EXERCISE 7.4

Integrate the functions in Exercises 1 to 23.

1. $\dfrac{3x^2}{x^6+1}$

Solution:-

Let us assume that $x^3 = t$

Then, $3x^2\,dx = dt$

By applying integrals, we get,

$$\int \frac{3x^2}{x^6+1}dx = \int \frac{dt}{t^2+1}$$

On integrating,

$= \tan^{-1}t + C$

No, Substitute the value of t,

$= \tan^{-1}(x^3) + C$

2.

$\dfrac{1}{\sqrt{1+4x^2}}$

Solution:

Take $2x = t$

We get $2x\,dx = dt$

Integrating both sides

$$\int \frac{1}{\sqrt{1+4x^2}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}}$$

Using the formula

$$\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|$$

We get

$$= \frac{1}{2}\left[\log\left|t+\sqrt{t^2+1}\right|\right] + C$$

Substituting the value of t

$$= \frac{1}{2}\log\left|2x + \sqrt{4x^2+1}\right| + C$$

3.

$$\frac{1}{\sqrt{(2-x)^2+1}}$$

Solution:

Take 2 - x = t

We get - dx = dt

Integrating both sides

$$\int \frac{1}{\sqrt{(2-x)^2+1}}dx = -\int \frac{1}{\sqrt{t^2+1}}dt$$

Using the formula

$$\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x + \sqrt{x^2+a^2}\right|$$

We get

$$= -\log\left|t + \sqrt{t^2+1}\right| + C$$

Substituting the value of t

$$= -\log\left|2 - x + \sqrt{(2-x)^2+1}\right| + C$$

$$= \log\left|\frac{1}{(2-x)+\sqrt{x^2-4x+5}}\right| + C$$

4.

$$\frac{1}{\sqrt{9-25x^2}}$$

Solution:

Take 5x = t

We get 5dx = dt

Integrating both sides

$$\int \frac{1}{\sqrt{9-25x^2}}\,dx = \frac{1}{5}\int \frac{1}{\sqrt{9-t^2}}dt$$

We get

$$= \frac{1}{5}\int \frac{1}{\sqrt{3^2 - t^2}}\, dt$$

On further calculation

$$= \frac{1}{5}\sin^{-1}\left(\frac{t}{3}\right) + C$$

Substituting the value of t

$$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$$

5.

$$\frac{3x}{1+2x^4}$$

Solution:

Take $\sqrt{2}\, x^2 = t$

We get $2\sqrt{2}\, x\, dx = dt$

Integrating both sides

$$\int \frac{3x}{1+2x^4}dx = \frac{3}{2\sqrt{2}}\int \frac{dt}{1+t^2}$$

On further calculation

$$= \frac{3}{2\sqrt{2}}\left[\tan^{-1} t\right] + C$$

Substituting the value of t

$$= \frac{3}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x^2\right) + C$$

6.

$$\frac{x^2}{1-x^6}$$

Solution:

Take $x^3 = t$

We get $3x^2\, dx = dt$

Integrating both sides

$$\int \frac{x^2}{1-x^6}dx = \frac{1}{3}\int \frac{dt}{1-t^2}$$

On further calculation

$$= \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|\right] + C$$

Substituting the value of t

$$= \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C$$

7.
$$\frac{x-1}{\sqrt{x^2-1}}$$
Solution:

By separating the terms

$$\int \frac{x-1}{\sqrt{x^2-1}}\, dx = \int \frac{x}{\sqrt{x^2-1}}\, dx - \int \frac{1}{\sqrt{x^2-1}}\, dx \qquad \ldots(1)$$

Take

$$\int \frac{x}{\sqrt{x^2-1}} dx$$

If $x^2 - 1 = t$ we get $2x\, dx = dt$

$$\int \frac{x}{\sqrt{x^2-1}} dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$

It can be written as

$$= \frac{1}{2}\int t^{-\frac{1}{2}} dt$$

By integration

$$= \frac{1}{2}\left[2t^{\frac{1}{2}}\right]$$

$$= \sqrt{t}$$

Substituting the value of t

$$= \sqrt{x^2-1}$$

Using equation (1) we get

$$\int \frac{x-1}{\sqrt{x^2-1}} dx = \int \frac{x}{\sqrt{x^2-1}} dx - \int \frac{1}{\sqrt{x^2-1}} dx$$

