EXERCISE 13A
Evaluate the following integrals:
Very-Short-Answer Questions

1. \( \int (2x + 9)^5 \, dx \)
   Solution:
   Take \( 2x + 9 = t \)
   So we get
   \( 2 \, dx = dt \)
   It can be written as
   \[ \int t^5 \left( \frac{dt}{2} \right) = \frac{1}{2} \int t^5 \, dt \]
   By integrating w.r.t. \( t \)
   \[ = \frac{1}{2} \frac{t^6}{6} + c = \frac{t^6}{12} + c \]
   By substituting the value of \( t \)
   \[ = \frac{(2x + 9)^6}{12} + c \]

2. \( \int (7 - 3x)^4 \, dx \)
   Solution:
   Take \( 7 - 3x = t \)
   So we get
   \( -3 \, dx = dt \)
   It can be written as
   \[ \int t^4 \left( \frac{dt}{-3} \right) = \frac{1}{-3} \int t^4 \, dt \]
   By integrating w.r.t. \( t \)
   \[ = \frac{1}{-3} \frac{t^5}{5} + c = -\frac{t^5}{15} + c \]
   By substituting the value of \( t \)
   \[ = -\frac{(7 - 3x)^5}{15} + c \]

3. \( \int \sqrt{3x - 5} \, dx \)
   Solution:
   Take \( 3x - 5 = t \)
   So we get
   \( 3 \, dx = dt \)
   It can be written as
Solution:

Take $4x + 3 = t$
So we get
$4 \, dx = dt$
It can be written as

$$\int \frac{1}{4} \, dt$$

By integrating w.r.t. $t$

$$\frac{1}{4} \times \frac{t^{0.5}}{0.5} + c = \frac{2}{4} \times \frac{t^{0.5}}{1} + c$$

By substituting the value of $t$

$$= \frac{\sqrt{4x + 3}}{2} + c$$

5. $\int \frac{1}{\sqrt{3 - 4x}} \, dx$

Solution:

Take $3 - 4x = t$
So we get
$-4 \, dx = dt$
It can be written as

$$\int \frac{1}{-4} \, dt$$

By integrating w.r.t. $t$

$$= \frac{1}{-4} \times \frac{t^{0.5}}{0.5} + c = \frac{2}{-4} \times \frac{t^{0.5}}{1} + c$$

By substituting the value of $t$
6. $\int \frac{1}{(2x - 3)^{3/2}} \, dx$

Solution:

Take $2x - 3 = t$
So we get
$2 \, dx = dt$
It can be written as
$\int t^{-3/2} \left(\frac{dt}{2}\right) = \frac{1}{2} \int t^{-3/2} \, dt$
By integrating w.r.t. $t$
$= \frac{1}{2} \times t^{-1/2} + c = \frac{-2}{2} \times \frac{t^{-0.5}}{1} + c$
By substituting the value of $t$
$= -\frac{1}{\sqrt{2x - 3}} + c$

7. $\int e^{(2x - 1)} \, dx$

Solution:

Take $2x - 1 = t$
So we get
$2 \, dx = dt$
It can be written as
$\int e^t \left(\frac{dt}{2}\right) = \frac{1}{2} \int e^t \, dt$
By integrating w.r.t. $t$
$= \frac{1}{2} \times e^t + c = \frac{e^{2x-1}}{2} + c$
By substituting the value of $t$
$e^{(2x-1)} = \frac{e^{2x-1}}{2} + c$

8. $\int e^{(1-3x)} \, dx$

Solution:
Take $1 - 3x = t$
So we get
$-3\, dx = dt$
It can be written as
\[
\int e^t \left( \frac{dt}{-3} \right) = \frac{1}{-3} \int e^t \, dt
\]
By integrating w.r.t. $t$
\[
= \frac{1}{-3} \times e^t + c
\]
By substituting the value of $t$
\[
e^t = \frac{e^{1 - 3x}}{3} + c
\]

9. $\int \left(2 - 3x\right) \, dx$

Solution:

Take $2 - 3x = t$
So we get
$-3\, dx = dt$
It can be written as
\[
\int 3^t \left( \frac{dt}{-3} \right) = \frac{1}{-3} \int 3^t \, dt
\]
By integrating w.r.t. $t$
\[
= \frac{1}{-3} \times \left( \frac{3^t}{\log 3} \right) + c = \frac{3^t}{-3 \log 3} + c
\]
By substituting the value of $t$
\[
e^{2 - 3x} = \frac{3^{2 - 3x}}{3 \log 3} + c
\]

10. $\int \sin 3x \, dx$

Solution:

Take $3x = t$
So we get
$3\, dx = dt$
It can be written as
\[
\int \sin t \left( \frac{dt}{3} \right) = \frac{1}{3} \int \sin t \, dt
\]
By integrating w.r.t. $t$
\[ \int \cos (5 + 6x) \, dx \]

**Solution:**

Take \( 5 + 6x = t \)

So we get

\[ 6 \, dx = dt \]

It can be written as

\[ \int \cos t \left( \frac{dt}{6} \right) = \frac{1}{6} \int \cos t \, dt \]

By integrating w.r.t. \( t \)

\[ = \frac{1}{6} \times (\sin t) + c \]

By substituting the value of \( t \)

\[ = \frac{\sin(5 + 6x)}{6} + c \]

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\[ \int \sin x \sqrt{1 + \cos 2x} \, dx \]

**Solution:**

It is given that

\[ \int \sin x \sqrt{1 + \cos 2x} \, dx \]

Here we know that \( 1 + \cos 2x = 2 \cos^2 x \)

We can write it as

\[ \int \sin x \sqrt{1 + \cos 2x} \, dx = \int \sin x \sqrt{2} \cos x \, dx = \sqrt{2} \int \sin x \cos x \, dx \]

Take \( \sin x = t \)

So we get

\( \cos x \, dx = dt \)

It can be written as

\[ \int \sqrt{2} \sin x \cos x \, dx = \int \sqrt{2} t \, dt \]

By integrating w.r.t. \( t \)
Take $2x + 5 = t$
So we get
$2 \, dx = dt$
By integrating w.r.t. $t$

$$\int \frac{t^2}{2} + c$$

By substituting the value of $t$
$$\frac{(\sin x)^2}{\sqrt{2}} + c$$

13. $\int \csc^2(2x + 5) \, dx$

Solution:
Take $2x + 5 = t$
So we get
$2 \, dx = dt$
By integrating w.r.t. $t$
$$\int \csc^2 t \, dt = -\frac{1}{2} \cot t + c$$
By substituting the value of $t$
$$= -\frac{1}{2} \cot(2x + 5) + c$$

14. $\int \sin x \cos x \, dx$

Solution:
Take $\sin x = t$
So we get
$\cos x \, dx = dt$
By integrating w.r.t. $t$
$$\int t \, dt = \frac{t^2}{2} + c$$
By substituting the value of $t$
$$= \frac{(\sin x)^2}{2} + c$$

15. $\int \sin^3 x \cos x \, dx$

Solution:
Take $\sin x = t$
So we get
$\cos x \, dx = dt$
By integrating w.r.t. $t$
\[ \int t^3 \, dt = \frac{t^4}{4} + c \]

By substituting the value of \( t \)
\[ = \frac{(\sin x)^4}{4} + c \]

16. \[ \int \left( \sqrt{\cos x} \right) \sin x \, dx \]

Solution:

Take \( \cos x = t \)
So we get
- \( \sin x \, dx = dt \)

By integrating w.r.t. \( t \)
\[ \int t^{0.5} (-1) \, dt = -\frac{t^{1.5}}{1.5} + c \]

By substituting the value of \( t \)
\[ = -\frac{2(\cos x)^{3/2}}{3} + c \]

17. \[ \int \frac{\sin^{-1} x}{\sqrt{1-x^2}} \, dx \]

Solution:

Take \( \sin^{-1} x = t \)
So we get
- \( \frac{1}{\sqrt{1-x^2}} \, dx = dt \)

By integrating w.r.t. \( t \)
\[ \int t^1 \, dt = \frac{t^2}{2} + c \]

By substituting the value of \( t \)
\[ = \frac{(\sin^{-1} x)^2}{2} + c \]

18. \[ \int \frac{\sin \left( 2 \tan^{-1} x \right)}{\left( 1+x^2 \right)} \, dx \]
Solution:

Take \( \tan^{-1} x = t \)
So we get
\[
\frac{1}{1+x^2} \, dx = dt
\]
By integrating w.r.t. \( t \)
\[
\int \sin 2t \, dt = -\frac{\cos 2t}{2} + c
\]
By substituting the value of \( t \)
\[
= -\frac{\cos(2 \tan^{-1} x)}{2} + c
\]

