

EXERCISE 13.1
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1. Evaluate the following:

(i) i^{457}

(ii) i^{528}

(iii) $1/i^{58}$

(iv) $i^{37} + 1/i^{67}$

(v) $[i^{41} + 1/i^{257}]$

(vi) $(i^{77} + i^{70} + i^{87} + i^{414})^3$

(vii) $i^{30} + i^{40} + i^{60}$

(viii) $i^{49} + i^{68} + i^{89} + i^{110}$

Solution:

(i) i^{457}

Let us simplify we get,

$$\begin{aligned} i^{457} &= i^{4(114) + 1} \\ &= i^{4(114)} \times i \\ &= (1)^{114} \times i \\ &= i \text{ [since } i^4 = 1] \end{aligned}$$

(ii) i^{528}

Let us simplify we get,

$$\begin{aligned} i^{528} &= i^{4(132)} \\ &= (1)^{132} \\ &= 1 \text{ [since } i^4 = 1] \end{aligned}$$

(iii) $1/i^{58}$

Let us simplify we get,

$$\begin{aligned} 1/i^{58} &= 1/i^{56+2} \\ &= 1/i^{56} \times i^2 \\ &= 1/(i^4)^{14} \times i^2 \\ &= 1/i^2 \text{ [since, } i^4 = 1] \\ &= 1/-1 \text{ [since, } i^2 = -1] \\ &= -1 \end{aligned}$$

(iv) $i^{37} + 1/i^{67}$

Let us simplify we get,

$$\begin{aligned} i^{37} + 1/i^{67} &= i^{36+1} + 1/i^{64+3} \\ &= i + 1/i^3 \text{ [since, } i^4 = 1] \\ &= i + i/i^4 \end{aligned}$$

$$= i + i$$

$$= 2i$$

(v) $[i^{41} + 1/i^{257}]$

Let us simplify we get,

$$\begin{aligned} [i^{41} + 1/i^{257}] &= [i^{40+1} + 1/i^{256+1}] \\ &= [i + 1/i]^9 \text{ [since, } 1/i = -1] \\ &= [i - i] \\ &= 0 \end{aligned}$$

(vi) $(i^{77} + i^{70} + i^{87} + i^{414})^3$

Let us simplify we get,

$$\begin{aligned} (i^{77} + i^{70} + i^{87} + i^{414})^3 &= (i^{(76+1)} + i^{(68+2)} + i^{(84+3)} + i^{(412+2)})^3 \\ &= (i + i^2 + i^3 + i^2)^3 \text{ [since } i^3 = -i, i^2 = -1] \\ &= (i + (-1) + (-i) + (-1))^3 \\ &= (-2)^3 \\ &= -8 \end{aligned}$$

(vii) $i^{30} + i^{40} + i^{60}$

Let us simplify we get,

$$\begin{aligned} i^{30} + i^{40} + i^{60} &= i^{(28+2)} + i^{40} + i^{60} \\ &= (i^4)^7 i^2 + (i^4)^{10} + (i^4)^{15} \\ &= i^2 + 1^{10} + 1^{15} \\ &= -1 + 1 + 1 \\ &= 1 \end{aligned}$$

(viii) $i^{49} + i^{68} + i^{89} + i^{110}$

Let us simplify we get,

$$\begin{aligned} i^{49} + i^{68} + i^{89} + i^{110} &= i^{(48+1)} + i^{68} + i^{(88+1)} + i^{(116+2)} \\ &= (i^4)^{12} \times i + (i^4)^{17} + (i^4)^{11} \times i + (i^4)^{29} \times i^2 \\ &= i + 1 + i - 1 \\ &= 2i \end{aligned}$$

2. Show that $1 + i^{10} + i^{20} + i^{30}$ is a real number?

Solution:

Given:

$$\begin{aligned} 1 + i^{10} + i^{20} + i^{30} &= 1 + i^{(8+2)} + i^{20} + i^{(28+2)} \\ &= 1 + (i^4)^2 \times i^2 + (i^4)^5 + (i^4)^7 \times i^2 \\ &= 1 - 1 + 1 - 1 \text{ [since, } i^4 = 1, i^2 = -1] \\ &= 0 \end{aligned}$$

Hence, $1 + i^{10} + i^{20} + i^{30}$ is a real number.

3. Find the values of the following expressions:

(i) $i^{49} + i^{68} + i^{89} + i^{110}$

(ii) $i^{30} + i^{80} + i^{120}$

(iii) $i + i^2 + i^3 + i^4$

(iv) $i^5 + i^{10} + i^{15}$

(v) $[i^{592} + i^{590} + i^{588} + i^{586} + i^{584}] / [i^{582} + i^{580} + i^{578} + i^{576} + i^{574}]$

(vi) $1 + i^2 + i^4 + i^6 + i^8 + \dots + i^{20}$

(vii) $(1 + i)^6 + (1 - i)^3$

Solution:

(i) $i^{49} + i^{68} + i^{89} + i^{110}$

Let us simplify we get,

$$\begin{aligned} i^{49} + i^{68} + i^{89} + i^{110} &= i^{(48+1)} + i^{68} + i^{(88+1)} + i^{(108+2)} \\ &= (i^4)^{12} \times i + (i^4)^{17} + (i^4)^{11} \times i + (i^4)^{27} \times i^2 \\ &= i + 1 + i - 1 \text{ [since } i^4 = 1, i^2 = -1] \\ &= 2i \end{aligned}$$

$\therefore i^{49} + i^{68} + i^{89} + i^{110} = 2i$

(ii) $i^{30} + i^{80} + i^{120}$

Let us simplify we get,

$$\begin{aligned} i^{30} + i^{80} + i^{120} &= i^{(28+2)} + i^{80} + i^{120} \\ &= (i^4)^7 \times i^2 + (i^4)^{20} + (i^4)^{30} \\ &= -1 + 1 + 1 \text{ [since } i^4 = 1, i^2 = -1] \\ &= 1 \end{aligned}$$

$\therefore i^{30} + i^{80} + i^{120} = 1$

(iii) $i + i^2 + i^3 + i^4$

Let us simplify we get,

$$\begin{aligned} i + i^2 + i^3 + i^4 &= i + i^2 + i^2 \times i + i^4 \\ &= i - 1 + (-1) \times i + 1 \text{ [since } i^4 = 1, i^2 = -1] \\ &= i - 1 - i + 1 \\ &= 0 \end{aligned}$$

$\therefore i + i^2 + i^3 + i^4 = 0$

(iv) $i^5 + i^{10} + i^{15}$

Let us simplify we get,

$$\begin{aligned} i^5 + i^{10} + i^{15} &= i^{(4+1)} + i^{(8+2)} + i^{(12+3)} \\ &= (i^4)^1 \times i + (i^4)^2 \times i^2 + (i^4)^3 \times i^3 \end{aligned}$$

$$\begin{aligned}
 &= (i^4)^1 \times i + (i^4)^2 \times i^2 + (i^4)^3 \times i^2 \times i \\
 &= 1 \times i + 1 \times (-1) + 1 \times (-1) \times i \\
 &= i - 1 - i \\
 &= -1
 \end{aligned}$$

$$\therefore i^5 + i^{10} + i^{15} = -1$$

$$(v) [i^{592} + i^{590} + i^{588} + i^{586} + i^{584}] / [i^{582} + i^{580} + i^{578} + i^{576} + i^{574}]$$