From formula

$$\int \frac{1}{\sqrt{x^2-a^2}} dt = \log\left|x + \sqrt{x^2-a^2}\right|$$

We get

$$= \sqrt{x^2 - 1} - \log\left|x + \sqrt{x^2 - 1}\right| + C$$

8.

$$\frac{x^2}{\sqrt{x^6 + a^6}}$$

Solution:

Take $x^3 = t$

We get $3x^2\,dx = dt$

Integrating both sides

$$\int \frac{x^2}{\sqrt{x^6 + a^6}}dx = \frac{1}{3}\int \frac{dt}{\sqrt{t^2 + \left(a^3\right)^2}}$$

On further calculation

$$= \frac{1}{3}\log\left|t + \sqrt{t^2 + a^6}\right| + C$$

Substituting the value of t

$$= \frac{1}{3}\log\left|x^3 + \sqrt{x^6 + a^6}\right| + C$$

9.

$$\frac{\sec^2 x}{\sqrt{\tan^2 x + 4}}$$

Solution:

Take $\tan x = t$

We get $\sec^2 x\,dx = dt$

Integrating both sides

$$\int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}}dx = \int \frac{dt}{\sqrt{t^2 + 2^2}}$$

On further calculation

$$= \log\left|t + \sqrt{t^2 + 4}\right| + C$$

Substituting the value of t

$$= \log\left|\tan x + \sqrt{\tan^2 x + 4}\right| + C$$

10.
$$\frac{1}{\sqrt{x^2+2x+2}}$$
Solution:

It is given that

$$\int \frac{1}{\sqrt{x^2+2x+2}}dx = \int \frac{1}{\sqrt{(x+1)^2+(1)^2}}dx$$

Take x + 1 = t

We get dx = dt

Integrating both sides

$$\int \frac{1}{\sqrt{x^2+2x+2}}dx = \int \frac{1}{\sqrt{t^2+1}}dt$$

On further calculation

$$= \log\left|t+\sqrt{t^2+1}\right| + C$$

Substituting the value of t

$$= \log\left|(x+1)+\sqrt{(x+1)^2+1}\right| + C$$

So we get

$$= \log\left|(x+1)+\sqrt{x^2+2x+2}\right| + C$$

11.
$$\frac{1}{9x^2+6x+5}$$
Solution:

It is given that

$$\int \frac{1}{9x^2+6x+5}dx = \int \frac{1}{(3x+1)^2+(2)^2}dx$$

Take (3x + 1) = t

We get 3dx = dt

Integrating both sides

$$\int \frac{1}{(3x+1)^2+(2)^2}dx = \frac{1}{3}\int \frac{1}{t^2+2^2}dt$$

On further calculation

$$= \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right] + C$$

Substituting the value of t

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right) + C$$

12.

$$\frac{1}{\sqrt{7-6x-x^2}}$$

Solution:

It is given that

$$\frac{1}{\sqrt{7-6x-x^2}}$$

We can write it as

$7 - 6x - x^2 = 7 - (x^2 + 6x + 9 - 9)$

By further calculation

$= 16 - (x^2 + 6x - 9)$

We get

$= 16 - (x + 3)^2$

$= 4^2 - (x + 3)^2$

Here

$$\int \frac{1}{\sqrt{7-6x-x^2}}dx = \int \frac{1}{\sqrt{(4)^2 - (x+3)^2}}dx$$

Consider x + 3 = t

We get dx = dt

Integrating both sides

$$\int \frac{1}{\sqrt{(4)^2 - (x+3)^2}}dx = \int \frac{1}{\sqrt{(4)^2 - (t)^2}}dt$$

We get

$$= \sin^{-1}\left(\frac{t}{4}\right) + C$$

Substituting the value of t

$$= \sin^{-1}\left(\frac{x+3}{4}\right) + C$$

13.

$$\frac{1}{\sqrt{(x-1)(x-2)}}$$

Solution:

It is given that

$$\frac{1}{\sqrt{(x-1)(x-2)}}$$

We can write it as

$(x-1)(x-2) = x^2 - 3x + 2$

By further calculation

$= x^2 - 3x + 9/4 - 9/4 + 2$

We get

$$= \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$$

$$= \left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

Here

$$\int \frac{1}{\sqrt{(x-1)(x-2)}} dx = \int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx$$

Consider $x - 3/2 = t$

We get $dx = dt$

Integrating both sides

$$\int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx = \int \frac{1}{\sqrt{t^2 - \left(\frac{1}{2}\right)^2}} dt$$

We get

$$= \log\left|t + \sqrt{t^2 - \left(\frac{1}{2}\right)^2}\right| + C$$

Substituting the value of t

$$= \log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+C$$

14.

$$\frac{1}{\sqrt{8+3x-x^2}}$$

Solution:

It is given that

$$\frac{1}{\sqrt{8+3x-x^2}}$$

We can write it as

$8 + 3x - x^2 = 8 - (x^2 - 3x + 9/4 - 9/4)$

By further calculation

$$=\frac{41}{4}-\left(x-\frac{3}{2}\right)^2$$

Here

$$\int\frac{1}{\sqrt{8+3x-x^2}}dx=\int\frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx$$

Consider $x - 3/2 = t$

We get $dx = dt$

Integrating both sides

$$\int\frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx=\int\frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-t^2}}dt$$

We get

$$=\sin^{-1}\left(\frac{t}{\frac{\sqrt{41}}{2}}\right)+C$$

Substituting the value of t

$$= \sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{41}}{2}}\right)+C$$

On further calculation

$$= \sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C$$

15.

$$\frac{1}{\sqrt{(x-a)(x-b)}}$$

Solution:

It is given that

$$\frac{1}{\sqrt{(x-a)(x-b)}}$$

We can write it as

$(x - a)(x - b) = x^2 - (a + b)x + ab$

By further calculation

$$= x^2-(a+b)x+\frac{(a+b)^2}{4}-\frac{(a+b)^2}{4}+ab$$

Here

$$= \left[x-\left(\frac{a+b}{2}\right)\right]^2-\frac{(a-b)^2}{4}$$

Integrating both sides

$$\int\frac{1}{\sqrt{(x-a)(x-b)}}dx=\int\frac{1}{\sqrt{\left\{x-\left(\frac{a+b}{2}\right)\right\}^2-\left(\frac{a-b}{2}\right)^2}}dx$$

Consider

$$x-\left(\frac{a+b}{2}\right)=t$$

We get dx = dt

$$\int \frac{1}{\sqrt{\left\{x-\left(\frac{a+b}{2}\right)\right\}^2-\left(\frac{a-b}{2}\right)^2}}dx = \int \frac{1}{\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}}dt$$

It can be written as

$$= \log\left|t+\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}\right|+C$$

Substituting the value of t

$$= \log\left|\left\{x-\left(\frac{a+b}{2}\right)\right\}+\sqrt{(x-a)(x-b)}\right|+C$$

16.

$$\frac{4x+1}{\sqrt{2x^2+x-3}}$$

Solution:

Consider

$4x + 1 = A\ d/dx\ (2x^2 + x - 3) + B$

So we get

$4x + 1 = A\ (4x + 1) + B$

On further calculation

$4x + 1 = 4\ Ax + A + B$

By equating the coefficients of x and constant term on both sides

$4A = 4$

$A = 1$

$A + B = 1$

$B = 0$

Take $2x^2 + x - 3 = t$

By differentiation

$(4x + 1)\ dx = dt$

Integrating both sides

$$\int \frac{4x+1}{\sqrt{2x^2+x-3}}dx = \int \frac{1}{\sqrt{t}}dt$$

We get

$= 2\sqrt{t} + C$

Substituting the value of t

$= 2\sqrt{2x^2+x-3}+C$

17.

$$\frac{x+2}{\sqrt{x^2-1}}$$

Solution:

Consider

$$x+2 = A\frac{d}{dx}\left(x^2-1\right)+B \qquad ...(1)$$

It can be written as

x + 2 = A (2x) + B

Now equating the coefficients of x and constant term on both sides

2A = 1

A = ½

B = 2

Using equation (1) we get

$$(x+2)=\frac{1}{2}(2x)+2$$

Integrating both sides

$$\int\frac{x+2}{\sqrt{x^2-1}}dx=\int\frac{\frac{1}{2}(2x)+2}{\sqrt{x^2-1}}dx$$

Separating the terms

$$=\frac{1}{2}\int\frac{2x}{\sqrt{x^2-1}}dx+\int\frac{2}{\sqrt{x^2-1}}dx \qquad ...(2)$$

Take

$$\frac{1}{2}\int\frac{2x}{\sqrt{x^2-1}}dx$$

If $x^2 - 1 = t$ we get $2x\,dx = dt$

So we get

$$\frac{1}{2}\int\frac{2x}{\sqrt{x^2-1}}dx=\frac{1}{2}\int\frac{dt}{\sqrt{t}}$$

By integration

$$=\frac{1}{2}\left[2\sqrt{t}\right]$$

$$=\sqrt{t}$$

Substituting the value of t

$$=\sqrt{x^2-1}$$

We can write it as

$$\int\frac{2}{\sqrt{x^2-1}}dx=2\int\frac{1}{\sqrt{x^2-1}}dx=2\log\left|x+\sqrt{x^2-1}\right|$$

Using equation (2) we get

$$\int \frac{x+2}{\sqrt{x^2-1}}dx = \sqrt{x^2-1} + 2\log\left|x+\sqrt{x^2-1}\right| + C$$

18.
$$\frac{5x-2}{1+2x+3x^2}$$
Solution:

Consider

$$5x-2 = A\frac{d}{dx}\left(1+2x+3x^2\right)+B$$

It can be written as

5x - 2 = A (2 + 6x) + B

Now equating the coefficients of x and constant term on both sides

5 = 6A

A = 5/6

2A + B = - 2

B = - 11/3

Using equation (1) we get

$$5x-2 = \frac{5}{6}(2+6x) + \left(-\frac{11}{3}\right)$$

Integrating both sides

$$\int \frac{5x-2}{1+2x+3x^2}dx = \int \frac{\frac{5}{6}(2+6x) - \frac{11}{3}}{1+2x+3x^2}dx$$

Separating the terms

$$= \frac{5}{6}\int \frac{2+6x}{1+2x+3x^2}dx - \frac{11}{3}\int \frac{1}{1+2x+3x^2}dx$$

We know that

$$I_1 = \int \frac{2+6x}{1+2x+3x^2}dx \text{ and } I_2 = \int \frac{1}{1+2x+3x^2}dx$$

$$\int \frac{5x-2}{1+2x+3x^2}dx = \frac{5}{6}I_1 - \frac{11}{3}I_2 \qquad ...(1)$$

Take

$$I_1 = \int \frac{2+6x}{1+2x+3x^2}dx$$

If $1 + 2x + 3x^2 = t$ we get $(2 + 6x)\, dx = dt$

So we get

$$I_1 = \int \frac{dt}{t}$$

By integration

$$I_1 = \log|t|$$

Substituting the value of t

$$I_1 = \log\left|1+2x+3x^2\right| \qquad ...(2)$$

Take

$$I_2 = \int \frac{1}{1+2x+3x^2}dx$$

$1 + 2x + 3x^2 = 1 + 3\,(x^2 + 2/3\, x)$

By addition and subtraction of 1/9

$$=1+3\left(x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}\right)$$

We get

$$=1+3\left(x+\frac{1}{3}\right)^2-\frac{1}{3}$$

On further calculation

$$=\frac{2}{3}+3\left(x+\frac{1}{3}\right)^2$$

Here

$$=3\left[\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right]$$

$$=3\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]$$

By integration

$$I_2 = \frac{1}{3}\int \frac{1}{\left[\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2\right]}dx$$

So we get

$$= \frac{1}{3}\left[\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right)\right]$$

By taking LCM

$$= \frac{1}{3}\left[\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]$$

On further calculation

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) \qquad ...(3)$$

Now substituting the equations (2) and (3) in equation (1)

$$\int \frac{5x-2}{1+2x+3x^2}dx = \frac{5}{6}\left[\log\left|1+2x+3x^2\right|\right] - \frac{11}{3}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right] + C$$

We get

$$= \frac{5}{6}\log\left|1+2x+3x^2\right| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$$

19.