19. \( \int \frac{\cos (\log x)}{x} \, dx \)

Solution:

Take \( \log x = t \)
So we get
\[
\frac{1}{x} \, dx = dt
\]
By integrating w.r.t. \( t \)
\[
\int \cos t \, dt = \sin t + c
\]
By substituting the value of \( t \)
\[
= \sin (\log x) + c
\]

20. \( \int \frac{\csc^2(\log x)}{x} \, dx \)

Solution:

Take \( \log x = t \)
So we get
\[
\frac{1}{x} \, dx = dt
\]
By integrating w.r.t. \( t \)
\[
\int \csc^2 t \, dt = -\cot t + c
\]
By substituting the value of \( t \)
\[
= -\cot (\log x) + c
\]

21. \( \int \frac{1}{x \log x} \, dx \)

Solution:
Take \( \log x = t \)
So we get
\[ \frac{1}{x} \, dx = dt \]
By integrating w.r.t. \( t \)
\[ \int \frac{dt}{t} = \log t + c \]
By substituting the value of \( t \)
\[ = \log (\log x) + c \]

22. \[ \int \frac{(x+1)(x + \log x)^2}{x} \, dx \]
Solution:
It is given that
\[ \int \frac{(x+1)(x + \log x)^2}{x} \, dx \]
We can write it as
\[ = \int \frac{x+1}{x} \times \frac{(x + \log x)^2}{1} \, dx \]
On further simplification
\[ = \int \left(1 + \frac{1}{x}\right) \times \frac{(x + \log x)^2}{1} \, dx \]
Take \( x + \log x = t \)
So we get
\[ \left(1 + \frac{1}{x}\right) \, dx = dt \]
By integrating w.r.t. \( t \)
\[ \int t^2 \, dt = \frac{t^3}{3} + c \]
By substituting the value of \( t \)
\[ = \frac{(x + \log x)^3}{3} + c \]

23. \[ \int \frac{(\log x)^2}{x} \, dx \]
Solution:
Take \( \log x = t \)
So we get
\[ \frac{1}{x} \, dx = dt \]
By integrating w.r.t. t
\[ \int t^2 dt = \frac{t^3}{3} + c \]

By substituting the value of t
\[ \frac{(\log x)^3}{3} + c \]

24. \[ \int \frac{\cos \sqrt{x}}{\sqrt{x}} dx \]
Solution:
Take \( \sqrt{x} = t \)
So we get
\[ \frac{1}{2} \sqrt{x} dx = dt \]
By integrating w.r.t. t
\[ \int \cos 2t dt = 2 \sin t + c \]
By substituting the value of t
\[ = 2 \sin \sqrt{x} + c \]

25. \[ \int e^{\tan x} \sec^2 x dx \]
Solution:
Take \( \tan x = t \)
So we get
\[ \sec^2 x dx = dt \]
By integrating w.r.t. t
\[ \int e^t dt = e^t + c \]
By substituting the value of t
\[ = e^{\tan x} + c \]

26. \[ \int e^{\cos^2 x} \sin 2x dx \]
Solution:
Take \( \cos^2 x = t \)
So we get
\[ - \sin 2x dx = dt \]
By integrating w.r.t. t
\[ \int -e^t dt = -e^t + c \]
By substituting the value of t
\[ = -e^{\cos^2 x} + c \]

27. \[ \int \sin (ax + b) \cos (ax + b) dx \]
Solution:

Take $ax + b = t$
So we get
$a \, dx = dt$
It can be written as
$$\int \sin t \cos t \, \frac{dt}{a} = \frac{1}{a} \int \sin t \cos t \, dt$$

Take $\sin t = z$
So we get
$\cos t \, dt = dz$
By integrating w.r.t. $z$
$$\frac{1}{a} \int zdz = \frac{1}{a} \frac{z^2}{2} + c$$
By substituting the value of $z$ and $t$
$$= \frac{(\sin ax + b)^2}{2a} + c$$

28. $\int \cos^3 x \, dx$

Solution:

We know that
$\cos 3x = 4 \cos^3 x - 3 \cos x$
It can be written as
$$\int \left(\frac{3 \cos x}{4} + \cos 3x \right) dx = \frac{3 \sin x}{4} + \frac{\sin 3x}{4 \times 3} + c$$
So we get
$$= \frac{3 \sin x}{4} + \frac{\sin 3x}{12} + c$$

29. $\int \frac{1}{x^2} e^{-1/x} \, dx$

Solution:

Take $-1/x = t$
So we get
$\frac{1}{x^2} \, dx = dt$
By integrating w.r.t. $t$
$$\int e^t \, dt = e^t + c$$
By substituting the value of $t$
$$= e^{-\frac{1}{x}} + c$$
30. $\int \frac{\cos\left(\frac{1}{x}\right)}{x^2} \, dx$

Solution:

Take $-1/x = t$
So we get $1/x^2 \, dx = dt$
By integrating w.r.t. $t$
$\int \cos t \, dt = \sin t + c$
By substituting the value of $t$
$= -\sin 1/x + c$

31. $\int \frac{dx}{e^x + e^{-x}}$

Solution:

It is given that
$\int \frac{dx}{e^x + e^{-x}}$

By taking LCM we get
$\int \frac{e^x}{1 + e^{2x}} \, dx$

Take $e^x = t$
So we get $e^x \, dx = dt$
By integrating w.r.t. $t$
$\int \frac{1}{1 + t^2} \, dt = \tan^{-1} t + c$
By substituting the value of $t$
$= \tan^{-1}(e^x) + c$

32. $\int \frac{e^{2x}}{(e^{2x} - 2)} \, dx$

Solution:

Take $e^{2x} - 2 = t$
So we get $2 e^{2x} \, dx = dt$
By integrating w.r.t. $t$
\[
\int \frac{1}{t} \left(\frac{dt}{2}\right) = \frac{1}{2} \int \frac{1}{t} \, dt = \frac{1}{2} \log t + c
\]

By substituting the value of \( t \)
\[= \frac{1}{2} \log(e^{2x} - 2) + c \]

33. \( \int \cot x \log(\sin x) \, dx \)

Solution:
Take \( \log(\sin x) = t \)
So we get \( \frac{\cos x}{\sin x} \, dx = dt \)
By cross multiplication \( \cot x \, dx = dt \)
By integrating w.r.t. \( t \)
\[
\int \frac{1}{t} \, dt = \log t + c
\]
By substituting the value of \( t \)
\[= \frac{1}{2} \log(\sin x) \]

34. \( \int \cot x \log(\sin x) \, dx \)

Solution:
Take \( \log(\sin x) = t \)
So we get \( \frac{\cos x}{\sin x} \, dx = dt \)
By cross multiplication \( \cot x \, dx = dt \)
By integrating w.r.t. \( t \)
\[
\int t \, dt = \frac{t^2}{2} + c
\]
By substituting the value of \( t \)
\[= \frac{(\log\sin x)^2}{2} + c \]

35. \( \int 2x \sin(x^2 + 1) \, dx \)

Solution:
Take \( x^2 + 1 = t \)
So we get \( 2x \, dx = dt \)
By integrating w.r.t. \( t \)
\[
\int \sin t \, dt = -\cos t + c
\]
By substituting the value of t
\[ t = - \cos (x^2 + 1) + c \]
EXERCISE 13B
Evaluate the following integrals:

1. (i) \( \int \sin^2 x \, dx \)
(ii) \( \int \cos^2 x \, dx \)

Solution:

(i) \( \int \sin^2 x \, dx \)
Here, we know that \( 1 - \cos 2x = 2 \sin^2 x \)
It can be written as
\[
\int \sin^2 x \, dx = \frac{1}{2} \left(1 - \cos 2x\right) dx
\]
So we get
\[
\frac{1}{2} \left( \int dx - \int \cos 2x \, dx \right)
\]
By integrating w.r.t x,
\[
x \sin 2x = \frac{1}{2} \left( \sin 2x \right) + c
\]
On further calculation
\[
x \sin 2x = \frac{1}{2} \left( \sin 2x \right) + c
\]

(ii) \( \int \cos^2 x \, dx \)
Here, we know that \( 1 + \cos 2x = 2 \cos^2 x \)
It can be written as
\[
\int \cos^2 x \, dx = \frac{1}{2} \left(1 + \cos 2x\right) dx
\]
So we get
\[
\frac{1}{2} \left( \int dx + \int \cos 2x \, dx \right)
\]
By integrating w.r.t x,
\[
x \sin 2x = \frac{1}{2} \left( \sin 2x \right) + c
\]
On further calculation
\[
x \sin 2x = \frac{1}{2} \left( \sin 2x \right) + c
\]

2. (i) \( \cos^2 \left(\frac{x}{2}\right) \, dx \)
(ii) \( \cot^2 \left(\frac{x}{2}\right) \, dx \)