Let us simplify we get,

$$\begin{aligned}
 &[i^{592} + i^{590} + i^{588} + i^{586} + i^{584}] / [i^{582} + i^{580} + i^{578} + i^{576} + i^{574}] \\
 &= [i^{10} (i^{582} + i^{580} + i^{578} + i^{576} + i^{574})] / (i^{582} + i^{580} + i^{578} + i^{576} + i^{574}) \\
 &= i^{10} \\
 &= i^8 i^2 \\
 &= (i^4)^2 i^2 \\
 &= (1)^2 (-1) \text{ [since } i^4 = 1, i^2 = -1] \\
 &= -1
 \end{aligned}$$

$$\therefore [i^{592} + i^{590} + i^{588} + i^{586} + i^{584}] / [i^{582} + i^{580} + i^{578} + i^{576} + i^{574}] = -1$$

$$(vi) 1 + i^2 + i^4 + i^6 + i^8 + \dots + i^{20}$$

Let us simplify we get,

$$\begin{aligned}
 1 + i^2 + i^4 + i^6 + i^8 + \dots + i^{20} &= 1 + (-1) + 1 + (-1) + 1 + \dots + 1 \\
 &= 1
 \end{aligned}$$

$$\therefore 1 + i^2 + i^4 + i^6 + i^8 + \dots + i^{20} = 1$$

$$(vii) (1 + i)^6 + (1 - i)^3$$

Let us simplify we get,

$$\begin{aligned}
 (1 + i)^6 + (1 - i)^3 &= \{(1 + i)^2\}^3 + (1 - i)^2 (1 - i) \\
 &= \{1 + i^2 + 2i\}^3 + (1 + i^2 - 2i)(1 - i) \\
 &= \{1 - 1 + 2i\}^3 + (1 - 1 - 2i)(1 - i) \\
 &= (2i)^3 + (-2i)(1 - i) \\
 &= 8i^3 + (-2i) + 2i^2 \\
 &= -8i - 2i - 2 \text{ [since } i^3 = -i, i^2 = -1] \\
 &= -10i - 2 \\
 &= -2(1 + 5i) \\
 &= -2 - 10i
 \end{aligned}$$

$$\therefore (1 + i)^6 + (1 - i)^3 = -2 - 10i$$

EXERCISE 13.2
PAGE NO: 13.31
1. Express the following complex numbers in the standard form $a + ib$:

(i) $(1 + i)(1 + 2i)$

(ii) $(3 + 2i) / (-2 + i)$

(iii) $1/(2 + i)^2$

(iv) $(1 - i) / (1 + i)$

(v) $(2 + i)^3 / (2 + 3i)$

(vi) $[(1 + i)(1 + \sqrt{3}i)] / (1 - i)$

(vii) $(2 + 3i) / (4 + 5i)$

(viii) $(1 - i)^3 / (1 - i^3)$

(ix) $(1 + 2i)^{-3}$

(x) $(3 - 4i) / [(4 - 2i)(1 + i)]$

(xi)

$$\left(\frac{1}{1 - 4i} - \frac{2}{1 + i} \right) \left(\frac{3 - 4i}{5 + i} \right)$$

(xii) $(5 + \sqrt{2}i) / (1 - \sqrt{2}i)$

Solution:

(i) $(1 + i)(1 + 2i)$

 Let us simplify and express in the standard form of $(a + ib)$,

$$\begin{aligned} (1 + i)(1 + 2i) &= (1+i)(1+2i) \\ &= 1(1+2i)+i(1+2i) \\ &= 1+2i+i+2i^2 \\ &= 1+3i+2(-1) \text{ [since, } i^2 = -1\text{]} \\ &= 1+3i-2 \\ &= -1+3i \end{aligned}$$

 \therefore The values of a, b are $-1, 3$.

(ii) $(3 + 2i) / (-2 + i)$

 Let us simplify and express in the standard form of $(a + ib)$,

$$\begin{aligned} (3 + 2i) / (-2 + i) &= [(3 + 2i) / (-2 + i)] \times (-2-i) / (-2-i) \text{ [multiply and divide with } (-2-i)\text{]} \\ &= [3(-2-i) + 2i(-2-i)] / [(-2)^2 - (i)^2] \\ &= [-6 - 3i - 4i - 2i^2] / (4 - i^2) \\ &= [-6 - 7i - 2(-1)] / (4 - (-1)) \text{ [since, } i^2 = -1\text{]} \\ &= [-4 - 7i] / 5 \end{aligned}$$

 \therefore The values of a, b are $-4/5, -7i/5$

(iii) $1/(2 + i)^2$

 Let us simplify and express in the standard form of $(a + ib)$,

$$\begin{aligned}
 1/(2+i)^2 &= 1/(2^2 + i^2 + 2(2)(i)) \\
 &= 1/(4 - 1 + 4i) \text{ [since, } i^2 = -1] \\
 &= 1/(3 + 4i) \text{ [multiply and divide with } (3 - 4i)] \\
 &= 1/(3 + 4i) \times (3 - 4i)/(3 - 4i) \\
 &= (3-4i)/(3^2 - (4i)^2) \\
 &= (3-4i)/(9 - 16i^2) \\
 &= (3-4i)/(9 - 16(-1)) \text{ [since, } i^2 = -1] \\
 &= (3-4i)/25
 \end{aligned}$$

∴ The values of a, b are $3/25$, $-4i/25$

(iv) $(1 - i) / (1 + i)$

Let us simplify and express in the standard form of $(a + ib)$,

$$\begin{aligned}
 (1 - i) / (1 + i) &= (1 - i) / (1 + i) \times (1-i)/(1-i) \text{ [multiply and divide with } (1-i)] \\
 &= (1^2 + i^2 - 2(1)(i)) / (1^2 - i^2) \\
 &= (1 + (-1) - 2i) / (1 - (-1)) \\
 &= -2i/2 \\
 &= -i
 \end{aligned}$$

∴ The values of a, b are 0, -i

(v) $(2 + i)^3 / (2 + 3i)$

Let us simplify and express in the standard form of $(a + ib)$,

$$\begin{aligned}
 (2 + i)^3 / (2 + 3i) &= (2^3 + i^3 + 3(2)^2(i) + 3(i)^2(2)) / (2 + 3i) \\
 &= (8 + (i^2 \cdot i) + 3(4)(i) + 6i^2) / (2 + 3i) \\
 &= (8 + (-1)i + 12i + 6(-1)) / (2 + 3i) \\
 &= (2 + 11i) / (2 + 3i) \\
 &\text{ [multiply and divide with } (2-3i)] \\
 &= (2 + 11i)/(2 + 3i) \times (2-3i)/(2-3i) \\
 &= [2(2-3i) + 11i(2-3i)] / (2^2 - (3i)^2) \\
 &= (4 - 6i + 22i - 33i^2) / (4 - 9i^2) \\
 &= (4 + 16i - 33(-1)) / (4 - 9(-1)) \text{ [since, } i^2 = -1] \\
 &= (37 + 16i) / 13
 \end{aligned}$$

∴ The values of a, b are $37/13$, $16i/13$

(vi) $[(1 + i) (1 + \sqrt{3}i)] / (1 - i)$

Let us simplify and express in the standard form of $(a + ib)$,

$$\begin{aligned}
 [(1 + i) (1 + \sqrt{3}i)] / (1 - i) &= [1(1 + \sqrt{3}i) + i(1 + \sqrt{3}i)] / (1 - i) \\
 &= (1 + \sqrt{3}i + i + \sqrt{3}i^2) / (1 - i) \\
 &= (1 + (\sqrt{3}+1)i + \sqrt{3}(-1)) / (1 - i) \text{ [since, } i^2 = -1] \\
 &= [(1 - \sqrt{3}) + (1 + \sqrt{3})i] / (1 - i)
 \end{aligned}$$

$$\begin{aligned}
 & \text{[multiply and divide with } (1+i)\text{]} \\
 & = [(1-\sqrt{3}) + (1+\sqrt{3})i] / (1-i) \times (1+i)/(1+i) \\
 & = [(1-\sqrt{3})(1+i) + (1+\sqrt{3})i(1+i)] / (1^2 - i^2) \\
 & = [1-\sqrt{3} + (1-\sqrt{3})i + (1+\sqrt{3})i + (1+\sqrt{3})i^2] / (1-(-1)) \text{ [since, } i^2 = -1\text{]} \\
 & = [(1-\sqrt{3})+(1-\sqrt{3}+1+\sqrt{3})i+(1+\sqrt{3})(-1)] / 2 \\
 & = (-2\sqrt{3} + 2i) / 2 \\
 & = -\sqrt{3} + i
 \end{aligned}$$

∴ The values of a, b are $-\sqrt{3}, i$

(vii) $(2 + 3i) / (4 + 5i)$

Let us simplify and express in the standard form of $(a + ib)$,

$$\begin{aligned}
 (2 + 3i) / (4 + 5i) &= \text{[multiply and divide with } (4-5i)\text{]} \\
 &= (2 + 3i) / (4 + 5i) \times (4-5i)/(4-5i) \\
 &= [2(4-5i) + 3i(4-5i)] / (4^2 - (5i)^2) \\
 &= [8 - 10i + 12i - 15i^2] / (16 - 25i^2) \\
 &= [8+2i-15(-1)] / (16 - 25(-1)) \text{ [since, } i^2 = -1\text{]} \\
 &= (23 + 2i) / 41
 \end{aligned}$$

