EXERCISE 7.10

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Evaluate the integrals in Exercise 1 to 8 by substitution.

1. $\int_0^1 \frac{x}{x^2+1} dx$

Solution:

Given integral: $\int_0^1 \frac{x}{x^2+1} dx$

Let's take $x^2 + 1 = t$

Then, $2x\, dx = dt$

$x\, dx = ½\, dt$

When $x = 0$, $t = 1$ and when $x = 1$, $t = 2$

Now,

$$\int_0^1 \frac{x}{x^2+1} dx = \int_1^2 \frac{dt}{2t}$$

$$= \frac{1}{2}\int_1^2 \frac{dt}{t}$$

$$= \frac{1}{2}[\log|t|]_1^2$$

$$= \frac{1}{2}[\log 2 - \log 1]$$

$$= \frac{1}{2}\log 2$$

2. $\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\, d\phi$

Solution:

Given integral: $\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\, d\phi$

Let's consider $I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\, d\phi = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^4\phi\, \cos\phi\, d\phi$

$$I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,(\cos^2\phi)^2\ \cos\phi\ d\phi = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\left(1-\sin^2\phi\right)^2\ \cos\phi\ d\phi$$

Also, let $\sin\phi = t \Rightarrow \cos\phi d\phi = dt$

So when, $\phi = 0,\ t = 0$ and when $\phi = \frac{\pi}{2},\ t = 1$

Hence,

$$I = \int_0^1 \sqrt{t}\left(1 - t^2\right)^2\ dt$$

Expanding and splitting the integrals, we have

$$= \int_0^1 t^{\frac{1}{2}}\left(1 + t^4 - 2t^2\right)\ dt$$

$$= \int_0^1 (t^{\frac{1}{2}} + t^{\frac{9}{2}} - 2t^{\frac{5}{2}})dt$$

Integrating the terms individually by standard form, we get

$$= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{11}{2}}}{\frac{11}{2}} + \frac{2t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1$$

$$= \frac{2}{3} + \frac{2}{11} - \frac{4}{7}$$

$$= \frac{154 + 42 - 132}{231} = \frac{64}{231}$$

Therefore, $\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\ d\phi = 64/231$

3. $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Solution:

Given integral: $\int_0^1 \sin^{-1}\left(\frac{2x}{x^2+1}\right)dx$

Let us take x = tan θ ⇒ dx = sec² θ d θ

So when, x = 0, θ = 0 and when x = 1, θ = π /4

Let $I = \int_0^1 \sin^{-1}\left(\frac{2x}{x^2+1}\right)dx$

Now, by substitution I becomes

$$I = \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{\tan^2\theta+1}\right)\sec^2\theta d\theta$$

Transforming the trigonometric ratio into its simple form, we have

$$I = \int_0^{\frac{\pi}{4}} \sin^{-1}(\sin 2\theta)\sec^2\theta d\theta$$

Applying the inverse trigonometric ratio, we get

$$I = \int_0^{\frac{\pi}{4}} 2\theta\sec^2\theta d\theta$$

$$I = 2\int_0^{\frac{\pi}{4}} \theta\sec^2\theta d\theta$$

Now, by applying product rule as:

$$\int u.vdx = u.\int vdx - \int \frac{du}{dx}.\left\{\int vdx\right\}dx$$

$$I = 2\left[\theta\int\sec^2\theta d\theta - \int\frac{d}{d\theta}\theta.\left\{\int\sec^2\theta d\theta\right\}d\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\theta\tan\theta - \int 1.\tan\theta d\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\theta\tan\theta - \log|\sec\theta|\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\frac{\pi}{4}\tan\frac{\pi}{4} - \log\left|\sec\frac{\pi}{4}\right| - 0 + \log\left|\sec 0\right|\right]$$

$$= 2\left[\frac{\pi}{4} - \log\left(\sqrt{2}\right) + \log 1\right]$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\log\left(2\right)\right]$$

$$= \frac{\pi}{2} + \log\left(2\right)$$

Therefore, $\int_0^1 \sin^{-1}\left(\frac{2x}{x^2+1}\right)dx = \frac{\pi}{2} + \log\left(2\right)$

4. $\int_0^2 x\sqrt{x+2}$ (Put $x + 2 = t^2$)

Solution:

Given integral: $\int_0^2 x\sqrt{x+2}\,dx$

Let's take $x + 2 = t^2 \Rightarrow dx = 2t\,dt$

And, $x = t^2 - 2$

So when, $x = 0$, $t = \sqrt{2}$ and when $x = 2$, $t = 2$

Hence, after substitution the given integral can be written as:

$$\int_0^2 x\sqrt{x+2}\,dx = \int_{\sqrt{2}}^2 \left(t^2 - 2\right)\sqrt{t^2}\,2t\,dt$$