Solution:

(i) \( \cos^2 \left(\frac{x}{2}\right) \, dx \)
Here, we know that \( 1 + \cos x = 2 \cos^2 \left(\frac{x}{2}\right) \)
It can be written as
\[
\int \cos^2 \left(\frac{x}{2}\right) dx = \int \frac{(1 + \cos x)dx}{2}
\]

So we get
\[
= \frac{1}{2} \left( \int dx + \int \cos x dx \right)
\]
By integrating w.r.t. x
\[
= \frac{x}{2} + \frac{\sin 2x}{2} + C
\]

(ii) \(\cot^2 \left(\frac{x}{2}\right) dx\)
Here, we know that \(\csc^2 x - \cot^2 x = 1\)
It can be written as
\[
\int \cot^2 \left(\frac{x}{2}\right) dx = \int \left(\csc^2 \left(\frac{x}{2}\right) - 1\right) dx
\]
So we get
\[
\int \left(\csc^2 \left(\frac{x}{2}\right) - 1\right) dx = \int \csc^2 \left(\frac{x}{2}\right) dx - \int 1 dx
\]
By integrating w.r.t. x
\[
\int \csc^2 \left(\frac{x}{2}\right) dx = -\cot \left(\frac{x}{2}\right) x + C
\]
We get
\[
\int \cot^2 \left(\frac{x}{2}\right) dx = -2\cot \left(\frac{x}{2}\right) x + C
\]

3. (i) \(\int \sin^3 nx dx\)
(ii) \(\int \sin^5 x dx\)
Solution:

(i) \(\int \sin^3 nx dx\)
Here, we know that \(1 - \cos 2nx = 2 \sin^2 nx\)
It can be written as
\[
\int \sin^2 nx dx = \int \frac{(1 - \cos 2nx)dx}{2}
\]
So we get
\[
= \frac{1}{2} \left( \int dx - \int \cos 2nx dx \right)
\]
By integrating w.r.t. x
\[
= \frac{x}{2} - \frac{\sin 2nx}{2n} + C
\]
We get
\[
= \frac{x}{2} - \frac{\sin 2nx}{4n} + C
\]

(ii) \(\int \sin^5 x dx\)
Here, we know that $1 - \cos^2 x = \sin^2 x$
It can be written as
\[
\int \sin^5 x \, dx = \int (1 - \cos^2 x)^2 \sin x \, dx
\]
Take $\cos x = t$
So we get
\[-\sin x \, dx = dt\]
By considering $\cos x = t$ we get
\[
\int (1 - \cos^2 x)^2 \sin x \, dx = -\int (1 - t^2)^2 \, dt
\]
We get
\[-\int (1 - t^2)^2 \, dt = -\int dt + \int 2t^2 \, dt - \int t^4 \, dt
\]
By integrating w.r.t $t$
\[-\int (1 - t^2)^2 \, dt = -t + \frac{2t^3}{3} - \frac{t^5}{5} + c\]
By substituting the value of $t$
\[
\int \sin^5 x \, dx = -\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5} + c
\]

4. $\int \cos^3 (3x + 5) \, dx$

Solution:

It is given that
\[
\int \cos^3 (3x + 5) \, dx
\]
By substituting $3x + 5 = u$
We get $3 \, dx = du$
Here $dx = du/3$

So we get
\[
\int \cos^3 (3x + 5) \, dx = \frac{1}{3} \int \cos^3 (u) \, du
\]
Here, we know that $1 - \cos^2 x = \sin^2 x$
\[
\frac{1}{3} \int \cos^3 (u) \, du = \frac{1}{3} \int (1 - \sin^2 (u)) \cos u \, du
\]
Now by substituting $\sin u = t$
We get $\cos u \, du = dt$
We get
\[
\frac{1}{3} \int (1 - \sin^2 (u)) \cos u \, du = \frac{1}{3} \int (1 - t^2) \, dt
\]
It can be written as
\[
= \frac{1}{3} \int dt - \frac{1}{3} \int t^2 \, dt
\]
By integrating w.r.t $t$
\[
= \frac{t}{3} - \frac{t^3}{9} + c
\]
So we get
\[
= \frac{t}{3} - \frac{t^3}{9} + c
\]
Now by re-substituting the value of $t = \sin u$ and $u = 3x + 5$
5. \[ \int \sin^7(3-2x) \, dx \]

Solution:

We can write it as

\[ = \int \sin^7(2x - 3) \, dx \]

By substituting the value of \( 2x - 3 = u \)

So we get

\[ 2 \, dx = du \text{ where } dx = du/2 \]

\[ = -\left(\frac{1}{2}\right) \int \sin^7(u) \, du \]

Here, we know that \( 1 - \cos^2 x = \sin^2 x \)

\[ = -\left(\frac{1}{2}\right) \left(1 - \cos^2(u)\right)^3 \sin u \, du \]

Consider \( \cos u = t \)

We get \( -\sin x \, du = dt \)

\[ = \left(\frac{1}{2}\right) \int (1 - t^2)^3 \, dt \]

We can write it as

\[ = \left(\frac{1}{2}\right) \int (1 - t^6 - 3t^2 + 3t^4) \, dt \]

By further simplification

\[ = \left(\frac{1}{2}\right) \left[ \int dt - \int t^6 \, dt - \int 3t^2 \, dt + \int 3t^4 \, dt \right] \]

By integrating w.r.t \( t \)

\[ = \left(\frac{1}{2}\right) \left[ t - \frac{t^7}{7} - \frac{3t^3}{3} + \frac{3t^5}{5} \right] + c \]

We get

\[ = \left(\frac{1}{2}\right) \left[ t - \frac{t^7}{7} - t^3 + \frac{3t^5}{5} \right] + c \]

Now by re-substituting the value of \( t = \cos u \) and \( u = 2x - 3 \)

\[ = \left(\frac{1}{2}\right) \left[ \cos(2x - 3) - \frac{\cos^7(2x-3)}{7} - \cos^3(2x - 3) + \frac{3\cos^5(2x-3)}{5} \right] + c \]

By further simplification

\[ = \frac{\cos(2x-3)}{2} - \frac{\cos^7(2x-3)}{14} - \frac{\cos^3(2x-3)}{2} + \frac{3\cos^5(2x-3)}{10} + c \]

We know that \( \cos(-x) = \cos x \)

\[ \cos(3-2x) = \cos (3-2x) \]

\[ = \frac{\cos(3-2x)}{2} - \frac{\cos^7(3-2x)}{14} - \frac{\cos^3(3-2x)}{2} + \frac{3\cos^5(3-2x)}{10} + c \]
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6. (i) \[ \int \frac{1 - \cos 2x}{1 + \cos 2x} \, dx \]

Solution:

(i) \[ \int \frac{1 - \cos 2x}{1 + \cos 2x} \, dx \]

We know that

1 - \cos 2x = 2 \sin^2 x \quad \text{and} \quad 1 + \cos 2x = 2 \cos^2 x

It can be written as

\[ \int \frac{1 - \cos 2x}{1 + \cos 2x} \, dx = \int \frac{2\sin^2 x}{2\cos^2 x} \, dx \]

We get

\[ \int \frac{1 - \cos 2x}{1 + \cos 2x} \, dx = \int \tan^2 x \, dx \]

We know that \( \sec^2 x - 1 = \tan^2 x \)

\[ \int \frac{1 - \cos 2x}{1 + \cos 2x} \, dx = \int (\sec^2 x - 1) \, dx \]

On further simplification

\[ \int \frac{1 - \cos 2x}{1 + \cos 2x} \, dx = \int \sec^2 x \, dx - \int dx \]

By integrating w.r.t. x

\[ \int \frac{1 - \cos 2x}{1 + \cos 2x} \, dx = \tan x + x + C \]

(ii) \[ \int \frac{1 + \cos 2x}{1 - \cos 2x} \, dx \]

We know that

1 - \cos 2x = 2 \sin^2 x \quad \text{and} \quad 1 + \cos 2x = 2 \cos^2 x

It can be written as

\[ \int \frac{1 + \cos 2x}{1 - \cos 2x} \, dx = \int \frac{2\cos^2 x}{2\sin^2 x} \, dx \]

We get

\[ \int \frac{1 + \cos 2x}{1 - \cos 2x} \, dx = \int \cot^2 x \, dx \]

We know that cosec^2 x - 1 = cot^2 x

\[ \int \frac{1 + \cos 2x}{1 - \cos 2x} \, dx = \int (\cosec^2 x - 1) \, dx \]

On further simplification
\[
\int \frac{1+\cos x}{1-\cos 2x} \, dx = \int \csc^2 x \, dx - \int dx
\]