∴ The values of a, b are $23/41, 2i/41$

(viii) $(1 - i)^3 / (1 - i^3)$

Let us simplify and express in the standard form of $(a + ib)$,

$$\begin{aligned}
 (1 - i)^3 / (1 - i^3) &= [1^3 - 3(1)^2i + 3(1)(i)^2 - i^3] / (1 - i^2 \cdot i) \\
 &= [1 - 3i + 3(-1) - i^2 \cdot i] / (1 - (-1)i) \text{ [since, } i^2 = -1\text{]} \\
 &= [-2 - 3i - (-1)i] / (1+i) \\
 &= [-2-4i] / (1+i) \\
 & \text{[Multiply and divide with } (1-i)\text{]} \\
 &= [-2-4i] / (1+i) \times (1-i)/(1-i) \\
 &= [-2(1-i)-4i(1-i)] / (1^2 - i^2) \\
 &= [-2+2i-4i+4i^2] / (1 - (-1)) \\
 &= [-2-2i+4(-1)] / 2 \\
 &= (-6-2i)/2 \\
 &= -3 - i
 \end{aligned}$$

∴ The values of a, b are $-3, -i$

(ix) $(1 + 2i)^{-3}$

Let us simplify and express in the standard form of $(a + ib)$,

$$\begin{aligned}
 (1 + 2i)^{-3} &= 1/(1 + 2i)^3 \\
 &= 1/(1^3+3(1)^2(2i)+2(1)(2i)^2 + (2i)^3) \\
 &= 1/(1+6i+4i^2+8i^3)
 \end{aligned}$$

$$\begin{aligned}
 &= 1/(1+6i+4(-1)+8i^2.i) \text{ [since, } i^2 = -1] \\
 &= 1/(-3+6i+8(-1)i) \text{ [since, } i^2 = -1] \\
 &= 1/(-3-2i) \\
 &= -1/(3+2i) \\
 &\text{[Multiply and divide with } (3-2i)] \\
 &= -1/(3+2i) \times (3-2i)/(3-2i) \\
 &= (-3+2i)/(3^2 - (2i)^2) \\
 &= (-3+2i) / (9-4i^2) \\
 &= (-3+2i) / (9-4(-1)) \\
 &= (-3+2i) / 13
 \end{aligned}$$

∴ The values of a, b are $-3/13, 2i/13$

(x) $(3 - 4i) / [(4 - 2i)(1 + i)]$

Let us simplify and express in the standard form of $(a + ib)$,

$$\begin{aligned}
 (3 - 4i) / [(4 - 2i)(1 + i)] &= (3-4i) / [4(1+i)-2i(1+i)] \\
 &= (3-4i) / [4+4i-2i-2i^2] \\
 &= (3-4i) / [4+2i-2(-1)] \text{ [since, } i^2 = -1] \\
 &= (3-4i) / (6+2i) \\
 &\text{[Multiply and divide with } (6-2i)] \\
 &= (3-4i) / (6+2i) \times (6-2i)/(6-2i) \\
 &= [3(6-2i)-4i(6-2i)] / (6^2 - (2i)^2) \\
 &= [18 - 6i - 24i + 8i^2] / (36 - 4i^2) \\
 &= [18 - 30i + 8(-1)] / (36 - 4(-1)) \text{ [since, } i^2 = -1] \\
 &= [10-30i] / 40 \\
 &= (1 - 3i) / 4
 \end{aligned}$$

∴ The values of a, b are $1/4, -3i/4$

(xi)

$$\left(\frac{1}{1-4i} - \frac{2}{1+i} \right) \left(\frac{3-4i}{5+i} \right)$$

Let us simplify and express in the standard form of $(a + ib)$,

$$\begin{aligned}
 \left(\frac{1}{1-4i} - \frac{2}{1+i} \right) \left(\frac{3-4i}{5+i} \right) &= \left(\frac{1+i-2(1-4i)}{(1-4i)(1+i)} \right) \left(\frac{3-4i}{5+i} \right) \\
 &= \left(\frac{1+i-2+8i}{1(1+i)-4i(1+i)} \right) \left(\frac{3-4i}{5+i} \right) \\
 &= \left(\frac{-1+9i}{1+i-4i-4i^2} \right) \left(\frac{3-4i}{5+i} \right) \\
 &= \left(\frac{-1+9i}{1-3i-4(-1)} \right) \left(\frac{3-4i}{5+i} \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{(-1+9i)(3-4i)}{(5-3i)(5+i)} \\
 &= \frac{-1(3-4i)+9i(3-i)}{5(5+i)-3i(5+i)} \\
 &= \frac{-3+4i+27i-9i^2}{25+5i-15i-3i^2} \\
 &= \frac{-3+31i-9(-1)}{25-10i-3(-1)} \\
 &= \frac{6+31i}{28-10i}
 \end{aligned}$$

[Multiply and divide with (28+10i)]

$$\begin{aligned}
 &= \frac{6+31i}{28-10i} \times \frac{28+10i}{28+10i} \\
 &= \frac{6(28+10i)+31i(28+10i)}{28^2-(10i)^2} \\
 &= \frac{168+60i+868i+310i^2}{784-100i^2} \\
 &= \frac{168+928i+310(-1)}{784-100(-1)} \\
 &= \frac{478+928i}{884}
 \end{aligned}$$

∴ The values of a, b are $478/884$, $928i/884$

(xii) $(5 + \sqrt{2}i) / (1 - \sqrt{2}i)$

Let us simplify and express in the standard form of $(a + ib)$,

$$\begin{aligned}
 (5 + \sqrt{2}i) / (1 - \sqrt{2}i) &= [\text{Multiply and divide with } (1 + \sqrt{2}i)] \\
 &= (5 + \sqrt{2}i) / (1 - \sqrt{2}i) \times (1 + \sqrt{2}i) / (1 + \sqrt{2}i) \\
 &= [5(1 + \sqrt{2}i) + \sqrt{2}i(1 + \sqrt{2}i)] / (1^2 - (\sqrt{2})^2) \\
 &= [5 + 5\sqrt{2}i + \sqrt{2}i + 2i^2] / (1 - 2i^2) \\
 &= [5 + 6\sqrt{2}i + 2(-1)] / (1 - 2(-1)) \quad [\text{since, } i^2 = -1] \\
 &= [3 + 6\sqrt{2}i] / 3 \\
 &= 1 + 2\sqrt{2}i
 \end{aligned}$$

∴ The values of a, b are 1, $2\sqrt{2}i$

2. Find the real values of x and y, if

(i) $(x + iy)(2 - 3i) = 4 + i$

(ii) $(3x - 2iy)(2 + i)^2 = 10(1 + i)$

(iii) $\frac{(1 + i)x - 2i}{3 + i} + \frac{(2 - 3i)y + i}{3 - i} = i$

$$(iv) (1 + i)(x + iy) = 2 - 5i$$

Solution:

$$(i) (x + iy)(2 - 3i) = 4 + i$$

Given:

$$(x + iy)(2 - 3i) = 4 + i$$

Let us simplify the expression we get,

$$x(2 - 3i) + iy(2 - 3i) = 4 + i$$

$$2x - 3xi + 2yi - 3yi^2 = 4 + i$$

$$2x + (-3x+2y)i - 3y(-1) = 4 + i \text{ [since, } i^2 = -1]$$

$$2x + (-3x+2y)i + 3y = 4 + i \text{ [since, } i^2 = -1]$$

$$(2x+3y) + i(-3x+2y) = 4 + i$$

Equating Real and Imaginary parts on both sides, we get

$$2x+3y = 4 \dots (i)$$

$$\text{And } -3x+2y = 1 \dots (ii)$$

Multiply (i) by 3 and (ii) by 2 and add

On solving we get,

$$6x - 6x - 9y + 4y = 12 + 2$$

$$13y = 14$$

$$y = 14/13$$

Substitute the value of y in (i) we get,

$$2x+3y = 4$$

$$2x + 3(14/13) = 4$$

$$2x = 4 - (42/13)$$

$$= (52-42)/13$$

$$2x = 10/13$$

$$x = 5/13$$

$$x = 5/13, y = 14/13$$

∴ The real values of x and y are 5/13, 14/13

$$(ii) (3x - 2iy)(2 + i)^2 = 10(1 + i)$$

Given:

$$(3x - 2iy)(2+i)^2 = 10(1+i)$$

$$(3x - 2iy)(2^2+i^2+2(2)(i)) = 10+10i$$

$$(3x - 2iy)(4 + (-1)+4i) = 10+10i \text{ [since, } i^2 = -1]$$

$$(3x - 2iy)(3+4i) = 10+10i$$

Let us divide with 3+4i on both sides we get,

$$(3x - 2iy) = (10+10i)/(3+4i)$$

$$= \text{Now multiply and divide with } (3-4i)$$

$$= [10(3-4i) + 10i(3-4i)] / (3^2 - (4i)^2)$$

$$\begin{aligned}
 &= [30-40i+30i-40i^2] / (9 - 16i^2) \\
 &= [30-10i-40(-1)] / (9-16(-1)) \\
 &= [70-10i]/25
 \end{aligned}$$

