1. If \( \left( \frac{x}{3} + 1, \frac{y - 2}{3} \right) = \left( \frac{5}{3}, \frac{1}{3} \right) \), find the values of \( x \) and \( y \).

Solution:

Given,

\[ \left( \frac{x}{3} + 1, \frac{y - 2}{3} \right) = \left( \frac{5}{3}, \frac{1}{3} \right) \]

As the ordered pairs are equal, the corresponding elements should also be equal.

Thus,

\[ \frac{x}{3} + 1 = \frac{5}{3} \]
\[ \frac{y - 2}{3} = \frac{1}{3} \]

Solving, we get

\[ \frac{x}{3} = \frac{2}{3} \]
\[ y - 2 = 1 \]

Therefore,

\[ x = 2 \quad \text{and} \quad y = 1 \]

2. If the set \( A \) has 3 elements and the set \( B = \{3, 4, 5\} \), then find the number of elements in \( (A \times B) \)?

Solution:

Given, set \( A \) has 3 elements and the elements of set \( B \) are \( \{3, 4, 5\} \).

So, the number of elements in \( B = 3 \)

Then, the number of elements in \( (A \times B) = (\text{Number of elements in } A) \times (\text{Number of elements in } B) \)

\[ = 3 \times 3 = 9 \]

Therefore, the number of elements in \((A \times B)\) will be 9.

3. If \( G = \{7, 8\} \) and \( H = \{5, 4, 2\} \), find \( G \times H \) and \( H \times G \).

Solution:

Given, \( G = \{7, 8\} \) and \( H = \{5, 4, 2\} \)

We know that,

The Cartesian product of two non-empty sets \( P \) and \( Q \) is given as

\[ P \times Q = \{(p, q) : p \in P, q \in Q\} \]

So,

\[ G \times H = \{(7, 5), (7, 4), (7, 2), (8, 5), (8, 4), (8, 2)\} \]
\[ H \times G = \{(5, 7), (5, 8), (4, 7), (4, 8), (2, 7), (2, 8)\} \]

4. State whether each of the following statements are true or false. If the statement is false, rewrite the given statement correctly.

(i) If \( P = \{m, n\} \) and \( Q = \{n, m\} \), then \( P \times Q = \{(m, n), (n, m)\} \).

(ii) If \( A \) and \( B \) are non-empty sets, then \( A \times B \) is a non-empty set of ordered pairs \( (x, y) \) such that \( x \in A \) and \( y \in B \).

(iii) If \( A = \{1, 2\} \), \( B = \{3, 4\} \), then \( A \times (B \cap \Phi) = \Phi \).

Solution:

(i) Corrected statement: If \( P = \{m, n\} \) and \( Q = \{n, m\} \), then \( P \times Q = \{(m, n), (n, m)\} \).

(ii) Corrected statement: If \( A \) and \( B \) are non-empty sets, then \( A \times B \) is a non-empty set of ordered pairs \( (x, y) \) such that \( x \in A \) and \( y \in B \).

(iii) Corrected statement: If \( A = \{1, 2\} \), \( B = \{3, 4\} \), then \( A \times (B \cap \Phi) = \Phi \)
(i) The statement is False. The correct statement is:
If P = \{m, n\} and Q = \{n, m\}, then
P × Q = \{(m, m), (m, n), (n, m), (n, n)\}
(ii) True
(iii) True

5. If A = \{-1, 1\}, find A × A × A.
Solution:
The A × A × A for a non-empty set A is given by
A × A × A = \{(a, b, c): a, b, c ∈ A\}
Here, given A = \{-1, 1\}
So,
A × A × A = \{(-1, -1, -1), (-1, -1, 1), (-1, 1, -1), (-1, 1, 1), (1, -1, -1), (1, -1, 1), (1, 1, -1), (1, 1, 1)\}

