

$$\frac{d^2y}{dx^2} = 9e^{3x}\sin 4x + 12e^{3x}\cos 4x + 12e^{3x}\cos 4x - 16e^{3x}\sin 4x$$

It can be written as

$$\frac{d^2y}{dx^2} = 24e^{3x}\cos 4x - 7e^{3x}\sin x$$

So we get

$$\frac{d^2y}{dx^2} = e^{3x}(24\cos x - 7\sin x)$$

13. Find the second derivative of sin 3x cos 5x.
Solution:

We know that

y = sin 3x cos 5x

It can be written as

$$y = \frac{1}{2}[\sin(5x+3x) + \sin(5x-3x)] = \frac{1}{2}\sin 8x + \frac{1}{2}\sin 2x$$

By differentiating w.r.t. x

$$\frac{dy}{dx} = \frac{8}{2}\cos 8x + \frac{2}{2}\cos 2x$$

So we get

$$\frac{dy}{dx} = 4\cos 8x + \cos 2x$$

Again by differentiating w.r.t. x

$$\frac{d^2y}{dx^2} = -32\sin 8x - 2\sin 2x$$

Therefore, it is proved.

14. If $y = e^{\tan x}$, prove that $(\cos^2 x)\frac{d^2y}{dx^2} - (1+\sin 2x)\frac{dy}{dx} = 0$.
Solution:

We know that

$y = e^{\tan x}$

By differentiating w.r.t. x

$$\frac{dy}{dx} = \sec^2 x\, e^{\tan x}$$

So we get

$$\frac{1}{\sec^2 x}\frac{dy}{dx} = e^{\tan x}$$

It can be written as

$$\cos^2 x\ \frac{dy}{dx} = e^{\tan x}$$

Again by differentiating w.r.t. x

$$(\cos^2 x)\frac{d^2y}{dx^2} - \ (2\cos x \sin x)\frac{dy}{dx} = \sec^2 x\, e^{\tan x}$$

It can be written as

$$(\cos^2 x)\ \frac{d^2y}{dx^2} - \sin 2x\frac{dy}{dx} = \frac{dy}{dx}$$

On further calculation

$$(\cos^2 x)\ \frac{d^2y}{dx^2} - \sin 2x\frac{dy}{dx} - \frac{dy}{dx} = 0$$

By further simplification

$$(\cos^2 x)\ \frac{d^2y}{dx^2} - \ (\sin 2x + \ 1)\ \frac{dy}{dx} = \ 0$$

Therefore, it is proved.

15. If $y = \frac{\log x}{x}$, prove that $\frac{d^2y}{dx^2} = \frac{(2\log x - 3)}{x^3}$.

Solution:

We know that

y = log x/ x

By differentiating w.r.t. x

$$\frac{dy}{dx} = \frac{\frac{1}{x} \times x - \log x}{x^2}$$

So we get

$$\frac{dy}{dx} = \frac{1-\log x}{x^2}$$

Again by differentiating w.r.t. x

$$\frac{d^2y}{dx^2} = \frac{\frac{-1}{x} \times x^2 - 2x(1-\log x)}{x^4}$$

It can be written as

$$\frac{d^2y}{dx^2} = \frac{-x-2x(1-\log x)}{x^4}$$

On further calculation

$$\frac{d^2y}{dx^2} = \frac{-1-2+2\log x}{x^3}$$

By further simplification

$$\frac{d^2y}{dx^2} = \frac{(2\log x - 3)}{x^3}$$

Therefore, it is proved.