Now, equating Real and Imaginary parts on both sides we get

$$3x = 70/25 \text{ and } -2y = -10/25$$

$$x = 70/75 \text{ and } y = 1/5$$

$$x = 14/15 \text{ and } y = 1/5$$

∴ The real values of x and y are 14/15, 1/5

$$(iii) \frac{(1+i)x - 2i}{3+i} + \frac{(2-3i)y + i}{3-i} = i$$

Given:

$$\frac{(1+i)x - 2i}{3+i} + \frac{(2-3i)y + i}{3-i} = i$$

$$\frac{((1+i)x-2i)(3-i) + ((2-3i)y+i)(3+i)}{(3+i)(3-i)} = i$$

$$\frac{((1+i)(3-i)x) - (2i)(3-i) + ((2-3i)(3+i)y) + (i)(3+i)}{3^2 - i^2} = i$$

$$\frac{(3-i+3i-i^2)x - 6i+2i^2 + (6+2i-9i-3i^2)y + 3i+i^2}{9-(-1)} = i$$

$$\frac{(3+2i-(-1))x - 6i+2(-1) + (6-7i-3(-1))y + 3i+(-1)}{10} = i \text{ [since, } i^2 = -1]$$

$$(4+2i)x - 3i - 3 + (9-7i)y = 10i$$

$$(4x+9y-3) + i(2x-7y-3) = 10i$$

Now, equating Real and Imaginary parts on both sides we get,

$$4x+9y-3 = 0 \dots (i)$$

$$\text{And } 2x-7y-3 = 10$$

$$2x-7y = 13 \dots (ii)$$

Multiply (i) by 7 and (ii) by 9 and add

On solving these equations we get

$$28x + 18x + 63y - 63y = 117 + 21$$

$$46x = 117 + 21$$

$$46x = 138$$

$$x = 138/46$$

$$= 3$$

Substitute the value of x in (i) we get,

$$4x+9y-3 = 0$$

$$9y = -9$$

$$y = -9/9$$

$$= -1$$

$$x = 3 \text{ and } y = -1$$

∴ The real values of x and y are 3 and -1

$$(iv) (1 + i)(x + iy) = 2 - 5i$$

Given:

$$(1 + i)(x + iy) = 2 - 5i$$

Divide with (1+i) on both the sides we get,

$$(x + iy) = (2 - 5i)/(1+i)$$

Multiply and divide by (1-i)

$$= (2 - 5i)/(1+i) \times (1-i)/(1-i)$$

$$= [2(1-i) - 5i(1-i)] / (1^2 - i^2)$$

$$= [2 - 7i + 5(-1)] / 2 \text{ [since, } i^2 = -1]$$

$$= (-3-7i)/2$$

Now, equating Real and Imaginary parts on both sides we get

$$x = -3/2 \text{ and } y = -7/2$$

∴ The real values of x and y are -3/2, -7/2

3. Find the conjugates of the following complex numbers:

(i) $4 - 5i$

(ii) $1 / (3 + 5i)$

(iii) $1 / (1 + i)$

(iv) $(3 - i)^2 / (2 + i)$

(v) $[(1 + i)(2 + i)] / (3 + i)$

(vi) $[(3 - 2i)(2 + 3i)] / [(1 + 2i)(2 - i)]$

Solution:

(i) $4 - 5i$

Given:

$$4 - 5i$$

We know the conjugate of a complex number (a + ib) is (a - ib)

So,

∴ The conjugate of (4 - 5i) is (4 + 5i)

(ii) $1 / (3 + 5i)$

Given:

$$1 / (3 + 5i)$$

Since the given complex number is not in the standard form of (a + ib)

Let us convert to standard form by multiplying and dividing with $(3 - 5i)$

We get,

$$\begin{aligned} \frac{1}{3+5i} &= \frac{1}{3+5i} \times \frac{3-5i}{3-5i} \\ &= \frac{3-5i}{3^2-(5i)^2} \\ &= \frac{3-5i}{9-25i^2} \\ &= \frac{3-5i}{9-25(-1)} \quad [\text{Since, } i^2 = -1] \\ &= \frac{3-5i}{34} \end{aligned}$$

We know the conjugate of a complex number $(a + ib)$ is $(a - ib)$

So,

∴ The conjugate of $(3 - 5i)/34$ is $(3 + 5i)/34$

(iii) $1 / (1 + i)$

Given:

$$1 / (1 + i)$$

Since the given complex number is not in the standard form of $(a + ib)$

Let us convert to standard form by multiplying and dividing with $(1 - i)$

We get,

$$\begin{aligned} \frac{1}{1+i} &= \frac{1}{1+i} \times \frac{1-i}{1-i} \\ &= \frac{1-i}{1^2-i^2} \\ &= \frac{1-i}{1-(-1)} \quad [\text{since, } i^2 = -1] \\ &= \frac{1-i}{2} \end{aligned}$$

We know the conjugate of a complex number $(a + ib)$ is $(a - ib)$

So,

∴ The conjugate of $(1-i)/2$ is $(1+i)/2$

(iv) $(3 - i)^2 / (2 + i)$

Given:

$$(3 - i)^2 / (2 + i)$$

Since the given complex number is not in the standard form of $(a + ib)$

Let us convert to standard form,

$$\begin{aligned}\frac{(3-i)^2}{2+i} &= \frac{3^2+i^2-2(3)(i)}{2+i} \\ &= \frac{9+(-1)-6i}{2+i} \quad [\text{Since, } i^2 = -1] \\ &= \frac{8-6i}{2+i}\end{aligned}$$

Now, let us multiply and divide with $(2 - i)$ we get,

$$\begin{aligned}\frac{8-6i}{2+i} &= \frac{8-6i}{2+i} \times \frac{2-i}{2-i} \\ &= \frac{8(2-i)-6i(2-i)}{2^2-i^2} \\ &= \frac{16-8i-12i+6i^2}{4-(-1)} \quad [\text{Since, } i^2 = -1] \\ &= \frac{16-20i+6(-1)}{5} \\ &= \frac{10-20i}{5} \\ &= 10/5 - 20i/5 \\ &= 2 - 4i\end{aligned}$$

We know the conjugate of a complex number $(a + ib)$ is $(a - ib)$

So,

\therefore The conjugate of $(2 - 4i)$ is $(2 + 4i)$

(v) $[(1 + i)(2 + i)] / (3 + i)$

Given:

$$[(1 + i)(2 + i)] / (3 + i)$$

Since the given complex number is not in the standard form of $(a + ib)$

Let us convert to standard form,

$$\begin{aligned}\frac{(1+i)(2+i)}{3+i} &= \frac{1(2+i)+i(2+i)}{3+i} \\ &= \frac{2+i+2i+i^2}{3+i} \\ &= \frac{2+3i+(-1)}{3+i} \quad [\text{Since, } i^2 = -1] \\ &= \frac{1+3i}{3+i}\end{aligned}$$

Now, let us multiply and divide with $(3 - i)$ we get,

$$\frac{1+3i}{3+i} = \frac{1+3i}{3+i} \times \frac{3-i}{3-i}$$

$$\begin{aligned}
 &= \frac{1(3-i)+3i(3-i)}{3^2-i^2} \\
 &= \frac{3-i+9i-3i^2}{9-(-1)} \quad [\text{Since, } i^2 = -1] \\
 &= \frac{3+8i-3(-1)}{10} \\
 &= \frac{6+8i}{10} \\
 &= \frac{3}{5} + \frac{4i}{5}
 \end{aligned}$$