6. If A × B = \{(a, x), (a, y), (b, x), (b, y)\}. Find A and B.
Solution:
Given,
A × B = \{(a, x), (a, y), (b, x), (b, y)\}
We know that the Cartesian product of two non-empty sets P and Q is given by:
P × Q = \{(p, q): p ∈ P, q ∈ Q\}
Hence, A is the set of all first elements and B is the set of all second elements.
Therefore, A = \{a, b\} and B = \{x, y\}

7. Let A = \{1, 2\}, B = \{1, 2, 3, 4\}, C = \{5, 6\} and D = \{5, 6, 7, 8\}. Verify that
(i) A × (B ∩ C) = (A × B) ∩ (A × C)
(ii) A × C is a subset of B × D
Solution:
Given,
A = \{1, 2\}, B = \{1, 2, 3, 4\}, C = \{5, 6\} and D = \{5, 6, 7, 8\}
(i) To verify: A × (B ∩ C) = (A × B) ∩ (A × C)
Now, B ∩ C = \{1, 2, 3, 4\} ∩ \{5, 6\} = \emptyset
Thus,
L.H.S. = A × (B ∩ C) = A × \emptyset = \emptyset
Next,
A × B = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4)\}
A × C = \{(1, 5), (1, 6), (2, 5), (2, 6)\}
Thus,
R.H.S. = (A × B) ∩ (A × C) = \emptyset
Therefore, L.H.S. = R.H.S
- Hence verified

(ii) To verify: A × C is a subset of B × D
First,
A × C = {(1, 5), (1, 6), (2, 5), (2, 6)}
And,
B × D = {(1, 5), (1, 6), (1, 7), (1, 8), (2, 5), (2, 6), (2, 7), (2, 8), (3, 5), (3, 6), (3, 7), (3, 8), (4, 5), (4, 6), (4, 7), (4, 8)}
Now, it’s clearly seen that all the elements of set A × C are the elements of set B × D.
Thus, A × C is a subset of B × D.
- Hence verified

8. Let A = {1, 2} and B = {3, 4}. Write A × B. How many subsets will A × B have? List them.
Solution:
Given,
A = {1, 2} and B = {3, 4}
So,
A × B = {(1, 3), (1, 4), (2, 3), (2, 4)}
Number of elements in A × B is n(A × B) = 4
We know that,
If C is a set with n(C) = m, then n[P(C)] = 2^m.
Thus, the set A × B has 2^4 = 16 subsets.
And, these subsets are as below:
Φ, {(1, 3)}, {(1, 4)}, {(2, 3)}, {(2, 4)}, {(1, 3), (1, 4)}, {(1, 3), (2, 3)}, {(1, 3), (2, 4)}, {(1, 4), (2, 3)}, {(1, 4), (2, 4)}, {(2, 3), (2, 4)}, {(1, 3), (1, 4), (2, 3)}, {(1, 3), (1, 4), (2, 4)}, {(1, 3), (2, 3), (2, 4)}, {(1, 4), (2, 3), (2, 4)}, {(1, 3), (1, 4), (2, 3), (2, 4)}

9. Let A and B be two sets such that n(A) = 3 and n(B) = 2. If (x, 1), (y, 2), (z, 1) are in A × B, find A and B, where x, y and z are distinct elements.
Solution:
Given,
n(A) = 3 and n(B) = 2; and (x, 1), (y, 2), (z, 1) are in A × B.
We know that,
A = Set of first elements of the ordered pair elements of A × B
B = Set of second elements of the ordered pair elements of A × B.
So, clearly x, y, and z are the elements of A; and 1 and 2 are the elements of B.
As n(A) = 3 and n(B) = 2, it is clear that set A = {x, y, z} and set B = {1, 2}.

