EXERCISE 7.5

Integrate the rational functions in Exercises 1 to 21.

1.

$$\frac{x}{(x+1)(x+2)}$$

Solution:

Consider

$$\frac{x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$$

We get

x = A (x + 2) + B (x + 1)

Now by equating the coefficients of x and constant term, we get

A + B = 1

2A + B = 0

By solving the equations we get

A = - 1 and B = 2

Substituting the values of A and B

$$\frac{x}{(x+1)(x+2)} = \frac{-1}{(x+1)} + \frac{2}{(x+2)}$$

By integrating both sides w.r.t x

$$\int \frac{x}{(x+1)(x+2)} dx = \int \frac{-1}{(x+1)} + \frac{2}{(x+2)} dx$$

So we get

= - log |x + 1| + 2 log |x + 2| + c

We can write it as

= log $(x + 2)^2$ – log |x + 1| + c

$$= \log \frac{(x+2)^2}{(x+1)} + C$$

2.

$$\frac{1}{(x+3)(x-3)}$$

Solution:

Consider

$$\frac{1}{(x+3)(x-3)}=\frac{A}{(x+3)}+\frac{B}{(x-3)}$$

We get

1 = A (x - 3) + B (x + 3)

Now by equating the coefficients of x and constant term, we get

A + B = 1

-3A + 3B = 0

By solving the equations we get

A = - 1/6 and B = 1/6

Substituting the values of A and B

$$\frac{1}{(x+3)(x-3)}=\frac{-1}{6(x+3)}+\frac{1}{6(x-3)}$$

By integrating both sides w.r.t x

$$\int\frac{1}{(x^2-9)}dx=\int\left(\frac{-1}{6(x+3)}+\frac{1}{6(x-3)}\right)dx$$

So we get

$$=-\frac{1}{6}\log|x+3|+\frac{1}{6}\log|x-3|+C$$

We can write it as

$$=\frac{1}{6}\log\left|\frac{(x-3)}{(x+3)}\right|+C$$

3.

$$\frac{3x-1}{(x-1)(x-2)(x-3)}$$

Solution:

Consider

$$\frac{3x-1}{(x-1)(x-2)(x-3)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}+\frac{C}{(x-3)}$$

We get

3x - 1 = A (x - 2) (x – 3) + B (x - 1) (x – 3) + C (x – 1) (x – 2) ….. (1)

By substituting the value of x in equation (1), we get

A = 1, B = -5 and C = 4

Substituting the values of A, B and C

$$\frac{3x-1}{(x-1)(x-2)(x-3)}=\frac{1}{(x-1)}-\frac{5}{(x-2)}+\frac{4}{(x-3)}$$

By integrating both sides w.r.t x

$$\int\frac{3x-1}{(x-1)(x-2)(x-3)}dx=\int\left\{\frac{1}{(x-1)}-\frac{5}{(x-2)}+\frac{4}{(x-3)}\right\}dx$$

So we get

= log |x – 1| - 5 log |x – 2| + 4 log |x – 3| + c

4.

$$\frac{x}{(x-1)(x-2)(x-3)}$$

Solution:

Consider

$$\frac{x}{(x-1)(x-2)(x-3)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}+\frac{C}{(x-3)}$$

We get

x = A (x - 2) (x – 3) + B (x - 1) (x – 3) + C (x – 1) (x – 2) ….. (1)

By substituting the value of x in equation (1), we get

A = 1/2, B = -2 and C = 3/2

Substituting the values of A, B and C

$$\frac{x}{(x-1)(x-2)(x-3)}=\frac{1}{2(x-1)}-\frac{2}{(x-2)}+\frac{3}{2(x-3)}$$

By integrating both sides w.r.t x

$$\int\frac{x}{(x-1)(x-2)(x-3)}dx=\int\left\{\frac{1}{2(x-1)}-\frac{2}{(x-2)}+\frac{3}{2(x-3)}\right\}dx$$

So we get

= 1/2 log |x – 1| - 2 log |x – 2| + 3/2 log |x – 3| + c

5.

$$\frac{2x}{x^2+3x+2}$$

Solution:

Consider

$$\frac{2x}{x^2+3x+2}=\frac{A}{(x+1)}+\frac{B}{(x+2)}$$

We get

2x = A (x + 2) + B (x + 1) ….. (1)

By substituting the value of x in equation (1), we get

A = - 2 and B = 4

Substituting the values of A and B

$$\frac{2x}{(x+1)(x+2)}=\frac{-2}{(x+1)}+\frac{4}{(x+2)}$$

By integrating both sides w.r.t x

$$\int\frac{2x}{(x+1)(x+2)}dx=\int\left\{\frac{4}{(x+2)}-\frac{2}{(x+1)}\right\}dx$$

So we get

= 4 log |x + 2| - 2 log |x + 1| + c

6.

$$\frac{1-x^2}{x(1-2x)}$$

Solution:

Consider

$$\frac{1-x^2}{x(1-2x)}=\frac{1}{2}+\frac{1}{2}\left(\frac{2-x}{x(1-2x)}\right)$$

We know that

$$\frac{2-x}{x(1-2x)}=\frac{A}{x}+\frac{B}{(1-2x)}$$

We get

(2 – x) = A (1 – 2x) + Bx ….. (1)

By substituting the value of x in equation (1), we get

A = 2 and B = 3

Substituting the values of A and B

$$\frac{2-x}{x(1-2x)}=\frac{2}{x}+\frac{3}{1-2x}$$

We get

$$\frac{1-x^2}{x(1-2x)}=\frac{1}{2}+\frac{1}{2}\left\{\frac{2}{x}+\frac{3}{(1-2x)}\right\}$$
By integrating both sides w.r.t x
$$\int\frac{1-x^2}{x(1-2x)}dx=\int\left\{\frac{1}{2}+\frac{1}{2}\left(\frac{2}{x}+\frac{3}{1-2x}\right)\right\}dx$$
By further calculation
$$=\frac{x}{2}+\log|x|+\frac{3}{2(-2)}\log|1-2x|+C$$
So we get
$$=\frac{x}{2}+\log|x|-\frac{3}{4}\log|1-2x|+C$$

7.
$$\frac{x}{(x^2+1)(x-1)}$$
Solution:

We know that
$$\frac{x}{(x^2+1)(x-1)}=\frac{Ax+B}{(x^2+1)}+\frac{C}{(x-1)}$$
It can be written as
$x = (Ax + B)(x – 1) + C(x^2 + 1)$
By multiplying the terms
$x = Ax^2 – Ax + Bx – B + Cx^2 + C$

Now by equating the coefficients of x^2, x and constant terms we get
$A + C = 0$
$-A + B = 1$
$-B + C = 0$

By solving the equations
$A = -½$, $B = ½$ and $C = ½$
Using equation (1)
$$\frac{x}{(x^2+1)(x-1)}=\frac{\left(-\frac{1}{2}x+\frac{1}{2}\right)}{x^2+1}+\frac{\frac{1}{2}}{(x-1)}$$

By integrating both sides w.r.t. x
$$\int\frac{x}{(x^2+1)(x-1)}=-\frac{1}{2}\int\frac{x}{x^2+1}dx+\frac{1}{2}\int\frac{1}{x^2+1}dx+\frac{1}{2}\int\frac{1}{x-1}dx$$

We get
$$=-\frac{1}{4}\int\frac{2x}{x^2+1}dx+\frac{1}{2}\tan^{-1}x+\frac{1}{2}\log|x-1|+C$$

Here

$\int \frac{2x}{x^2+1}dx$, let $\left(x^2+1\right)=t$

We get

2x dx = dt

Substituting the values

$\int \frac{2x}{x^2+1}dx = \int \frac{dt}{t}$

By integrating w.r.t t

= log |t|

Substituting the value of t

= log $|x^2 + 1|$

So we get

$\int \frac{x}{\left(x^2+1\right)\left(x-1\right)} = -\frac{1}{4}\log\left|x^2+1\right| + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log\left|x-1\right| + C$

We can write it as

$= \frac{1}{2}\log\left|x-1\right| - \frac{1}{4}\log\left|x^2+1\right| + \frac{1}{2}\tan^{-1}x + C$

8.