By integrating w.r.t. \(x\)
\[
\int \frac{1+\cos 2x}{1-\cos 2x} \, dx = \cot x - x + c
\]

7. (i) \(\int \frac{1-\cos x}{1+\cos x} \, dx\)

(ii) \(\int \frac{1+\cos x}{1-\cos x} \, dx\)

Solution:

(i) \(\int \frac{1-\cos x}{1+\cos x} \, dx\)

Here, we know that
\(1 - \cos x = 2 \sin^2 \frac{x}{2}\) and \(1 + \cos x = 2 \cos^2 \frac{x}{2}\)

\[
\int \frac{1-\cos x}{1+\cos x} \, dx = \int \frac{2\sin^2 \left(\frac{x}{2}\right)}{2\cos^2 \left(\frac{x}{2}\right)} \, dx
\]

We get
\[
\int \frac{1-\cos x}{1+\cos x} \, dx = \int \tan^2 \left(\frac{x}{2}\right) \, dx
\]

We know that \(\sec^2 \left(x/2\right) - 1 = \tan^2 \left(x/2\right)\)

\[
\int \frac{1-\cos x}{1+\cos x} \, dx = \int \left(\sec^2 \left(\frac{x}{2}\right) - 1\right) \, dx
\]

On further simplification
\[
\int \frac{1-\cos x}{1+\cos x} \, dx = \int \sec^2 \left(\frac{x}{2}\right) \, dx - \int dx
\]

By integrating w.r.t. \(x\)
\[
\int \frac{1-\cos x}{1+\cos x} \, dx = 2 \tan \left(x/2\right) - x + c
\]

(ii) \(\int \frac{1+\cos x}{1-\cos x} \, dx\)

Here, we know that
\(1 - \cos x = 2 \sin^2 \frac{x}{2}\) and \(1 + \cos x = 2 \cos^2 \frac{x}{2}\)

\[
\int \frac{1+\cos x}{1-\cos x} \, dx = \int \frac{2\cos^2 \left(\frac{x}{2}\right)}{2\sin^2 \left(\frac{x}{2}\right)} \, dx
\]

We get
\[
\int \frac{1+\cos x}{1-\cos x} \, dx = \int \cot^2 \left(\frac{x}{2}\right) \, dx
\]

We know that \(\csc^2 \left(x/2\right) - 1 = \cot^2 \left(x/2\right)\)

\[
\int \frac{1+\cos x}{1-\cos x} \, dx = \int \left(\csc^2 \left(\frac{x}{2}\right) - 1\right) \, dx
\]

On further simplification
8. \( \int \sin 3x \cos 4x \, dx \)
Solution:

It is given that
\[
\int \sin 3x \cos 4x \, dx
\]
Using the formula
\[
\sin x \times \cos y = \frac{1}{2} (\sin(x + y) - \sin(y - x))
\]
\[
\int \sin 3x \cos 4x \, dx = \frac{1}{2} \int (\sin 7x - \sin x) \, dx
\]
On further simplification
\[
\int \sin 3x \cos 4x \, dx = \frac{1}{2} \int \sin 7x \, dx - \frac{1}{2} \int \sin x \, dx
\]
By integrating w.r.t. \( x \)
\[
\int \sin 3x \cos 4x \, dx = \frac{-\cos 7x}{14} + \frac{\cos x}{2} + C
\]

9. \( \int \cos 4x \cos 3x \, dx \)
Solution:

It is given that
\[
\int \cos 4x \cos 3x \, dx
\]
Using the formula
\[
\cos x \times \cos y = \frac{1}{2} (\cos(x + y) + \cos(x - y))
\]
\[
\int \cos 4x \cos 3x \, dx = \frac{1}{2} \int (\cos 7x + \cos x) \, dx
\]
On further simplification
\[
\int \cos 4x \cos 3x \, dx = \frac{1}{2} \int \cos 7x \, dx + \frac{1}{2} \int \cos x \, dx
\]
By integrating w.r.t. \( x \)
\[
\int \cos 4x \cos 3x \, dx = \frac{\sin 7x}{14} + \frac{\sin x}{2} + C
\]

10. \( \int \sin 4x \sin 8x \, dx \)
Solution:

It is given that
\[
\int \sin 4x \sin 8x \, dx
\]
Using the formula
\[
\sin x \times \sin y = \frac{1}{2} (\cos(y - x) - \cos(y + x))
\]
\[
\int \sin 4x \sin 8x \, dx = \frac{1}{2} \int (\cos 4x - \cos 12x) \, dx
\]
On further simplification
\[
\int \sin 4x \sin 8x \, dx = \frac{1}{2} \int \cos 4x \, dx - \frac{1}{2} \int \cos 12x \, dx
\]
By integrating w.r.t. \( x \)
\[
\int \sin 4x \sin 8x \, dx = \frac{\sin 4x}{8} - \frac{\sin 12x}{24} + C
\]

11. \( \int \sin 6x \cos x \, dx \)
Solution:
It is given that
\[ \int \sin 6x \cos x \, dx \]
Using the formula \( \sin x \times \cos y = \frac{1}{2} (\sin(y + x) - \sin(y - x)) \)
\[ \int \sin 6x \cos x \, dx = \frac{1}{2} \int (\sin 7x - \sin (-5x)) \, dx \]
On further simplification
\[ \int \sin 6x \cos x \, dx = \frac{1}{2} \int \sin 7x \, dx + \frac{1}{2} \int \sin 5x \, dx \]
By integrating w.r.t. \( x \)
\[ \int \frac{-\cos 7x}{14} + \frac{\cos 5x}{10} \, dx = C \]

12. \[ \int \sin x \sqrt{1 + \cos 2x} \, dx \]
Solution:
We know that
\[ 1 + \cos 2x = 2 \cos^2 x \]
So it can be written as
\[ \int \sin x \sqrt{2 \cos^2 x} \, dx = \int \sin x \sqrt{2 \cos^2 x} \, dx \]
We get
\[ \int \sin x \sqrt{1 + \cos 2x} \, dx = \sqrt{2} \int \sin x \cos x \, dx \]
Consider \( \sin x = t \)
So we get \( \cos x \, dx = dt \)
\[ \int \sqrt{2} t \, dt = \sqrt{2} \int t \, dt \]
By integrating w.r.t. \( t \)
\[ = \frac{\sqrt{2} t^2}{2} + C = \frac{t^2}{\sqrt{2}} + C \]
Now re-substituting the value of \( t = \sin x \)
\[ = \frac{\sin^2 x}{\sqrt{2}} + C \]

13. \[ \int \cos^4 x \, dx \]
Solution:
It is given that
\[ \int \cos^4 x \, dx = \int \cos^2 x \cos^2 x \, dx \]
We know that
\[ (1 + \cos 2x)/2 = \cos^2 x \]
So we get
\[ = \int \left( \frac{1 + \cos 2x}{2} \right) \left( \frac{1 + \cos 2x}{2} \right) \, dx \]
By further simplification
\[ = \frac{1}{4} \int (1 + \cos 2x)^2 \, dx \]
It can be written as
14. \( \int \cos 2x \cos 4x \cos 6x \, dx \)

Solution:

It is given that
\[ \int \cos 2x \cos 4x \cos 6x \, dx \]

By multiplying and dividing by 2
\[ = \frac{1}{2} \int 2 \cos 2x \cos 4x \cos 6x \, dx \]

We can write it as
\[ = \frac{1}{2} \int \cos 2x [\cos (4x + 6x) + \cos (4x - 6x)] \, dx \]

So we get
\[ = \frac{1}{2} \int \cos 2x (\cos 10x + \cos 2x) \, dx \]

By further simplification
\[ = \frac{1}{4} \int [\cos (2x + 10x) + \cos (2x - 10x)] \, dx + \frac{1}{4} \int (1 + \cos 4x) \, dx \]

So we get
\[ = \frac{1}{4} \left[ \frac{\sin 12x}{12} + \frac{\sin 8x}{8} \right] + \frac{1}{4}x + \frac{\sin 4x}{16} + c \]

15. \( \int \sin^3 x \cos x \, dx \)

Solution:

It is given that
\[ \int \sin^3 x \cos x \, dx \]

By multiplying and dividing by 2
\[ = \frac{1}{2} \int \sin x (1 - \cos^2 x) \cos x \, dx \]

We can write it as
\[ = \frac{1}{2} \left[ \int \sin x \cos x \, dx + \int \sin x \cos^2 x \, dx \right] \]

By further simplification
\[ = \frac{1}{4} \left[ -\cos x \sin x + \frac{\cos^3 x}{3} \right] + \frac{1}{4} \left[ \frac{1}{4} \sin 8x + \frac{1}{4} \sin 4x \right] + c \]
\[ \int \sin^3 x \cos x \, dx \]