10. The Cartesian product A × A has 9 elements among which are found (−1, 0) and (0, 1). Find the set A and the remaining elements of A × A.
Solution:
We know that,
If n(A) = p and n(B) = q, then n(A × B) = pq.
Also, n(A × A) = n(A) × n(A)
Given,
n(A × A) = 9
So, \( n(A) \times n(A) = 9 \)
Thus, \( n(A) = 3 \)
Also given that, the ordered pairs \((-1, 0)\) and \((0, 1)\) are two of the nine elements of \(A \times A\).
And, we know in \(A \times A = \{(a, a) : a \in A\}\).
Thus, \(-1, 0,\) and \(1\) has to be the elements of \(A\).
As \(n(A) = 3\), clearly \(A = \{-1, 0, 1\}\).
Hence, the remaining elements of set \(A \times A\) are as follows:
\((-1, -1), (-1, 1), (0, -1), (0, 0), (1, -1), (1, 0),\) and \((1, 1)\)
1. Let $A = \{1, 2, 3, \ldots, 14\}$. Define a relation $R$ from $A$ to $A$ by $R = \{(x, y): 3x - y = 0, \text{ where } x, y \in A\}$. Write down its domain, codomain and range.

Solution:

The relation $R$ from $A$ to $A$ is given as:

$$R = \{(x, y): 3x - y = 0, \text{ where } x, y \in A\}$$

So,

$$R = \{(1, 3), (2, 6), (3, 9), (4, 12)\}$$

Now,

The domain of $R$ is the set of all first elements of the ordered pairs in the relation.
Hence, Domain of $R = \{1, 2, 3, 4\}$

The codomain of $R$ is the whole set $A$.
Hence, Codomain of $R = A = \{1, 2, 3, \ldots, 14\}$

The range of $R$ is the set of all second elements of the ordered pairs in the relation.
Hence, Range of $R = \{3, 6, 9, 12\}$

2. Define a relation $R$ on the set $N$ of natural numbers by $R = \{(x, y): y = x + 5, x \text{ is a natural number less than } 4; x, y \in N\}$. Depict this relationship using roster form. Write down the domain and the range.

Solution:

The relation $R$ is given by:

$$R = \{(x, y): y = x + 5, x \text{ is a natural number less than } 4; x, y \in N\}$$

The natural numbers less than 4 are 1, 2, and 3.

So,

$$R = \{(1, 6), (2, 7), (3, 8)\}$$

Now,

The domain of $R$ is the set of all first elements of the ordered pairs in the relation.
Hence, Domain of $R = \{1, 2, 3\}$

The range of $R$ is the set of all second elements of the ordered pairs in the relation.
Hence, Range of $R = \{6, 7, 8\}$

3. $A = \{1, 2, 3, 5\}$ and $B = \{4, 6, 9\}$. Define a relation $R$ from $A$ to $B$ by $R = \{(x, y): \text{the difference between } x \text{ and } y \text{ is odd}; x \in A, y \in B\}$. Write $R$ in roster form.

Solution:

Given,

$A = \{1, 2, 3, 5\}$ and $B = \{4, 6, 9\}$

The relation from $A$ to $B$ is given as:

$$R = \{(x, y): \text{the difference between } x \text{ and } y \text{ is odd}; x \in A, y \in B\}$$

Thus,

$$R = \{(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)\}$$

4. The figure shows a relationship between the sets $P$ and $Q$. Write this relation (i) in set-builder form (ii) in roster form.
What is its domain and range?
Solution:

From the given figure, it's seen that
P = \{5, 6, 7\}, Q = \{3, 4, 5\}
The relation between P and Q:
Set-builder form
(i) \(R = \{(x, y): y = x - 2; x \in P\}\) or \(R = \{(x, y): y = x - 2 \text{ for } x = 5, 6, 7\}\)
Roster form
(ii) \(R = \{(5, 3), (6, 4), (7, 5)\}\)
Domain of \(R = \{5, 6, 7\}\)
Range of \(R = \{3, 4, 5\}\)

5. Let \(A = \{1, 2, 3, 4, 6\}\). Let \(R\) be the relation on \(A\) defined by \((a, b): a, b \in A, b \text{ is exactly divisible by } a\).
   (i) Write \(R\) in roster form
   (ii) Find the domain of \(R\)
   (iii) Find the range of \(R\).
Solution:

Given,
\(A = \{1, 2, 3, 4, 6\}\) and relation \(R = \{(a, b): a, b \in A, b \text{ is exactly divisible by } a\}\)
Hence,
(i) \(R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6)\}\)
(ii) Domain of \(R = \{1, 2, 3, 4, 6\}\)
(iii) Range of \(R = \{1, 2, 3, 4, 6\}\)

6. Determine the domain and range of the relation \(R\) defined by \(R = \{(x, x + 5): x \in \{0, 1, 2, 3, 4, 5\}\}\).
Solution:

Given,
Relation \(R = \{(x, x + 5): x \in \{0, 1, 2, 3, 4, 5\}\}\)
Thus,
\(R = \{(0, 5), (1, 6), (2, 7), (3, 8), (4, 9), (5, 10)\}\)
So,
Domain of \(R = \{0, 1, 2, 3, 4, 5\}\) and
Range of \(R = \{5, 6, 7, 8, 9, 10\}\)

7. Write the relation \(R = \{(x, x^3): x \text{ is a prime number less than } 10\}\) in roster form.
Solution:

Given,
Relation \(R = \{(x, x^3): x \text{ is a prime number less than } 10\}\)
The prime numbers less than 10 are 2, 3, 5, and 7.
Therefore,
8. Let $A = \{x, y, z\}$ and $B = \{1, 2\}$. Find the number of relations from $A$ to $B$.

Solution:

Given, $A = \{x, y, z\}$ and $B = \{1, 2\}$.
Now, $A \times B = \{(x, 1), (x, 2), (y, 1), (y, 2), (z, 1), (z, 2)\}$
As $n(A \times B) = 6$, the number of subsets of $A \times B$ will be $2^6$.
Thus, the number of relations from $A$ to $B$ is $2^6$.

9. Let $R$ be the relation on $\mathbb{Z}$ defined by $R = \{(a, b): a, b \in \mathbb{Z}, a - b \text{ is an integer}\}$. Find the domain and range of $R$.

Solution:

Given,
Relation $R = \{(a, b): a, b \in \mathbb{Z}, a - b \text{ is an integer}\}$
We know that the difference between any two integers is always an integer.
Therefore,
Domain of $R = \mathbb{Z}$ and Range of $R = \mathbb{Z}$
Exercise 2.3

1. Which of the following relations are functions? Give reasons. If it is a function, determine its domain and range.
   (i) \{(2, 1), (5, 1), (8, 1), (11, 1), (14, 1), (17, 1)\}
   (ii) \{(2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6), (14, 7)\}
   (iii) \{(1, 3), (1, 5), (2, 5)\}

Solution:

(i) \{(2, 1), (5, 1), (8, 1), (11, 1), (14, 1), (17, 1)\}
As 2, 5, 8, 11, 14, and 17 are the elements of the domain of the given relation having their unique images, this relation can be called as a function.
Here, domain = \{2, 5, 8, 11, 14, 17\} and range = \{1\}

(ii) \{(2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6), (14, 7)\}
As 2, 4, 6, 8, 10, 12, and 14 are the elements of the domain of the given relation having their unique images, this relation can be called as a function.
Here, domain = \{2, 4, 6, 8, 10, 12, 14\} and range = \{1, 2, 3, 4, 5, 6, 7\}

(iii) \{(1, 3), (1, 5), (2, 5)\}
It’s seen that the same first element i.e., 1 corresponds to two different images i.e., 3 and 5, this relation cannot be called as a function.

2. Find the domain and range of the following real function:
   (i) \(f(x) = -|x|\)
   (ii) \(f(x) = \sqrt{9 - x^2}\)

Solution:

(i) Given,
\(f(x) = -|x|, x \in \mathbb{R}\)
We know that,
\(|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}\)
\[f(x) = -|x| = \begin{cases} -x, & x \geq 0 \\ x, & x < 0 \end{cases}\]
As \(f(x)\) is defined for \(x \in \mathbb{R}\), the domain of \(f\) is \(\mathbb{R}\).
It is also seen that the range of \(f(x) = -|x|\) is all real numbers except positive real numbers.
Therefore, the range of \(f\) is given by \((-\infty, 0]\).

