EXERCISE 7.8

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Evaluate the following definite integrals as limit of sums.

1. $\int_a^b x\,dx$

Solution:

Given:

$$\int_a^b x\,dx$$

We know that f(x) is continuous in [a, b]

Then we have,

$$\int_a^b f(x)dx = \lim_{n\to\infty} h\sum_{r=0}^{n-1} f(a+rh), \text{ where } h = \frac{b-a}{n}$$

By substituting the value of h in the above expression we get

$$\int_a^b (x)dx = \lim_{n\to\infty}\left(\frac{b-a}{n}\right)\sum_{r=0}^{n-1} f\left(a+\frac{(b-a)r}{n}\right)$$

Since, f (a) = a

$$= \lim_{n\to\infty}\left(\frac{b-a}{n}\right)\sum_{r=0}^{n-1}\left(\frac{(b-a)r}{n}\right)+a$$

By expanding the summation we get,

$$= \lim_{n\to\infty}\left(\frac{b-a}{n}\right)\left(\frac{(b-a)(n-1)(n)}{2n}+a(n-1)\right)$$

Upon simplification we get,

$$= \lim_{n\to\infty}\frac{(b-a)}{n}.\frac{(b-a)(n^2-n)+2an^2-2an}{2n}$$

$$= \lim_{n\to\infty}\frac{(b-a)}{n}.\frac{(b+a)n^2-(b+a)n}{2n}$$

$$= \lim_{n\to\infty} \frac{(b+a)(b-a)n^2 - (b+a)(b-a)n}{2n^2}$$

On computing we get,

$$= \lim_{n\to\infty} \left(\frac{(b+a)(b-a)}{2} - \frac{(b+a)(b-a)}{n} \right)$$

$$= \frac{(b+a)(b-a)}{2}$$

$$= \frac{b^2 - a^2}{2}$$

2. $\int_0^5 (x+1)\, dx$

Solution:

Given:

$$\int_0^5 (x+1)\, dx$$

We know that f(x) is continuous in [a, b] i.e., [0, 5]

Then we have,

$$\int_a^b f(x)dx = \lim_{n\to\infty} h \sum_{r=0}^{n-1} f(a+rh), \text{ where } h = \frac{b-a}{n}$$

Substituting the value of h in the above expression we get,

$$\int_0^5 (x+1)dx = \lim_{n\to\infty} \left(\frac{5}{n}\right) \sum_{r=0}^{n-1} f\left(\frac{5r}{n}\right)$$

Since, f (a) = a

$$= \lim_{n\to\infty} \left(\frac{5}{n}\right) \sum_{r=0}^{n-1} \left(\frac{5r}{n}\right) + 1$$

By expanding the summation we get,

$$= \lim_{n\to\infty} \left(\frac{5}{n}\right) \left(\frac{5(n-1)(n)}{2n} + (n-1) \right)$$

Upon simplification we get,

$$= \lim_{n\to\infty} \frac{5}{n} . \frac{5n^2 - 5n + 2n^2 - 2n}{2n}$$

$$= \lim_{n\to\infty} \frac{5}{n} . \frac{7n^2 - 7n}{2n}$$

$$= \lim_{n\to\infty} \frac{35n^2 - 35n}{2n^2}$$

$$= \lim_{n\to\infty} \frac{35}{2} - \left(\frac{35}{2n}\right)$$

$$= \frac{35}{2}$$

3. $\int_2^3 x^2 \, dx$

Solution:

Given:

$\int_2^3 x^2 \, dx$

We know that f(x) is continuous in [a, b] i.e., [2, 3]

Then we have,

$$\int_a^b f(x)dx = \lim_{n\to\infty} h \sum_{r=0}^{n-1} f(a + rh), \text{ where } h = \frac{b-a}{n}$$

Substituting the value of h in the above expression we get,

$$\int_2^3 (x^2)dx = \lim_{n\to\infty} \left(\frac{1}{n}\right) \sum_{r=0}^{n-1} f\left(2 + \left(\frac{r}{n}\right)\right)$$

Since, f (a) = a

$$= \lim_{n\to\infty} \left(\frac{1}{n}\right) \sum_{r=0}^{n-1} \left(2 + \left(\frac{r}{n}\right)\right)^2$$

By expanding the summation we get,

$$= \lim_{n\to\infty} \left(\frac{1}{n}\right) \sum_{r=0}^{n-1} \left(\frac{r^2}{n^2} + 4 + \frac{4r}{n}\right)$$