We know the conjugate of a complex number $(a + ib)$ is $(a - ib)$

So,

\therefore The conjugate of $(3 + 4i)/5$ is $(3 - 4i)/5$

(vi) $[(3 - 2i)(2 + 3i)] / [(1 + 2i)(2 - i)]$

Given:

$$[(3 - 2i)(2 + 3i)] / [(1 + 2i)(2 - i)]$$

Since the given complex number is not in the standard form of $(a + ib)$

Let us convert to standard form,

$$\begin{aligned}
 \frac{(3-2i)(2+3i)}{(1+2i)(2-i)} &= \frac{3(2+3i)-2i(2+3i)}{1(2-i)+2i(2-i)} \\
 &= \frac{6+9i-4i-6i^2}{2-i+4i-2i^2} \\
 &= \frac{6+5i-6(-1)}{2+3i-2(-1)} \\
 &= \frac{12+5i}{4+3i}
 \end{aligned}$$

Now, let us multiply and divide with $(4 - 3i)$ we get,

$$\begin{aligned}
 \frac{12+5i}{4+3i} &= \frac{12+5i}{4+3i} \times \frac{4-3i}{4-3i} \\
 &= \frac{12(4-3i)+5i(4-3i)}{4^2-(3i)^2} \\
 &= \frac{48-36i+20i-15i^2}{16-9i^2} \\
 &= \frac{48-16i-15(-1)}{16-9(-1)} \\
 &= \frac{63 - 16i}{25}
 \end{aligned}$$

We know the conjugate of a complex number $(a + ib)$ is $(a - ib)$

So,

∴ The conjugate of $(63 - 16i)/25$ is $(63 + 16i)/25$

4. Find the multiplicative inverse of the following complex numbers:

(i) $1 - i$

(ii) $(1 + i\sqrt{3})^2$

(iii) $4 - 3i$

(iv) $\sqrt{5} + 3i$

Solution:

(i) $1 - i$

Given:

$1 - i$

We know the multiplicative inverse of a complex number (Z) is Z^{-1} or $1/Z$

So,

$$Z = 1 - i$$

$$Z^{-1} = \frac{1}{1 - i}$$

Let us multiply and divide by $(1 + i)$ we get,

$$= \frac{1}{1 - i} \times \frac{1 + i}{1 + i}$$

$$= \frac{1 + i}{1^2 - (i)^2}$$

$$= \frac{1 + i}{1 - (-1)} \text{ [Since, } i^2 = -1]$$

$$= \frac{1 + i}{2}$$

∴ The multiplicative inverse of $(1 - i)$ is $(1 + i)/2$

(ii) $(1 + i\sqrt{3})^2$

Given:

$(1 + i\sqrt{3})^2$

$Z = (1 + i\sqrt{3})^2$

$$= 1^2 + (i\sqrt{3})^2 + 2(1)(i\sqrt{3})$$

$$= 1 + 3i^2 + 2i\sqrt{3}$$

$$= 1 + 3(-1) + 2i\sqrt{3} \text{ [since, } i^2 = -1]$$

$$= 1 - 3 + 2i\sqrt{3}$$

$$= -2 + 2i\sqrt{3}$$

We know the multiplicative inverse of a complex number (Z) is Z^{-1} or $1/Z$

So,

$Z = -2 + 2i\sqrt{3}$

$$Z^{-1} = \frac{1}{-2 + 2i\sqrt{3}}$$

Let us multiply and divide by $-2 - 2i\sqrt{3}$, we get

$$\begin{aligned} &= \frac{1}{-2+2\sqrt{3}i} \times \frac{-2-2\sqrt{3}i}{-2-2\sqrt{3}i} \\ &= \frac{-2-2\sqrt{3}i}{(-2)^2-(2\sqrt{3}i)^2} \\ &= \frac{-2-2\sqrt{3}i}{4-12i^2} \\ &= \frac{-2-2\sqrt{3}i}{4-12(-1)} \\ &= \frac{-2-2\sqrt{3}i}{16} \\ &= \frac{-1-i\sqrt{3}}{8} \end{aligned}$$

\therefore The multiplicative inverse of $(1 + i\sqrt{3})^2$ is $(-1-i\sqrt{3})/8$

(iii) $4 - 3i$

Given:

$$4 - 3i$$

We know the multiplicative inverse of a complex number (Z) is Z^{-1} or $1/Z$

So,

$$Z = 4 - 3i$$

$$Z^{-1} = \frac{1}{4 - 3i}$$

Let us multiply and divide by $(4 + 3i)$, we get

$$\begin{aligned} &= \frac{1}{4-3i} \times \frac{4+3i}{4+3i} \\ &= \frac{4+3i}{4^2-(3i)^2} \\ &= \frac{4+3i}{16-9i^2} \\ &= \frac{4+3i}{16-9(-1)} \\ &= \frac{4+3i}{25} \end{aligned}$$

\therefore The multiplicative inverse of $(4 - 3i)$ is $(4 + 3i)/25$

(iv) $\sqrt{5} + 3i$

Given:

$$\sqrt{5} + 3i$$

We know the multiplicative inverse of a complex number (Z) is Z^{-1} or $1/Z$

So,

$$Z = \sqrt{5} + 3i$$

$$Z^{-1} = \frac{1}{\sqrt{5} + 3i}$$

Let us multiply and divide by $(\sqrt{5} - 3i)$

$$= \frac{1}{\sqrt{5} + 3i} \times \frac{\sqrt{5} - 3i}{\sqrt{5} - 3i}$$

$$= \frac{\sqrt{5} - 3i}{(\sqrt{5})^2 - (3i)^2}$$

$$= \frac{\sqrt{5} - 3i}{5 - 9i^2}$$

$$= \frac{\sqrt{5} - 3i}{5 - 9(-1)}$$

$$= \frac{\sqrt{5} - 3i}{14}$$

\therefore The multiplicative inverse of $(\sqrt{5} + 3i)$ is $(\sqrt{5} - 3i)/14$

5. If $z_1 = 2 - i$, $z_2 = 1 + i$, find

$$\left| \frac{z_1 + z_2 + 1}{z_1 - z_2 + i} \right|$$

Solution:

Given:

$$z_1 = (2 - i) \text{ and } z_2 = (1 + i)$$

We know that, $|a/b| = |a| / |b|$

So,

$$\begin{aligned} \left| \frac{z_1 + z_2 + 1}{z_1 - z_2 + i} \right| &= \frac{|z_1 + z_2 + 1|}{|z_1 - z_2 + i|} \\ &= \frac{|2 - i + 1 + i + 1|}{|2 - i - (1 + i) + i|} \\ &= \frac{|4|}{|1 - i|} \end{aligned}$$

We know, $|a + ib|$ is $\sqrt{a^2 + b^2}$

So now,

$$\begin{aligned}
 &= \frac{\sqrt{4^2+0^2}}{\sqrt{1^2+(-1)^2}} \\
 &= \frac{4}{\sqrt{2}} \\
 &= 2\sqrt{2}
 \end{aligned}$$

∴ The value of $\left| \frac{z_1+z_2+1}{z_1-z_2+i} \right|$ is $2\sqrt{2}$

6. If $z_1 = (2 - i)$, $z_2 = (-2 + i)$, find

- (i) $\operatorname{Re} \left(\frac{z_1 z_2}{\bar{z}_1} \right)$
 (ii) $\operatorname{Im} \left(\frac{1}{z_1 \bar{z}_1} \right)$

Solution:

Given:

$$z_1 = (2 - i) \text{ and } z_2 = (-2 + i)$$

- (i) $\operatorname{Re} \left(\frac{z_1 z_2}{\bar{z}_1} \right)$

We shall rationalise the denominator, we get

$$\begin{aligned}
 \frac{z_1 z_2}{\bar{z}_1} &= \frac{z_1 z_2}{z_1} \times \frac{z_1}{z_1} \\
 &= \frac{(z_1)^2 z_2}{z_1 z_1} \\
 &= \frac{(2 - i)^2 (-2 + i)}{|z_1|^2} \quad [\text{since, } z\bar{z} = |z|^2] \\
 &= \frac{(2^2 + i^2 - 2 \times 2 \times i)(-2 + i)}{|2 - i|^2} \\
 &= \frac{(4 - 1 - 4i)(-2 + i)}{2^2 + (-1)^2} \\
 &= \frac{(3 - 4i)(-2 + i)}{4 + i} \\
 &= \frac{3(-2 + i) - 4i(-2 + i)}{4 + i} \\
 &= \frac{-6 + 3i + 8i + 4}{4 + i} \\
 &= \frac{-2 + 11i}{5}
 \end{aligned}$$