(ii) \(f(x) = \sqrt{9 - x^2}\)
As \(\sqrt{9 - x^2}\) is defined for all real numbers that are greater than or equal to \(-3\) and less than or equal to 3, for \(9 - x^2 \geq 0\).
So, the domain of \(f(x)\) is \(\{x: -3 \leq x \leq 3\}\) or \([-3, 3]\).
Now,
For any value of \(x\) in the range \([-3, 3]\), the value of \(f(x)\) will lie between 0 and 3.
Therefore, the range of \(f(x)\) is \(\{x: 0 \leq x \leq 3\}\) or \([0, 3]\).
3. A function \( f \) is defined by \( f(x) = 2x - 5 \). Write down the values of
(i) \( f(0) \),   (ii) \( f(7) \),   (iii) \( f(-3) \)
Solution:
Given, 
Function, \( f(x) = 2x - 5 \).
Therefore,
(i) \( f(0) = 2 \times 0 - 5 = 0 - 5 = -5 \)
(ii) \( f(7) = 2 \times 7 - 5 = 14 - 5 = 9 \)
(iii) \( f(-3) = 2 \times (-3) - 5 = -6 - 5 = -11 \)

4. The function ‘\( t \)’ which maps temperature in degree Celsius into temperature in degree Fahrenheit is defined by
\[ t(C) = \frac{9C}{5} + 32 \]
Find (i) \( t(0) \)   (ii) \( t(28) \)   (iii) \( t(-10) \)   (iv) The value of \( C \), when \( t(C) = 212 \)
Solution:
Given function, \( t(C) = \frac{9C}{5} + 32 \)
So,
(i) \( t(0) = \frac{9 \times 0}{5} + 32 = 0 + 32 = 32 \)
(ii) \( t(28) = \frac{9 \times 28}{5} + 32 = \frac{252 + 160}{5} = \frac{412}{5} \)
(iii) \( t(-10) = \frac{9 \times (-10)}{5} + 32 = 9 \times (-2) + 32 = -18 + 32 = 14 \)
(iv) Given that, \( t(C) = 212 \)
\[ \therefore 212 = \frac{9C}{5} + 32 \]
\[ \Rightarrow \frac{9C}{5} = 212 - 32 \]
\[ \Rightarrow \frac{9C}{5} = 180 \]
\[ \Rightarrow 9C = 180 \times 5 \]
\[ \Rightarrow C = \frac{180 \times 5}{9} = 100 \]
Therefore, the value of \( t \) when \( t(C) = 212 \), is 100.

5. Find the range of each of the following functions.
(i) \( f(x) = 2 - 3x, x \in \mathbb{R}, x > 0. \)
(ii) \( f(x) = x^2 + 2, x \) is a real number.
(iii) \( f(x) = x, x \) is a real number.
Solution:

(i) Given,
\[ f(x) = 2 - 3x, \ x \in \mathbb{R}, \ x > 0. \]
We have,
\[ x > 0 \]
So,
\[ 3x > 0 \]
\[ -3x < 0 \quad \text{[Multiplying by } -1 \text{ both the sides, the inequality sign changes]} \]
\[ 2 - 3x < 2 \]
Therefore, the value of \( 2 - 3x \) is less than 2.
Hence, Range = \( (-\infty, 2) \)

(ii) Given,
\[ f(x) = x^2 + 2, \ x \text{ is a real number} \]
We know that,
\[ x^2 \geq 0 \]
So,
\[ x^2 + 2 \geq 2 \quad \text{[Adding 2 both the sides]} \]
Therefore, the value of \( x^2 + 2 \) is always greater or equal to 2 for \( x \) is a real number.
Hence, Range = \( [2, \infty) \)

(iii) Given,
\[ f(x) = x, \ x \text{ is a real number} \]
Clearly, the range of \( f \) is the set of all real numbers.
Thus,
Range of \( f = \mathbb{R} \)
1. The relation \( f \) is defined by
\[
 f(x) = \begin{cases} 
 x^2, & 0 \leq x \leq 3 \\
 3x, & 3 \leq x \leq 10 
\end{cases}
\]
The relation \( g \) is defined by
\[
 g(x) = \begin{cases} 
 x^2, & 0 \leq x \leq 2 \\
 3x, & 2 \leq x \leq 10 
\end{cases}
\]
Show that \( f \) is a function and \( g \) is not a function.

