MISCELLANEOUS EXERCISE

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Integrate the functions in Exercises 1 to 24.

1. $\dfrac{1}{x-x^3}$

Solution:

Given: $\frac{1}{x-x^3}$

Let $I = \frac{1}{x-x^3} = \frac{1}{x(1-x^2)} = \frac{1}{x(1+x)(1-x)}$

Using partial differentiation

Let $\frac{1}{x(1+x)(1-x)} = \frac{A}{x} + \frac{B}{1+x} + \frac{C}{1-x}$ ... (1)

By taking LCM we get

$\Rightarrow \frac{1}{x(1+x)(1-x)} = \frac{A(1+x)(1-x)+B(x)(1-x)+C(x)(1+x)}{x(1+x)(1-x)}$

$\Rightarrow \frac{1}{x(1+x)(1-x)} = \frac{A(1-x^2)+Bx(1-x)+Cx(1+x)}{x(1+x)(1-x)}$

$\Rightarrow 1 = A - Ax^2 + Bx - Bx^2 + C\,x + Cx^2$

$\Rightarrow 1 = A + (B + C)\,x + (-A - B + C)\,x^2$

Equating the coefficients of x, x^2 and constant value. We get:

(a) A = 1

(b) B + C = 0 ⇒ B = -C

(c) -A – B + C =0

⇒ -1 – (-C) +C = 0

⇒ 2C = 1 ⇒ C = 1/2

So, B = -1/2

Put these values in equation (1)

$$\Rightarrow \frac{1}{x(1+x)(1-x)} = \frac{1}{x} + \frac{-\left(\frac{1}{2}\right)}{1+x} + \frac{\left(\frac{1}{2}\right)}{1-x}$$

$$\Rightarrow \int \frac{1}{x(1+x)(1-x)} dx = \int \frac{1}{x} dx - \frac{1}{2}\int \frac{1}{1+x} dx + \frac{1}{2}\int \frac{1}{1-x} dx$$

On integrating we get

$$= \log|x| - \frac{1}{2}\log|1+x| + \frac{1}{2}\log|1-x|$$

By using logarithmic formula the above equation can be written as

$$= \log|x| - \log\left|(1+x)^{\frac{1}{2}}\right| + \log\left|(1-x)^{\frac{1}{2}}\right|$$

$$= \log\left|\frac{x}{(1+x)^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}\right| + C$$

On simplification we get

$$= \log\left|\frac{(x^2)^{\frac{1}{2}}}{(1+x)(1-x)^{\frac{1}{2}}}\right| + C$$

$$= \log\left|\frac{(x^2)^{\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}}}\right| + C$$

$$= \log\left|\left(\frac{x^2}{1-x^2}\right)^{\frac{1}{2}}\right| + C$$

$$\Rightarrow I = \frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right| + C$$

2. $\dfrac{1}{\sqrt{x+a}+\sqrt{x+b}}$

Solution:

Given: $\frac{1}{\sqrt{x+a}+\sqrt{x+b}}$

Let $I = \frac{1}{\sqrt{x+a}+\sqrt{x+b}}$

Multiply and divide by, $\sqrt{x+a} - \sqrt{x+b}$

$$\Rightarrow I = \frac{1}{\sqrt{x+a}+\sqrt{x+b}} \times \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}$$

$$= \frac{\sqrt{x+a}-\sqrt{x+b}}{(\sqrt{x+a})^2-(\sqrt{x+b})^2}$$

On simplification we get

$$= \frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)}$$

$$= \frac{\sqrt{x+a}-\sqrt{x+b}}{a-b}$$

Applying integration

$$\Rightarrow \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}dx = \int \frac{\sqrt{x+a}-\sqrt{x+b}}{a-b}dx$$

$$= \frac{1}{a-b}\int (\sqrt{x+a} - \sqrt{x+b})dx$$

$$= \frac{1}{a-b}\int ((x+a)^{\frac{1}{2}} - (x+b)^{\frac{1}{2}})dx$$

On integrating we get

$$= \frac{1}{a-b}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right]$$

$$\Rightarrow I = \frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}}\right] + C$$

3. $\frac{1}{x\sqrt{ax-x^2}}$ [Hint: Put $x = \frac{a}{t}$]

Solution:

Given: $\frac{1}{x\sqrt{ax-x^2}}$

Let $I = \frac{1}{x\sqrt{ax-x^2}}$

Put $x = \frac{a}{t} \Rightarrow dx = -\frac{a}{t^2}dt$

$$\Rightarrow \int \frac{1}{x\sqrt{ax-x^2}}dx = \int \frac{1}{\frac{a}{t}\sqrt{\frac{a.a}{t} - \left(\frac{a}{t}\right)^2}} . -\frac{a}{t^2}dt$$

By taking a common we get

$$= \int \frac{-1}{at} . \frac{1}{\sqrt{\frac{1}{t} - \left(\frac{1}{t}\right)^2}}dt$$

Now by multiplying t we get

$$= -\frac{1}{a}\int \frac{1}{\sqrt{\frac{t^2}{t} - \left(\frac{t}{t}\right)^2}}dt$$

The above equation becomes

$$= -\frac{1}{a}\int \frac{1}{\sqrt{t-1}}dt$$

$$= -\frac{1}{a}\int (t-1)^{-\frac{1}{2}}dt$$

On integrating we get

$$= -\frac{1}{a}\left[\frac{\sqrt{(t-1)}}{\frac{1}{2}}\right] + C$$

$$= -\frac{2}{a}\left[\sqrt{\left(\frac{a}{x} - 1\right)}\right] + C \text{ because, } t = \frac{a}{x}$$

$$\Rightarrow I = -\frac{2}{a}\left[\sqrt{\left(\frac{a-x}{x}\right)}\right] + C$$

4. $\dfrac{1}{x^2(x^4+1)^{\frac{3}{4}}}$

Solution:

Given: $\frac{1}{x^2.(x^4+1)^{\frac{3}{4}}}$

Let $I = \frac{1}{x^2.(x^4+1)^{\frac{3}{4}}}$

Multiply and divide by x^{-3}, we get

$$\frac{x^{-3}}{x^2.x^{-3}(x^4+1)^{\frac{3}{4}}} = \frac{x^{-3}.(x^4+1)^{-\frac{3}{4}}}{x^2.x^{-3}}$$

$$= \frac{(x^4+1)^{-\frac{3}{4}}}{x^5.x^{-3\times\frac{4}{4}}}$$

On simplification the above equation can be written as

$$= \frac{(x^4+1)^{-\frac{3}{4}}}{x^5.(x^4)^{-\frac{3}{4}}}$$

$$= \frac{1}{x^5}.\left(\frac{x^4+1}{x^4}\right)^{-\frac{3}{4}}$$

On computing we get

$$= \frac{1}{x^5}.\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}$$

let, $\frac{1}{x^4} = t = (x)^{-4} \Rightarrow \frac{-4}{x^5}dx = dt \Rightarrow \frac{1}{x^5}dx = -\frac{dt}{4}$

$$\Rightarrow \int \frac{1}{x^2.(x^4+1)^{\frac{3}{4}}}.dx = \int \frac{1}{x^5}.\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}.dx$$

Substituting the above values we get

$$= \int (1+t)^{-\frac{3}{4}}.\left(-\frac{dt}{4}\right)$$

$$= -\frac{1}{4}\int (1+t)^{-\frac{3}{4}}.dt$$

On integrating

$$= -\frac{1}{4}\left[\frac{(1+t)^{\frac{1}{4}}}{\frac{1}{4}}\right] + C$$

Now by substituting the value of t we get

$$= -\frac{1}{4}\left[\frac{\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}}{\frac{1}{4}}\right] + C$$

$$= -\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}} + C$$

5. $\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}$ [Hint: $\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} = \frac{1}{x^{\frac{1}{3}}\left(1 + x^{\frac{1}{6}}\right)}$, put $x = t^6$]

Solution:

Given $\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}$

Given question can be written as,

$$\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}=\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}$$

Let x = $t^6 \Rightarrow dx = 6t^5dt$

$$\Rightarrow \int \frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}.dx = \int \frac{6t^5}{t^2(1+t)}.dt$$

On computing we get

$$=6.\int \frac{t^3}{(1+t)}.dt$$

After division we get,

$$\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}=6.\int \left[(t^2-t+1)-\frac{1}{(1+t)}\right].dt$$

Now by splitting the integrals and computing

$$=6.\left\{\int t^2.dt-\int t.dt+\int 1.dt-\int \left[\frac{1}{(1+t)}\right].dt\right\}$$

On integrating

$$=6\left[\left(\frac{t^3}{3}\right)-\left(\frac{t^2}{2}\right)+t-\log(1+t)\right]$$

Now by substituting the value of t we get

$$=6\left[\left(\frac{\left(x^{\frac{1}{6}}\right)^3}{3}\right)-\left(\frac{\left(x^{\frac{1}{6}}\right)^2}{2}\right)+\left(x^{\frac{1}{6}}\right)-\log\left(1+\left(x^{\frac{1}{6}}\right)\right)\right]+C$$

$$=\left[\left(2x^{\frac{1}{2}}\right)-\left(3x^{\frac{1}{3}}\right)+6.x^{\frac{1}{6}}-6.\log\left(1+x^{\frac{1}{6}}\right)\right]+C$$

$= 2\sqrt{x} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\log\left(1 + x^{\frac{1}{6}}\right) + C$

6. $\dfrac{5x}{(x+1)(x^2+9)}$

Solution:

Given: $\frac{5x}{(x+1)(x^2+9)}$

Let I $= \dfrac{5x}{(x+1)(x^2+9)}$

Using partial fraction

Let $\frac{5x}{(x+1)(x^2+9)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+9)}$ ... (1)

$\Rightarrow \dfrac{5x}{(x+1)(x^2+9)} = \dfrac{A(x^2+9)+(Bx+C)(x+1)}{(x+1)(x^2+9)}$

$\Rightarrow 5x = A(x^2+9) + (Bx+C)(x+1)$

$\Rightarrow 5x = Ax^2 + 9A + Bx^2 + Bx + Cx + C$

$\Rightarrow 5x = 9A + C + (B+C)x + (A+B)x^2$

Equating the coefficients of x, x^2 and constant value, we get

(a) $9A + C = 0 \Rightarrow C = -9A$

(b) $B+C = 5 \Rightarrow B = 5-C \Rightarrow B = 5 - (-9A) \Rightarrow B = 5 + 9A$

(c) $A + B = 0 \Rightarrow A = -B \Rightarrow A = -(5 + 9A) \Rightarrow 10A = -5 \Rightarrow A = -1/2$

And $C = 9/2$ and $B = 1/2$

Put these values in equation (1) we get

$\Rightarrow \dfrac{5x}{(x+1)(x^2+9)} = \dfrac{A}{(x+1)} + \dfrac{Bx+C}{(x^2+9)}$

$$\Rightarrow \frac{5x}{(x+1)(x^2+9)} = \frac{-\frac{1}{2}}{(x+1)} + \frac{\left(\frac{1}{2}\right)x + \frac{9}{2}}{(x^2+9)}$$