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}}$$

Solution:

It is given that

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}} = \frac{6x+7}{\sqrt{x^2-9x+20}}$$

Consider

$$6x+7 = A\frac{d}{dx}\left(x^2-9x+20\right) + B$$

It can be written as

6x + 7 = A (2x - 9) + B

Now equating the coefficients of x and constant term on both sides

2A = 6

A = 3

-9A + B = 7

B = 34

Using equation (1) we get

6x + 7 = 3 (2x – 9) + 34

Integrating both sides

$$\int \frac{6x+7}{\sqrt{x^2-9x+20}} = \int \frac{3(2x-9)+34}{\sqrt{x^2-9x+20}}dx$$

Separating the terms

$$= 3\int \frac{2x-9}{\sqrt{x^2-9x+20}}dx + 34\int \frac{1}{\sqrt{x^2-9x+20}}dx$$

We know that

$$I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}dx \text{ and } I_2 = \int \frac{1}{\sqrt{x^2-9x+20}}dx$$

$$\int \frac{6x+7}{\sqrt{x^2-9x+20}} = 3I_1 + 34I_2 \qquad ...(1)$$

Take

$$I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}dx$$

If $x^2 – 9x + 20 = t$ we get (2x - 9) dx = dt

So we get

$$I_1 = \frac{dt}{\sqrt{t}}$$

By integration

$$I_1 = 2\sqrt{t}$$

Substituting the value of t

$$I_1 = 2\sqrt{x^2-9x+20} \qquad ...(2)$$

Take

$$I_2 = \int \frac{1}{\sqrt{x^2 - 9x + 20}} dx$$

By addition and subtraction of 81/4

$x^2 - 9x + 20 = x^2 - 9x + 20 + 81/4 - 81/4$

$$= \left(x - \frac{9}{2}\right)^2 - \frac{1}{4}$$

We get

$$= \left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

By integration

$$I_2 = \int \frac{1}{\sqrt{\left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx$$

So we get

$$I_2 = \log\left|\left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20}\right| \qquad ...(3)$$

Now substituting the equations (2) and (3) in equation (1)

$$\int \frac{6x+7}{\sqrt{x^2 - 9x + 20}} dx = 3\left[2\sqrt{x^2 - 9x + 20}\right] + 34\log\left[\left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20}\right] + C$$

We get

$$= 6\sqrt{x^2 - 9x + 20} + 34\log\left[\left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20}\right] + C$$

20.

$$\frac{x+2}{\sqrt{4x - x^2}}$$

Solution:

Consider

$$x + 2 = A\frac{d}{dx}\left(4x - x^2\right) + B$$

It can be written as

x + 2 = A (4 – 2x) + B

Now equating the coefficients of x and constant term on both sides

-2A = 1
A = -1/2
4A + B = 2
B = 4
Using equation (1) we get

$$(x+2) = -\frac{1}{2}(4-2x)+4$$

Integrating both sides

$$\int \frac{x+2}{\sqrt{4x-x^2}}dx = \int \frac{-\frac{1}{2}(4-2x)+4}{\sqrt{4x-x^2}}dx$$

Separating the terms

$$= -\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}dx + 4\int \frac{1}{\sqrt{4x-x^2}}dx$$

We know that

$$I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx \text{ and } I_2 \int \frac{1}{\sqrt{4x-x^2}}dx$$

$$\int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}I_1 + 4I_2 \quad ...(1)$$