$\frac{x}{\left(x-1\right)^2\left(x+2\right)}$

Solution:

We know that

$\frac{x}{\left(x-1\right)^2\left(x+2\right)} = \frac{A}{\left(x-1\right)} + \frac{B}{\left(x-1\right)^2} + \frac{C}{\left(x+2\right)}$

It can be written as

$x = A (x – 1) (x + 2) + B (x + 2) + C (x – 1)^2$

Taking x = 1 we get

B = 1/3

Now by equating the coefficients of x^2 and constant terms we get

A + C = 0

-2A + 2B + C = 0

By solving the equations

A = 2/9 and C = -2/9

We get

$\frac{x}{\left(x-1\right)^2\left(x+2\right)} = \frac{2}{9\left(x-1\right)} + \frac{1}{3\left(x-1\right)^2} - \frac{2}{9\left(x+2\right)}$

By integrating both sides w.r.t. x

$\int \frac{x}{\left(x-1\right)^2\left(x+2\right)}dx = \frac{2}{9}\int \frac{1}{\left(x-1\right)}dx + \frac{1}{3}\int \frac{1}{\left(x-1\right)^2}dx - \frac{2}{9}\int \frac{1}{\left(x+2\right)}dx$

Here

$$= \frac{2}{9}\log|x-1| + \frac{1}{3}\left(\frac{-1}{x-1}\right) - \frac{2}{9}\log|x+2| + C$$

By further calculation

$$= \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + C$$

9.

$$\frac{3x+5}{x^3-x^2-x+1}$$

Solution:

It is given that

$$\frac{3x+5}{x^3-x^2-x+1} = \frac{3x+5}{(x-1)^2(x+1)}$$

We know that

$$\frac{3x+5}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$$

It can be written as

$3x + 5 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$

We get

$3x + 5 = A(x^2-1) + B(x+1) + C(x^2+1-2x)$ ….. (1)

By substituting the value of $x = 1$ in equation (1)

$B = 4$

Now by equating the coefficients of x^2 and x we get

$A + C = 0$

$B - 2C = 3$

By solving the equations

$A = -1/2$ and $C = 1/2$

We get

$$\frac{3x+5}{(x-1)^2(x+1)} = \frac{-1}{2(x-1)} + \frac{4}{(x-1)^2} + \frac{1}{2(x+1)}$$

By integrating both sides w.r.t. x

$$\int\frac{3x+5}{(x-1)^2(x+1)}dx = -\frac{1}{2}\int\frac{1}{x-1}dx + 4\int\frac{1}{(x-1)^2}dx + \frac{1}{2}\int\frac{1}{(x+1)}dx$$

Here

$$= -\frac{1}{2}\log|x-1| + 4\left(\frac{-1}{x-1}\right) + \frac{1}{2}\log|x+1| + C$$

By further calculation

$$= \frac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{4}{(x-1)} + C$$

10.
$$\frac{2x-3}{(x^2-1)(2x+3)}$$
Solution:

It is given that
$$\frac{2x-3}{(x^2-1)(2x+3)} = \frac{2x-3}{(x+1)(x-1)(2x+3)}$$
We know that
$$\frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(2x+3)}$$
It can be written as
(2x – 3) = A (x – 1) (2x + 3) + B (x + 1) (2x + 3) + C (x + 1) (x - 1)
(2x – 3) = A ($2x^2$ + x – 3) + B ($2x^2$ + 5x + 3) + C (x^2 – 1)
We get
(2x – 3) = (2A + 2B + C) x^2 + (A + 5B) x + (-3A + 3B – C) ….. (1)

Now by equating the coefficients of x^2 and x we get
B = -1/10, A = 5/2 and C = -24/5
We get
$$\frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{5}{2(x+1)} - \frac{1}{10(x-1)} - \frac{24}{5(2x+3)}$$

By integrating both sides w.r.t. x
$$\int \frac{2x-3}{(x^2-1)(2x+3)}dx = \frac{5}{2}\int \frac{1}{(x+1)}dx - \frac{1}{10}\int \frac{1}{x-1}dx - \frac{24}{5}\int \frac{1}{(2x+3)}dx$$

Here
$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{24}{5\times 2}\log|2x+3|$$
By further calculation
$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{12}{5}\log|2x+3| + C$$

