EXERCISE 7.11

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By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.

1. $\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$

Solution:

Given, $\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$

Let, $I = \int_0^{\frac{\pi}{2}} \cos^2 x\, dx \ldots..(1)$

We know that, $\left\{ \int_0^a f(x)dx = \int_0^a f(a-x)dx \right\}$

By using above formula, the given question can be written as

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \cos^2\left(\frac{\pi}{2} - x\right) dx$$

From the standard integration formulae we have

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin^2(x)\, dx \ldots.(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \left[\sin^2(x) + \cos^2(x)\right] dx$$

By using standard identities the above equation can be written as

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} [1] dx$$

Now by applying the limits we get

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2} - 0$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

2. $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

Solution:

Given: $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

Let, $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \ldots..(1)$

As we know that, $\left\{ \int_0^a f(x) dx = \int_0^a f(a-x) dx \right\}$

By using the above formula we get

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\,dx$$

By substituting the standard identities we get

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx\,(2)$$

Adding (1) and (2), we get

$$2I=\int_0^{\frac{\pi}{2}}\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}[1]dx$$

Integrating the above equation and applying the limits we get

$$\Rightarrow 2I=[x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I=\frac{\pi}{2}-0$$

$$\Rightarrow 2I=\frac{\pi}{2}$$

$$\Rightarrow I=\frac{\pi}{4}$$

3. $\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x\, dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}$

Solution:

Given $\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$

$$\text{let, } I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx \ldots..(1)$$

As we know that

$$\left\{ \int_0^a f(x)dx = \int_0^a f(a-x)dx \right\}$$

By substituting the above formula we get

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right)}{\sin^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right) + \cos^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right)} dx$$

Again by substituting the standard identities we get

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^{\frac{3}{2}} x}{\cos^{\frac{3}{2}} x + \sin^{\frac{3}{2}} x} dx\,(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} \, dx$$

The above equation can be written as

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} [1] dx$$

Integrating and applying the limit we get

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2} - 0$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

4. $\int_0^{\frac{\pi}{2}} \frac{\cos^5 x \, dx}{\sin^5 x + \cos^5 x}$

Solution:

Given: $\int_0^{\frac{\pi}{2}} \frac{\cos^5 x}{\sin^5 x + \cos^5 x} \, dx$

$$\text{let, } I = \int_0^{\frac{\pi}{2}} \frac{\cos^5 x}{\sin^5 x + \cos^5 x} \, dx \ldots..(1)$$

As we know that

$$\left\{\int_0^a f(x)dx = \int_0^a f(a-x)dx\right\}$$

By substituting the above formula we get

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^5\left(\frac{\pi}{2}-x\right)}{\sin^5\left(\frac{\pi}{2}-x\right)+\cos^5\left(\frac{\pi}{2}-x\right)}\ dx$$

The above equation can be written as

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x}{\cos^5 x + \sin^5 x}\ dx \ldots..(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x + \cos^5 x}{\sin^5 x + \cos^5 x}\ dx$$

The above equation becomes

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} [1]dx$$

Now by integrating and applying the limits we get

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2} - 0$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

5. $\int_{-5}^{5} |x+2|\, dx$

Solution:

Given: $\int_{-5}^{5} |x+2|\ dx$

As we can see that (x+2) ≤ 0 on [-5, -2] and (x + 2) ≥ 0 on [-2, 5]

As we know that

$$\left\{\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\right\}$$

Now by substituting the formula we get

$$\Rightarrow I = \int_{-5}^{-2} -(x+2)dx + \int_{-2}^{5} (x+2)dx$$

Integrating and applying the limits we get

$$\Rightarrow I = -\left[\frac{x^2}{2} + 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$$

On simplifying

$$\Rightarrow I = -\left[\frac{(-2)^2}{2} + 2(-2) - \frac{(-5)^2}{2} - 2(-5)\right] + \left[\frac{(5)^2}{2} + 2(5) - \frac{(-2)^2}{2} - 2(-2)\right]$$

$$\Rightarrow I=-\left[2-4-\frac{25}{2}+10\right]+\left[\frac{25}{2}+10-2+4\right]$$