Take

$$I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx$$

If $4x - x^2 = t$ we get $(4 – 2x)\, dx = dt$

So we get

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t}$$

Substituting the value of t

$$= 2\sqrt{4x-x^2} \;\ldots\ldots (2)$$

Take

$$I_2 = \int \frac{1}{\sqrt{4x-x^2}}dx$$

$4x - x^2 = - (-4x + x^2)$

By addition and subtraction of 4

$4x - x^2 = (-4x + x^2 + 4 - 4)$

It can be written as

$= 4 - (x - 2)^2$

$= (2)^2 - (x - 2)^2$

By integration

$$I_2 = \int \frac{1}{\sqrt{(2)^2 - (x-2)^2}} dx$$

So we get

$$= \sin^{-1}\left(\frac{x-2}{2}\right) \qquad ...(3)$$

Now substituting the equations (2) and (3) in equation (1)

$$\int \frac{x+2}{\sqrt{4x-x^2}} dx = -\frac{1}{2}\left(2\sqrt{4x-x^2}\right) + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

We get

$$= -\sqrt{4x-x^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

21.

$$\frac{(x+2)}{\sqrt{x^2+2x+3}}$$

Solution:

It is given that

$$\int \frac{(x+2)}{\sqrt{x^2+2x+3}} dx$$

By multiplying and dividing by 2

$$= \frac{1}{2}\int \frac{2(x+2)}{\sqrt{x^2+2x+3}} dx$$

Multiplying the terms

$$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+2x+3}} dx$$

Separating the terms

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}} dx + \frac{1}{2}\int \frac{2}{\sqrt{x^2+2x+3}} dx$$

We get

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

We know that

$$I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}\, dx \text{ and } I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}I_1 + I_2 \qquad ...(1)$$

Take

$$I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}\, dx$$

Here $x^2 + 2x + 3 = t$

We get $(2x + 2)\, dx = dt$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t}$$

Substituting the value of t

$$= 2\sqrt{x^2+2x+3} \qquad ...(2)$$

Take

$$I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

We can write it as

$x^2 + 2x + 3 = x^2 + 2x + 1 + 2$

$$= (x+1)^2 + \left(\sqrt{2}\right)^2$$

So we get

$$I_2 = \int \frac{1}{\sqrt{(x+1)^2 + \left(\sqrt{2}\right)^2}}\, dx$$

By integration

$$= \log\left|(x+1) + \sqrt{x^2+2x+3}\right| \qquad ...(3)$$

By using equations (2) and (3) in (1) we get

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\left[2\sqrt{x^2+2x+3}\right] + \log\left|(x+1)+\sqrt{x^2+2x+3}\right| + C$$

So we get

$$= \sqrt{x^2+2x+3} + \log\left|(x+1)+\sqrt{x^2+2x+3}\right| + C$$

22.

$\frac{x+3}{x^2-2x-5}$

Solution:

Consider

$$(x+3) = A\frac{d}{dx}(x^2-2x-5)+B$$

It can be written as

x + 3 = A (2x – 2) + B

Now equating the coefficients of x and constant term on both sides

2A = 1

A = 1/2

-2A + B = 3

B = 4

Using equation (1) we get

$$(x+3) = \frac{1}{2}(2x-2)+4$$

Integrating both sides

$$\int \frac{x+3}{x^2-2x-5}dx = \int \frac{\frac{1}{2}(2x-2)+4}{x^2-2x-5}dx$$

Separating the terms

$$= \frac{1}{2}\int \frac{2x-2}{x^2-2x-5}dx + 4\int \frac{1}{x^2-2x-5}dx$$

We know that

$$I_1 = \int \frac{2x-2}{x^2-2x-5}dx \text{ and } I_2 = \int \frac{1}{x^2-2x-5}dx$$

$$\int \frac{x+3}{(x^2-2x-5)}dx = \frac{1}{2}I_1 + 4I_2 \qquad ...(1)$$

Take

$$I_1 = \int \frac{2x-2}{x^2-2x-5}dx$$

If $x^2 - 2x - 5 = t$ we get $(2x - 2)\, dx = dt$

So we get

$$I_1 = \int \frac{dt}{t} = \log|t|$$

Substituting the value of t

$$= \log|x^2-2x-5| \quad \ldots\ldots (2)$$

Take

$$I_2 = \int \frac{1}{x^2-2x-5}dx$$

We can write it as

$$= \int \frac{1}{(x^2-2x+1)-6}dx$$

By separating the terms

$$= \int \frac{1}{(x-1)^2-(\sqrt{6})^2}dx$$

By integration

$$= \frac{1}{2\sqrt{6}}\log\left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right) \quad \ldots(3)$$