Solution:

The given relation \( f \) is defined as:
\[
 f(x) = \begin{cases} 
 x^2, & 0 \leq x \leq 3 \\
 3x, & 3 \leq x \leq 10 
\end{cases}
\]
It is seen that, for \( 0 \leq x < 3 \),
\( f(x) = x^2 \) and for \( 3 < x \leq 10 \),
\( f(x) = 3x \)
Also, at \( x = 3 \)
\( f(x) = 3^2 = 9 \) or \( f(x) = 3 \times 3 = 9 \)
i.e., at \( x = 3 \), \( f(x) = 9 \) [Single image]
Hence, for \( 0 \leq x \leq 10 \), the images of \( f(x) \) are unique.
Therefore, the given relation is a function.

Now, In the given relation \( g \) is defined as
\[
 g(x) = \begin{cases} 
 x^2, & 0 \leq x \leq 2 \\
 3x, & 2 \leq x \leq 10 
\end{cases}
\]
It is seen that, for \( x = 2 \)
\( g(x) = 2^2 = 4 \) and \( g(x) = 3 \times 2 = 6 \)
Thus, element 2 of the domain of the relation \( g \) corresponds to two different images i.e., 4 and 6.
Therefore, this relation is not a function.

2. If \( f(x) = x^2 \), find \( \frac{f(1.1) - f(1)}{(1.1-1)} \)

Solution:

Given,
\( f(x) = x^2 \)
Hence,
\[
\frac{f(1.1) - f(1)}{(1.1-1)} = \frac{(1.1)^2 - (1)^2}{0.1} = \frac{1.21 - 1}{0.1} = \frac{0.21}{0.1} = 2.1
\]

3. Find the domain of the function \( f(x) = \frac{x^2 + 2x + 1}{x^2 - 8x + 12} \)

Solution:
Given function,
\[ f(x) = \frac{x^2 + 2x + 1}{x^2 - 8x + 12}. \]
It can be seen that the function \( f \) is defined for all real numbers except at \( x = 6 \) and \( x = 2 \) as the denominator becomes zero otherwise.
Therefore, the domain of \( f \) is \( \mathbb{R} - \{2, 6\} \).

4. Find the domain and the range of the real function \( f \) defined by \( f(x) = \sqrt{x - 1} \).
Solution:
Given real function, \( f(x) = \sqrt{x - 1} \)
Clearly, \( \sqrt{x - 1} \) is defined for \( (x - 1) \geq 0 \).
So, the function \( f(x) = \sqrt{x - 1} \) is defined for \( x \geq 1 \).
Thus, the domain of \( f \) is the set of all real numbers greater than or equal to 1.
Domain of \( f \) = \( [1, \infty) \).

Now, \( x \geq 1 \Rightarrow (x - 1) \geq 0 \Rightarrow \sqrt{x - 1} \geq 0 \)
Thus, the range of \( f \) is the set of all real numbers greater than or equal to 0.
Range of \( f \) = \( [0, \infty) \).

5. Find the domain and the range of the real function \( f \) defined by \( f(x) = |x - 1| \).
Solution:
Given real function, \( f(x) = |x - 1| \)
Clearly, the function \( |x - 1| \) is defined for all real numbers.
Hence, Domain of \( f \) = \( \mathbb{R} \)
Also, for \( x \in \mathbb{R} \), \( |x - 1| \) assumes all real numbers.
Therefore, the range of \( f \) is the set of all non-negative real numbers.