On computing we get

$$\Rightarrow I=-2+4+\frac{25}{2}-10+\frac{25}{2}+10-2+4$$

$$\Rightarrow I=29$$

6. $\int_{2}^{8}|x-5|dx$

Solution:

Given $\int_{2}^{8}|x-5|\ dx$

As we can see that (x - 5) ≤ 0 on [2, 5] and (x + 2) ≥ 0 on [5, 8]

As we know that

$$\left\{\int_{a}^{b}f(x)dx=\int_{a}^{c}f(x)dx+\int_{c}^{b}f(x)dx\right\}$$

By applying the above formula we get

$$\Rightarrow I=\int_{2}^{5}-(x-5)dx+\int_{5}^{8}(x-5)dx$$

Now by integrating the above equation

$$\Rightarrow I=-\left[\frac{x^2}{2}-5x\right]_2^5+\left[\frac{x^2}{2}-5x\right]_5^8$$

Now by applying the limits we get

$$\Rightarrow I = -\left[\frac{(5)^2}{2} - 5(5) - \frac{(2)^2}{2} + 5(2)\right] + \left[\frac{(8)^2}{2} - 5(8) - \frac{(5)^2}{2} + 5(5)\right]$$

On computing

$$\Rightarrow I = -\left[\frac{25}{2} - 25 - 2 + 10\right] + \left[\frac{64}{2} - 40 - \frac{25}{2} + 25\right]$$

$$\Rightarrow I = -\frac{25}{2} + 17 + 32 - 15 - \frac{25}{2}$$

On simplifying we get

$$\Rightarrow I = 34 - 25$$

$\Rightarrow$ I = 9

7. $\int_0^1 x(1-x)^n dx$

Solution:

Given: $\int_0^1 x(1-x)^n \ dx$

$$\text{let, } I = \int_0^1 x(1-x)^n \ dx$$

As we know that

$$\left\{\int_0^a f(x)dx = \int_0^a f(a-x)dx\right\}$$

By using the above formula we get

$$\Rightarrow I = \int_0^1 (1-x)(1-(1-x))^n \ dx$$

The above equation can be written as

$$\Rightarrow I = \int_0^1 (1-x)(x)^n \ dx$$

By multiplying we get

$$\Rightarrow I = \int_0^1 (x)^n - (x)^{n+1} \ dx$$

On integrating

$$\Rightarrow I = \left[\frac{(x)^{n+1}}{n+1} - \frac{(x)^{n+2}}{n+2}\right]_0^1$$

Now by applying the limits we get

$$\Rightarrow I = \left[\frac{1}{n+1} - \frac{1}{n+2}\right]$$

$$\Rightarrow I = \left[\frac{(n+2)-(n+1)}{(n+1)(n+2)}\right]$$

On simplification

$$\Rightarrow I = \left[\frac{1}{(n+1)(n+2)}\right]$$

8. $\int_0^{\frac{\pi}{4}} \log(1+\tan x)\, dx$

Solution:

Given: $\int_0^{\frac{\pi}{4}} \log(1+\tan x)dx$

$$\text{let, } I = \int_0^{\frac{\pi}{4}} \log(1+\tan x)dx \ldots..(1)$$

As we know that

$$\left\{\int_0^a f(x)dx = \int_0^a f(a-x)dx\right\}$$

By using the above formula we get

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx$$

Again we know the standard formula

$$\left\{\tan(A-B) = \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}\right\}$$

By substituting the above formula we get

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\left[1+\frac{\tan\left(\frac{\pi}{4}\right)-\tan(x)}{1+\tan\left(\frac{\pi}{4}\right)\tan(x)}\right]dx$$

Applying the values we get

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\left[1+\frac{1-\tan(x)}{1+\tan(x)}\right]dx$$

On simplification the above equation can be written as

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\left[\frac{2}{1+\tan(x)}\right]dx$$

Now by applying log formula we get

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log[2]dx - \int_0^{\frac{\pi}{4}} \log[1+\tan(x)]dx$$

From equation (1) we can write as

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log[2]dx - I$$

On integration

$$\Rightarrow 2I = [x\log 2]_0^{\frac{\pi}{4}}$$

Now by applying the limits we get

$$\Rightarrow 2I = \frac{\pi}{4}\log 2 - 0$$

$$\Rightarrow I = \frac{\pi}{8}\log 2$$

9. $\int_0^2 x\sqrt{2-x}\,dx$

Solution:

Given: $\int_0^2 x\sqrt{2-x}\,dx$

$$\text{let, } I = \int_0^2 x\sqrt{2-x}\,dx \ldots..(1)$$

As we know that

$$\left\{\int_0^a f(x)dx = \int_0^a f(a-x)dx\right\}$$

By using the above formula we get

$$\Rightarrow I = \int_0^2 (2-x)\sqrt{2-(2-x)}dx$$

On simplification the above equation can be written as

$$\Rightarrow I = \int_0^2 (2-x)\sqrt{(x)}dx$$

On multiplication we get

$$\Rightarrow I = \int_0^2 \left(2x^{\frac{1}{2}} - x^{\frac{3}{2}}\right)dx$$

On integration

$$\Rightarrow I = \left[2\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) - \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^2$$

$$\Rightarrow I = \left[\frac{4}{3}\left(x^{\frac{3}{2}}\right) - \frac{2}{5}\left(x^{\frac{5}{2}}\right)\right]_0^2$$

Now by applying the limits the above equation can be written as

$$\Rightarrow I = \left[\frac{4}{3}\left((2)^{\frac{3}{2}}\right) - \frac{2}{5}\left((2)^{\frac{5}{2}}\right)\right]$$

By computing

$$\Rightarrow I = \frac{4}{3} \times 2\sqrt{2} - \frac{2}{5} \times 4\sqrt{2}$$

$$\Rightarrow I = \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5}$$

On simplification

$$\Rightarrow I = \frac{40\sqrt{2} - 24\sqrt{2}}{15}$$

$$\Rightarrow I = \frac{16\sqrt{2}}{15}$$

10. $\int_0^{\frac{\pi}{2}} (2\log \sin x - \log \sin 2x)\, dx$

Solution:

Given: $\int_0^{\frac{\pi}{2}} (2\log \sin x - \log \sin 2x)\, dx$

$$\text{let, } I = \int_0^{\frac{\pi}{2}} (2\log \sin x - \log \sin 2x)\, dx$$

Now by applying Sin 2x formula we get

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{2\log \sin x - \log(2\sin x \cos x)\}\, dx$$

Applying log formula we can write above equation as

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left\{ 2\log \sin x - \log(2) - \log(\sin x) - \log(\cos x) \right\} dx$$

On simplification

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{ \log \sin x - \log 2 - \log \cos x \} dx \ldots .(1)$$

As we know that

$$\left\{ \int_0^a f(x) dx = \int_0^a f(a-x) dx \right\}$$

By using the above formula we get

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left\{ \log \sin\left(\frac{\pi}{2} - x\right) - \log 2 - \log \cos\left(\frac{\pi}{2} - x\right) \right\} dx$$

Using allied angles formulae, the above equation becomes

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{ \log \cos x - \log 2 - \log \sin x \} dx \ldots .(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} (-\log 2 - \log 2) \ dx$$

By taking common

$$2I = -2\log 2\int_0^{\frac{\pi}{2}}(1)\ dx$$

On integrating we get

$$\Rightarrow 2I = -2\log 2[x]_0^{\frac{\pi}{2}}$$

Now by applying the limits

$$\Rightarrow 2I = -2\log 2\left[\frac{\pi}{2} - 0\right]$$

$$\Rightarrow 2I = -2\log 2\left(\frac{\pi}{2}\right)$$

On simplification we get

$$\Rightarrow I = \frac{\pi}{2}(-\log 2)$$

$$\Rightarrow I = \frac{\pi}{2}\left(\log\frac{1}{2}\right)$$

11. $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\, dx$

Solution:

As we can see f (x) = $\sin^2 x$ and f (-x) = $\sin^2$ (-x) = $(\sin(-x))^2$ = $(-\sin x)^2$ = $\sin^2 x$.

That is f (x) = f (-x)

So, $\sin^2 x$ is an even function.