Now substituting the equations (2) and (3) in equation (1)

$$\int \frac{x+3}{x^2-2x-5}dx = \frac{1}{2}\log|x^2-2x-5| + \frac{4}{2\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

We get

$$= \frac{1}{2}\log|x^2-2x-5| + \frac{2}{\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

23.

$$\frac{5x+3}{\sqrt{x^2+4x+10}}$$

Solution:

Consider

$$5x+3 = A\frac{d}{dx}\left(x^2+4x+10\right)+B$$

It can be written as

5x + 3 = A (2x + 4) + B

Now equating the coefficients of x and constant term on both sides

2A = 5

A = 5/2

4A + B = 3

B = -7

Using equation (1) we get

$$5x+3 = \frac{5}{2}\left(2x+4\right)-7$$

Integrating both sides

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \int \frac{\frac{5}{2}\left(2x+4\right)-7}{\sqrt{x^2+4x+10}}dx$$

Separating the terms

$$= \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}dx - 7\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

We know that

$$I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}dx \text{ and } I_2 = \int \frac{1}{\sqrt{x^2+4x+10}}dx$$

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}I_1 - 7I_2 \qquad \text{...(1)}$$

Take

$$I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}dx$$

If $x^2 + 4x + 10 = t$ we get $(2x + 4)\, dx = dt$

So we get

$$I_1 = \int \frac{dt}{t} = 2\sqrt{t}$$

Substituting the value of t

$$= 2\sqrt{x^2+4x+10} \quad \text{..... (2)}$$

Take

$$I_2 = \int \frac{1}{\sqrt{x^2+4x+10}}dx$$

We can write it as

$$= \int \frac{1}{\sqrt{(x^2+4x+4)+6}}dx$$

By separating the terms

$$= \int \frac{1}{(x+2)^2+(\sqrt{6})^2}dx$$

By integration

$$= \log\left|x+2+\sqrt{x^2+4x+10}\right| \quad ...(3)$$

Now substituting the equations (2) and (3) in equation (1)

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}\left[2\sqrt{x^2+4x+10}\right] - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$

We get

$$= 5\sqrt{x^2+4x+10} - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$

Choose the correct answer in Exercises 24 and 25.

24.

$\int \frac{dx}{x^2+2x+2}$ equals

(A) $x \tan^{-1}(x+1) + C$ **(B)** $\tan^{-1}(x+1) + C$

(C) $(x+1)\tan^{-1} x + C$ **(D)** $\tan^{-1} x + C$

Solution:

It is given that

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x^2+2x+1)+1}$$

By separating the terms

$$= \int \frac{1}{(x+1)^2+(1)^2}dx$$

By integrating we get

$$= \left[\tan^{-1}(x+1)\right] + C$$

Therefore, B is the correct answer.

25.

$\int \frac{dx}{\sqrt{9x-4x^2}}$ equals

(A) $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$ (B) $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$

(C) $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$ (D) $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{9}\right)+C$

Solution:

It is given that

$$\int \frac{dx}{\sqrt{9x-4x^2}}$$

We can write it as

$$= \int \frac{1}{\sqrt{-4\left(x^2-\frac{9}{4}x\right)}}dx$$

By further calculation we get

$$= \int \frac{1}{\sqrt{-4\left(x^2-\frac{9x}{4}+\frac{81}{64}-\frac{81}{64}\right)}}dx$$

Separating the terms we get

$$= \int \frac{1}{\sqrt{-4\left[\left(x-\frac{9}{8}\right)^2-\left(\frac{9}{8}\right)^2\right]}}dx$$

On further simplification

$$= \frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{9}{8}\right)^2-\left(x-\frac{9}{8}\right)^2}}dx$$

Using the formula

$$\int \frac{dy}{\sqrt{a^2-y^2}} = \sin^{-1}\frac{y}{a}+C$$

$$= \frac{1}{2}\left[\sin^{-1}\left(\frac{x-\frac{9}{8}}{\frac{9}{8}}\right)\right]+C$$

Taking LCM

$$=\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$$

Therefore, B is the correct answer.