11.
$$\frac{5x}{(x+1)(x^2-4)}$$
Solution:

It is given that
$$\frac{5x}{(x+1)(x^2-4)} = \frac{5x}{(x+1)(x+2)(x-2)}$$
We know that
$$\frac{5x}{(x+1)(x+2)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x-2)}$$

It can be written as
5x = A (x + 2) (x – 2) + B (x + 1) (x – 2) + C (x + 1) (x + 2) ….. (1)

By substituting x = - 1, - 2 and 2 in equation (1)
A = 5/3, B = - 5/2 and C = 5/6

We get

$$\frac{5x}{(x+1)(x+2)(x-2)} = \frac{5}{3(x+1)} - \frac{5}{2(x+2)} + \frac{5}{6(x-2)}$$

By integrating both sides w.r.t. x

$$\int \frac{5x}{(x+1)(x^2-4)}dx = \frac{5}{3}\int \frac{1}{(x+1)}dx - \frac{5}{2}\int \frac{1}{(x+2)}dx + \frac{5}{6}\int \frac{1}{(x-2)}dx$$

By further calculation

$$= \frac{5}{3}\log|x+1| - \frac{5}{2}\log|x+2| + \frac{5}{6}\log|x-2| + C$$

12.

$$\frac{x^3+x+1}{x^2-1}$$

Solution:

It is given that

$$\frac{x^3+x+1}{x^2-1} = x + \frac{2x+1}{x^2-1}$$

We know that

$$\frac{2x+1}{x^2-1} = \frac{A}{(x+1)} + \frac{B}{(x-1)}$$

It can be written as
2x + 1 = A (x - 1) + B (x + 1) ….. (1)

By substituting x = 1 and -1 in equation (1)
A = 1/2 and B = 3/2

We get

$$\frac{x^3+x+1}{x^2-1} = x + \frac{1}{2(x+1)} + \frac{3}{2(x-1)}$$

By integrating both sides w.r.t. x

$$\int \frac{x^3+x+1}{x^2-1}dx = \int x\,dx + \frac{1}{2}\int \frac{1}{(x+1)}dx + \frac{3}{2}\int \frac{1}{(x-1)}dx$$

By further calculation

$$= \frac{x^2}{2} + \frac{1}{2}\log|x+1| + \frac{3}{2}\log|x-1| + C$$

13.

$$\frac{2}{(1-x)(1+x^2)}$$

Solution:

We know that

$$\frac{2}{(1-x)(1+x^2)} = \frac{A}{(1-x)} + \frac{Bx+C}{(1+x^2)}$$

It can be written as

$2 = A(1 + x^2) + (Bx + C)(1 - x)$

$2 = A + Ax^2 + Bx - Bx^2 + C - Cx$ ….. (1)

Now by equating the coefficient of x^2, x and constant terms

$A - B = 0$

$B - C = 0$

$A + C = 2$

Solving the equations

$A = 1$, $B = 1$ and $C = 1$

We get

$$\frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$$

By integrating both sides w.r.t. x

$$\int \frac{2}{(1-x)(1+x^2)} dx = \int \frac{1}{1-x} dx + \int \frac{x}{1+x^2} dx + \int \frac{1}{1+x^2} dx$$

Multiplying and dividing by 2 in the second term

$$= -\int \frac{1}{x-1} dx + \frac{1}{2} \int \frac{2x}{1+x^2} dx + \int \frac{1}{1+x^2} dx$$

By further calculation

$$= -\log|x-1| + \frac{1}{2}\log|1+x^2| + \tan^{-1} x + C$$

14.

$$\frac{3x-1}{(x+2)^2}$$

Solution:

We know that

$$\frac{3x-1}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$$

It can be written as

$3x - 1 = A(x + 2) + B$ ….. (1)

Now by equating the coefficient of x and constant terms
A = 3
2A + B = - 1

Solving the equations
B = - 7

We get
$$\frac{3x-1}{(x+2)^2}=\frac{3}{(x+2)}-\frac{7}{(x+2)^2}$$

By integrating both sides w.r.t. x
$$\int\frac{3x-1}{(x+2)^2}dx=3\int\frac{1}{(x+2)}dx-7\int\frac{x}{(x+2)^2}dx$$