The above equation can be written as

$$\Rightarrow \frac{5x}{(x+1)(x^2+9)} = -\frac{1}{2}.\frac{1}{(x+1)} + \frac{1}{2}.\left(\frac{x+9}{(x^2+9)}\right)$$

Now by applying integrals on both sides we get

$$\Rightarrow \int \frac{5x}{(x+1)(x^2+9)}dx = -\frac{1}{2}.\int \frac{1}{(x+1)}dx + \frac{1}{2}.\int \frac{x}{(x^2+9)}dx + \frac{9}{2}\int \frac{1}{(x^2+9)}dx$$

$$\Rightarrow \int \frac{5x}{(x+1)(x^2+9)}dx = -\frac{1}{2}.\int \frac{1}{(x+1)}dx + I_1 + \frac{9}{2}\int \frac{1}{(x^2+(3^2)}dx$$

$$\Rightarrow \int \frac{5x}{(x+1)(x^2+9)}dx = -\frac{1}{2}.\log|x+1| + I_1 + \frac{9}{2}.\left(\frac{1}{3}\tan^{-1}\frac{x}{3}\right) \quad \ldots.(2)$$

Now solving for I_1we get

$$I_1 = \frac{1}{2}.\int \frac{x}{(x^2+9)}dx$$

Put $x^2 = t \Rightarrow 2xdx = dt$

$$\Rightarrow I_1 = \frac{1}{2}.\int \frac{1}{(t+9)}.\frac{dt}{2}$$

$$\Rightarrow I_1 = \frac{1}{4}\log|t+9|$$

$$\Rightarrow I_1 = \frac{1}{4}\log\ |x^2+9|$$

Put the value in equation (2)

$$\Rightarrow \int \frac{5x}{(x+1)(x^2+9)}dx = -\frac{1}{2}.\log|x+1| + \frac{1}{4}\log|x^2+9| + \frac{3}{2}.\left(\tan^{-1}\frac{x}{3}\right) + C$$

7. $\dfrac{\sin x}{\sin(x-a)}$

Solution:

Given: $\frac{\sin x}{\sin(x-a)}$

Let I $= \dfrac{\sin x}{\sin(x-a)}$

Let x – a = t ⇒ x = t + a ⇒ dx = dt

$$\Rightarrow \int \frac{\sin x}{\sin(x-a)}dx = \int \frac{\sin(t+a)}{\sin(t)}dt$$

As we know that, {sin (A+B) = sin A cos B + cos A sin B}

$$\Rightarrow \int \frac{\sin x}{\sin(x-a)}dx = \int \frac{\sin t \cos a + \cos t \sin a}{\sin(t)}dt$$

The above equation becomes

$$= \int \frac{\sin t \cos a}{\sin t} + \frac{\cos t \sin a}{\sin t}dt$$

On simplification

$$= \int (\cos a + \cot t \sin a)\, dt$$

Now by splitting the integrals we get

$$= \int (\cos a)dt + \int (\cot t \sin a)dt$$

$$= (\cos a)\int 1.\, dt + \sin a . \int (\cot t)dt$$

On integrating we get

$$= (\cos a).t + \sin a \,.\log|\sin t| + C$$

Now by substituting the value of t we get

$$= (\cos a).(x - a) + \sin a\,.\log|\sin(x - a)| + C$$

$$= \sin a\,.\log|\sin(x - a)| + x.\cos a - a.\cos a + C$$

$$= \sin a\,.\log|\sin(x - a)| + x.\cos a + C_2$$

8. $\dfrac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$

Solution:

Given $\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$

$$\text{let}, I = \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$$

Now by taking common and above equation can be written as

$$\Rightarrow \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} = \frac{e^{4\log x}(e^{\log x} - 1)}{e^{2\log x}(e^{\log x} - 1)}$$

On simplification

$$= e^{2\log x}$$

$$= e^{\log x^2}$$

$$= x^2$$

Applying integrals

$$\Rightarrow \int \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} dx = \int x^2 dx$$

$$= \frac{x^3}{3} + C$$

9. $\frac{\cos x}{\sqrt{4-\sin^2 x}}$

Solution:

Given: $\frac{\cos x}{\sqrt{4-\sin^2 x}}$

$$\text{let } I = \frac{\cos x}{\sqrt{4-\sin^2 x}}$$

Put sin x = t ⇒ cos x dx = dt

The given equation can be written as

$$\Rightarrow \int \frac{\cos x}{\sqrt{4-\sin^2 x}} dx = \int \frac{1}{\sqrt{4-t^2}} dt$$

$$= \int \frac{1}{\sqrt{(2^2-t^2)}} dt$$

On integrating we get

$$= \sin^{-1}\left(\frac{t}{2}\right) + C$$

$$\Rightarrow I = \sin^{-1}\left(\frac{\sin x}{2}\right) + C$$

10. $\frac{\sin^8 - \cos^8 x}{1-2\sin^2 x\cos^2 x}$

Solution:

Given: $\frac{\sin^8 x - \cos^8 x}{1-2\sin^2 x.\cos^2 x}$

$$\text{let, } I = \frac{\sin^8 x - \cos^8 x}{1-2\sin^2 x.\cos^2 x}$$

As we know that $a^2 - b^2 = (a + b)(a - b)$

Now by using this formula we get

$$\Rightarrow \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x.\cos^2 x} = \frac{(\sin^4 x + \cos^4 x)(\sin^4 x - \cos^4 x)}{\sin^2 x + \cos^2 x - \sin^2 x.\cos^2 x - \sin^2 x.\cos^2 x}$$

$$= \frac{(\sin^4 x + \cos^4 x)(\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)}{(\sin^2 x - \sin^2 x.\cos^2 x) + (\cos^2 x - \sin^2 x.\cos^2 x)}$$

We know that $\cos^2 + \sin^2 x = 1$, using this in above equation

$$= \frac{(\sin^4 x + \cos^4 x)(\sin^2 x - \cos^2 x).(1)}{\sin^2 x(1 - \cos^2 x) + \cos^2 x(1 - \sin^2 x)}$$

$$= \frac{-(\sin^4 x + \cos^4 x)(\cos^2 x - \sin^2 x)}{\sin^2 x(\sin^2 x) + \cos^2 x(\cos^2 x)}$$

On simplification we get

$$= \frac{-(\sin^4 x + \cos^4 x)(\cos^2 x - \sin^2 x)}{(\sin^4 x + \cos^4 x)}$$

$$= (\sin^2 x - \cos^2 x)$$

$$= -\cos 2x$$

$$\Rightarrow \int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x.\cos^2 x} dx = \int -\cos 2x\, dx$$

On integrating

$$\Rightarrow I = -\frac{\sin 2x}{2} + C$$

11. $\dfrac{1}{\cos(x+a)\cos(x+b)}$

Solution:

Given: $\frac{1}{\cos(x+a)\cos(x+b)}$

$$\text{let, } I = \frac{1}{\cos(x+a)\cos(x+b)}$$

Multiply and divide by sin (a - b), we get

$$I = \frac{1}{\sin(a-b)}.\left(\frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}\right)$$

Now by adding and subtracting x from the numerator

$$= \frac{1}{\sin(a-b)}.\left(\frac{\sin(a-b+x-x)}{\cos(x+a)\cos(x+b)}\right)$$

By grouping we get

$$= \frac{1}{\sin(a-b)}.\left(\frac{\sin[(x+a)-(x+b)]}{\cos(x+a)\cos(x+b)}\right)$$

As we know that {sin (A-B) = sin A cos B - cos A sin B}

By using this formula we get

$$\Rightarrow I = \frac{1}{\sin(a-b)}.\left(\frac{\sin(x+a).\cos(x+b)-\cos(x+a).\sin(x+b)}{\cos(x+a)\cos(x+b)}\right)$$

$$= \frac{1}{\sin(a-b)}.\left(\frac{\sin(x+a).\cos(x+b)}{\cos(x+a)\cos(x+b)} - \frac{\cos(x+a).\sin(x+b)}{\cos(x+a)\cos(x+b)}\right)$$

On simplification we get

$$= \frac{1}{\sin(a-b)}.\left(\frac{\sin(x+a)}{\cos(x+a)} - \frac{\sin(x+b)}{\cos(x+b)}\right)$$

$$= \frac{1}{\sin(a-b)}.[\tan(x+a)-\tan(x+b)]$$

Taking integrals on both sides we get

$$\Rightarrow \int \frac{1}{\cos(x+a)\cos(x+b)}dx = \int \frac{1}{\sin(a-b)}.[\tan(x+a)-\tan(x+b)]dx$$

$$=\frac{1}{\sin(a-b)}\left\{\int \tan(x+a)\,dx-\int \tan(x+b)\,dx\right\}$$

On integrating we get

$$=\frac{1}{\sin(a-b)}[-\log|\cos(x+a)|-(-\log|\cos(x+a)|)]$$

$$=\frac{1}{\sin(a-b)}[-\log|\cos(x+a)|+\log|\cos(x+a)|]$$

$$\Rightarrow I=\frac{1}{\sin(a-b)}.\log\left|\frac{\cos(x+b)}{\cos(x+a)}\right|+C$$

12. $\frac{x^3}{\sqrt{1-x^8}}$

Solution:

Given: $\frac{x^3}{\sqrt{1-x^8}}$

$$\text{let } I=\frac{x^3}{\sqrt{1-x^8}}$$

Now, let $x^4 = t \Rightarrow 4x^3\,dx = dt$

And $x^3\,dx = dt/4$

Substituting these values in given question we get

$$\Rightarrow \int \frac{x^3}{\sqrt{1-x^8}}dx=\int \frac{1}{\sqrt{1-t^2}}\left(\frac{dt}{4}\right)$$

$$=\frac{1}{4}\int \frac{1}{\sqrt{1^2-t^2}}.\,dt$$

On integrating we get

$$=\frac{1}{4}\sin^{-1}t+C$$

Now by substituting t value we get

$$\Rightarrow I = \frac{1}{4}\sin^{-1}(x^4) + C$$

13. $\dfrac{e^x}{(1+e^x)(2+e^x)}$

Solution:

Given: $\frac{e^x}{(1+e^x)(2+e^x)}$

$$\text{let, } I = \frac{e^x}{(1+e^x)(2+e^x)}$$

Let $e^x = t \Rightarrow e^x\,dx = dt$

Now substituting these values in given question we get

$$\Rightarrow \int \frac{e^x}{(1+e^x)(2+e^x)}dx = \int \frac{1}{(1+t)(2+t)}dt$$

$$= \int \left[\frac{1}{(1+t)} - \frac{1}{(2+t)}\right]dt$$

Now by splitting the integrals we get

$$= \int \left[\frac{1}{(1+t)}\right]dt - \int \left[\frac{1}{(2+t)}\right]dt$$

On integrating we get

$$= \log|(1+t)| - \log|(2+t)| + C$$

$$= \log\left|\frac{1+t}{2+t}\right| + C$$

$$\Rightarrow I = \log\left|\frac{1+e^x}{2+e^x}\right| + C$$

14. $\frac{1}{(x^2+1)(x^2+4)}$

Solution:

Given: $\frac{1}{(x^2+1)(x^2+4)}$

Let $I = \frac{1}{(x^2+1)(x^2+4)}$

Using partial fraction method, we get

let $\frac{1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+4)}$ ... (1)

$\Rightarrow \frac{1}{(x+1)(x^2+9)} = \frac{(Ax+B)(x^2+4)+(Cx+D)(x^2+1)}{(x+1)(x^2+9)}$

$\Rightarrow 1 = (Ax + B)(x^2 + 4) + (Cx + D)(x^2 + 1)$

$\Rightarrow 1 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + Dx^2 + D$

$\Rightarrow 1 = (A + C)x^3 + (B + D)x^2 + (4A + C)x + (4B + D)$

Equating the coefficients of x, x^2, x^3 and constant value. We get:

(a) $A + C = 0 \Rightarrow C = -A$

(b) $B + D = 0 \Rightarrow B = -D$

(c) $4A + C = 0 \Rightarrow 4A = -C \Rightarrow 4A = A \Rightarrow 3A = 0 \Rightarrow A = 0 \Rightarrow C = 0$

(d) $4B + D = 1 \Rightarrow 4B - B = 1 \Rightarrow B = 1/3 \Rightarrow D = -1/3$

Put these values in equation (1)

$\Rightarrow \frac{1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+4)}$

$$\Rightarrow \frac{1}{(x^2+1)(x^2+4)} = \frac{(0)x+\frac{1}{3}}{(x^2+1)} + \frac{(0)x+\left(-\frac{1}{3}\right)}{(x^2+4)}$$

$$\Rightarrow \frac{1}{(x^2+1)(x^2+4)} = \frac{\frac{1}{3}}{(x^2+1)} + \frac{\left(-\frac{1}{3}\right)}{(x^2+4)}$$

Now by taking integrals on both sides we get

$$\Rightarrow \int \frac{1}{(x^2+1)(x^2+4)}dx = \frac{1}{3}.\int \frac{1}{(x^2+1)}dx - \frac{1}{3}.\int \frac{1}{(x^2+4)}dx$$

$$\Rightarrow \int \frac{1}{(x^2+1)(x^2+4)}dx = \frac{1}{3}.\int \frac{1}{(x^2+1^2)}dx - \frac{1}{3}.\int \frac{1}{(x^2+2^2)}dx$$

On integrating we get

$$= \frac{1}{3}.\tan^{-1}x - \frac{1}{3}.\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$\Rightarrow I = \frac{1}{3}.\tan^{-1}x - \frac{1}{6}\tan^{-1}\frac{x}{2} + C$$

15. $\cos^3 x\ e^{\log\ \sin x}$

Solution:

Given: $\cos^3 x e^{\log \sin x}$

Let $I = \cos^3 x e^{\log \sin x}$

Logarithmic and exponential functions cancels each other in above equation then we get

$$= \cos^3 x.\sin x$$

Let cos x = t ⇒ -sin x dx = dt ⇒ sin x dx = dt

Substituting these values in given question we get

$$\Rightarrow \int \cos^3 x e^{\log \sin x} dx = \int \cos^3 x.\sin x\, dx$$

$$= \int t^3.(-dt)$$

$$= -\int t^3.dt$$

On integrating

$$= -\frac{t^4}{4} + C$$

Now by substituting the value of t we get

$$= -\frac{\cos^4 x}{4} + C$$

16. $e^{3\log x}(x^4 + 1)^{-1}$

Solution:

Given: $e^{3\log x}(x^4 + 1)^{-1}$

Let $I = e^{3\log x}(x^4 + 1)^{-1}$

$$= e^{\log x^3}(x^4 + 1)^{-1}$$

Logarithmic and exponential functions cancels each other in above equation then we get

$$= \frac{x^3}{x^4 + 1}$$

Let $x^4 = t \Rightarrow 4x^3\,dx = dt \Rightarrow x^3\,dx = dt/4$

Now by substituting these values in given question we get

$$\Rightarrow \int e^{3\log x}(x^4 + 1)^{-1} = \int \frac{x^3}{x^4 + 1}dx$$

$$= \int \frac{1}{t + 1}.\frac{dt}{4}$$

$$= \frac{1}{4} . \int \frac{1}{t+1} . dt$$

On integration we get

$$= \frac{1}{4} \log(t+1) + C$$

Now by substituting the values of t we get

$$\Rightarrow I = \frac{1}{4} \log(x^4 + 1) + C$$

17. $f'(ax+b)\ [f(ax+b)]^n$

Solution:

Given: $f'(ax + b)\ [f(ax + b)]^n$

Let $f(ax + b) = t \Rightarrow a . f'(ax + b)\, dx = dt$

Now by substituting these values in given question we get

$$\Rightarrow \int f'(ax+b)[f(ax+b)^n] = \int t^n \left(\frac{dt}{a}\right)$$

$$= \frac{1}{a} \int t^n dt$$

On integrating

$$= \frac{1}{a} . \frac{t^{n+1}}{n+1} + C$$

Here by substituting the value of t we get

$$= \frac{1}{a} . \frac{(f(ax+b))^{n+1}}{n+1} + C$$

$$= \frac{1}{a(n+1)} . (f(ax+b))^{n+1} + C$$

18. $\dfrac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$

Solution:

Given: $\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$

$$\text{let I} = \frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$$

As we know that, {sin (A+B) = sin A cos B + cos A sin B}

Using this formula we get

$$\Rightarrow \text{I} = \frac{1}{\sqrt{\sin^3 x(\sin x \cos\alpha + \cos x \sin\alpha)}}$$

Multiplying and dividing by sin x to denominator we get

$$\Rightarrow \text{I} = \frac{1}{\sqrt{\sin^3 x(\sin x \cos\alpha + \cos x . \frac{\sin x}{\sin x} \sin\alpha)}}$$

On rearranging we get

$$= \frac{1}{\sqrt{\sin^3 x(\sin x \cos\alpha + \sin x . \frac{\cos x}{\sin x} \sin\alpha)}}$$

Simplifying we get

$$= \frac{1}{\sqrt{\sin^4 x(\cos\alpha + \cot x \sin\alpha)}}$$

$$= \frac{1}{\sin^2 x\sqrt{(\cos\alpha + \cot x \sin\alpha)}}$$

$$= \frac{\operatorname{cosec}^2 x}{\sqrt{(\cos\alpha + \cot x \sin\alpha)}}$$

now, let $(\cos\alpha + \cot x \sin\alpha) = t \Rightarrow -\operatorname{cosec}^2 x. \sin\alpha\, dx = dt$

Now by substituting these values in given question we get

$$\Rightarrow \int \frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} dx = \int \frac{\operatorname{cosec}^2 x}{\sqrt{(\cos\alpha + \cot x \sin\alpha)}} dx$$

$$= \int \frac{1}{\sqrt{t}} . -\frac{dt}{\sin\alpha}$$

$$= -\frac{1}{\sin\alpha} \int \frac{1}{\sqrt{t}} . dt$$

$$= -\frac{1}{\sin\alpha} \int t^{-\frac{1}{2}} . dt$$

On integrating we get

$$= -\frac{1}{\sin\alpha} \left[\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right] + C$$

$$= -\frac{2}{\sin\alpha} [\sqrt{t}] + C$$

Now by substituting the value of t

$$= -\frac{2}{\sin\alpha} \left[\sqrt{(\cos\alpha + \cot x \sin\alpha)}\right] + C$$

Computing and simplifying

$$= -\frac{2}{\sin\alpha} \left[\sqrt{\left(\cos\alpha + \frac{\cos x}{\sin x} \sin\alpha\right)}\right] + C$$

$$= -\frac{2}{\sin\alpha} \left[\sqrt{\frac{(\cos\alpha \sin x + \cos x \sin\alpha)}{\sin x}}\right] + C$$

$$\Rightarrow I = -\frac{2}{\sin\alpha} \left[\sqrt{\frac{\sin(x+\alpha)}{\sin x}}\right] + C$$

19. $\dfrac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}, x \in [0, 1]$

Solution:

Given: $\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}$

Let $I = \frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}$ ... (1)

As we know, $\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x} = \frac{\pi}{2}$

Now using this identity we get

$$\Rightarrow I = \frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}} = \frac{\left(\frac{\pi}{2}-\cos^{-1}\sqrt{x}\right)-\cos^{-1}\sqrt{x}}{\left(\frac{\pi}{2}\right)}$$

$$\Rightarrow \int \frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}dx = \int \frac{\left(\frac{\pi}{2}-\cos^{-1}\sqrt{x}\right)-\cos^{-1}\sqrt{x}}{\left(\frac{\pi}{2}\right)}dx$$

$$= \left(\frac{2}{\pi}\right)\int \left(\frac{\pi}{2} - 2\cos^{-1}\sqrt{x}\right)dx$$

Now by splitting the integral we get

$$= \left(\frac{2}{\pi}\right)\int \left(\frac{\pi}{2}.dx\right) - \left(\frac{2}{\pi}\right)\int 2.\left(\cos^{-1}\sqrt{x}.dx\right)$$

$$= \int (1.dx) - \left(\frac{4}{\pi}\right)\int \left(\cos^{-1}\sqrt{x}.dx\right)$$

On integration we get

$$\Rightarrow I = x - \left(\frac{4}{\pi}\right)I_1 \;....(2)$$

Now, first solve for I_1:

as, $I_1 = \int \left(\cos^{-1}\sqrt{x}.dx\right)$

let $\sqrt{x} = t \Rightarrow \frac{1}{2}x^{-\frac{1}{2}}dx = dt \Rightarrow \frac{dx}{\sqrt{x}} = 2.dt \Rightarrow dx = 2.tdt$

$$\Rightarrow I_1 = \int (\cos^{-1}t.2t.dt)$$

$$= 2\int t.\cos^{-1}t\ dt$$

Because, $\int u.v\,dx = u.\int v\,dx - \int \frac{du}{dx}.\{\int vdx\}\ dx$

$$\Rightarrow 2\int t.\cos^{-1}t\ dt = 2.\left[\cos^{-1}t.\int t\,dt - \int \frac{d(\cos^{-1}t)}{dt}.\left\{\int tdt\right\} dt\right]$$

$$= 2.\cos^{-1}t.\frac{t^2}{2} - 2.\int \left(-\frac{1}{\sqrt{1-t^2}}\right).\left\{\frac{t^2}{2}\right\} dt$$

$$= t^2.\cos^{-1}t - \int \left(\frac{-t^2}{\sqrt{1-t^2}}\right).\ dt$$

Now by adding and subtracting 1 to numerator we get

$$= t^2.\cos^{-1}t - \int \left(\frac{-1+1-t^2}{\sqrt{1-t^2}}\right).\ dt$$

Splitting the denominator

$$= t^2.\cos^{-1}t - \int \left(\frac{-1}{\sqrt{1-t^2}} + \frac{1-t^2}{\sqrt{1-t^2}}\right).\ dt$$