\therefore The real value of $\left(\frac{z_1 z_2}{\bar{z}_1}\right)$ is $\frac{-2}{5}$

$$(ii) \operatorname{Im}\left(\frac{1}{z_1 \bar{z}_1}\right)$$

$$\begin{aligned} \frac{1}{z_1 \bar{z}_1} &= \frac{1}{|z_1|^2} \\ &= \frac{1}{|2 - i|^2} \\ &= \frac{1}{2^2 + (-1)^2} \\ &= \frac{1}{4 + 1} \\ &= \frac{1}{5} \end{aligned}$$

\therefore The imaginary value of $\left(\frac{1}{z_1 \bar{z}_1}\right)$ is 0

7. Find the modulus of $[(1 + i)/(1 - i)] - [(1 - i)/(1 + i)]$

Solution:

Given:

$$[(1 + i)/(1 - i)] - [(1 - i)/(1 + i)]$$

So,

$$Z = [(1 + i)/(1 - i)] - [(1 - i)/(1 + i)]$$

Let us simplify, we get

$$\begin{aligned} &= [(1+i)(1+i) - (1-i)(1-i)] / (1^2 - i^2) \\ &= [1^2 + i^2 + 2(1)(i) - (1^2 + i^2 - 2(1)(i))] / (1 - (-1)) \text{ [Since, } i^2 = -1] \\ &= 4i/2 \\ &= 2i \end{aligned}$$

We know that for a complex number $Z = (a+ib)$ it's magnitude is given by $|z| = \sqrt{a^2 + b^2}$

So,

$$\begin{aligned} |Z| &= \sqrt{0^2 + 2^2} \\ &= 2 \end{aligned}$$

\therefore The modulus of $[(1 + i)/(1 - i)] - [(1 - i)/(1 + i)]$ is 2.

8. If $x + iy = (a+ib)/(a-ib)$, prove that $x^2 + y^2 = 1$

Solution:

Given:

$$x + iy = (a+ib)/(a-ib)$$

We know that for a complex number $Z = (a+ib)$ it's magnitude is given by $|z| = \sqrt{(a^2 + b^2)}$

So,

$$|a/b| \text{ is } |a| / |b|$$

Applying Modulus on both sides we get,

$$|x + iy| = \left| \frac{a+ib}{a-ib} \right|$$

$$\sqrt{x^2 + y^2} = \frac{|a+ib|}{|a-ib|}$$

$$= \frac{\sqrt{a^2+b^2}}{\sqrt{a^2+(-b)^2}}$$

$$= \frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}}$$

$$= 1$$

Squaring on both sides we get,

$$(\sqrt{x^2 + y^2})^2 = 1^2$$

$$x^2+y^2=1$$

∴ Hence Proved.

9. Find the least positive integral value of n for which $[(1+i)/(1-i)]^n$ is real.

Solution:

Given:

$$[(1+i)/(1-i)]^n$$

$$Z = [(1+i)/(1-i)]^n$$

Now let us multiply and divide by $(1+i)$, we get

$$= \frac{1+i}{1-i} \times \frac{1+i}{1+i}$$

$$= \frac{(1+i)^2}{1^2-i^2}$$

$$= \frac{1^2+i^2+2(1)(i)}{1-(-1)}$$

$$= \frac{1-1+2i}{2}$$

$$= \frac{2i}{2}$$

$$= i \text{ [which is not real]}$$

For $n = 2$, we have

$$\begin{aligned} [(1+i)/(1-i)]^2 &= i^2 \\ &= -1 \text{ [which is real]} \end{aligned}$$

So, the smallest positive integral 'n' that can make $[(1+i)/(1-i)]^n$ real is 2.

∴ The smallest positive integral value of 'n' is 2.

10. Find the real values of θ for which the complex number $(1 + i \cos \theta) / (1 - 2i \cos \theta)$ is purely real.

Solution:

Given:

$$(1 + i \cos \theta) / (1 - 2i \cos \theta)$$

$$Z = (1 + i \cos \theta) / (1 - 2i \cos \theta)$$

Let us multiply and divide by $(1 + 2i \cos \theta)$

$$\begin{aligned} &= \frac{1+i\cos\theta}{1-2i\cos\theta} \times \frac{1+2i\cos\theta}{1+2i\cos\theta} \\ &= \frac{1(1+2i\cos\theta)+i\cos\theta(1+2i\cos\theta)}{1^2-(2i\cos\theta)^2} \\ &= \frac{1+2i\cos\theta+i\cos\theta+2i^2\cos^2\theta}{1-4i^2\cos^2\theta} \\ &= \frac{1+3i\cos\theta+2(-1)\cos^2\theta}{1-4(-1)\cos^2\theta} \\ &= \frac{1-2\cos^2\theta+3i\cos\theta}{1+4\cos^2\theta} \end{aligned}$$

For a complex number to be purely real, the imaginary part should be equal to zero.

So,

$$\frac{3\cos\theta}{1+4\cos^2\theta} = 0$$

$$3\cos\theta = 0 \text{ (since, } 1 + 4\cos^2\theta \geq 1)$$

$$\cos\theta = 0$$

$$\cos\theta = \cos\pi/2$$

$$\theta = [(2n+1)\pi] / 2, \text{ for } n \in \mathbb{Z}$$

$$= 2n\pi \pm \pi/2, \text{ for } n \in \mathbb{Z}$$

∴ The values of θ to get the complex number to be purely real is $2n\pi \pm \pi/2$, for $n \in \mathbb{Z}$

11. Find the smallest positive integer value of n for which $(1+i)^n / (1-i)^{n-2}$ is a real number.

Solution:

Given:

$$(1+i)^n / (1-i)^{n-2}$$

$$Z = (1+i)^n / (1-i)^{n-2}$$

Let us multiply and divide by $(1 - i)^2$

$$\begin{aligned}
 &= \frac{(1+i)^n}{(1-i)^{n-2}} \times \frac{(1-i)^2}{(1-i)^2} \\
 &= \left(\frac{1+i}{1-i}\right)^n \times (1-i)^2 \\
 &= \left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^n \times (1^2 + i^2 - 2(1)(i)) \\
 &= \left(\frac{(1+i)^2}{1^2-i^2}\right)^n \times (1 + i^2 - 2i) \\
 &= \left(\frac{1^2+i^2+2(1)(i)}{1-(-1)}\right)^n \times (1 + (-1) - 2i) \\
 &= \left(\frac{1-1+2i}{2}\right)^n \times (-2i) \\
 &= \left(\frac{2i}{2}\right)^n \times (-2i) \\
 &= i^n \times (-2i) \\
 &= -2i^{n+1}
 \end{aligned}$$

For $n = 1$,

$$Z = -2i^{1+1}$$

$$= -2i^2$$

$= 2$, which is a real number.

\therefore The smallest positive integer value of n is 1.

12. If $[(1+i)/(1-i)]^3 - [(1-i)/(1+i)]^3 = x + iy$, find (x, y)

Solution:

Given:

$$[(1+i)/(1-i)]^3 - [(1-i)/(1+i)]^3 = x + iy$$

Let us rationalize the denominator, we get

$$\left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^3 - \left(\frac{1-i}{1+i} \times \frac{1-i}{1-i}\right)^3 = x + iy$$

$$\left(\frac{(1+i)^2}{1^2-i^2}\right)^3 - \left(\frac{(1-i)^2}{1^2-i^2}\right)^3 = x + iy$$

$$\left(\frac{1^2+i^2+2(1)(i)}{1-(-1)}\right)^3 - \left(\frac{1^2+i^2-2(1)(i)}{1-(-1)}\right)^3 = x + iy$$

$$\left(\frac{1-1+2i}{2}\right)^3 - \left(\frac{1-1-2i}{2}\right)^3 = x + iy$$

$$\left(\frac{2i}{2}\right)^3 - \left(\frac{-2i}{2}\right)^3 = x + iy$$

$$i^3 - (-i)^3 = x + iy$$

$$2i^3 = x + iy$$

$$2i^2 \cdot i = x + iy$$

$$2(-1)i = x + iy$$

$$-2i = x + iy$$

Equating Real and Imaginary parts on both sides we get

$$x = 0 \text{ and } y = -2$$

\therefore The values of x and y are 0 and -2.