So we get
$$=3\log|x+2|-7\left(\frac{-1}{(x+2)}\right)+C$$

By further calculation
$$=3\log|x+2|+\frac{7}{(x+2)}+C$$

15.
$$\frac{1}{(x^4-1)}$$
Solution:

It is given that
$$\frac{1}{(x^4-1)}=\frac{1}{(x^2-1)(x^2+1)}=\frac{1}{(x+1)(x-1)(1+x^2)}$$
We know that
$$\frac{1}{(x+1)(x-1)(1+x^2)}=\frac{A}{(x+1)}+\frac{B}{(x-1)}+\frac{Cx+D}{(x^2+1)}$$
So we get
$1 = A (x - 1) (x^2 + 1) + B (x + 1) (x^2 + 1) + (Cx + D) (x^2 - 1)$
By multiplying the terms
$1 = A (x^3 + x - x^2 - 1) + B (x^3 + x + x^2 + 1) + Cx^3 + Dx^2 - Cx - D$
It can be written as
$1 = (A + B + C) x^3 + (-A + B + D) x^2 + (A + B - C) x + (-A + B - D)$ ….. (1)

Now by equating the coefficient of x^3, x^2, x and constant terms
A + B + C = 0
-A + B + D = 0
A + B - C = 0
- A + B - D = 1

Solving the equations
A = -1/4, B = 1/4, C = 0 and D = - 1/2

We get

$$\frac{1}{x^4-1}=\frac{-1}{4(x+1)}+\frac{1}{4(x-1)}-\frac{1}{2(x^2+1)}$$

By integrating both sides w.r.t. x

$$\int\frac{1}{x^4-1}dx=-\frac{1}{4}\log|x+1|+\frac{1}{4}\log|x-1|-\frac{1}{2}\tan^{-1}x+C$$

So we get

$$=\frac{1}{4}\log\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\tan^{-1}x+C$$

16.

$$\frac{1}{x(x^n+1)}$$

Solution:

By multiplying both numerator and denominator by x^{n-1}

$$\frac{1}{x(x^n+1)}=\frac{x^{n-1}}{x^{n-1}x(x^n+1)}=\frac{x^{n-1}}{x^n(x^n+1)}$$

Here $x^n = t$ we get
$nx^{n-1}\,dx = dt$
So we get

$$\int\frac{1}{x(x^n+1)}dx=\int\frac{x^{n-1}}{x^n(x^n+1)}dx=\frac{1}{n}\int\frac{1}{t(t+1)}dt$$

We know that

$$\frac{1}{t(t+1)}=\frac{A}{t}+\frac{B}{(t+1)}$$

It can be written as
1 = A (1 + t) + Bt ….. (1)

By substituting t = 0, -1 in equation (1)
A = 1 and B = -1

We get

$$\frac{1}{t(t+1)}=\frac{1}{t}-\frac{1}{(1+t)}$$

By integrating both sides w.r.t. x

$$\int \frac{1}{x(x^n+1)}dx = \frac{1}{n}\int\left\{\frac{1}{t} - \frac{1}{(t+1)}\right\}dx$$

So we get

$$= \frac{1}{n}\left[\log|t| - \log|t+1|\right] + C$$

Substituting the value of t

$$= -\frac{1}{n}\left[\log|x^n| - \log|x^n+1|\right] + C$$

It can be written as

$$= \frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right| + C$$

17.

$$\frac{\cos x}{(1-\sin x)(2-\sin x)}$$

Solution:

It is given that

$$\frac{\cos x}{(1-\sin x)(2-\sin x)}$$

Consider

sin x = t

By differentiating w.r.t t

cos x dx = dt

Integrating w.r.t x

$$\int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx = \int \frac{dt}{(1-t)(2-t)}$$

Here we can write it as

$$\frac{1}{(1-t)(2-t)} = \frac{A}{(1-t)} + \frac{B}{(2-t)}$$

We get

1 = A (2 – t) + B (1 – t) …….. (1)

By substituting t = 2 and t = 1 in equation (1)

A = 1 and B = - 1

$$\frac{1}{(1-t)(2-t)} = \frac{1}{(1-t)} - \frac{1}{(2-t)}$$

Integrating w.r.t t

$$\int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx = \int\left\{\frac{1}{1-t} - \frac{1}{(2-t)}\right\}dt$$

So we get

= - log |1 – t| + log |2 – t| + C

It can be written as

$$= \log\left|\frac{2-t}{1-t}\right| + C$$

Substituting the value of t

$$= \log\left|\frac{2-\sin x}{1-\sin x}\right| + C$$

18.