Upon simplification we get,

$$=\lim_{n\to\infty}\frac{1}{n}\left(\frac{(n-1)(n)(2n-1)}{6n^2}+4n+\frac{4(n-1)(n)}{2n}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\left(\frac{(n^2-n)(2n-1)}{6n^2}+4n+\frac{2(n^2-n)}{n}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\left(\frac{(2n^3-2n^2-n^2+n)}{6n^2}+4n+\frac{2(n^2-n)}{n}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\left(\frac{(2n^3-3n^2+n)+(24n^3)+(12n^3-12n^2)}{6n^2}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\left(\frac{38n^3-15n^2+n}{6n^2}\right)$$

$$=\lim_{n\to\infty}\left(\frac{38n^3-15n^2+n}{6n^3}\right)$$

On computing we get,

$$=\lim_{n\to\infty}\left(\frac{38}{6}\right)-\left(\frac{15}{6n}\right)+\left(\frac{1}{6n^2}\right)$$

$$=\frac{38}{6}$$

$$=\frac{19}{3}$$

4. $\int_1^4 (x^2-x)\,dx$

Solution:

Given:

$$\int_1^4 (x^2-x)\,dx$$

We know that f(x) is continuous in [a, b] i.e., [1, 4]

Then we have,

$$\int_a^b f(x)dx = \lim_{n\to\infty} h\sum_{r=0}^{n-1} f(a+rh), \text{where } h = (b-a)/n$$

Substituting the value of h in the above expression we get,

$$\int_1^4 (x^2 - x)dx = \lim_{n\to\infty}\left(\frac{3}{n}\right)\sum_{r=0}^{n-1} f\left(\left(1+\frac{3r}{n}\right)\right)$$

Since, f (a) = a

$$= \lim_{n\to\infty}\left(\frac{3}{n}\right)\sum_{r=0}^{n-1}(\left(1+\frac{3r}{n}\right)^2 - \left(1+\frac{3r}{n}\right)$$

By expanding the summation we get,

$$= \lim_{n\to\infty}\left(\frac{3}{n}\right)\sum_{r=0}^{n-1}(1+\frac{9r^2}{n^2}+\frac{6r}{n}-1-\frac{3r}{n})$$

$$= \lim_{n\to\infty}\left(\frac{3}{n}\right)\sum_{r=0}^{n-1}\left(\frac{9r^2}{n^2}+\frac{3r}{n}\right)$$

Upon simplification we get,

$$= \lim_{n\to\infty}\frac{3}{n}\left(\frac{9(n-1)(n)(2n-1)}{6n^2}+\frac{3n(n-1)}{2n}\right)$$

$$= \lim_{n\to\infty}\frac{3}{n}\left(\frac{9(n^2-n)(2n-1)}{6n^2}+\frac{3n(n-1)}{2n}\right)$$

$$= \lim_{n\to\infty}\frac{3}{n}\left(\frac{9(2n^3-2n^2-n^2+n)}{6n^2}+\frac{3n(n-1)}{2n}\right)$$

$$= \lim_{n\to\infty}\frac{3}{n}\left(\frac{(18n^3-27n^2+9n)+(9n^3-9n^2)}{6n^2}\right)$$

$$= \lim_{n\to\infty}\frac{3}{n}\left(\frac{27n^3-36n^2+9n}{6n^2}\right)$$

On computing we get,

$$= \lim_{n\to\infty}\left(\frac{81n^3-108n^2+27n}{6n^3}\right)$$

$$= \lim_{n\to\infty}\left(\frac{81}{6}\right)-\left(\frac{108}{6n}\right)+\left(\frac{27}{6n^2}\right)$$

= 27/2

5. $\int_{-1}^{1} e^x \, dx$

Solution:

Given:

$$\int_{-1}^{1} e^x \, dx$$

We know that f(x) is continuous in [a, b] i.e., [-1, 1]

Then we have,

$$\int_a^b f(x)dx = \lim_{n\to\infty} h \sum_{r=0}^{n-1} f(a+rh), \text{where } h = \frac{b-a}{n}$$

Substituting the value of h in the above expression we get,

$$\int_0^2 (e^x)dx = \lim_{n\to\infty} \left(\frac{2}{n}\right) \sum_{r=0}^{n-1} f\left(-1+\frac{2r}{n}\right)$$

Since, f (a) = a

$$= \lim_{n\to\infty} \left(\frac{2}{n}\right) \sum_{r=0}^{n-1} e^{\frac{2r}{n}-1}$$

By expanding the summation we get,

$$= \lim_{n\to\infty} \left(\frac{2}{ne}\right) (e^0 + e^h + e^{2h} + \cdots \ldots \ldots \ldots \ldots . + e^{nh}$$

sum of $= e^0 + e^h + e^{2h} + \cdots \ldots \ldots \ldots \ldots . + e^{nh}$

Whose g.p has common ratio with $e^{1/n}$.