By taking \( \sin x = t \)

We get \( \cos x = \frac{dt}{dx} \)

It can be written as \( \cos x \, dx = dt \)

So we get

\[ = \int t^3 \, dt \]

By integrating w.r.t. \( t \)

\[ = \frac{t^4}{4} + c \]

By substituting the value of \( t \)

\[ = \sin^4 x/4 + c \]

16. \( \int \sec^4 x \, dx \)

Solution:

It is given that

\( \int \sec^4 x \, dx \)

We can write it as

\[ = \int \sec^2 x \, \sec^2 x \, dx \]

So we get

\[ = \int (1 + \tan^2 x) \, \sec^2 x \, dx \]

Take \( \tan x = t \)

By differentiation we get \( \sec^2 x \, dx = dt \)

It can be written as

\[ = \int (1 + t^2) \, dt \]

By integrating w.r.t. \( t \)

\[ = t + t^3/3 + c \]

By substituting the value of \( t \)

\[ = \tan x + \tan^3 x/3 + c \]

17. \( \int \cos^3 x \sin^4 x \, dx \)

Solution:

It is given that

\( \int \cos^3 x \sin^4 x \, dx \)

We can write it as

\[ = \int \sin^4 x \cos^2 x \cos x \, dx \]

By taking \( \sin^4 x \) as common

\[ = \int \sin^4 x (1 - \sin^2 x) \cos x \, dx \]

Take \( \sin x = t \)

We get \( \cos x = dx/dt \)

Here \( \cos x \, dx = dt \)

It can be written as

\[ = \int t^4 (1 - t^2) \, dt \]

On further simplification

\[ = \int t^4 \, dt - \int t^6 \, dt \]

By integration w.r.t. \( t \)

\[ = t^5/5 - t^7/7 + c \]
By substituting the value of $t = \sin \frac{x}{5} + \sin \frac{x}{7} + c$
EXERCISE 13C

Evaluate the following integrals:

1. \( \int x e^x \, dx \)

Solution:

It is given that
\( \int x e^x \, dx \)
We can write it as
\[
\int x e^x \, dx = x \int e^x \, dx - \int \left( \int e^x \, dx \right) \, dx
\]
By integrating w.r.t \( x \)
\[
= x e^x - \int 1 \, e^x \, dx
\]
So we get
\[
= x e^x - e^x + c
\]
By taking \( e^x \) as common
\[
= e^x (x - 1) + c
\]

2. \( \int x \cos x \, dx \)

Solution:

It is given that
\( \int x \cos x \, dx \)
We can write it as
\[
\int x \cos x \, dx = x \int \cos x \, dx - \int \left( \frac{dx}{dx} \right) \left( \int \cos x \, dx \right) \, dx
\]
By integrating w.r.t \( x \)
\[
= x \sin x - \int 1 \sin x \, dx
\]
So we get
\[
= x \sin x + \cos x + c
\]

3. \( \int x e^{2x} \, dx \)

Solution:

It is given that
\( \int x e^{2x} \, dx \)
We can write it as
\[
\int x e^{2x} \, dx = x \int e^{2x} \, dx - \int \left( \frac{dx}{dx} \right) \left( \int e^{2x} \, dx \right) \, dx
\]
By integrating w.r.t \( x \)
\[
= x \frac{e^{2x}}{2} - \int 1 \cdot \frac{e^{2x}}{2} \, dx
\]
So we get
4. \[ \int x \sin 3x \, dx \]

Solution:

It is given that
\[ \int x \sin 3x \, dx \]

We can write it as
\[ \int x \sin 3x \, dx = x \int \sin 3x \, dx - \int \left[ \frac{dx}{dx} \cdot \int \sin 3x \, dx \right] \, dx \]

By integrating w.r.t. \( x \)

\[ = x \left( -\frac{\cos 3x}{3} \right) - \int 1 \left( -\frac{\cos 3x}{3} \right) \, dx \]

So we get
\[ = x \left( -\frac{\cos 3x}{3} \right) + \frac{\sin 3x}{9} + c \]

On further calculation
\[ = x \left( -\frac{\cos 3x}{3} \right) + \frac{\sin 3x}{9} + c \]

5. \[ \int x \cos 2x \, dx \]

Solution:

It is given that
\[ \int x \cos 2x \, dx \]

We can write it as
\[ \int x \cos 2x \, dx = x \int \cos 2x \, dx - \int \left[ \frac{dx}{dx} \cdot \int \cos 2x \, dx \right] \, dx \]

By integrating w.r.t. \( x \)

\[ = x \left( \frac{\sin 2x}{2} \right) - \int 1 \left( \frac{\sin 2x}{2} \right) \, dx \]

So we get
6. \( \int x \log 2x \, dx \)

**Solution:**

It is given that

\[ \int x \log 2x \, dx \]

We can write it as

\[ \int x \left( \frac{\sin 2x}{2} \right) + \left( \frac{\cos 2x}{2} \right) + c \]

On further calculation

\[ \int x \left( \frac{\sin 2x}{2} \right) + \left( \frac{\cos 2x}{4} \right) + c \]


7. \( \int x \csc^2 x \, dx \)

**Solution:**

It is given that

\[ \int x \csc^2 x \, dx \]

We can write it as

\[ \int x \csc^2 x \, dx = \log 2x \int dx - \int \left[ \frac{d}{dx} \log 2x \cdot \int \frac{1}{2x} \, dx \right] \, dx \]

By integrating w.r.t x

\[ = \log 2x \frac{x^2}{2} - \int \left[ \frac{1 \cdot 2x^2}{2x} \right] \, dx \]

So we get

\[ = \frac{x^2}{2} \log 2x - \int \frac{x}{2} \, dx \]

Again by integration

\[ = \frac{x^2}{2} \log 2x - \frac{x^2}{2 \times 2} + c \]

On further calculation

\[ = \frac{x^2}{2} \log 2x - \frac{x^2}{4} + c \]


7. \( \int x \csc^2 x \, dx \)

**Solution:**

It is given that

\[ \int x \csc^2 x \, dx \]

We can write it as

\[ \int x \csc^2 x \, dx = x \int \csc^2 x \, dx - \int \left[ \frac{dx}{dx} \cdot \frac{1}{2} \int \csc^2 x \, dx \right] \, dx \]

By integrating w.r.t x

\[ = x \left( - \cot x \right) - \int 1 \left( - \cot x \right) \, dx \]

So we get
= - x cot x + \int \cot x \, dx

Again by integration we get

= - x cot x + \log |\sin x| + c

8. \int x^2 \cos x \, dx

Solution:

It is given that

\int x^2 \cos x \, dx

We can write it as

\int x^2 \cos x \, dx = x^2 \int \cos x \, dx - \int 2x \sin x \, dx

So we get

\int x^2 \cos x \, dx = x^2 \sin x - 2 \int x \sin x \, dx

Now apply by the part method

\int x^2 \cos x \, dx = x^2 \sin x - 2 \left( x (- \cos x) - \int (- \cos x) \, dx \right)

Again by integration

\int x^2 \cos x \, dx = x^2 \sin x - 2 \left( x (- \cos x) - \int (- \cos x) \, dx \right)

By integrating w.r.t x

= x^2 \sin x - 2 \left( -x \cos x + \sin x \right) + c

On further simplification

= x^2 \sin x + 2x \cos x - 2 \sin x + c

9. \int x \sin^2 x \, dx

Solution:

It is given that

\int x \sin^2 x \, dx

We can write it as

\sin^2 x = \frac{(1 + \cos 2x)}{2}

By integration

\int x \sin^2 x \, dx = \int x \left( \frac{1 - \cos 2x}{2} \right) \, dx

On further simplification

= \int \left( \frac{x}{2} - \frac{x \cos 2x}{2} \right) \, dx

It can be written as
10. ∫ x \tan^2 x \, dx

Solution:

It is given that

\[ \int x \tan^2 x \, dx \]

We can write it as

\[ \tan^2 x = \sec^2 x - 1 \]

So we get

\[ \int x \sec^2 x \, dx - \int x \, dx \]

By further simplification

\[ = \left\{ \int x \sec^2 x \, dx - \int \frac{x \tan^2 x}{2} \, dx \right\} - \frac{x^2}{2} \]
By integration w.r.t x
\[ = \int (x \tan x - \tan x \, dx) - x^2/2 \]
It can be written as
\[ = x \tan x - \log |\sec x| - x^2/2 + c \]
So we get
\[ = x \tan x - \log |\sec x| - x^2/2 + c \]

11. \[ \int x^2 e^x \, dx \]
Solution:

It is given that
\[ \int x^2 e^x \, dx \]
By integrating w.r.t x
\[ = x^2 e^x - 2 \int x e^x \, dx \]
By further simplification
\[ = x^2 e^x - 2 \int x e^x \, dx \]
On integration w.r.t x
\[ = x^2 e^x - 2 \left[ x e^x - \int e^x \, dx \right] \]
We get
\[ = x^2 e^x - 2 \left[ x e^x - e^x \right] \]
By further simplification
\[ = x^2 e^x - 2 \left[ x e^x - e^x \right] + c \]
By multiplication
\[ = x^2 e^x - 2x e^x + 2 e^x + c \]
By taking e^x as common
\[ = e^x (x^2 - 2x + 2) + c \]

13. \[ \int x^2 e^{3x} \, dx \]
Solution:

It is given that
\[ \int x^2 e^{3x} \, dx \]
We can write it as
14. \[ \int x^2 \sin^2 x \, dx \]

Solution:

It is given that
\[ \int x^2 \sin^2 x \, dx \]

We know that
\[ \sin^2 x = \frac{1 - \cos 2x}{2} \]

So we get

\[ \int x^2 \left( \frac{1 - \cos 2x}{2} \right) \, dx = \int \frac{x^2}{2} - \frac{x^2 \cos 2x}{2} \, dx \]

By further simplification

\[ = \int \frac{x^2}{2} \, dx - \int \frac{x^2 \cos 2x}{2} \, dx \]

By integrating w.r.t. \( x \)

\[ = \frac{x^3}{3 \times 2} - \frac{1}{2} \int x^2 \cos 2x \, dx \]

We can write it as

\[ = \frac{x^3}{6} - \frac{1}{2} \left( x^2 \int \cos 2x \, dx - \int \left(\frac{dx}{dx} \cdot \int \cos 2x \, dx \right) \, dx \right) \]

By integrating w.r.t. \( x \)

\[ = \frac{x^3}{6} - \frac{1}{2} \left( x^2 \cdot \frac{\sin 2x}{2} \right) \]

On further calculation

\[ = \frac{x^3}{6} - \frac{1}{2} \left( x^2 \cdot \frac{\sin 2x}{2} - \int x \cdot \sin 2x \, dx \right) \]

Now by integrating the second part we get

\[ = \frac{x^3}{6} - \frac{1}{2} \left( x^2 \cdot \frac{\sin 2x}{2} - \left[ x \int \sin 2x \, dx - \int \left( \frac{dx}{dx} \cdot \int \sin 2x \, dx \right) \, dx \right] \right) \]

We get

\[ = \frac{x^3}{6} - \frac{1}{2} \left( x^2 \cdot \frac{\sin 2x}{2} - \left[ x \cdot \frac{-\cos 2x}{2} - \int 1 \cdot \frac{-\cos 2x}{2} \, dx \right] \right) \]

By further integration we get

\[ = \frac{x^3}{6} - \frac{1}{2} \left( x^2 \cdot \frac{\sin 2x}{2} + \frac{x \cos 2x}{2} - \frac{\sin 2x}{4} \right) + c \]

So we get

\[ = \frac{x^3}{6} - \frac{x^2 \sin 2x}{4} - \frac{x \cos 2x}{4} + \frac{\sin 2x}{8} + c \]
15. \[ \int x^3 \log 2x \, dx \]
Solution:

It is given that 

\[ \int x^3 \log 2x \, dx \]

We can write it as

\[
\log 2x \int x^3 \, dx - \int \left( \frac{d}{dx} \log 2x \right) \int x^3 \, dx \, dx
\]

By integrating w.r.t x

\[ = \log 2x \cdot \frac{x^4}{4} - \frac{1}{2 \cdot 2x} \cdot \frac{x^4}{4} \, dx \]

So we get

\[ = \log 2x \cdot \frac{x^4}{4} - \frac{1}{4} \int x^3 \, dx \]

On further integration we get

\[ = \log 2x \cdot \frac{x^4}{4} - \frac{1}{4} \cdot \frac{x^4}{4} + c \]

By further calculation

\[ = \log 2x \cdot \frac{x^4}{4} - \frac{x^4}{16} + c \]

16. \[ \int x \log (x + 1) \, dx \]
Solution:

It is given that
\[ \int x \log (x + 1) \, dx = \log (x + 1) \int x \, dx - \int \left( \frac{d \log (x + 1)}{dx} \right) \cdot \int x \, dx \, dx \]

We can write it as
\[ = \log (x + 1) \frac{x^2}{2} - \int \frac{1}{x + 1} \times \frac{x^2}{2} \, dx \]

So we get
\[ = \log (x + 1) \frac{x^2}{2} - \frac{1}{2} \int \frac{x^2 - 1 + 1}{x + 1} \, dx \]

By adding and subtracting 1 from the numerator
\[ = \log (x + 1) \frac{x^2}{2} - \frac{1}{2} \left[ \left( \int \frac{x^2 - 1}{x + 1} + \frac{1}{x + 1} \right) \, dx \right] \]

We can write it as
\[ = \log (x + 1) \frac{x^2}{2} - \frac{1}{2} \left[ \left( \int (x + 1)(x - 1) + \frac{1}{x + 1} \right) \, dx \right] \]

By further calculation
\[ = \log (x + 1) \frac{x^2}{2} - \frac{1}{2} \left[ \left( \int (x - 1) + \frac{1}{x + 1} \right) \, dx \right] \]

By integrating w.r.t x
\[ = \log (x + 1) \frac{x^2}{2} - \frac{1}{2} \left[ \frac{x^2}{2} - x + \log (x + 1) \right] + c \]

It can be written as
\[ = \log (x + 1) \frac{x^2}{2} - \frac{x^2}{4} + \frac{x}{2} - \frac{\log (x + 1)}{2} + c \]

We get
\[ = \log (x + 1) \frac{x^2 - 1}{2} - \frac{x^2}{4} + \frac{x}{2} + c \]

17. \[ \int \frac{\log x}{x^n} \, dx \]

Solution:

It is given that
\[
\int x^{-n} \log x \, dx = \log x \int x^{-n} \, dx - \left( \frac{1}{x} \frac{d\log x}{dx} \int x^{-n} \, dx \right) \, dx
\]

By integrating w.r.t x

\[
= \log x \left( \frac{x^{-n-1}}{-n+1} \right) - \int \frac{1}{x} \frac{x^{-n-1}}{-n+1} \, dx
\]

We can write it as

\[
= \frac{x^{-n-1} \log x}{1-n} + \frac{1}{1-n} \int \frac{x^{-n}x}{x} \, dx
\]

Again by integrating the second part

\[
= \frac{x^{-n-1} \log x}{1-n} + \frac{1}{1-n} \frac{x^{-n+1}}{-n+1} + c
\]

So we get

\[
= \frac{x^{-n-1} \log x}{1-n} - \frac{x^{-n+1}}{(1-n)^2} + c
\]

18. \[\int 2x^3 e^{x^2} \, dx \]

Solution:

It is given that

\[\int 2x^3 e^{x^2} \, dx \]

We can write it as

\[\int 2x \cdot x^2 \cdot e^{x^2} \, dx \]

Take \(x^2 = t\)

So we get \(2x \, dx = dt\)

It can be written as

\[\int 2x \cdot x^2 \cdot e^{x^2} \, dx = \int t \cdot e^t \, dt \]

By integrating w.r.t. t taking the first function as t and second function as \(e^t\)

\[\int t \cdot e^t \, dt = t \int e^t \, dt - \int \left( \frac{dt}{dt} \right) \cdot \int e^t \, dt \, dt \]

We can write it as

\[= t \cdot e^t - \int 1 \cdot e^t \, dt \]

So we get

\[= t \cdot e^t - e^t + C \]

Now by replacing t with \(x^2\)
19. \[ \int x \sin^3 x \, dx \]

Solution:

It is given that 
\[ \int x \sin^3 x \, dx \]

We know that 
\[ \sin 3x = 3 \sin x - 4 \sin^3 x \]

It can be written as 
\[ \sin^3 x = (3 \sin x - \sin 3x)/4 \]

We get 
\[ \frac{1}{4} \int 3x \sin x - x \sin 3x \, dx \]

By further simplification
\[ = \frac{3}{4} \int x \sin x \, dx - \frac{1}{4} \int x \sin 3x \, dx \]

Take the first function as \( x \) and second function as \( \sin x \) and \( \sin 3x \)

\[ = \frac{3}{4} \int x \sin x \, dx - \frac{1}{4} \int x \sin 3x \, dx \]

By integrating w.r.t. \( x \)
\[ = \frac{3}{4} \left( x \int \sin x \, dx - \int \left( \frac{dx}{dx} \cdot \int \sin x \, dx \right) \, dx \right) - \frac{1}{4} \left( s \int 3x \, dx - \int \left( \frac{dx}{dx} \cdot \int \sin 3x \, dx \right) \, dx \right) \]