13. If $(1+i)^2 / (2-i) = x + iy$, find $x + y$

Solution:

Given:

$$(1+i)^2 / (2-i) = x + iy$$

Upon expansion we get,

$$\frac{1^2 + i^2 + 2(1)(i)}{2-i} = x + iy$$

$$\frac{1 + (-1) + 2i}{2-i} = x + iy$$

$$\frac{2i}{2-i} = x + iy$$

Now, let us multiply and divide by $(2+i)$, we get

$$\frac{2i}{2-i} \times \frac{2+i}{2+i} = x + iy$$

$$\frac{4i + 2i^2}{2^2 - i^2} = x + iy$$

$$\frac{2(-1) + 4i}{4 - (-1)} = x + iy$$

$$\frac{-2 + 4i}{5} = x + iy$$

Let us equate real and imaginary parts on both sides we get,

$$x = -2/5 \text{ and } y = 4/5$$

so,

$$x + y = -2/5 + 4/5$$

$$= (-2+4)/5$$

$$= 2/5$$

\therefore The value of $(x + y)$ is $2/5$

EXERCISE 13.3
PAGE NO: 13.39
1. Find the square root of the following complex numbers.

(i) $-5 + 12i$

(ii) $-7 - 24i$

(iii) $1 - i$

(iv) $-8 - 6i$

(v) $8 - 15i$

(vi) $-11 - 60\sqrt{-1}$

(vii) $1 + 4\sqrt{-3}$

(viii) $4i$

(ix) $-i$

Solution:

$$\text{if } b > 0, \sqrt{a + ib} = \pm \left[\left(\frac{a + \sqrt{a^2 + b^2}}{2} \right)^{\frac{1}{2}} + i \left(\frac{-a + \sqrt{a^2 + b^2}}{2} \right)^{\frac{1}{2}} \right]$$

$$\text{if } b < 0, \sqrt{a + ib} = \pm \left[\left(\frac{a + \sqrt{a^2 + b^2}}{2} \right)^{\frac{1}{2}} - i \left(\frac{-a + \sqrt{a^2 + b^2}}{2} \right)^{\frac{1}{2}} \right]$$

(i) $-5 + 12i$

Given:

$-5 + 12i$

 We know, $Z = a + ib$

So, $\sqrt{a + ib} = \sqrt{-5 + 12i}$

 Here, $b > 0$

Let us simplify now,

$$\begin{aligned} \sqrt{-5 + 12i} &= \pm \left[\left(\frac{-5 + \sqrt{(-5)^2 + 12^2}}{2} \right)^{\frac{1}{2}} + i \left(\frac{5 + \sqrt{(-5)^2 + 12^2}}{2} \right)^{\frac{1}{2}} \right] \quad [\text{Since, } b > 0] \\ &= \pm \left[\left(\frac{-5 + \sqrt{25 + 144}}{2} \right)^{\frac{1}{2}} + i \left(\frac{5 + \sqrt{25 + 144}}{2} \right)^{\frac{1}{2}} \right] \\ &= \pm \left[\left(\frac{-5 + \sqrt{169}}{2} \right)^{\frac{1}{2}} + i \left(\frac{5 + \sqrt{169}}{2} \right)^{\frac{1}{2}} \right] \\ &= \pm \left[\left(\frac{-5 + 13}{2} \right)^{\frac{1}{2}} + i \left(\frac{5 + 13}{2} \right)^{\frac{1}{2}} \right] \end{aligned}$$

$$\begin{aligned}
 &= \pm \left[\left(\frac{8}{2}\right)^{\frac{1}{2}} + i \left(\frac{18}{2}\right)^{\frac{1}{2}} \right] \\
 &= \pm \left[4^{\frac{1}{2}} + i 9^{\frac{1}{2}} \right] \\
 &= \pm [2 + 3i]
 \end{aligned}$$

∴ Square root of $(-5 + 12i)$ is $\pm[2 + 3i]$

(ii) $-7 - 24i$

Given:

$$-7 - 24i$$

We know, $Z = -7 - 24i$

So, $\sqrt{(a + ib)} = \sqrt{(-7 - 24i)}$

Here, $b < 0$

Let us simplify now,

$$\begin{aligned}
 \sqrt{-7 - 24i} &= \pm \left[\left(\frac{-7 + \sqrt{(-7)^2 + (-24)^2}}{2} \right)^{\frac{1}{2}} - i \left(\frac{7 + \sqrt{(-7)^2 + (-24)^2}}{2} \right)^{\frac{1}{2}} \right] \quad [\text{Since, } b < 0] \\
 &= \pm \left[\left(\frac{-7 + \sqrt{49 + 576}}{2} \right)^{\frac{1}{2}} - i \left(\frac{7 + \sqrt{49 + 576}}{2} \right)^{\frac{1}{2}} \right] \\
 &= \pm \left[\left(\frac{-7 + \sqrt{625}}{2} \right)^{\frac{1}{2}} - i \left(\frac{7 + \sqrt{625}}{2} \right)^{\frac{1}{2}} \right] \\
 &= \pm \left[\left(\frac{-7 + 25}{2} \right)^{\frac{1}{2}} - i \left(\frac{7 + 25}{2} \right)^{\frac{1}{2}} \right] \\
 &= \pm \left[\left(\frac{18}{2} \right)^{\frac{1}{2}} - i \left(\frac{32}{2} \right)^{\frac{1}{2}} \right] \\
 &= \pm [9^{1/2} - i16^{1/2}] \\
 &= \pm [3 - 4i]
 \end{aligned}$$

∴ Square root of $(-7 - 24i)$ is $\pm [3 - 4i]$

(iii) $1 - i$

Given:

$$1 - i$$

We know, $Z = (1 - i)$

So, $\sqrt{(a + ib)} = \sqrt{(1 - i)}$

Here, $b < 0$

Let us simplify now,

$$\begin{aligned}
 \sqrt{1-i} &= \pm \left[\left(\frac{1+\sqrt{(1)^2+(-1)^2}}{2} \right)^{\frac{1}{2}} - i \left(\frac{-1+\sqrt{(1)^2+(-1)^2}}{2} \right)^{\frac{1}{2}} \right] \quad [\text{Since, } b < 0] \\
 &= \pm \left[\left(\frac{1+\sqrt{1+1}}{2} \right)^{\frac{1}{2}} - i \left(\frac{-1+\sqrt{1+1}}{2} \right)^{\frac{1}{2}} \right] \\
 &= \pm \left[\left(\frac{1+\sqrt{2}}{2} \right)^{\frac{1}{2}} - i \left(\frac{-1+\sqrt{2}}{2} \right)^{\frac{1}{2}} \right] \\
 &= \pm \left[\left(\sqrt{\frac{\sqrt{2}+1}{2}} \right) - i \left(\sqrt{\frac{\sqrt{2}-1}{2}} \right) \right]
 \end{aligned}$$

\therefore Square root of $(1 - i)$ is $\pm \left[\left(\sqrt{\frac{\sqrt{2}+1}{2}} \right) - i \left(\sqrt{\frac{\sqrt{2}-1}{2}} \right) \right]$

(iv) $-8 - 6i$

Given:

$-8 - 6i$

We know, $Z = -8 - 6i$

So, $\sqrt{a + ib} = -8 - 6i$

Here, $b < 0$

Let us simplify now,

$$\begin{aligned}
 \sqrt{-8-6i} &= \pm \left[\left(\frac{-8+\sqrt{(-8)^2+(-6)^2}}{2} \right)^{\frac{1}{2}} - i \left(\frac{8+\sqrt{(-8)^2+(-6)^2}}{2} \right)^{\frac{1}{2}} \right] \quad [\text{Since, } b < 0] \\
 &= \pm \left[\left(\frac{-8+\sqrt{64+36}}{2} \right)^{\frac{1}{2}} - i \left(\frac{8+\sqrt{64+36}}{2} \right)^{\frac{1}{2}} \right] \\
 &= \pm \left[\left(\frac{-8+\sqrt{100}}{2} \right)^{\frac{1}{2}} - i \left(\frac{8+\sqrt{100}}{2} \right)^{\frac{1}{2}} \right] \\
 &= \pm \left[\left(\frac{-8+10}{2} \right)^{\frac{1}{2}} - i \left(\frac{8+10}{2} \right)^{\frac{1}{2}} \right] \\
 &= \pm \left[\left(\frac{2}{2} \right)^{\frac{1}{2}} - i \left(\frac{18}{2} \right)^{\frac{1}{2}} \right] \\
 &= [1^{1/2} - i 9^{1/2}]
 \end{aligned}$$