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$$

Solution:

We know that

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 - \frac{(4x^2+10)}{(x^2+3)(x^2+4)}$$

It can be written as

$$\frac{4x^2+10}{(x^2+3)(x^2+4)} = \frac{Ax+B}{(x^2+3)} + \frac{Cx+D}{(x^2+4)}$$

So we get

$4x^2 + 10 = (Ax + B)(x^2 + 4) + (Cx + D)(x^2 + 3)$

Multiplying the terms

$4x^2 + 10 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + 3Cx + Dx^2 + 3D$

Grouping the terms

$4x^2 + 10 = (A + C) x^3 + (B + D) x^2 + (4A + 3C) x + (4B + 3D)$

Now by equating the coefficients of x^3, x^2, x and constant terms

A + C = 0

B + D = 4

4A + 3C = 0

4B + 3D = 10

By solving these equations

A = 0, B = - 2, C = 0 and D = 6

Substituting the values

$$\frac{4x^2+10}{(x^2+3)(x^2+4)} = \frac{-2}{(x^2+3)} + \frac{6}{(x^2+4)}$$

We can write it as

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 - \left(\frac{-2}{(x^2+3)} + \frac{6}{(x^2+4)}\right)$$

Integrating both sides w.r.t x

$$\int\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}dx = \int\left\{1+\frac{2}{(x^2+3)}-\frac{6}{(x^2+4)}\right\}dx$$

So we get

$$= \int\left\{1+\frac{2}{x^2+(\sqrt{3})^2}-\frac{6}{x^2+2^2}\right\}$$

Here

$$= x+2\left(\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right)-6\left(\frac{1}{2}\tan^{-1}\frac{x}{2}\right)+C$$

By further calculation

$$= x+\frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}-3\tan^{-1}\frac{x}{2}+C$$

19.

$$\frac{2x}{(x^2+1)(x^2+3)}$$

Solution:

It is given that

$$\frac{2x}{(x^2+1)(x^2+3)}$$

Consider $x^2 = t$

So we get

2x dx = dt

Integrating both sides

$$\int\frac{2x}{(x^2+1)(x^2+3)}dx = \int\frac{dt}{(t+1)(t+3)}$$

We can write it as

$$\frac{1}{(t+1)(t+3)}=\frac{A}{(t+1)}+\frac{B}{(t+3)}$$

1 = A (t + 3) + B (t + 1) ….. (1)

Now by substituting t = - 3 and t = - 1 in equation (1)

A = 1/2 and B = - 1/2

Substituting the values

$$\frac{1}{(t+1)(t+3)}=\frac{1}{2(t+1)}-\frac{1}{2(t+3)}$$

Integrating w.r.t t

$$\int \frac{2x}{(x^2+1)(x^2+3)}dx = \int \left\{ \frac{1}{2(t+1)} - \frac{1}{2(t+3)} \right\} dt$$

So we get

$$= \frac{1}{2}\log|(t+1)| - \frac{1}{2}\log|t+3| + C$$

It can be written as

$$= \frac{1}{2}\log\left|\frac{t+1}{t+3}\right| + C$$

Substituting the value of t

$$= \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + C$$

20.

$$\frac{1}{x(x^4-1)}$$

Solution:

It is given that

$$\frac{1}{x(x^4-1)}$$

By multiplying both numerator and denominator by x^3

$$\frac{1}{x(x^4-1)} = \frac{x^3}{x^4(x^4-1)}$$

Integrating both sides

$$\int \frac{1}{x(x^4-1)}dx = \int \frac{x^3}{x^4(x^4-1)}dx$$

Consider $x^4 = t$

So we get $4x^3\, dx = dt$

We can write it as

$$\int \frac{1}{x(x^4-1)}dx = \frac{1}{4}\int \frac{dt}{t(t-1)}$$

So we get

$$\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{(t-1)}$$

$1 = A(t-1) + Bt$ ….. (1)