Splitting the integral we get

$$= t^2.\cos^{-1}t + \int \left(\frac{1}{\sqrt{1-t^2}}dt\right) - \int \left(\sqrt{1-t^2}\right).dt$$

$$= t^2.\cos^{-1}t + \int \left(\frac{1}{\sqrt{1-t^2}}dt\right) - \frac{t}{2}.\sqrt{1-t^2}$$

as, $\int \left(\sqrt{a^2-x^2}\right).dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

$$\Rightarrow I_1 = t^2.\cos^{-1}t + \sin^{-1}t - \frac{t}{2}\sqrt{1-t^2} - \frac{1}{2}\sin^{-1}(t)$$

$$\Rightarrow I_1 = t^2.\cos^{-1}t - \frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1}t$$

Put it in equation. (2)

$$\Rightarrow I = x - \left(\frac{4}{\pi}\right)\left[t^2.\cos^{-1}t - \frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1}t\right] \ldots.(2)$$

Now substitute the value of t we get

$$\Rightarrow I = x - \left(\frac{4}{\pi}\right)\left[(\sqrt{x})^2.\cos^{-1}\sqrt{x} - \frac{\sqrt{x}}{2}\sqrt{1-(\sqrt{x})^2} + \frac{1}{2}\sin^{-1}\sqrt{x}\right]$$

Computing and simplifying we get

$$= x - \left(\frac{4}{\pi}\right)\left[x.\cos^{-1}\sqrt{x} - \frac{\sqrt{x}}{2}\sqrt{1-x} + \frac{1}{2}\sin^{-1}\sqrt{x}\right]$$

$$= x - \left(\frac{4}{\pi}\right)\left[x.\left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right) - \frac{\left(\sqrt{x-x^2}\right)}{2} + \frac{1}{2}\sin^{-1}\sqrt{x}\right]$$

$$= x - 2x + \frac{4x}{\pi}\sin^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x-x^2} - \frac{2}{\pi}\sin^{-1}\sqrt{x}$$

$$= -x + \frac{2}{\pi}\left[(2x-1)\sin^{-1}\sqrt{x}\right] + \frac{2}{\pi}\sqrt{x-x^2} + C$$

$$\Rightarrow I = \frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x-x^2} - x + C$$

20. $\sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}$

Solution:

Given: $\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$

Let $I = \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$

Let x= $\cos^2\theta \Rightarrow dx = -2\sin\theta \cos\theta\, d\theta$

$\Rightarrow \sqrt{x} = \cos\theta \text{ or } \theta = \cos^{-1}\sqrt{x}$

Substituting these values in given question we get

$$\Rightarrow I = \int \sqrt{\frac{1-\sqrt{\cos^2\theta}}{1+\sqrt{\cos^2\theta}}}(-2\sin\theta\cos\theta)d\theta$$

$$= \int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-2\sin\theta\cos\theta)d\theta$$

Substituting the standard formulae we get

$$= \int - \sqrt{\frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)}}(2\sin\theta\cos\theta)d\theta$$

Multiplying and dividing by 2 we get

$$= \int - \sqrt{\frac{\sin^2\left(\frac{\theta}{2}\right)}{\cos^2\left(\frac{\theta}{2}\right)}}\left(2\sin 2\frac{\theta}{2}\cos 2\frac{\theta}{2}\right)d\theta$$

Using standard identities the above equation can be written as

$$= \int -\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}.(2).\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right).\left(2\cos^2\left(\frac{\theta}{2}\right)-1\right)d\theta$$

$$\Rightarrow \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx = \int -4.\left[\sin^2\left(\frac{\theta}{2}\right)\right]\left(2\cos^2\left(\frac{\theta}{2}\right)-1\right)d\theta$$

$$= \int -4.\left\{\left[2.\sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)\right] - \sin^2\left(\frac{\theta}{2}\right)\right\} d\theta$$

Splitting the integrals we get

$$= \int -2.\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2 d\theta + 4\int \sin^2\left(\frac{\theta}{2}\right) d\theta$$

Again by using standard identities above equation can be written as

$$= -2.\int \sin^2\theta d\theta + 4\int \sin^2\left(\frac{\theta}{2}\right) d\theta$$

$$= -2.\int \frac{1-\cos 2\theta}{2} d\theta + 4\int \frac{1-\cos\theta}{2} d\theta$$

On integrating we get

$$= -2\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right] + 4\left[\frac{\theta}{2} - \frac{\sin\theta}{2}\right] + C$$

$$= -\theta + \frac{\sin 2\theta}{2} + 2\theta - 2\sin\theta + C$$

Computing and simplifying

$$= \theta + \frac{2.\sin\theta.\cos\theta}{2} - 2\sin\theta + C$$

$$= \theta + \frac{2.\sqrt{1-\cos^2\theta}.\cos\theta}{2} - 2\sqrt{1-\cos^2\theta} + C$$

Substituting the values we get

$$= \cos^{-1}\sqrt{x} + \sqrt{1-x}.\sqrt{x} - 2\sqrt{1-x} + C$$

$$= \cos^{-1}\sqrt{x} + \sqrt{x(1-x)} - 2\sqrt{1-x} + C$$

$$\Rightarrow I = \cos^{-1}\sqrt{x} + \sqrt{x-x^2} - 2\sqrt{1-x} + C$$

21. $\dfrac{2+\sin 2x}{1+\cos 2x} e^x$

Solution:

$$\text{let } I = \frac{2+\sin 2x}{1+\cos 2x} e^x$$

Subsisting the sin 2x = 2 sin x cos x formula we get

$$= \left(\frac{2+2\sin x\cos x}{2\cos^2 x}\right) e^x$$

Now by taking 2 common

$$= 2.\left(\frac{1+\sin x\cos x}{2\cos^2 x}\right) e^x$$

On simplification

$$= \left(\frac{1}{\cos^2 x} + \frac{\sin x\cos x}{\cos^2 x}\right) e^x$$

$$= (\sec^2 x + \tan x)e^x$$

Substituting integrals both the sides we get

$$\Rightarrow \int \frac{2+\sin 2x}{1+\cos 2x} e^x dx = \int (\sec^2 x + \tan x)e^x dx$$

Now let tan x = f(x)

$\Rightarrow f'(x) = \sec^2 x \, dx$

$$\Rightarrow \int \frac{2+\sin 2x}{1+\cos 2x} e^x dx = \int (f(x) + f'(x))e^x dx$$

On integrating we get

$$= e^x f(x) + C$$

$$\Rightarrow I = e^x \tan x + C$$

22. $\dfrac{x^2+x+1}{(x+1)^2\,(x+2)}$

Solution:

Given: $\frac{x^2+x+1}{(x+1)^2(x+2)}$

Let $I = \frac{x^2+x+1}{(x+1)^2(x+2)}$

Using partial fraction we get

Let $\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+2)}$ ... (1)

$$\Rightarrow \frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A(x+1)(x+2) + B(x+2) + C(x+1)^2}{(x+1)^2(x+2)}$$

$$\Rightarrow \frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A(x^2+3x+2) + B(x+2) + C(x^2+2x+1)}{(x+1)^2(x+2)}$$

$\Rightarrow x^2 + x + 1 = Ax^2 + 3Ax + 2A + Bx + 2B + Cx^2 + 2Cx + C$

$\Rightarrow x^2 + x + 1 = (2A+2B+C) + (3A+B+2C)x + (A+C)x^2$

Equating the coefficients of x, x^2 and constant value. We get:

(a) $A + C = 1$

(b) $3A + B + 2C = 1$

(c) $2A+2B+C = 1$

After solving the above equations we get

$A = -2$, $B = 1$ and $C = 3$

Substituting the values of A, B and C we get

$$\Rightarrow \frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{(x+2)}$$

Taking integrals on both sides

$$\Rightarrow \int \frac{x^2+x+1}{(x+1)^2(x+2)} dx = \int \left(\frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{(x+2)}\right) dx$$

Splitting the integrals we get

$$= -2.\int\left(\frac{1}{x+1}\right)dx + \int\left(\frac{1}{(x+1)^2}\right)dx + 3.\int\left(\frac{1}{(x+2)}\right)dx$$

$$= -2.\int\left(\frac{1}{x+1}\right)dx + \int((x+1)^{-2})dx + 3.\int\left(\frac{1}{(x+2)}\right)dx$$

On integrating we get

$$= -2\log|x+1| + \left(\frac{(x+1)^{-1}}{(-1)}\right) + 3\log|x+1| + C$$

$$= -2\log|x+1| - \frac{1}{(x+1)} + 3\log|x+1| + C$$

23. $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

Solution:

Given: $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

$$\text{let } I = \tan^{-1}\sqrt{\frac{1-x}{1+x}}$$

Let x= cos θ ⇒ dx = -sin θ d θ

$$\Rightarrow \theta = \cos^{-1}x$$

Now by substituting these values in given question we get

$$\Rightarrow I = \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}dx = \int \tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-\sin\theta)d\theta$$

Using standard identities the above equation can be written as

$$= -\int \tan^{-1}\sqrt{\frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)}}(\sin\theta)\,d\theta$$

$$= -\int \tan^{-1}\sqrt{\tan^2\left(\frac{\theta}{2}\right)}(\sin\theta)\,d\theta$$

On simplification we get

$$= -\int \tan^{-1}\tan\frac{\theta}{2}.(\sin\theta)\,d\theta$$

$$= -\frac{1}{2}\int \theta.(\sin\theta)\,d\theta$$

Now by using product rule

$$\int u.v\,dx = u.\int v\,dx - \int \frac{du}{dx}.\left\{\int vdx\right\}dx$$

$$= -\frac{1}{2}\int \theta.(\sin\theta)\,d\theta \ = -\frac{1}{2}\left[\theta.\int \sin\theta\,d\theta - \int \frac{d\theta}{d\theta}.\left\{\int \sin v\,d\theta\right\}d\theta\right]$$

Computing and integrating we get

$$= -\frac{1}{2}\left[\theta.(-\cos\theta) - \int 1.(-\cos\theta)\ d\theta\right]$$

$$= -\frac{1}{2}[-\theta.\cos\theta + \sin\theta\,]$$

Substituting the values we get

$$= \frac{1}{2}\theta.\cos\theta - \frac{1}{2}\sqrt{(1-\cos^2\theta}$$

$$= \frac{1}{2}\cos^{-1}x.x - \frac{1}{2}\sqrt{(1-x^2} + C$$

$$= \frac{1}{2}\left(x.\cos^{-1}x - \sqrt{(1-x^2)}\right) + C$$

24. $\dfrac{\sqrt{x^2+1}\left[\log(x^2+1)-2\log x\right]}{x^4}$

Solution:

Given: $\frac{\sqrt{x^2+1}[\log(x^2+1)-2\log x]}{x^4}$

$$\text{let } I = \frac{\sqrt{x^2+1}[\log(x^2+1)-2\log x]}{x^4}$$

$$= \frac{\sqrt{x^2+1}}{x^4}[\log(x^2+1)-\log x^2]$$

Using logarithmic identities we get

$$= \frac{1}{x^3}\sqrt{\frac{x^2+1}{x^2}}\left[\log\left(\frac{x^2+1}{x^2}\right)\right]$$

On computing

$$= \frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\left[\log\left(1+\frac{1}{x^2}\right)\right]$$

$$\text{now let } 1+\frac{1}{x^2} = t \Rightarrow -\frac{2}{x^3}dx = dt$$

Substituting these values in given question we get

$$\Rightarrow \int \frac{\sqrt{x^2+1}[\log(x^2+1)-2\log x]}{x^4}dx = \int \frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\left[\log\left(1+\frac{1}{x^2}\right)\right]dx$$

$$= \int -\frac{1}{2}.\sqrt{t}[\log(t)]dt$$

By using product rule

$$\int u.v\,dx = u.\int v\,dx - \int \frac{du}{dx}.\left\{\int v dx\right\} dx$$

$$= \int -\frac{1}{2}.\sqrt{t}[\log(t)]dt = -\frac{1}{2}\left[\log t.\int \sqrt{t}\,dt - \int \frac{d}{dt}\log t.\left\{\int \sqrt{t}dt\right\} dt\right]$$

Computing and simplifying we get

$$= -\frac{1}{2}\left[\log t.\frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \int \frac{1}{t}.\left\{\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right\}dt\right]$$

$$= -\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t - \int \left\{\frac{t^{\frac{3}{2}-1}}{\frac{3}{2}}\right\}dt\right]$$

$$= -\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t - \frac{2}{3}\int t^{\frac{1}{2}}\,dt\right]$$

On integration we get

$$= -\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t - \frac{2}{3}.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]$$

$$= \left[-\frac{1}{2}.\frac{2}{3}t^{\frac{3}{2}}\log t + \frac{1}{2}.\frac{2}{3}.\frac{2}{3}.t^{\frac{3}{2}}\right]$$

$$= -\frac{1}{3}t^{\frac{3}{2}}\left[\log t - \frac{2}{3}\right]$$

Substituting the value of t we get

$$\Rightarrow I = -\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}\left[\log\left(1+\frac{1}{x^2}\right) - \frac{2}{3}\right] + C$$

Evaluate the definite integrals in Exercises 25 to 33.

25. $\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$

Solution:

Given: $\int_{-\frac{\pi}{2}}^{\pi}(e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$

$$\text{let, } I = \int_{-\frac{\pi}{2}}^{\pi}(e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$$

Substituting the standard identities for 1 – sin x and 1 – cos x we get

$$= \int_{-\frac{\pi}{2}}^{\pi}(e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\left(\frac{x}{2}\right)}\right)dx$$

Now splitting the denominator

$$= \int_{-\frac{\pi}{2}}^{\pi}(e^x\left(\frac{1}{2\sin^2\left(\frac{x}{2}\right)} - \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\left(\frac{x}{2}\right)}\right)dx$$

$$= \int_{-\frac{\pi}{2}}^{\pi}(e^x\left(\frac{1}{2}\mathrm{cosec}^2\left(\frac{x}{2}\right) - \cot\frac{x}{2}\right)dx$$

$$\text{now let } f(x) = -\cot\frac{x}{2}$$

Substituting these values we get

$$\Rightarrow f'(x) = -\left(-\frac{1}{2}\mathrm{cosec}^2\left(\frac{x}{2}\right)\right) = \frac{1}{2}\mathrm{cosec}^2\left(\frac{x}{2}\right)$$

$$\Rightarrow \int_{-\frac{\pi}{2}}^{\pi}(e^x\left(\frac{1}{2}\mathrm{cosec}^2\left(\frac{x}{2}\right) - \cot\frac{x}{2}\right)dx = \int_{-\frac{\pi}{2}}^{\pi}(f(x) + f'(x))e^x dx$$

On integration we get

$$= [e^x f(x)]_{-\frac{\pi}{2}}^{\pi}$$

$$= \left[e^x\left(-\cot\frac{x}{2}\right)\right]_{-\frac{\pi}{2}}^{\pi}$$

Now by applying the limits we get

$$= -\left[e^{\pi}\left(\cot\frac{\pi}{2}\right) - e^{\frac{\pi}{2}}\left(\cot\frac{\pi}{4}\right)\right]$$

$$= -\left[e^{\pi}(0) - e^{\frac{\pi}{2}}(1)\right]$$

$$= -\left[0 - e^{\frac{\pi}{2}}\right]$$

On simplification we get

$$\Rightarrow I = e^{\frac{\pi}{2}}$$

26. $\int_0^{\frac{\pi}{4}} \frac{\sin x \ \cos x}{\cos^4 x + \sin^4 x} dx$

Solution:

Given: $\int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$

$$\text{let, } I = \int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$$

Taking $\cos^4 x$ as common we get

$$= \int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x\left(1 + \frac{\sin^4 x}{\cos^4 x}\right)} dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\tan x \sec^2 x}{(1 + \tan^4 x)} dx$$

Now let $\tan^2 x = t \Rightarrow 2 \tan x \sec^2 x \, dx = dt$

And when x=0 then t=0 and when x=π /4 then t=1

Now by substituting these values in above equation we get

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\tan x \sec^2 x}{(1 + \tan^4 x)} dx = \int_0^1 \frac{1}{(1 + t^2)} \left(\frac{dt}{2}\right)$$

On integration

$$\Rightarrow I = \frac{1}{2}[\tan^{-1} t]_0^1$$

Now by applying the limits we get

$$= \frac{1}{2}[\tan^{-1} 1 - \tan^{-1} 0]$$

$$\Rightarrow I = \frac{1}{2}.\frac{\pi}{4}$$

$$\Rightarrow I = \frac{\pi}{8}$$

27. $\int_0^{\frac{\pi}{2}} \frac{\cos^2 x\, dx}{\cos^2 x + 4\sin^2 x}$

Solution:

Given: $\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x}\, dx$

$$\text{let}, I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x}\, dx \quad \ldots..(1)$$

Substituting $\sin^2 x$ formula we get

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4(1 - \cos^2 x)}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4(1) - (4\cos^2 x)}\, dx$$

On computing we get

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{4 - 3\cos^2 x}\, dx$$

Now multiplying and dividing by 3 to the numerator we get

$$= \int_0^{\frac{\pi}{2}} \frac{\frac{1}{3}.3\cos^2 x}{4 - 3\cos^2 x}\, dx$$

Again by adding and subtracting 4 to the numerator we get

$$= -\frac{1}{3}.\int_0^{\frac{\pi}{2}} \frac{-3\cos^2 x + 4 - 4}{4 - 3\cos^2 x}\, dx$$

The above equation can be written as

$$= -\frac{1}{3}.\int_0^{\frac{\pi}{2}} \frac{4 - 3\cos^2 x - 4}{4 - 3\cos^2 x}\, dx$$

Now splitting the integrals we get

$$= -\frac{1}{3}.\int_0^{\frac{\pi}{2}} \frac{4 - 3\cos^2 x}{4 - 3\cos^2 x}\, dx + \frac{1}{3}.\int_0^{\frac{\pi}{2}} \frac{4}{4 - 3\cos^2 x}\, dx$$

$$= -\frac{1}{3}.\int_0^{\frac{\pi}{2}} (1)\, dx + \frac{1}{3}.\int_0^{\frac{\pi}{2}} \frac{4}{4 - 3\left(\frac{1}{\sec^2 x}\right)}\, dx$$

Applying the limits we get

$$= -\frac{1}{3}.[x]_0^{\frac{\pi}{2}} + \frac{1}{3}.\int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4\sec^2 x - 3}\, dx$$

$$= -\frac{1}{3}.\left[\frac{\pi}{2}\right] + \frac{1}{3}.\int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4(1 + \tan^2 x) - 3}\, dx$$

$$\Rightarrow I = -\frac{\pi}{6} + \frac{2}{3}.\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1 + 4\tan^2 x}\, dx$$

$$\Rightarrow I = -\frac{\pi}{6} + I_1 \quad \ldots.(2)$$

First solve for I_1:

$$I_1 = \frac{2}{3}.\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1 + 4\tan^2 x}\,dx$$

Let 2 tan x = t ⇒ 2 sec^{2}x dx dt

When x = 0 then t = 0 and when x = π /2 then t = ∞

Substituting these values for above equation we get

$$\Rightarrow \frac{2}{3}.\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1 + 4\tan^2 x}\,dx = \frac{2}{3}.\int_0^{\infty} \frac{1}{1 + t^2}\,dt$$

Integrating and applying the limits we get

$$\Rightarrow I_1 = \frac{2}{3}[\tan^{-1} t]_0^{\infty}$$

$$= \frac{2}{3}[\tan^{-1}\infty - \tan^{-1}0]$$

$$\Rightarrow I_1 = \frac{2}{3}.\frac{\pi}{2}$$

$$\Rightarrow I_1 = \frac{\pi}{3}$$

Put this value in equation (2)

$$\Rightarrow I = -\frac{\pi}{6} + \frac{\pi}{3}$$

$$\Rightarrow I = \frac{\pi}{6}$$

28. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\,dx$

Solution:

Given: $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{\sin 2x}}\, dx$

$$\text{let, I} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\, dx$$

On rearranging we get

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{-(-\sin 2x)}}\, dx$$

Now by substituting the sin 2x formula we get

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{-(-1 + 1 - 2\sin x \cos x)}}\, dx$$

1 can be written as $\sin^2 x + \cos^2 x$

Substituting this in above equation we get

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{1 - (\sin^2 x + \cos^2 x - 2\sin x \cos x)}}\, dx$$

As we know $(a + b)^2 = a^2 + b^2$ using this in above equation we get

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{(1 - (\sin x - \cos x)^2)}}\, dx$$

Now let $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)\, dx = dt$

$$\text{when } x = \frac{\pi}{6} \Rightarrow t = \frac{1}{2} - \frac{\sqrt{3}}{2} = \frac{1 - \sqrt{3}}{2} \text{ and when } x = \frac{\pi}{3} \Rightarrow t = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3} - 1}{2}$$

Substituting these values in above equation we get

$$\Rightarrow \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{(1 - (\sin x - \cos x)^2)}}\, dx = \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{1}{\sqrt{(1 - (t)^2)}}\, dt$$

$$= \int_{-\left(\frac{\sqrt{3}-1}{2}\right)}^{\frac{\sqrt{3}-1}{2}} \frac{1}{\sqrt{(1-(t)^2)}}\, dt$$

$$\text{let } f(x) = \frac{1}{\sqrt{(1-(t)^2)}} \text{ and } f(-x) = \frac{1}{\sqrt{(1-(-t)^2)}} = \frac{1}{\sqrt{(1-(t)^2)}} = f(x)$$

That is f (x) = f (-x)

So, f(x) is an even function.