6. Let \( f = \left\{ \left( x, \frac{x^2}{1 + x^2} \right) : x \in \mathbb{R} \right\} \) be a function from \( \mathbb{R} \) into \( \mathbb{R} \). Determine the range of \( f \).
Solution:
Given function, \( f = \left\{ \left( x, \frac{x^2}{1 + x^2} \right) : x \in \mathbb{R} \right\} \)
Substituting values and determining the images, we have
\[\left\{(0, 0), \left(\pm 0.5, \frac{1}{5}\right), \left(\pm 1, \frac{1}{2}\right), \left(\pm 1.5, \frac{9}{13}\right), \left(\pm 2, \frac{4}{5}\right), \left(3, \frac{9}{10}\right), \left(4, \frac{16}{17}\right), \ldots\right\}\]

The range of \(f\) is the set of all second elements. It can be observed that all these elements are greater than or equal to 0 but less than 1.

Or,
We know that, \(f\) or \(x \in \mathbb{R}\),
\[x^2 \geq 0\]
Then,
\[x^2 + 1 \geq x^2\]
\[1 \geq x^2 / (x^2 + 1)\]
Therefore, the range of \(f = [0, 1)\)

7. Let \(f, g: \mathbb{R} \to \mathbb{R}\) be defined, respectively by \(f(x) = x + 1, \ g(x) = 2x − 3\). Find \(f + g, f − g\) and \(f/g\).

Solution:
Given, the functions \(f, g: \mathbb{R} \to \mathbb{R}\) is defined as
\[f(x) = x + 1, \ g(x) = 2x − 3\]
Now,
\[(f + g) (x) = f(x) + g(x) = (x + 1) + (2x − 3) = 3x − 2\]
Thus, \((f + g) (x) = 3x − 2\)
\[(f − g) (x) = f(x) − g(x) = (x + 1) − (2x − 3) = x + 1 − 2x + 3 = −x + 4\]
Thus, \((f − g) (x) = −x + 4\)
\[f/g(x) = f(x)/g(x), g(x) \neq 0, x \in \mathbb{R}\]
\[f/g(x) = x + 1/2x − 3, 2x − 3 \neq 0\]
Thus, \(f/g(x) = x + 1/2x − 3, x \neq 3/2\)

8. Let \(f = \{(1, 1), (2, 3), (0, -1), (-1, -3)\}\) be a function from \(\mathbb{Z}\) to \(\mathbb{Z}\) defined by \(f(x) = ax + b\), for some integers \(a, b\). Determine \(a, b\).

Solution:
Given, \(f = \{(1, 1), (2, 3), (0, -1), (-1, -3)\}\)
And the function defined as, \(f(x) = ax + b\)
For \((1, 1) \in f\)
We have, \(f(1) = 1\)
So, \(a \times 1 + b = 1\)
\[a + b = 1 \ldots (i)\]
And for \((0, -1) \in f\)
We have \(f(0) = -1\)
\[a \times 0 + b = -1\]
\[b = -1\]
On substituting \(b = -1\) in \((i)\), we get
\[a + (-1) = 1 \Rightarrow a = 1 + 1 = 2.\]
Therefore, the values of $a$ and $b$ are 2 and $-1$ respectively.

9. Let $R$ be a relation from $\mathbb{N}$ to $\mathbb{N}$ defined by $R = \{(a, b): a, b \in \mathbb{N} and a = b^2\}$. Are the following true?
   (i) $(a, a) \in R$, for all $a \in \mathbb{N}$
   (ii) $(a, b) \in R$, implies $(b, a) \in R$
   (iii) $(a, b) \in R$, $(b, c) \in R$ implies $(a, c) \in R$.
   Justify your answer in each case.
   Solution:

   Given relation $R = \{(a, b): a, b \in \mathbb{N} and a = b^2\}$
   (i) It can be seen that $2 \in \mathbb{N}$; however, $2 \neq 2^2 = 4$.
       Thus, the statement “$(a, a) \in R$, for all $a \in \mathbb{N}$” is not true.

   (ii) It’s clearly seen that $(9, 3) \in \mathbb{N}$ because $9, 3 \in \mathbb{N} and 9 = 3^2$.
       Now, $3 \neq 9^2 = 81$; therefore, $(3, 9) \notin \mathbb{N}$
       Thus, the statement “$(a, b) \in R$, implies $(b, a) \in R$” is not true.