Whose sum is:

$$= \frac{e^h(1-e^{nh})}{1-e^h}$$

Upon simplification we get,

$$= \lim_{n\to\infty}\left(\frac{2}{ne}\right)\left(\frac{e^h(1-e^{nh})}{1-e^h}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2}{ne}\right).\frac{e^h(1-e^{nh})}{\frac{1-e^h.h}{h}}$$

$$= \lim_{h\to 0}\frac{1-e^h}{h}$$

= -1

$$= \lim_{n\to\infty}\left(\frac{2}{ne}\right)\left(\frac{e^h(1-e^{nh})}{-h}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2}{ne}\right)\left(\frac{e^{\left(\frac{2}{n}\right)}\left(1-e^{n\times\left(\frac{2}{n}\right)}\right)}{-\frac{2}{n}}\right) \text{[Since, h = 2/n]}$$

$$= \frac{e^2-1}{e}$$

$= e\text{-}e^{-1}$

6. $\int_0^4 (x+e^{2x})\,dx$

Solution:

Given:

$$\int_0^4 (x+e^{2x})\,dx$$

$h(x) = \int_0^4 x.\,dx$

$$g(x) = \int_0^4 e^{2x}.dx$$

So, f (x) = h (x) + g (x)

Now let us solve for h (x)

We know that h (x) is continuous in [0, 4]

Then we have,

$$\int_a^b h(x)dx = \lim_{n\to\infty} h\sum_{r=0}^{n-1} f(a+rh), \text{where } h = \frac{b-a}{n}$$

Substituting the value of h in the above expression we get,

$$\int_0^4 (x)dx = \lim_{n\to\infty}\left(\frac{4}{n}\right)\sum_{r=0}^{n-1} f\left(\frac{4r}{n}\right)$$

Since, f (a) = a

$$= \lim_{n\to\infty}\left(\frac{4}{n}\right)\sum_{r=0}^{n-1}\left(\frac{4r}{n}\right)$$

By expanding the summation we get,

$$= \lim_{n\to\infty}\left(\frac{4}{n}\right)\left(\frac{2(n-1)(n)}{n}\right)$$

Upon simplification we get,

$$= \lim_{n\to\infty}\frac{4}{n}.\frac{2n^2-2n}{n}$$

$$= \lim_{n\to\infty}\frac{4}{n}\frac{2n^2-2n}{n}$$

$$= \lim_{n\to\infty}\frac{8n^2-8n}{n^2}$$

$$= \lim_{n\to\infty} 8-\left(\frac{8}{n}\right)$$

=8

Now let us solve for g(x)

We know that g(x) is continuous in [0, 4]

Then we have,

$$\int_a^b f(x)dx = \lim_{n\to\infty} h\sum_{r=0}^{n-1} f(a+rh), \text{where } h = \frac{b-a}{n}$$

Substituting the value of h in the above expression we get,

$$\int_0^4 (e^{2x})dx = \lim_{n\to\infty}\left(\frac{4}{n}\right)\sum_{r=0}^{n-1} f\left(\frac{4r}{n}\right)$$

Since, f (a) = a

$$= \lim_{n\to\infty} \left(\frac{4}{n}\right) \sum_{r=0}^{n-1} e^{\frac{4r}{n}}$$

By expanding the summation we get,

$$= \lim_{n\to\infty} \left(\frac{4}{n}\right) (e^0 + e^h + e^{2h} + \cdots \dots \dots \dots \dots . + e^{nh}$$

sum of $= e^0 + e^h + e^{2h} + \cdots \dots \dots \dots \dots . + e^{nh}$

Whose g.p is common with ratio $e^{1/n}$

Whose sum is:

$$= \frac{e^h(1-e^{nh})}{1-e^h}$$

Upon simplification we get,

$$= \lim_{n\to\infty} \left(\frac{4}{n}\right) \left(\frac{e^h(1-e^{nh})}{1-e^h}\right)$$

$$= \lim_{n\to\infty} \left(\frac{4}{n}\right) \left(\frac{e^h(1-e^{nh})}{\frac{1-e^h.h}{h}}\right)$$

$$= \lim_{n\to\infty} \left(\frac{4}{n}\right) \left(\frac{e^h(1-e^{nh})}{-h}\right) \text{[Since,} \lim_{h\to 0} \frac{1-e^h}{h} = -1\text{]}$$

$$= \lim_{n\to\infty} \left(\frac{4}{n}\right) \left(\frac{e^{\left(\frac{4}{n}\right)}\left(1-e^{n\times\left(\frac{4}{n}\right)}\right)}{-\frac{4}{n}}\right) \text{[Since, } h = 4/n\text{]}$$

$$= (e^8-1)$$

On computing we get,

f (x) = h (x) + g (x)

$= 8 + e^8-1$