It is also known that if f(x) is an even function then, we have

$$\left\{ \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \right\}$$

By using the above formula we get

$$\Rightarrow I = 2.\int_0^{\frac{\sqrt{3}-1}{2}} \frac{1}{\sqrt{(1-(t)^2)}}\, dt$$

On integrating

$$\Rightarrow I = [2.\sin^{-1} t]_0^{\frac{\sqrt{3}-1}{2}}$$

Now by applying the limits

$$\Rightarrow I = 2.\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

29. $\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$

Solution:

Given: $\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$

$$\text{let}, I = \int_0^1 \frac{dx}{\sqrt{1+x} - \sqrt{x}}$$

Now multiply and divide $\sqrt{1+x} + \sqrt{x}$ to the above equation we get

$$= \int_0^1 \frac{1}{\sqrt{1+x} - \sqrt{x}} \times \frac{\sqrt{1+x} + \sqrt{x}}{\sqrt{1+x} + \sqrt{x}} dx$$

$$= \int_0^1 \frac{\sqrt{1+x} + \sqrt{x}}{1+x-x} dx$$

On simplification

$$= \int_0^1 \frac{\sqrt{1+x} + \sqrt{x}}{1} dx$$

Now by splitting the integrals we get

$$= \int_0^1 \sqrt{1+x} dx + \int_0^1 \sqrt{x} dx$$

$$= \int_0^1 ((1+x)^{\frac{1}{2}}) dx + \int_0^1 (x)^{\frac{1}{2}} dx$$

On integrating we get

$$\Rightarrow I = \left[\frac{(1+x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 + \left[\frac{(x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1$$

Now by applying the limits we get

$$= \frac{2}{3}.[(1+1)^{\frac{3}{2}} - (1+0)^{\frac{3}{2}}] + \frac{2}{3}.[(1)^{\frac{3}{2}}]$$

Computing and simplifying we get

$$= \frac{2}{3}.[(2)^{\frac{3}{2}} - (1)^{\frac{3}{2}}] + \frac{2}{3}.[(1)^{\frac{3}{2}}]$$

$= \frac{2}{3}.[(2)^{\frac{3}{2}} - 1] + \frac{2}{3}.[1]$

$= \frac{2}{3}.\left[(2)^{\frac{3}{2}}\right] - \frac{2}{3}.[1] + \frac{2}{3}.[1]$

$= \frac{2}{3}.\left[2\sqrt{2}\right]$

$\Rightarrow \mathrm{I} = \frac{4\sqrt{2}}{3}$

30. $\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$

Solution:

Let $\mathrm{I} = \int_0^{\frac{\pi}{4}} \frac{\mathrm{sinx+cosx}}{9+16\,\mathrm{sin2x}} \mathrm{dx}$

Also, let sin x – cos x = t

Differentiating both sides, we get,

(Cos x + sin x) dx = dt

When x = 0, t = -1

And when $\mathrm{x} = \frac{\pi}{4}$, t = 0

Now, (Sin x – cos x)^2 = t^2

1 – 2 sin x cos x = t^2

Sin 2x = 1 – t^2

Putting all the values, we get the integral,

$\mathrm{I} = \int_{-1}^{0} \frac{\mathrm{dt}}{9 + 16(1 - \mathrm{t}^2)}$

$$I = \int_{-1}^{0} \frac{dt}{25 - 16t^2}$$

The above equation can be written as

$$I = \int_{-1}^{0} \frac{dt}{(5)^2 - (4t)^2}$$

On integrating we get

$$I = \frac{1}{4}\left[\frac{1}{2(5)} \log\left|\frac{5 + 4t}{5 - 4t}\right|\right]_{-1}^{0}$$

Now by applying the limits we get

$$I = \frac{1}{40}\left[\log 1 - \log\frac{1}{9}\right]$$

$$I = \frac{1}{40} \log 9$$

31. $\int_{0}^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\, dx$

Solution:

Given: $\int_{0}^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\ dx$

$$\text{let}, I = \int_{0}^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\ dx$$

$$= \int_{0}^{\frac{\pi}{2}} 2 \sin x \cos x . \tan^{-1}(\sin x)\ dx$$

Let sin x = t ⇒ cos x dx = dt

When x =0 then t = 0 and when x = π /2 then t = 1

Now by substituting these values in above equation we get

$$\Rightarrow \int_0^{\frac{\pi}{2}} 2\sin x\cos x.\tan^{-1}(\sin x)\ dx = \int_0^1 2t.\tan^{-1}(t)\ dt$$

Using product rule

$$\int u.v\,dx = u.\int v\,dx - \int \frac{du}{dx}.\left\{\int vdx\right\}dx$$

$$\Rightarrow 2\int_0^1 t.\tan^{-1}(t)\ dt = 2\left[\tan^{-1}(t).\int t\,dt - \int \frac{d}{dt}(\tan^{-1}(t)).\left\{\int t.dt\right\}dt\right]$$

Computing using product rule we get

$$= 2\left[\tan^{-1}(t).\frac{t^2}{2} - \int \frac{1}{1+t^2}.\frac{t^2}{2}\ dt\right]$$

$$= 2\left[\tan^{-1}(t).\frac{t^2}{2} - \frac{1}{2}.\int \frac{-1+1+t^2}{1+t^2}\ dt\right]$$

Splitting the integrals we get

$$= 2\left[\tan^{-1}(t).\frac{t^2}{2} - \frac{1}{2}.\left\{\int -\frac{1}{1+t^2}\ dt + \int \frac{1+t^2}{1+t^2}\ dt\right\}\right]$$

On simplification we get

$$= 2\left[\tan^{-1}(t).\frac{t^2}{2} - \frac{1}{2}.\left\{\int -\frac{1}{1+t^2}\ dt + \int 1\ dt\right\}\right]$$

$$= 2\left[\tan^{-1}(t).\frac{t^2}{2} - \frac{1}{2}.\{-\tan^{-1}(t) + t\}\right]$$

$$= [t^2.\tan^{-1}(t).-\{-\tan^{-1}(t) + t\}]$$

Computing we get

$$\Rightarrow 2\int_0^1 t.\tan^{-1}(t)\ dt = [t^2.\tan^{-1}(t).-\{-\tan^{-1}(t) + t\}]_0^1$$

Now by applying the limits

$$= [1^2.\tan^{-1}(1).-\{-\tan^{-1}(1) + 1\}] - [0^2.\tan^{-1}(0).-\{-\tan^{-1}(0) + 0\}]$$

$$= \left[1.\frac{\pi}{4}. - \left\{-\frac{\pi}{4} + 1\right\}\right]$$

$$= \left[\frac{\pi}{4} + \frac{\pi}{4} - 1\right]$$

$$\Rightarrow I = \left[\frac{\pi}{2} - 1\right]$$

32. $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$

Solution:

Given: $\int_0^{\pi} \frac{x\tan x}{\sec x + \tan x} dx$

$$\text{let}, I = \int_0^{\pi} \frac{x\tan x}{\sec x + \tan x} dx \quad \dots.(1)$$

As we know that

$$\left\{\int_0^a f(x)dx = \int_0^a f(a-x)dx\right\}$$

Using this in above equation we get

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)} dx$$

Using standard allied angles the above equation can be written as

$$= \int_0^{\pi} \frac{(\pi - x)(-\tan(x))}{(-\sec x) + (-\tan x)} dx$$

$$= \int_0^{\pi} \frac{-(\pi - x)(\tan(x))}{-[(\sec x) + (\tan x)]} dx$$

$$= \int_0^{\pi} \frac{(\pi - x)(\tan(x))}{\sec x + \tan x} dx \quad \dots.(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{x\tan x}{\sec x + \tan x} + \frac{(\pi - x)(\tan(x))}{\sec x + \tan x}\, dx$$

Now by adding we get

$$2I = \int_0^{\pi} \frac{\pi\tan x}{\sec x + \tan x}\, dx$$

Tan x can be written as

$$= \int_0^{\pi} \frac{\pi.\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}}\, dx$$

$$2I = \pi. \int_0^{\pi} \frac{(\sin x)}{(1 + \sin x)}\, dx$$

$$= \pi. \int_0^{\pi} \frac{(-1 + 1 + \sin x)}{(1 + \sin x)}\, dx$$

Now by splitting the integrals we get

$$= \pi. \int_0^{\pi} \frac{(-1)}{(1 + \sin x)}\, dx + \pi. \int_0^{\pi} \frac{(1 + \sin x)}{(1 + \sin x)}\, dx$$

Again by multiplying and dividing above equation by 1 – sin x we get

$$= \pi. \int_0^{\pi} \frac{(-1)}{(1 + \sin x)} \times \frac{(1 - \sin x)}{(1 - \sin x)} dx + \pi. \int_0^{\pi} 1.\, dx$$

Splitting the integrals

$$= -\pi. \int_0^{\pi} \frac{(1 - \sin x)}{(1 - \sin^2 x} dx + \pi. \int_0^{\pi} 1.\, dx$$

$$2I = -\pi. \int_0^{\pi} \frac{(1 - \sin x)}{\cos^2 x}\, dx + \pi. \int_0^{\pi} 1.\, dx$$

$$2I = -\pi. \int_0^{\pi} \left\{\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right\} dx + \pi. \int_0^{\pi} 1.\, dx$$

$$2I = -\pi.\int_0^{\pi}\{\sec^2 x - \tan x \sec x\}\,dx + \pi.\int_0^{\pi} 1.\,dx$$

On integrating we get

$$\Rightarrow 2I = -\pi.[\tan x - \sec x]_0^{\pi} + [x]_0^{\pi}$$

Now by applying the limits we get

$$\Rightarrow 2I = -\pi.[\tan\pi - \sec\pi - \tan 0 + \sec 0] + \pi.[\pi - 0]$$

$$\Rightarrow 2I = -\pi.[0 - (-1) - 0 + 1] + \pi.[\pi]$$

$$\Rightarrow 2I = \pi.[-2 + \pi]$$

$$\Rightarrow I = \frac{\pi}{2}.[\pi - 2]$$

33. $\int_1^4 [|x-1| + |x-2| + |x-3|]\,dx$

Solution:

Given: $\int_1^4 [|x-1| + |x-2| + |x-3|]\,dx$

Let,

$$\Rightarrow I = \int_1^4 [|x-1| + |x-2| + |x-3|]\,dx$$

Now by splitting the integrals we get

$$\Rightarrow I = \int_1^4 [|x-1|]\,dx + \int_1^4 [|x-2|]\,dx + \int_1^4 [|x-3|]\,dx$$

let $I = I_1 + I_2 + I_3$

First solve for I_1:

$$I_1 = \int_1^4 [|x-1|]\,dx$$

As we can see that (x - 1) ≥ 0 when 1 ≤ x ≤ 4

$$\Rightarrow I_1 = \int_1^4 (x-1)\,dx$$

On integrating we get

$$\Rightarrow I_1 = \left[\frac{x^2}{2} - x\right]_0^1$$

Now by applying the limits we get

$$\Rightarrow I_1 = \left[\frac{(4)^2}{2} - 4 - \frac{(1)^2}{2} + 1\right]$$

$$\Rightarrow I_1 = \left[8 - 4 - \frac{1}{2} + 1\right]$$

$$\Rightarrow I_1 = \left[5 - \frac{1}{2}\right]$$

$$\Rightarrow I_1 = \frac{9}{2}$$

Now solve for I_2:

$$I_2 = \int_1^4 [|x-2|]\,dx$$

As we can see that (x - 2) ≤ 0 when 1 ≤ x ≤ 2 and (x – 2) ≥ 0 when 2 ≤ x ≤ 4

As we know that

$$\left\{\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\right\}$$

By using this we get

$$\Rightarrow I_2 = \int_1^2 -(x-2)dx + \int_2^4 (x-2)dx$$

On integrating

$$\Rightarrow I_2 = -\left[\frac{x^2}{2} - 2x\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^4$$

Now by applying the limits we get

$$\Rightarrow I_2 = -\left[\frac{(2)^2}{2} - 2(2) - \frac{(1)^2}{2} + 2(1)\right] + \left[\frac{(4)^2}{2} - 2(4) - \frac{(2)^2}{2} + 2(2)\right]$$

$$\Rightarrow I_2 = -\left[2 - 4 - \frac{1}{2} + 2\right] + [8 - 8 - 2 + 4]$$

$$\Rightarrow I_2 = \left[\frac{1}{2} + 2\right]$$

$$\Rightarrow I_2 = \frac{5}{2}$$

Now solve for I_3:

$$I_3 = \int_1^4 [|x - 3|]\,dx$$

As we can see that (x - 3) ≤ 0 when 1 ≤ x ≤ 3 and (x - 3) ≥ 0 when 3 ≤ x ≤ 4

As we know that

$$\left\{\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\right\}$$

By using above formula we get

$$\Rightarrow I_3 = \int_1^3 -(x-3)dx + \int_3^4 (x-3)dx$$

On integrating we get

$$\Rightarrow I_3 = -\left[\frac{x^2}{2} - 3x\right]_1^3 + \left[\frac{x^2}{2} - 3x\right]_3^4$$

Now by applying the limits

$$\Rightarrow I_3 = -\left[\frac{(3)^2}{2} - 3(3) - \frac{(1)^2}{2} + 3(1)\right] + \left[\frac{(4)^2}{2} - 3(4) - \frac{(3)^2}{2} + 3(3)\right]$$

$$\Rightarrow I_3 = -\left[\frac{9}{2} - 9 - \frac{1}{2} + 3\right] + \left[8 - 12 - \frac{9}{2} + 9\right]$$

$$\Rightarrow I_3 = \left[2 + \frac{1}{2}\right]$$

$$\Rightarrow I_3 = \frac{5}{2}$$

as $I = I_1 + I_2 + I_3$

Substituting the above all values we get

$$\Rightarrow I = \frac{9}{2} + \frac{5}{2} + \frac{5}{2}$$

$$\Rightarrow I = \frac{19}{2}$$

Prove the following (Exercises 34 to 39)

34. $\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log\frac{2}{3}$

Solution:

Given: $\int_1^3 \frac{dx}{(x^2)(x+1)}$

To Prove : $\int_1^3 \frac{dx}{(x^2)(x+1)} = \frac{2}{3} + \log\frac{2}{3}$

Let I $= \frac{dx}{(x^2)(x+1)}$

Using partial fraction

let $\frac{1}{(x^2)(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$ ... (1)

$$\Rightarrow \frac{1}{(x^2)(x+1)} = \frac{A(x)(x+1) + B(x+1) + C(x^2)}{(x+1)(x^2)}$$

$$\Rightarrow 1 = A(x^2 + x) + (Bx + B) + Cx^2$$

$\Rightarrow 1 = Ax^2 + Ax + B + Bx + Cx^2$

$\Rightarrow 1 = B + (A + B)\,x + (A + C)\,x^2$

Equating the coefficients of x, x^2 and constant value. We get

(a) B = 1

(b) $A + B = 0 \Rightarrow A = -B \Rightarrow A = -1$

(c) $A + C = 0 \Rightarrow C = -A \Rightarrow C = 1$

Put these values in equation (1)

$$\Rightarrow \frac{1}{(x^2)(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\Rightarrow \frac{1}{(x^2)(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}$$

Taking integrals on both side we get

$$\Rightarrow \int \frac{1}{(x^2)(x+1)}\,dx = \int -\frac{1}{x}\,dx + \int \frac{1}{(x^2)}\,dx + \int \frac{1}{(x+1)}\,dx$$

$$\Rightarrow \int_1^3 \frac{1}{(x^2)(x+1)}\,dx = [-\log|x| - x^{-1} + \log|x+1|]_1^3$$

$$\Rightarrow \int_1^3 \frac{1}{(x^2)(x+1)}\,dx = \left[-\frac{1}{x} + \log\left|\frac{x+1}{x}\right|\right]_1^3$$

Now by applying the limits we get

$$= \left[-\frac{1}{3} + \log\left|\frac{3+1}{3}\right| - \left(-\frac{1}{1} + \log\left|\frac{1+1}{1}\right|\right)\right]$$

$$= \left[-\frac{1}{3} + \log\left|\frac{4}{3}\right| + \left(1 - \log\left|\frac{2}{1}\right|\right)\right]$$

Computing and simplifying we get

$$= \left[-\frac{1}{3} + 1 + \log\left|\frac{4}{3} \times \frac{1}{2}\right|\right]$$

$$\Rightarrow I = \left[\frac{2}{3} + \log\left|\frac{2}{3}\right|\right]$$

$\Rightarrow$ L.H.S = R.H.S

Hence proved.

35. $\int_0^1 x\, e^x dx = 1$

Solution:

Given: $\int_0^1 xe^x dx$

To Prove : $\int_0^1 xe^x dx = 1$

Let $I = \int_0^1 xe^x dx$

Using product rule we get

$$\int u.v\, dx = u.\int v\, dx - \int \frac{du}{dx}.\left\{\int v dx\right\} dx$$

$$\Rightarrow \int_0^1 xe^x dx = x.\int_0^1 e^x dx - \int_0^1 \frac{dx}{dx}.\left\{\int e^x dx\right\}. dx$$

On integrating

$$\Rightarrow \int_0^1 xe^x dx = [xe^x]_0^1 - \int_0^1 1 . e^x dx$$

Now by applying the limits we get

$$\Rightarrow \int_0^1 xe^x dx = [xe^x]_0^1 - [e^x]_0^1$$

$$\Rightarrow \int_0^1 xe^x dx = [1.e^1 - 0.e^0] - [e^1 - e^0]$$

$$\Rightarrow \int_0^1 xe^x dx = e - 0 - e + 1$$

$$\Rightarrow \int_0^1 xe^x dx = 1$$

Therefore L.H.S = R.H.S

Hence Proved.

36. $\int_{-1}^{1} x^{17} \cos^4 x \, dx = 0$

Solution:

Given: $\int_{-1}^{1} x^{17}.\cos^4 x dx$

To Prove : $\int_{-1}^{1} x^{17}.\cos^4 x dx = 0$

Let I $= \int_{-1}^{1} x^{17}.\cos^4 x dx$

As we can see f(x) $= x^{17}.\cos^4 x$ and f (-x) = $(-x)^{17}.\cos^4(-x) = -x^{17}.\cos^4 x$

That is f (x) = -f (-x)
so, it is an odd function.

It is also known that if f(x) is an odd function then we have

$$\left\{ \int_{-a}^{a} f(x) dx = 0 \right\}$$

$$\Rightarrow I = \int_{-1}^{1} x^{17}.\cos^4 x dx = 0$$

Hence proved.

37. $\int_0^{\frac{\pi}{2}} \sin^3 x \, dx = \frac{2}{3}$

Solution:

Given: $\int_0^{\frac{\pi}{2}} \sin^3 x dx$

To Prove : $\int_0^{\frac{\pi}{2}} \sin^3 x dx = \frac{2}{3}$

Let I $= \int_0^{\frac{\pi}{2}} \sin^3 x dx$ ... (1)

Above equation can be written as

$$= \int_0^{\frac{\pi}{2}} \sin x. \sin^2 x dx$$

$$= \int_0^{\frac{\pi}{2}} \sin x. (1 - \cos^2 x) dx$$

Now by splitting the integrals

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin x \, dx - \int_0^{\frac{\pi}{2}} \sin x. \cos^2 x dx$$

$$\Rightarrow I = [-\cos x]_0^{\pi/2} - I_1 \quad ... (2)$$

First solve for I_1:

$$\Rightarrow I_1 = \int_0^{\frac{\pi}{2}} \sin x. \cos^2 x dx$$

Let cos x = t ⇒ -sin x dx = dt ⇒ sin x dx = - dt

When x = 0 then t = 1 and when x = π /2 then t = 0

$$\Rightarrow I_1 = \int_1^0 t^2(-dt)$$

$$= -\int_1^0 t^2(dt)$$

On integrating we get

$$= -\left[\frac{t^3}{3}\right]_1^0$$

Now by applying the limits we get

$$= -\left\{-\frac{1}{3}\right\}$$

$$\Rightarrow I_1 = \frac{1}{3}$$

Substitute in equation (2)

$$\Rightarrow I = [-\cos x]_0^{\pi/2} - \frac{1}{3}$$

$$\Rightarrow I = -\left\{\cos\frac{\pi}{2} - \cos 0\right\} - \frac{1}{3}$$

$$\Rightarrow I = 1 - \frac{1}{3}$$

$$\Rightarrow I = \frac{2}{3}$$

L.H.S = R.H.S

Hence Proved.