Taking the square root we have,

$$= 2\int_{\sqrt{2}}^2 \left(t^2 - 2\right)t.t\,dt$$

$$= 2\int_{\sqrt{2}}^2 \left(t^2 - 2\right)t^2\,dt$$

$$= 2\int_{\sqrt{2}}^{2}\left(t^4 - 2t^2\right)dt$$

On integrating the terms separately, we get

$$= 2\left[\frac{t^5}{5} - \frac{2t^3}{3}\right]_{\sqrt{2}}^{2}$$

Applying the limits after integration, we have

$$= 2\left[\frac{(2)^5}{5} - \frac{2(2)^3}{3} - \frac{\left(\sqrt{2}\right)^5}{5} + \frac{2\left(\sqrt{2}\right)^3}{3}\right]_{\sqrt{2}}^{2}$$

$$= 2\left[\frac{32}{5} - \frac{16}{3} - \frac{4\sqrt{2}}{5} + \frac{4\sqrt{2}}{3}\right]$$

$$= 2\left[\frac{96 - 80 - 12\sqrt{2} + 20\sqrt{2}}{15}\right]$$ [Taking L.C.M for addition]

$$= 2\left[\frac{16 + 8\sqrt{2}}{15}\right]$$

$$= \left[\frac{16(2 + \sqrt{2}}{15}\right]$$ [After taking common terms]

$$= \frac{16\sqrt{2}\left(\sqrt{2} + 1\right)}{15}$$

Therefore, $\int_{0}^{2} x\sqrt{x + 2}\,dx = \frac{16\sqrt{2}\left(\sqrt{2} + 1\right)}{15}$

5. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x}\,dx$

Solution:

Given integral: $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx$

Let cos x = t

On differentiating,

-sin xdx = dt

sin xdx = -dt

So, when x = 0, t = 1 and when x = π /2, t = 0

Hence, the given integration upon substitution will change as

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx = -\int_1^0 \frac{dt}{1+t^2}$$

On integrating, we have

$$-\int_1^0 \frac{dt}{1+t^2} = -\left[\frac{1}{1}.\tan^{-1} t\right]_1^0$$

[As w.k.t $\int \frac{dt}{x^2+a^2} = \frac{1}{a}.\tan^{-1}\frac{x}{a} + C$]

$$= -\left[\tan^{-1} 0 - \tan^{-1} 1\right]$$

$$= -\left[0 - \frac{\pi}{4}\right]$$

$$= -\left[-\frac{\pi}{4}\right]$$

$$= \frac{\pi}{4}$$

Therefore, $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx = \frac{\pi}{4}$

6. $\int_0^2 \frac{dx}{x+4-x^2}$

Solution:

Given integral: $\int_0^2 \frac{dx}{x+4-x^2}$

$$\int_0^2 \frac{dx}{x+4-x^2} = \int_0^2 \frac{dx}{-(x^2-x-4)}$$

The given integral can be written as,

$$\int_0^2 \frac{dx}{-(x^2-x+\frac{1}{4}-\frac{1}{4}-4)}$$

[By completing its square method]

$$= \int_0^2 \frac{dx}{-\left[\left(x-\frac{1}{2}\right)^2-\frac{17}{4}\right]}$$

$$= \int_0^2 \frac{dx}{\left[\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2\right]}$$

Now, taking suitable substitution

Let $x-\frac{1}{2} = t \Rightarrow dx = dt$

So when $x = 0,\ t = -\frac{1}{2}$ and when $x = 2,\ t = \frac{3}{2}$

After substitution, the integral changes as:

$$\int_0^2 \frac{dx}{\left[\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2\right]} = \int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{dt}{\left[\left(\frac{\sqrt{17}}{2}\right)^2-(t)^2\right]}$$

[As w.k.t, $\int \frac{dx}{\left[(a)^2-(x)^2\right]} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$]

On integrating, we have

$$\int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{dt}{\left[\left(\frac{\sqrt{17}}{2}\right)^2 - (t)^2\right]} = \left[\frac{1}{2\left(\frac{\sqrt{17}}{2}\right)} \log \frac{\left(\frac{\sqrt{17}}{2} + t\right)}{\frac{\sqrt{17}}{2} - t}\right]_{-\frac{1}{2}}^{\frac{3}{2}}$$