We get
\[ = \frac{3}{4} \left( -x \cos x + \int \cos x \, dx \right) - \frac{1}{4} \left( \frac{-x \cos 3x}{3} + \int \frac{\cos 3x}{3} \, dx \right) \]

Now by integrating the second part w.r.t \( x \)
\[ = \frac{3}{4} \left( -x \cos x + \sin x \right) - \frac{1}{4} \left( -\frac{x \cos 3x}{3} + \frac{\sin 3x}{9} \right) + c \]

So we get
20. \( \int x \cos^3 x \, dx \)
Solution:

It is given that
\( \int x \cos^3 x \, dx \)
We know that
\( \cos^3 x = (\cos 3x + 3 \cos x)/4 \)
It can be written as

\[
\int x \cos^3 x \, dx = \int x \left( \frac{\cos 3x + 3 \cos x}{4} \right) \, dx
\]

So we get

\[
= \frac{1}{4} \int x \cos 3x \, dx + \frac{3}{4} \int x \cos x \, dx
\]

Take the first function as \( x \) and second function as \( \cos x \) and \( \cos 3x \)

\[
= \frac{1}{4} \left( x \int \cos 3x \, dx - \int \left( \frac{dx}{dx} \int \cos 3x \, dx \right) \, dx \right) + \frac{3}{4} \left( x \int \cos x \, dx - \int \left( \frac{dx}{dx} \int \cos x \, dx \right) \, dx \right)
\]

By integrating w.r.t \( x \)

\[
= \frac{1}{4} \left( x \frac{\sin 3x}{3} - \int \frac{\sin 3x}{3} \, dx \right) + \frac{3}{4} \left( x \sin x - \int \sin x \, dx \right)
\]

Now by integrating the second term

\[
= \frac{1}{4} \left( x \frac{\sin 3x}{3} + \frac{\cos 3x}{9} \right) + \frac{3}{4} \left( x \sin x + \cos x \right) + c
\]

So we get

\[
= \frac{x \sin 3x}{12} + \frac{\cos 3x}{36} + \frac{3x \sin x}{4} + \frac{3 \cos x}{4} + c
\]

21. \( \int x^3 \cos x^2 \, dx \)
Solution:

It is given that
\( \int x^3 \cos x^2 \, dx \)
We can write it as
\( \int x^2 \cos x^2 \, dx \)
Take \( x^2 = t \)
So we get \( 2x \, dx = dt \)
\( x \, dx = \frac{dt}{2} \)
22. \[\int \sin x \log (\cos x) \, dx\]
Solution:

It is given that
\[\int \sin x \log (\cos x) \, dx\]

Consider first function as \(\log (\cos x)\) and second function as \(\sin x\)

So we get
\[= \frac{1}{2} (t \sin t - \int \sin t \, dt)\]

Again by integrating the second term
\[= \frac{1}{2} (t \sin t + \cos t) + c\]

Substituting the value of \(t\) as \(x^2\)
\[= \frac{1}{2} x^2 \sin x^2 + \frac{1}{2} \cos x^2 + c\]

23. \[\int x \sin x \cos x \, dx\]
Solution:

It is given that
\[\int x \sin x \cos x \, dx\]

We know that
\[\sin 2x = 2 \sin x \cos x\]

It can be written as
\[\int x \sin x \cos x \, dx = \frac{1}{2} x \sin 2x\]

Consider first function as \(x\) and second function as \(\sin 2x\)
24. \( \int \cos \sqrt{x} \, dx \)
Solution:

It is given that
\( \int \cos \sqrt{x} \, dx \)
Take \( \sqrt{x} = t \)
So we get
\( \frac{1}{2} \sqrt{x} \, dx = dt \)
By cross multiplication
\( dx = 2\sqrt{x} \, dt \)
Here \( dx = 2t \, dt \)
It can be written as
\( \int \cos \sqrt{x} \, dx = 2 \int t \cos t \, dt \)
Take first function as \( t \) and second function as \( \cos t \)

\[ 2 \int t \cos t \, dt = 2 \left( t \int \cos t \, dt - \int \left( \frac{dt}{dt} \right) \int \cos t \, dt \right) \, dt \]

By integrating w.r.t \( t \)
\[ = 2 \left( t \sin t - \int \sin t \, dt \right) \]
We get
\[ = 2t \sin t + 2 \cos t + c \]
Now by substituting \( t \) as \( \sqrt{x} \)
\[ = 2\sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} + c \]
Taking 2 as common
\[ = 2 (\cos \sqrt{x} + \sqrt{x} \sin \sqrt{x}) + c \]

25. \( \int \cosec^3 x \, dx \)
Solution:

It is given that
\( \int \cosec^3 x \, dx \)
We can write it as
\[ \int \csc^3 x \, dx = \int \csc x \cot x \csc^2 x \, dx \]

So we get

\[ \int \cos ecx \cdot \cos ec^2 x \, dx = \cos ecx \int \csc^2 x \, dx - \int \left( \frac{d}{dx} \csc x \right) \cdot \csc^2 x \, dx \]

By integration we get

\[ = \csc x (- \cot x) - \int (- \csc x \cot x) (- \cot x) \, dx \]

It can be written as

\[ = - \csc x \cot x - \int \csc x \cot^2 x \, dx \]

Here \( \cot^2 x = \csc^2 x - 1 \)

We get

\[ = - \csc x \cot x - \int \csc^3 x \, dx + \int \csc x \, dx \]

On further calculation

\[ 2 \int \csc^3 x \, dx = - \csc x \cot x + \int \csc x \, dx \]

By integration we get

\[ 2 \int \csc^3 x \, dx = - \frac{1}{2} \csc x \cot x + \frac{1}{2} \log |\tan x/2| + c \]

Here

\[ \int \csc^3 x \, dx = - \frac{1}{2} \csc x \cot x + \frac{1}{2} \log |\tan x/2| + c \]

27. \( \int \sin x \log (\cos x) \, dx \)

Solution:

It is given that

\[ \int \sin x \log (\cos x) \, dx \]

Take \( \cos x = t \)

So we get

\[ - \sin x \, dx = dt \]

It can be written as

\[ \int \sin x \log (\cos x) \, dx = - \int \log t \, dt = - \int \log t \, dt \]

Consider first function as \( \log t \) and second function as \( 1 \)

By integrating w.r.t \( t \)

\[ \int \log t \, dt = t \int 1 \, dt - \int \left( \frac{d}{dt} \log t \right) \cdot \int 1 \, dt \, dt \]

By integrating w.r.t \( t \)

\[ = - t \log t + \int 1 \, dt \]

Again by integrating the second term

\[ = - t \log t + t + c \]

Now replace \( t \) as \( \cos x \)

\[ = t (- \log t + 1) + c \]

We get

\[ = \cos x (1 - \log (\cos x)) + c \]

28. \( \int \frac{\log (\log x)}{x} \, dx \)
Solution:

Take \( \log x = t \)
So we get
\( \frac{1}{x} \, dx = dt \)
Here
\[
\int \frac{\log(\log x)}{x} \, dx = \int \log t \, dt = \int 1 \cdot \log t \, dt
\]
Consider the first function as \( \log t \) and second function as 1
\[
\int 1 \cdot \log t \, dt = \log t \int 1 \, dt - \left( \frac{d \log t}{dt} \right) \int 1 \, dt \, dt
\]
By integrating w.r.t. \( t \)
\[
= t \log t - \int t \, dt
\]
Again by integrating the second term we get
\[
= t \log t - t + c
\]
Replace \( t \) with \( \log x \)
\[
= \log x \cdot \log(\log x) - \log x + c
\]
By taking \( \log x \) as common
\[
= \log x (\log(\log x) - 1) + c
\]

29. \( \int \log (2 + x^2) \, dx \)
Solution:
It is given that \( \int \log (2 + x^2) \, dx \)
We can write it as
\( = \int 1 \cdot \log (2 + x^2) \, dx \)
Consider first function as \( \log (2 + x^2) \) and second function as 1
\[
\int 1 \cdot \log (2 + x^2) \, dx = \log (2 + x^2) \int 1 \, dx - \left( \frac{d \log (2 + x^2)}{dx} \right) \int 1 \, dx \, dx
\]
By integrating w.r.t. \( x \)
\[
= \log (2 + x^2) \cdot x - \int \frac{1.2x}{2 + x^2} \cdot x \, dx
\]
We get
\[
= x \log (2 + x^2) - \int \frac{2x^2}{2 + x^2} \, dx
\]
It can be written as
\[
= x \log (2 + x^2) - 2 \int \frac{x^2 + 2 - 2}{2 + x^2} \, dx
\]
So we get

\[ = x \log(2 + x^2) - 2 \left[ \int \log x \, dx - \int \frac{2}{2 + x^2} \, dx \right] \]