38. $\int_0^{\frac{\pi}{4}} 2\tan^3 x\, dx = 1 - \log 2$

Solution:

Given: $\int_0^{\frac{\pi}{4}} 2\tan^3 x dx$

To Prove : $\int_0^{\frac{\pi}{4}} 2\tan^3 x dx = 1 - \log 2$

Let $I = \int_0^{\frac{\pi}{4}} 2\tan^3 x dx$ ... (1)

The above equation can be written as

$$= \int_0^{\frac{\pi}{4}} 2.\tan x.\tan^2 x dx$$

Substituting $\tan^2 x$ formula we get

$$= 2.\int_0^{\frac{\pi}{4}} \tan x.(\sec^2 x - 1) dx$$

Now by splitting the integral we get

$$\Rightarrow I = 2\left\{-\int_0^{\frac{\pi}{4}} \tan x\, dx + \int_0^{\frac{\pi}{4}} \tan x.\sec^2 x\, dx\right\}$$

$$\Rightarrow I = -[2\log \sec x]_0^{\pi/4} + 2.I_1 \quad ...(2)$$

First solve for I_1:

$$\Rightarrow I_1 = \int_0^{\frac{\pi}{4}} \tan x.\sec^2 x\, dx$$

Let $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

When x=0 then t= 0 and when x = π /2 then t = 1

$$\Rightarrow I_1 = \int_0^1 t.\, dt$$

On integrating we get

$$= \left[\frac{t^2}{2}\right]_0^1$$

Applying the limits we get

$$\Rightarrow I_1 = \frac{1}{2}$$

Substitute in equation (2)

$$\Rightarrow I = [2\log\cos x]_0^{\pi/4} + 2.\frac{1}{2}$$

On simplification we get

$$\Rightarrow I = 2\left\{\log\cos\frac{\pi}{4} - \log\cos 0\right\} + 1$$

Substituting the values of cos 0 = 1 we get

$$\Rightarrow I = 2\left\{\log\frac{1}{\sqrt{2}} - \log 1\right\} + 1$$

$$\Rightarrow I = \left\{\log\left(\frac{1}{\sqrt{2}}\right)^2 - \log(1)^2\right\} + 1$$

$$\Rightarrow I = 1 - \log 2 + \log 1$$

$$\Rightarrow I = 1 - \log 2$$

L.H.S = R.H.S

Hence the proof.

39. $\int_0^1 \sin^{-1} x\, dx = \frac{\pi}{2} - 1$

Solution:

Given: $\int_0^1 \sin^{-1} x \, dx$

To Prove : $\int_0^1 \sin^{-1} x \, dx = \frac{\pi}{2} - 1$

Let $I = \int_0^1 \sin^{-1} x . 1dx$

Using product rule we get

$$\int u.v\,dx = u.\int v\,dx - \int \frac{du}{dx}.\left\{\int vdx\right\} dx$$

$$\Rightarrow \int_0^1 xe^x dx = \sin^{-1} x . \int_0^1 1.dx - \int_0^1 \frac{d}{dx}\sin^{-1} x . \left\{\int 1.dx\right\}.dx$$

On integrating we get

$$\Rightarrow \int_0^1 xe^x dx = [\sin^{-1} x . x]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}}.x\,dx$$

$$\Rightarrow I = [\sin^{-1} x . x]_0^1 - I_1 \quad ... (2)$$

First solve for I_1:

$$\Rightarrow I_1 = \int_0^1 \frac{1}{\sqrt{1-x^2}}.x\,dx$$

Let $1 - x^2 = t \Rightarrow -2\,x\,dx = dt$

When $x = 0$ then $t = 1$ and when $x = 1$ then $t = 0$

$$\Rightarrow I_1 = \int_1^0 \frac{1}{\sqrt{t}}.\frac{-dt}{2}$$

$$= -\frac{1}{2}\left[\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^0$$

$$\Rightarrow I_1 = \sqrt{1}$$

$\Rightarrow I_1 = 1$

Substitute in equation (2)

$\Rightarrow I = [\sin^{-1} x . x]_0^1 - 1$

$\Rightarrow I = \sin^{-1}(1) - 0 - 1$

$\Rightarrow I = \frac{\pi}{2} - 1$

L.H.S = R.H.S

Hence Proved.

40. Evaluate $\int_0^1 e^{2-3x} dx$ as a limit of a sum.

Solution:

Given: $\int_0^1 e^{2-3x} dx$

Let $I = \int_0^1 e^{2-3x} dx$

because, $\int_a^b f(x) dx = (b-a) \lim_{n\to\infty} \frac{1}{n}[f(a) + f(a+h) + \cdots + f(a+(n-1)h)]$

where, $h = \frac{b-a}{n}$

Here, a = 0, b = 1, and f (x) = e^{2-3x} and h

$= \lim_{n\to\infty} \frac{1}{n}[e^2 + e^2 . e^{3h} + + e^2 . e^{-6h} \ldots + e^2 . e^{-3(n-1)h}]$

$= \lim_{n\to\infty} \frac{1}{n}[e^2\{1 + e^{3h} + e^{-6h} + \cdots + . e^{-3(n-1)h}\}]$

$= \lim_{n\to\infty} \frac{1}{n}\left[e^2\left\{\frac{1-(e^{-3h})^n}{1-(e^{-3h})}\right\}\right]$

$$= \lim_{n\to\infty} \frac{1}{n}\left[e^2\left\{\frac{1-\left(e^{-\frac{3}{n}}\right)^n}{1-\left(e^{-\frac{3}{n}}\right)}\right\}\right] \text{ as, } h = \frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[e^2\left\{\frac{(e^{-3})-1}{\left(e^{-\frac{3}{n}}\right)-1}\right\}\right]$$

$$= e^2.(e^{-3}-1)\lim_{n\to\infty}\frac{1}{n}.\left(-\frac{n}{3}\right)\left[\left\{\frac{-\frac{3}{n}}{\left(e^{-\frac{3}{n}}\right)-1}\right\}\right]$$

On simplification we get

$$= -\frac{(e^2.(e^{-3}-1))}{3}\lim_{n\to\infty}\left[\left\{\frac{-\frac{3}{n}}{\left(e^{-\frac{3}{n}}\right)-1}\right\}\right]$$

We know that

$$\lim_{n\to\infty}\left[\frac{x}{(e^x)-1}\right] = 1$$

Substituting this in above equation we get

$$= \frac{-e^{-1}+e^2}{3}(1)$$

$$\Rightarrow I = \frac{1}{3}\left(e^2 - \frac{1}{e}\right)$$

Choose the correct answers in Exercises 41 to 44.

41. $\int \frac{dx}{e^x + e^{-x}}$ is equal to

(A) $\tan^{-1}(e^x) + C$ (B) $\tan^{-1}(e^{-x}) + C$

(C) $\log(e^x - e^{-x}) + C$ (D) $\log(e^x + e^{-x}) + C$

Solution:

(A) $\tan^{-1}(e^x) + C$

Explanation:

Given: $\int \frac{dx}{e^x+e^{-x}}$

$$\text{let } I = \int \frac{dx}{e^x + e^{-x}}$$

The above equation can be written as

$$= \int \frac{dx}{e^{-x}(e^{2x} + 1)}$$

$$= \int \frac{e^x dx}{(e^{2x} + 1)}$$

Put $e^x = t \Rightarrow e^x\, dx = dt$

$$\Rightarrow \int \frac{e^x dx}{(e^{2x} + 1)} = \int \frac{dt}{(t^2 + 1)}$$

$$= \tan^{-1} t + C$$

$$= \tan^{-1}(e^x) + C$$

Hence, correct option is (A).

42. $\int \frac{\cos 2x}{(\sin x + \cos x)^2} dx$ is equal to

(A) $\frac{-1}{\sin x + \cos x} + C$

(B) $\log|\sin x + \cos x| + C$

(C) $\log|\sin x - \cos x| + C$

(D) $\frac{1}{(\sin x + \cos x)^2}$

Solution:

(B) log |sin x + cos x| + C

Explanation:

Given: $\int \frac{\cos 2x}{(\sin x+\cos x)^2}dx$

$$\text{let } I = \int \frac{\cos 2x}{(\sin x + \cos x)^2}dx$$

Substituting cos 2x formula we get

$$= \int \frac{\cos^2 x - \sin^2 x}{(\sin x + \cos x)^2}dx$$

By using $a^2 - b^2 = (a + b)(a - b)$ we get

$$= \int \frac{(\cos x - \sin x)(\cos x + \sin x)}{(\sin x + \cos x)^2}dx$$

On simplification

$$= \int \frac{(\cos x - \sin x)}{(\sin x + \cos x)}dx$$

Put $\sin x + \cos x = t \Rightarrow \cos x - \sin x = dt$

$$\Rightarrow \int \frac{(\cos x - \sin x)}{(\sin x + \cos x)}dx = \int \frac{dt}{t}$$

$$= \log|t| + C$$

$$= \log|\sin x + \cos x| + C$$

Hence, correct option is (B).

43. If $f(a+b-x) = f(x)$, then $\int_a^b x\, f(x)\, dx$ is equal to

(A) $\frac{a+b}{2}\int_a^b f(b-x)\,dx$ (B) $\frac{a+b}{2}\int_a^b f(b+x)\,dx$

(C) $\frac{b-a}{2}\int_a^b f(x)\,dx$ (D) $\frac{a+b}{2}\int_a^b f(x)\,dx$

Solution:

(D) $\frac{a+b}{2}\int_a^b f(x)\,dx$

Explanation:

Given: $\int_a^b x\, f(x)\, dx$

$$\text{let}, I = \int_a^b x\, f(x)\, dx$$

As we know that

$\{ f(x) = f(a+b-x)\}$

Using this we get

$$\Rightarrow I = \int_a^b (a+b-x)\, f(a+b-x)\, dx$$

$$\Rightarrow I = \int_a^b (a+b-x)\, f(x)\, dx$$

Now by splitting the integral we get

$$\Rightarrow I = \int_a^b (a+b)\, f(x)\, dx - \int_a^b (x)\, f(x)\, dx$$

$$\Rightarrow I = \int_a^b (a+b)\, f(x)\, dx - I$$

$$\Rightarrow 2I = \int_a^b (a+b)\, f(x)\, dx$$

$$\Rightarrow I = \frac{(a+b)}{2}\int_a^b f(x)\, dx$$

Hence, correct option is (D).

44. The value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$ is

(A) 1 (B) 0 (C) –1 (D) π

Solution:

(B) 0

Explanation:

Given: $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$

$$\text{Let } I = \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$$

The above equation can be written as

$$= \int_0^1 \tan^{-1}\left(\frac{x+x-1}{1+x(1-x)}\right)dx$$

$$= \int_0^1 \tan^{-1}\left(\frac{x-(1-x)}{1+x(1-x)}\right)dx$$

As we know that

$$\tan^{-1}\left(\frac{A-B}{1+AB)}\right) = \tan^{-1}(A)\tan^{-1}(B)$$

By using this formula we get

$$= \int_0^1 [\tan^{-1}(x) - \tan^{-1}(1-x)]\,dx \;\ldots(1)$$

Again as we know that

$$\left\{\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right\}$$

By using this we can write as

$$= \int_0^1 [\tan^{-1}(1-x) - \tan^{-1}(1-(1-x))]\,dx$$

$$= \int_0^1 [\tan^{-1}(1-x) - \tan^{-1}(x)]\,dx \ \dots (2)$$

Adding (1) and (2), we get

$$2I = \int_0^1 [\tan^{-1}(x) - \tan^{-1}(1-x)]\,dx + \int_0^1 [\tan^{-1}(1-x) - \tan^{-1}(x)]\,dx$$

$$2I = \int_0^1 [\tan^{-1}(x) - \tan^{-1}(1-x) + \tan^{-1}(1-x) - \tan^{-1}(x)]\,dx$$

$$\Rightarrow 2I = 0$$

$$\Rightarrow I = 0$$

Hence, correct option is (B).