Applying limits,

$$= \frac{1}{\sqrt{17}}\left[\log \frac{\left(\frac{\sqrt{17}}{2} + \frac{3}{2}\right)}{\frac{\sqrt{17}}{2} - \frac{3}{2}} - \log \frac{\left(\frac{\sqrt{17}}{2} - \frac{1}{2}\right)}{\frac{\sqrt{17}}{2} + \frac{1}{2}}\right]$$

$$= \frac{1}{\sqrt{17}}\left[\log \frac{\left(\sqrt{17} + 3\right)}{\sqrt{17} - 3} - \log \frac{\left(\sqrt{17} - 1\right)}{\sqrt{17} + 1}\right]$$

$$= \frac{1}{\sqrt{17}}\left[\log \left\{\frac{\left(\sqrt{17} + 3\right)}{\sqrt{17} - 3} \times \frac{\left(\sqrt{17} + 1\right)}{\sqrt{17} - 1}\right\}\right]$$

[Using logarithmic properties]

$$= \frac{1}{\sqrt{17}}\left[\log \left\{\frac{\left(\sqrt{17} + 3\right)\left(\sqrt{17} + 1\right)}{\left(\sqrt{17} - 3\right)\left(\sqrt{17} - 1\right)}\right\}\right]$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{17 + 3 + 4\sqrt{17}}{17 + 3 - 4\sqrt{17}}\right]$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{20 + 4\sqrt{17}}{20 - 4\sqrt{17}}\right]$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{5 + \sqrt{17}}{5 - \sqrt{17}}\right]$$

$$= \frac{1}{\sqrt{17}}\log\left[\frac{\left(5+\sqrt{17}\right)\left(5+\sqrt{17}\right)}{\left(5-\sqrt{17}\right)\left(5+\sqrt{17}\right)}\right]$$

[Rationalising the surd]

$$= \frac{1}{\sqrt{17}}\log\left[\frac{\left(25+17+10\sqrt{17}\right)}{25-17}\right]$$

$$= \frac{1}{\sqrt{17}}\log\left[\frac{\left(42+10\sqrt{17}\right)}{8}\right] = \frac{1}{\sqrt{17}}\log\left[\frac{\left(21+5\sqrt{17}\right)}{4}\right]$$

7. $\int_{-1}^{1}\frac{dx}{x^2+2x+5}$

Solution:

Given integral: $\int_{-1}^{1}\frac{dx}{x^2+2x+5}$

$$= \int_{-1}^{1}\frac{dx}{(x^2+2x+1)+4}$$

$$= \int_{-1}^{1}\frac{dx}{(x+1)^2+(2)^2}$$

[By completing the square]

Taking substitution, x + 1 = t

So, dx = dt

When x = -1, t = 0 and when x = 1, t = 2

Hence, the given integral is now changed as

$$\int_{-1}^{1}\frac{dx}{(x+1)^2+(2)^2} = \int_{0}^{2}\frac{dt}{(t)^2+(2)^2}$$

[As w.k.t $\int\frac{dt}{x^2+a^2} = \frac{1}{a}.\tan^{-1}\frac{x}{a} + C$]

$$\int_0^2 \frac{dt}{(t)^2 + (2)^2} = \left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]_0^2$$

$$= \frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}0$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}$$

Therefore, $\int_{-1}^{1} \frac{dx}{x^2 + 2x + 5} = \frac{\pi}{8}$

8. $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx$

Solution:

Given integral: $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx$

Taking substitution, $2x = t \Rightarrow 2\, dx = dt$

So when $x = 1, t = 2$ and when $x = 2, t = 4$

Hence, the given integral will change as:

$$\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx = \int_2^4 \left(\frac{1}{\left(\frac{t}{2}\right)} - \frac{1}{2\left(\frac{t}{2}\right)^2}\right) e^t \left(\frac{dt}{2}\right)$$

$$= \frac{1}{2}\int_2^4 \left(\frac{2}{t} - \frac{2}{t^2}\right) e^t dt$$

$$= \int_2^4 \frac{1}{2}.(2)\left(\frac{1}{t} - \frac{1}{t^2}\right) e^t dt$$

[Taking common and simplifying]

$$= \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2}\right) e^t dt$$

Further, let 1/t = f(t)

Then we have, f' (t) = -1/t²

Converting the integral into the required form,

$$\int_2^4 \left(\frac{1}{t}-\frac{1}{t^2}\right)e^t dt = \int_2^4 (f(t) + f'(t))e^t dt$$

$$\left[\text{As, w.k.t } \int (f(x) + f'(x))e^x dx = e^x f(x) + C\right]$$

Up to integration, we get

$$\int_2^4 (f(t) + f'(t))e^t dt = \left[e^t f(t)\right]_2^4$$

$$= \left[e^t.\frac{1}{t}\right]_2^4$$

$$= \frac{e^4}{4}-\frac{e^2}{2}$$

$$= \frac{e^4-2e^2}{4} = \frac{e^2(e^2-2)}{4}$$

Therefore, $\int_1^2 \left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x} dx = \frac{e^2(e^2-2)}{4}$

Choose the correct answer in Exercise 9 and 10.