   (iii) It’s clearly seen that $(16, 4) \in \mathbb{N}$, $(4, 2) \in \mathbb{N}$ because $16, 4, 2 \in \mathbb{N}$ and $16 = 4^2 and 4 = 2^2$.
       Now, $16 \neq 2^2 = 4$; therefore, $(16, 2) \notin \mathbb{N}$
       Thus, the statement “$(a, b) \in R$, $(b, c) \in R$ implies $(a, c) \in R$” is not true.

10. Let $A = \{1, 2, 3, 4\}$, $B = \{1, 5, 9, 11, 15, 16\}$ and $f = \{(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)\}$. Are the following true?
   (i) $f$ is a relation from $A$ to $B$
   (ii) $f$ is a function from $A$ to $B$.
   Justify your answer in each case.
   Solution:

   Given,
   $A = \{1, 2, 3, 4\}$ and $B = \{1, 5, 9, 11, 15, 16\}$
   So,
   $A \times B = \{(1, 1), (1, 5), (1, 9), (1, 11), (1, 15), (1, 16), (2, 1), (2, 5), (2, 9), (2, 11), (2, 15), (2, 16), (3, 1),
   (3, 5), (3, 9), (3, 11), (3, 15), (3, 16), (4, 1), (4, 5), (4, 9), (4, 11), (4, 15), (4, 16)\}$
   Also given that,
   $f = \{(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)\}$
   (i) A relation from a non-empty set $A$ to a non-empty set $B$ is a subset of the Cartesian product $A \times B$.
      It’s clearly seen that $f$ is a subset of $A \times B$.
      Therefore, $f$ is a relation from $A$ to $B$.

   (ii) As the same first element i.e., 2 corresponds to two different images (9 and 11), relation $f$ is not a function.

11. Let $f$ be the subset of $\mathbb{Z} \times \mathbb{Z}$ defined by $f = \{(ab, a + b): a, b \in \mathbb{Z}\}$. Is $f$ a function from $\mathbb{Z}$ to $\mathbb{Z}$: justify your answer.
   Solution:
Given relation \( f \) is defined as 
\[ f = \{(ab, a + b): a, b \in \mathbb{Z}\} \]
We know that a relation \( f \) from a set \( A \) to a set \( B \) is said to be a function if every element of set \( A \) has unique images in set \( B \).

As \( 2, 6, -2, -6 \in \mathbb{Z} \), \( (2 \times 6, 2 + 6), (-2 \times -6, -2 + (-6)) \in f \)
i.e., \( (12, 8), (12, -8) \in f \)
It’s clearly seen that, the same first element, 12 corresponds to two different images (8 and -8).
Therefore, the relation \( f \) is not a function.

12. Let \( A = \{9, 10, 11, 12, 13\} \) and let \( f: A \rightarrow \mathbb{N} \) be defined by \( f(n) = \) the highest prime factor of \( n \). Find the range of \( f \).
**Solution:**

Given,
\( A = \{9, 10, 11, 12, 13\} \)
Now, \( f: A \rightarrow \mathbb{N} \) is defined as
\[ f(n) = \text{The highest prime factor of } n \]

So,
Prime factor of 9 = 3
Prime factors of 10 = 2, 5
Prime factor of 11 = 11
Prime factors of 12 = 2, 3
Prime factor of 13 = 13
Thus, it can be expressed as
\[ f(9) = \text{The highest prime factor of } 9 = 3 \]
\[ f(10) = \text{The highest prime factor of } 10 = 5 \]
\[ f(11) = \text{The highest prime factor of } 11 = 11 \]
\[ f(12) = \text{The highest prime factor of } 12 = 3 \]
\[ f(13) = \text{The highest prime factor of } 13 = 13 \]
The range of \( f \) is the set of all \( f(n) \), where \( n \in A \).
Therefore,
Range of \( f = \{3, 5, 11, 13\} \)