It is also known that if f(x) is an even function then, we have

$$\left\{\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx\right\}$$

Now by using this formula the given question can be written as

$$\Rightarrow I = 2.\int_{0}^{\frac{\pi}{2}} \left(\sin^2 x\right) dx$$

Now by substituting $\sin^2 x$ formula we get

$$\Rightarrow I = 2.\int_{0}^{\frac{\pi}{2}} \frac{1-\cos 2x}{2}\,dx$$

$$\Rightarrow I = \int_{0}^{\frac{\pi}{2}} (1-\cos 2x)\,dx$$

On integrating we get

$$\Rightarrow I = \left[x - \frac{\sin 2x}{2}\right]_{0}^{\frac{\pi}{2}}$$

Now by applying the limits

$$\Rightarrow I = \frac{\pi}{2} - \sin\pi - 0 + \sin 0$$

$$\Rightarrow I = \frac{\pi}{2}$$

12. $\int_0^{\pi} \frac{x\,dx}{1+\sin x}$

Solution:

Given: $\int_0^{\pi} \frac{x}{1+\sin x}\, dx$

$$\text{let, } I = \int_0^{\pi} \frac{x}{1+\sin x}\, dx \ldots..(1)$$

As we know that

$$\left\{ \int_0^a f(x)dx = \int_0^a f(a-x)dx \right\}$$

By using above formula we get

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1+\sin(\pi - x)}\, dx$$

Now by multiplying and simplifying the equation we get

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1+\sin x}\, dx \ldots\ldots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{(\pi - x) + x}{1+\sin x}\, dx$$

$$2I = \int_0^{\pi} \frac{\pi}{1+\sin x}\, dx$$

Now by multiplying and dividing the above equation by (1 – sin x) we get

$$2I = \pi \int_0^{\pi} \frac{(1-\sin x)}{(1+\sin x)(1-\sin x)}\, dx$$

On simplification we get

$$2I = \pi\int_0^{\pi}\frac{(1-\sin x)}{\cos^2 x}\,dx$$

By splitting the numerator we get

$$2I = \pi\int_0^{\pi}\left\{\frac{1}{\cos^2 x}-\frac{\sin x}{\cos^2 x}\right\}dx$$

The above equation can be written as

$$2I = \pi\int_0^{\pi}\{\sec^2 x - \tan x\sec x\}\,dx$$

$$\Rightarrow 2I = \pi[\tan x - \sec x]_0^{\pi}$$

$$\Rightarrow 2I = \pi[2]$$

$$\Rightarrow I = \pi$$

13. $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\,dx$

Solution:

Given: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin^7 x)\,dx$

$$\text{let, } I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin^7 x)\,dx$$

As we can see $f(x) = \sin^7 x$ and $f(-x) = \sin^7(-x) = (\sin(-x))^7 = (-\sin x)^7 = -\sin^7 x$.

That is f (x) = -f (-x)

So, $\sin^2 x$ is an odd function.

It is also known that if f(x) is an odd function then,

$$\left\{\int_{-a}^{a} f(x)dx = 0\right\}$$

$$\Rightarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\sin^7 x\right)dx = 0$$

14. $\int_0^{2\pi} \cos^5 x\, dx$

Solution:

$$\text{let, } I = \int_0^{2\pi} \left(\cos^5 x\right)dx$$

As we see, $f(x) = \cos^5 x$ and $f(2\pi - x) = \cos^5(2\pi - x) = \cos^5 x = f(x)$

$$\text{because, } \int_0^{2a} f(x)dx = 2.\int_0^{a} f(x)dx, \text{if } f(2a-x) = f(x)$$

$$\text{and } \int_0^{2a} f(x)dx = 0, \text{if } f(2a-x) = -f(x)$$

$$\Rightarrow I = 2.\int_0^{\pi} \left(\cos^5 x\right)dx$$

$$\text{Now } \left\{\cos^5(\pi - x) = -\cos^5 x\right\}$$

$$\Rightarrow I = 0$$

15. $\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$

Solution:

Given: $\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} \, dx$

$$\text{let, } I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} \, dx \ldots..(1)$$

As we know that

$$\left\{ \int_0^a f(x)dx = \int_0^a f(a-x)dx \right\}$$

By using the above formula in given equation it can be written as

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)} \, dx$$

Now by applying allied angle formula we get

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \cos x \sin x} \, dx \ldots..(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x + \cos x - \sin x}{1 + \sin x \cos x} \, dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{0}{1+\sin x \cos x}\, dx$$

$$\Rightarrow I = 0$$

16. $\int_0^{\pi} \log(1+\cos x)\, dx$

Solution:

Given: $\int_0^{\pi} \log(1+\cos x)\, dx$

$$\text{let, } I = \int_0^{\pi} \log(1+\cos x)\, dx \ldots..(1)$$

As we know that

$$\left\{ \int_0^a f(x)dx = \int_0^a f(a-x)dx \right\}$$

Now by using the above formula we get

$$\Rightarrow I = \int_0^{\pi} \log(1+\cos(\pi - x))\, dx$$

Here by allied angle formula we get

$$\Rightarrow I = \int_0^{\pi} \log(1-\cos x)\, dx \ldots\ldots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \{\log(1+\cos x) + \log(1-\cos x)\}\, dx$$

The above equation can be written as

$$2I = \int_0^{\pi} \log\left(1 - \cos^2 x\right)\ dx$$

By using trigonometric identities we get

$$2I = \int_0^{\pi} \log\left(\sin^2 x\right)\ dx$$

$$2I = \int_0^{\pi} 2.\log\left(\sin x\right)\ dx$$

$$2I = 2.\int_0^{\pi} \log\left(\sin x\right)\ dx$$

$$I = \int_0^{\pi} \log\left(\sin x\right)\ dx \ldots..(3)$$

$$\text{because, } \int_0^{2a} f\left(x\right)dx = 2.\int_0^{a} f\left(x\right)dx, \text{if } f\left(2a - x\right) = f\left(x\right)$$

Here, if f (x) = log (sin x) and f (π – x) = log (sin (π – x)) = log (sin x) = f (x)

$$\Rightarrow I = 2.\int_0^{\frac{\pi}{2}} \log \sin x dx \ldots(4)$$

$$\Rightarrow I = 2.\int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right)dx$$

By using trigonometric equation we get

$$\Rightarrow I = 2.\int_0^{\frac{\pi}{2}} \log \cos x dx \ldots..(5)$$

Adding (1) and (2), we get

$$\Rightarrow 2I = 2.\int_0^{\frac{\pi}{2}}(\log\sin x + \log\cos x)dx$$

Now by adding and subtracting log 2 we get

$$\Rightarrow I = \int_0^{\frac{\pi}{2}}(\log\sin x + \log\cos x + \log 2 - \log 2)dx$$

The above equation can be written as

$$\Rightarrow I = \int_0^{\frac{\pi}{2}}(\log(2\sin x\cos x) - \log 2)dx$$

Now by splitting the integral we get

$$\Rightarrow I = \int_0^{\frac{\pi}{2}}(\log(\sin 2x)dx - \int_0^{\frac{\pi}{2}}\log 2dx$$

Let $2x = t \Rightarrow 2dx = dt$

When $x = 0$, $t = 0$ and when $x = \pi/2$, $t = \pi$

$$\Rightarrow I = \left[\frac{1}{2}\int_0^{\pi}(\log(\sin t)dt\right] - \left(\frac{\pi}{2}\log 2\right)$$

$$\Rightarrow I = \left[\frac{I}{2}\right] - \left(\frac{\pi}{2}\log 2\right)$$

$$\Rightarrow I = -\left(\frac{\pi}{2}\log 2\right)$$

$$\Rightarrow I = -(\pi\log 2)$$

17. $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx$

Solution:

Given: $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx$

$$\text{let, } I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx \ldots..(1)$$

As we know that

$$\left\{\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right\}$$

By using the above formula we get

$$\Rightarrow I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\,dx \ldots..(2)$$

Adding (1) and (2), we get

$$2I = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}\,dx$$

The above equation becomes,

$$\Rightarrow 2I = \int_0^a [1]\,dx$$

On integrating we get

$$\Rightarrow 2I = [x]_0^a$$

Now by applying the limits

$\Rightarrow 2I = a - 0$

$\Rightarrow 2I = a$

$\Rightarrow I = \frac{a}{2}$

18. $\int_0^4 |x-1|\, dx$

Solution:

Given: $\int_0^4 |x-1|\, dx$

As we can see that (x-1) ≤ 0 when 0 ≤ x ≤ 1 and (x - 1) ≥ 0 when 1 ≤ x ≤ 4

As we know that

$$\left\{\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\right\}$$

By substituting the above formula we get

$$\Rightarrow I = \int_0^1 -(x-1)dx + \int_1^4 (x-1)dx$$

On integration

$$\Rightarrow I = -\left[\frac{x^2}{2} - x\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^4$$

Now by applying the limit we get

$$\Rightarrow I = -\left[\frac{(1)^2}{2} - 1 - \frac{(0)^2}{2} + 0\right] + \left[\frac{(4)^2}{2} - 4 - \frac{(1)^2}{2} + 1\right]$$

$$\Rightarrow I = -\left[\frac{1}{2} - 1\right] + \left[8 - 4 - \frac{1}{2} + 1\right]$$

$$\Rightarrow I = \frac{1}{2} + 5 - \frac{1}{2}$$

$$\Rightarrow I = 5$$

19. Show that $\int_0^a f(x)g(x)\,dx = 2\int_0^a f(x)\,dx$, if f and g are defined as $f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$

Solution:

Given: $\int_0^a f(x)g(x)\, dx$

let, $I = \int_0^a f(x)g(x)\, dx \ldots..(1)$

As we know that

$$\left\{\int_0^a f(x)dx = \int_0^a f(a-x)dx\right\}$$

By using the above formula we get

$$\Rightarrow I = \int_0^a f(a-x)g(a-x)\, dx$$

$$\Rightarrow I = \int_0^a f(x)g(a-x)\, dx \ldots..(2)$$

Adding (1) and (2), we get

$$2I = \int_0^a \{f(x)g(x) + f(x)g(a-x)\}\ dx$$

$$\Rightarrow 2I = \int_0^a f(x)\{g(x) + g(a-x)\}\ dx$$

$$\Rightarrow 2I = \int_0^a f(x)\{4\}\ dx \text{ as, } \{g(x) + g(a-x) = 4\}$$

$$\Rightarrow I = \frac{1}{2}\int_0^a f(x) \times 4\ dx$$

$$\Rightarrow I = 2.\int_0^a f(x) dx$$

Choose the correct answer in Exercises 20 and 21.

20. The value of $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1)\, dx$ is

(A) 0 (B) 2 (C) π (D) 1

Solution:
(C) π

Explanation:

Given: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1) dx$

$$\text{let, } I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1) dx$$

Now by splitting the integrals we get

$$\Rightarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3)dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x\cos x)dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\tan^5 x)dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1)dx$$

It is also known that if f(x) is an even function then,

$$\left\{\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx\right\}$$

It is also known that if f(x) is an odd function then,

$$\Rightarrow I = 0+0+0+2.\int_{0}^{\frac{\pi}{2}}(1)dx \quad \left\{\int_{-a}^{a} f(x)dx = 0\right\}$$

$$\Rightarrow I = 2.[x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow I = 2.\frac{\pi}{2}$$

$$\Rightarrow I = \pi$$

Correct answer is C

21. The value of $\int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$ is

(A) 2 (B) 3/4 (C) 0 (D) –2

Solution:
(C) 0

Explanation:

Given: $\int_{0}^{2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$

$$\text{let, } I = \int_{0}^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx \ldots..(1)$$

As we know that

$$\left\{\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx\right\}$$

By using the above formula we get

$$\Rightarrow I = \int_{0}^{\frac{\pi}{2}} \log\left(\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)}\right) dx$$

By applying allied angles formulae we get

$$\Rightarrow \ I = \int_{0}^{\frac{\pi}{2}} \log\left(\frac{4+3\cos x}{4+3\sin}\right) dx \ldots..(2)$$

Adding (1) and (2), we get

$$2I = \int_{0}^{\frac{\pi}{2}} \left\{\log\left(\frac{4+3\sin x}{4+3\cos x}\right) + \left(\frac{4+3\cos x}{4+3\sin}\right)\right\} dx$$

$$\Rightarrow 2I = \int_{0}^{\frac{\pi}{2}} \log 1 dx$$

Substituting log 1 = 0 we get

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 0.dx$$

$$\Rightarrow I = 0$$

Correct answer is (c)