By integration

\[ = x \log(2 + x^2) - 2 \left[ x - \left( 2 \int \frac{1}{2 + x} \, dx \right) \right] \]

We get

\[ = x \log(2 + x^2) - 2 \left[ x - 2 \left( \frac{1}{\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} \right) \right] + c \]

It can be written as

\[ = x \log(2 + x^2) - 2x + 2\sqrt{2} \tan^{-1} \frac{x}{\sqrt{2}} + c \]

\[ 31. \int \left\{ \frac{1}{\log x} - \frac{1}{(\log x)^2} \right\} \, dx \]

**Solution:**

Consider \( \log x = t \)

Where \( x = e^t \)

So we get

\( dx = e^t \, dt \)

We can write it as
32. \( \int e^{-x} \cos 2x \cos 4x \, dx \)

**Solution:**

Using the formula
\[
\cos A \cos B = \frac{1}{2} [\cos (A + B) + \cos (A - B)]
\]
We get
\[
\cos 4x \cos 2x = \frac{1}{2} [\cos (4x + 4x) + \cos (4x - 4x)] = \frac{1}{2} [\cos 6x + \cos 2x]
\]
By applying in the original equation
\[
\int e^{-x} \cos 2x \cos 4x \, dx = \int e^{-x} \left( \frac{1}{2} [\cos 6x + \cos 2x] \right) \, dx
\]
\[
= \frac{1}{2} \left[ \int e^{-x} \cos 6x \, dx + \int e^{-x} \cos 2x \, dx \right]
\]
Taking first function as \( \cos 6x \) and \( \cos 2x \) and second function as \( e^{-x} \)

Now by solving both parts separately
\[ \int e^{-x} \cos 6x \, dx = \cos 6x \int e^{-x} \, dx - \int \left( \frac{d}{dx} \left( \cos 6x \right) \cdot e^{-x} \right) \, dx \]

By integrating w.r.t \( x \)
\[ I = \cos 6x \left( -e^{-x} \right) - \int (-6 \sin 6x) \cdot (-e^{-x}) \, dt \]

So we get
\[ I = -\cos 6x \cdot e^{-x} - 6 \int \sin 6x \cdot e^{-x} \, dx \]

It can be written as
\[ I = -e^{-x} \cos 6x - 6 \left[ \sin 6x \int e^{-x} \, dx - \int \left( \frac{d}{dx} \left( \sin 6x \right) \cdot e^{-x} \right) \, dx \right] \]

Again by integration we get
\[ I = -e^{-x} \cos 6x - 6 \left[ \sin 6x \left( -e^{-x} \right) - \int (6 \cos 6x) \cdot (-e^{-x}) \, dt \right] \]

We get
\[ I = -e^{-x} \cos 6x - 6 \left[ -e^{-x} \sin 6x + 6 \int e^{-x} \cos 6x \, dx \right] \]

We can write it as
\[ I = -e^{-x} \cos 6x - 6 \left[ -e^{-x} \sin 6x + 6I \right] \]

By further multiplication
\[ I = -e^{-x} \cos 6x + 6 e^{-x} \sin 6x - 36I \]

So we get
\[ 37I = e^{-x} \left( 6 \sin 6x - \cos 6x \right) \]

Here
\[ I = \frac{e^{-x} \left( 6 \sin 6x - \cos 6x \right)}{37} \]

Now solving the next part
\[ \int e^{-x} \cos 2x \, dx = \cos 2x \int e^{-x} \, dx - \int \left( \frac{d}{dx} \left( \cos 2x \right) \cdot e^{-x} \right) \, dx \]

We know that
\[ J = \cos 2x \left( -e^{-x} \right) - \int (-2 \sin 2x) \cdot (-e^{-x}) \, dt \]

It can be written as
\[ J = -\cos 2x \cdot e^{-x} - 2 \int \sin 2x \cdot e^{-x} \, dx \]
By further integration

\[
J = -e^{-x} \cos 2x - 2 \left[ \sin 2x \int e^{-x} \, dx - \int \left( \frac{d}{dx} \sin 2x \right) \left( -e^{-x} \right) \, dx \right]
\]

We get

\[
J = -e^{-x} \cos 2x - 2 \left[ \sin 2x (-e^{-x}) - \int (2 \cos 2x) (-e^{-x}) \, dt \right]
\]

By simplification

\[
J = -e^{-x} \cos 2x - 2 \left[ -e^{-x} \sin 2x + 2 \int e^{-x} \cos 2x \, dx \right]
\]

Here

\[
J = -e^{-x} \cos 2x - 2 \left[ -e^{-x} \sin 2x + 2J \right]
\]

By multiplication

\[
J = -e^{-x} \cos 2x + 2e^{-x} \sin 2x - 4J
\]

So we get

\[
5J = e^{-x} \left( 2 \sin 2x - \cos 2x \right)
\]

\[
J = \frac{e^{-x} \left( 2 \sin 2x - \cos 2x \right)}{5}
\]

Substituting in the equation

\[
= \frac{1}{2} \left[ \frac{e^{-x} (6 \sin 6x - \cos 6x)}{37} + \frac{e^{-x} (2 \sin 2x - \cos 2x)}{5} \right] + c
\]

On further calculation

\[
= \frac{e^{-x} (6 \sin 6x - \cos 6x)}{74} + \frac{e^{-x} (2 \sin 2x - \cos 2x)}{10} + c
\]

Taking \( e^{-x} \) as common

\[
= e^{-x} \left( \frac{6 \sin 6x - \cos 6x}{74} + \frac{2 \sin 2x - \cos 2x}{10} \right) + c
\]

33. \( \int e^{-\sqrt{x}} \, dx \)

Solution:

Consider \( \sqrt{x} = t \)

We get

\[
\frac{1}{2 \sqrt{x}} \, dx = dt
\]

It can be written as
\[ dx = 2 \sqrt{x} \, dt \]

where \( dx = 2t \, dt \)

We can replace it in the equation
\[ \int e^{\sqrt{x}} \, dx = \int e^{t} 2t \, dt \]

So we get
\[ = 2 \int t \, e^t \, dt \]

Consider the first function as \( t \) and second function as \( e^t \)

By integration w.r.t. \( t \)
\[ = 2 \left( t \, e^t - \int 1 \, e^t \, dt \right) \]

We get
\[ = 2 \left( t \, e^t - e^t \right) + c \]

Taking \( e^t \) as common
\[ = 2 \left( e^t \right) (t - 1) + c \]

Substituting the value of \( t \) we get
\[ = 2 e^{\sqrt{x}} (\sqrt{x} - 1) + c \]

34. \( \int e^{\sin x} \sin 2x \, dx \)

Solution:

We know that
\( \sin 2x = 2 \sin x \cos x \)

So we get
\[ \int e^{\sin x} \sin 2x \, dx = 2 \int e^{\sin x} \sin x \cos x \, dx \]

Take \( \sin x = t \)

So we get \( \cos x \, dx = dt \)

It can be written as
\[ 2 \int e^{\sin x} \sin x \cos x \, dx = 2 \int e^t \, dt \]

Consider first function as \( t \) and second function as \( e^t \)

By integrating w.r.t. \( t \)
\[ = 2 \left( t \, e^t - \int 1 \, e^t \, dt \right) \]

We get
\[ = 2 \left( t \, e^t - e^t \right) + c \]

Here
\[ = 2 e^t \left( t - 1 \right) + c \]

By substituting the value of \( t \)
\[ = 2 e^{\sin x} \left( \sin x - 1 \right) + C \]
35. \( \int \frac{x \sin^{-1} x}{\sqrt{1 - x^2}} \, dx \)

Solution:

Take \( \sin^{-1} x = t \)
Here \( x = \sin t \)
So we get

\[ \frac{1}{\sqrt{1 - x^2}} \, dx = dt \]

Substituting this in the equation we get

\[ \int \frac{x \sin^{-1} x}{\sqrt{1 - x^2}} \, dx = \int t \sin t \, dt \]

Consider the first function as \( t \) and second function as \( \sin t \)

\[ \int t \sin t \, dt = t \int \sin t \, dt - \int \frac{dt}{\cos t} \int \sin t \, dt \, dt \]

By integration we get

\[ = t (-\cos t) - \int (\cos t) \, dt \]

We get

\[ = -t \cos t + \sin t + c \]

It can be written as

\[ \cos t = \sqrt{1 - \sin^2 t} \]

Here we get

\[ = -t (\sqrt{1 - \sin^2 t}) + \sin t + c \]

By replacing \( t \) as \( \sin^{-1} x \)

\[ = -\sin^{-1} x (\sqrt{1 - x^2}) + x + c \]