9. The value of the integral $\int_{\frac{1}{3}}^{1} \frac{(x-x^3)^{\frac{1}{3}}}{x^4} dx$ is

(A) 6 (B) 0 (C) 3 (D) 4

Solution:

Given integral: $\int_{\frac{1}{3}}^{1} \left(\frac{(x-x^3)^{\frac{1}{3}}}{x^4}\right) dx$

Let $I = \int_{\frac{1}{3}}^{1} \left(\frac{(x - x^3)^{\frac{1}{3}}}{x^4} \right) dx$

Now, taking x = sin θ ⇒ dx = cos θ d θ

So when, $x = \frac{1}{3},\ \theta = \sin^{-1}\left(\frac{1}{3}\right)$ and when x = 1, θ = π/2

Hence, after substitution the given integral will become:

$$I = \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \left(\frac{(\sin\theta - \sin^3\theta)^{\frac{1}{3}}}{\sin^4\theta} \right) \cos\theta d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \left(\frac{(\sin\theta)^{\frac{1}{3}}(1 - \sin^2\theta)^{\frac{1}{3}}}{\sin^4\theta} \right) \cos\theta d\theta$$

[Taking common]

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \left(\frac{(\sin\theta)^{\frac{1}{3}}(\cos^2\theta)^{\frac{1}{3}}}{\sin^4\theta} \right) \cos\theta d\theta$$

[Simplifying by using trigonometric identity]

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \left(\frac{(\sin\theta)^{\frac{1}{3}}(\cos\theta)^{\frac{2}{3}}}{\sin^2\theta.\sin^2\theta} \right) \cos\theta d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \left(\frac{(\cos\theta)^{\frac{2}{3}+1}}{(\sin\theta)^{2-\frac{1}{3}}} \right) . \frac{1}{\sin^2\theta} d\theta$$

[Simplifying by using exponents properties]

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \left(\frac{(\cos\theta)^{\frac{5}{3}}}{(\sin\theta)^{\frac{5}{3}}} \right) .\operatorname{cosec}^2\theta d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \left((\cot\theta)^{\frac{5}{3}} \right).\text{cosec}^2\theta d\theta$$

.... (i)

Now, let cot θ = t ⇒ - cosec^2 θ d θ

So when, $\theta = \sin^{-1}\left(\frac{1}{3}\right)$, $t = 2\sqrt{2}$ and when $\theta = \frac{\pi}{2}$, $t = 0$

After substitution, (i) becomes:

$$= \int_{2\sqrt{2}}^{0} -(t)^{\frac{5}{3}}.dt$$

On integrating and applying limits, we have

$$= -\left[\frac{(t)^{\frac{5}{3}+1}}{\frac{5}{3}+1}\right]_{2\sqrt{2}}^{0}$$

$$= -\left[\frac{(t)^{\frac{8}{3}}}{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$

$$= -\frac{3}{8}\left[(0)^{\frac{8}{3}} - \left(2\sqrt{2}\right)^{\frac{8}{3}}\right]$$

$$= -\frac{3}{8}\left[-\left(\sqrt{8}\right)^{\frac{8}{3}}\right] = \frac{3}{8}\left[(8)^{\frac{4}{3}}\right]$$

$$= \frac{3}{8}[16]$$

= 6

Therefore, the correct option is (A).

If $f(x) = \int_0^x t \sin t \, dt$, then $f'(x)$ is

(A) $\cos x + x \sin x$ (B) $x \sin x$

10. (C) $x \cos x$ (D) $\sin x + x \cos x$

Solution:

Given integral function: $f(x) = \int_0^x t \sin t dt$

Applying product rule, we have

$$\int u.v dx = u.\int v dx - \int \frac{du}{dx}.\{\int v dx\} dx$$

So,

$$f(x) = t\int_0^x \sin t dt - \int_0^x \left\{\left(\frac{d}{dt}t\right).\int \sin t dt\right\} dt = \left[t(-\cos t)\right]_0^x - \int_0^x (-\cos t) dt$$

Applying the limits, we get

$$= \left[-t(\cos t) + \sin t\right]_0^x$$

= -x cos x + sin x – 0

Thus, f(x) = -x cos x + sin x

On differentiating, we have

$$f'(x) = -\left[x.\frac{d}{dx}\cos x + \cos x.\frac{d}{dx}x + \frac{d}{dx}\sin x\right]$$

f'(x) = -[{x (-sinx)} + cosx] + cosx

= x sin x – cos x + cos x

= x sin x

Therefore, the correct option is (B).