Now by substituting t = 0 in equation (1)

A = -1 and B = 1

Substituting the values

$$\frac{1}{t(t+1)}=\frac{-1}{t}+\frac{1}{t-1}$$

Integrating w.r.t t

$$\int\frac{1}{x(x^4-1)}dx=\frac{1}{4}\int\left\{\frac{-1}{t}+\frac{1}{t-1}\right\}dt$$

So we get

$$=\frac{1}{4}\left[-\log|t|+\log|t-1|\right]+C$$

It can be written as

$$=\frac{1}{4}\log\left|\frac{t-1}{t}\right|+C$$

Substituting the value of t

$$=\frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right|+C$$

21.

$$\frac{1}{(e^x-1)}$$

Solution:

It is given that

$$\frac{1}{(e^x-1)}$$

Consider $e^x = t$

So we get $e^x\,dx = dt$

We can write it as

$$\int\frac{1}{e^x-1}dx=\int\frac{1}{t-1}\times\frac{dt}{t}=\int\frac{1}{t(t-1)}dt$$

So we get

$$\frac{1}{t(t-1)}=\frac{A}{t}+\frac{B}{t-1}$$

1 = A (t - 1) + Bt ….. (1)

Now by substituting t = 1 and t = 0 in equation (1)

A = -1 and B = 1

Substituting the values

$$\frac{1}{t(t+1)}=\frac{-1}{t}+\frac{1}{t-1}$$

Integrating w.r.t t

$$\int \frac{1}{t(t-1)} dt = \log\left|\frac{t-1}{t}\right| + C$$

Substituting the value of t

$$= \log\left|\frac{e^x - 1}{e^x}\right| + C$$

Choose the correct answer in each of the Exercises 22 and 23.

22. $\int \frac{x dx}{(x-1)(x-2)}$ *equals*

(A) $log|\frac{(x-1)^2}{x-2}| + C$

(B) $log|\frac{(x-2)^2}{x-1}| + C$

(C) $log|(\frac{x-1}{x-2})^2| + C$

(D) $log|(x-1)(x-2)| + C$

Solution:

We know that

$$\frac{x}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$$

It can be written as

x = A (x – 2) + B (x – 1) …… (1)

Now by substituting x = 1 and 2 in equation (1)

A = - 1 and B = 2

Substituting the value of A and B

$$\frac{x}{(x-1)(x-2)} = -\frac{1}{(x-1)} + \frac{2}{(x-2)}$$

Integrating both sides w.r.t x

$$\int \frac{x}{(x-1)(x-2)} dx = \int \left\{\frac{-1}{(x-1)} + \frac{2}{(x-2)}\right\} dx$$

We get

= - log |x – 1| + 2 log |x – 2| + C

We can write it as

$$= \log\left|\frac{(x-2)^2}{x-1}\right| + C$$

Therefore, B is the correct answer.

23. $\int \frac{dx}{x(x^2+1)}$ *equals*

(A) $log|x| - \frac{1}{2}log(x^2+1) + C$

(B) $log|x| + \frac{1}{2}log(x^2+1) + C$

(C) $-log|x| + \frac{1}{2}log(x^2+1) + C$

(D) $\frac{1}{2}log|x| + log(x^2+1) + C$

Solution:

We know that

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

It can be written as

$1 = A(x^2+1) + (Bx+C)x$ …… (1)

Now by equating the coefficients of x^2, x and constant terms

A + B = 0

C = 0

A = 1

By solving the equations we get

A = 1, B = - 1 and C = 0

Substituting the value of A and B

$$\frac{1}{x(x^2+1)} = \frac{1}{x} + \frac{-x}{x^2+1}$$

Integrating both sides w.r.t x

$$\int \frac{1}{x(x^2+1)}dx = \int \left\{\frac{1}{x} - \frac{x}{x^2+1}\right\}dx$$

We get

$$= \log|x| - \frac{1}{2}\log|x^2+1| + C$$

Therefore, A is the correct answer.