$$= \pm [1 - 3i]$$

\therefore Square root of $(-8 - 6i)$ is $\pm [1 - 3i]$

(v) $8 - 15i$

Given:

$$8 - 15i$$

We know, $Z = 8 - 15i$

So, $\sqrt{(a + ib)} = 8 - 15i$

Here, $b < 0$

Let us simplify now,

$$\begin{aligned} \sqrt{8 - 15i} &= \pm \left[\left(\frac{8 + \sqrt{8^2 + (-15)^2}}{2} \right)^{\frac{1}{2}} - i \left(\frac{-8 + \sqrt{8^2 + (-15)^2}}{2} \right)^{\frac{1}{2}} \right] \quad [\text{Since, } b < 0] \\ &= \pm \left[\left(\frac{8 + \sqrt{64 + 225}}{2} \right)^{\frac{1}{2}} - i \left(\frac{-8 + \sqrt{64 + 225}}{2} \right)^{\frac{1}{2}} \right] \\ &= \pm \left[\left(\frac{8 + \sqrt{289}}{2} \right)^{\frac{1}{2}} - i \left(\frac{-8 + \sqrt{289}}{2} \right)^{\frac{1}{2}} \right] \\ &= \pm \left[\left(\frac{8 + 17}{2} \right)^{\frac{1}{2}} - i \left(\frac{-8 + 17}{2} \right)^{\frac{1}{2}} \right] \\ &= \pm \left[\left(\frac{25}{2} \right)^{\frac{1}{2}} - i \left(\frac{9}{2} \right)^{\frac{1}{2}} \right] \\ &= \pm \left[\frac{5}{\sqrt{2}} - \frac{i3}{\sqrt{2}} \right] \\ &= \pm \frac{1}{\sqrt{2}} (5 - 3i) \end{aligned}$$

\therefore Square root of $(8 - 15i)$ is $\pm \frac{1}{\sqrt{2}} (5 - 3i)$

(vi) $-11 - 60\sqrt{-1}$

Given:

$$-11 - 60\sqrt{-1}$$

We know, $Z = -11 - 60\sqrt{-1}$

So, $\sqrt{(a + ib)} = -11 - 60\sqrt{-1}$
 $= -11 - 60i$

Here, $b < 0$

Let us simplify now,

$$\begin{aligned}
 \sqrt{-11 - 60i} &= \pm \left[\left(\frac{-11 + \sqrt{(-11)^2 + (-60)^2}}{2} \right)^{\frac{1}{2}} - i \left(\frac{11 + \sqrt{(-11)^2 + (60)^2}}{2} \right)^{\frac{1}{2}} \right] \quad [\text{Since, } b < 0] \\
 &= \pm \left[\left(\frac{-11 + \sqrt{121 + 3600}}{2} \right)^{\frac{1}{2}} - i \left(\frac{11 + \sqrt{121 + 3600}}{2} \right)^{\frac{1}{2}} \right] \\
 &= \pm \left[\left(\frac{-11 + \sqrt{3721}}{2} \right)^{\frac{1}{2}} - i \left(\frac{11 + \sqrt{3721}}{2} \right)^{\frac{1}{2}} \right] \\
 &= \pm \left[\left(\frac{-11 + 61}{2} \right)^{\frac{1}{2}} - i \left(\frac{11 + 61}{2} \right)^{\frac{1}{2}} \right] \\
 &= \pm \left[\left(\frac{50}{2} \right)^{\frac{1}{2}} - i \left(\frac{72}{2} \right)^{\frac{1}{2}} \right] \\
 &= \pm \left[25^{\frac{1}{2}} - i 36^{\frac{1}{2}} \right] \\
 &= \pm (5 - 6i)
 \end{aligned}$$

∴ Square root of $(-11 - 60\sqrt{-1})$ is $\pm (5 - 6i)$

(vii) $1 + 4\sqrt{-3}$

Given:

$$1 + 4\sqrt{-3}$$

We know, $Z = 1 + 4\sqrt{-3}$

$$\begin{aligned}
 \text{So, } \sqrt{a + ib} &= 1 + 4\sqrt{-3} \\
 &= 1 + 4(\sqrt{3})(\sqrt{-1}) \\
 &= 1 + 4\sqrt{3}i
 \end{aligned}$$

Here, $b > 0$

Let us simplify now,

$$\begin{aligned}
 \sqrt{1 + 4\sqrt{3}i} &= \pm \left[\left(\frac{1 + \sqrt{1^2 + (4\sqrt{3})^2}}{2} \right)^{\frac{1}{2}} + i \left(\frac{-1 + \sqrt{1^2 + (4\sqrt{3})^2}}{2} \right)^{\frac{1}{2}} \right] \quad [\text{Since, } b > 0] \\
 &= \pm \left[\left(\frac{1 + \sqrt{1 + 48}}{2} \right)^{\frac{1}{2}} + i \left(\frac{-1 + \sqrt{1 + 48}}{2} \right)^{\frac{1}{2}} \right] \\
 &= \pm \left[\left(\frac{1 + \sqrt{49}}{2} \right)^{\frac{1}{2}} + i \left(\frac{-1 + \sqrt{49}}{2} \right)^{\frac{1}{2}} \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \pm \left[\left(\frac{1+7}{2} \right)^{\frac{1}{2}} + i \left(\frac{-1+7}{2} \right)^{\frac{1}{2}} \right] \\
 &= \pm \left[\left(\frac{8}{2} \right)^{\frac{1}{2}} + i \left(\frac{6}{2} \right)^{\frac{1}{2}} \right] \\
 &= \pm \left[4^{\frac{1}{2}} + i 3^{\frac{1}{2}} \right] \\
 &= \pm [2 + \sqrt{3}i]
 \end{aligned}$$

∴ Square root of $(1 + 4\sqrt{-3})$ is $\pm (2 + \sqrt{3}i)$

(viii) $4i$

Given:

$4i$

We know, $Z = 4i$

So, $\sqrt{a + ib} = 4i$

Here, $b > 0$

Let us simplify now,

$$\begin{aligned}
 \sqrt{4i} &= \pm \left[\left(\frac{0+\sqrt{0^2+4^2}}{2} \right)^{\frac{1}{2}} + i \left(\frac{0+\sqrt{0^2+4^2}}{2} \right)^{\frac{1}{2}} \right] \quad [\text{Since, } b > 0] \\
 &= \pm \left[\left(\frac{0+\sqrt{0+16}}{2} \right)^{\frac{1}{2}} + i \left(\frac{0+\sqrt{0+16}}{2} \right)^{\frac{1}{2}} \right] \\
 &= \pm \left[\left(\frac{0+\sqrt{16}}{2} \right)^{\frac{1}{2}} + i \left(\frac{0+\sqrt{16}}{2} \right)^{\frac{1}{2}} \right] \\
 &= \pm \left[\left(\frac{0+4}{2} \right)^{\frac{1}{2}} + i \left(\frac{0+4}{2} \right)^{\frac{1}{2}} \right] \\
 &= \pm \left[\left(\frac{4}{2} \right)^{\frac{1}{2}} + i \left(\frac{4}{2} \right)^{\frac{1}{2}} \right] \\
 &= \pm \left[2^{\frac{1}{2}} + i 2^{\frac{1}{2}} \right] \\
 &= \pm [\sqrt{2} + \sqrt{2}i] \\
 &= \pm \sqrt{2} (1 + i)
 \end{aligned}$$

∴ Square root of $4i$ is $\pm \sqrt{2} (1 + i)$

(ix) $-i$

Given:

$-i$

We know, $Z = -i$

So, $\sqrt{a + ib} = -i$

Here, $b < 0$

Let us simplify now,

$$\sqrt{-i} = \pm \left[\left(\frac{0 + \sqrt{0^2 + (-1)^2}}{2} \right)^{\frac{1}{2}} - i \left(\frac{0 + \sqrt{0^2 + (-1)^2}}{2} \right)^{\frac{1}{2}} \right] \quad [\text{Since, } b < 0]$$

$$= \pm \left[\left(\frac{0 + \sqrt{0+1}}{2} \right)^{\frac{1}{2}} - i \left(\frac{0 + \sqrt{0+1}}{2} \right)^{\frac{1}{2}} \right]$$

$$= \pm \left[\left(\frac{0 + \sqrt{1}}{2} \right)^{\frac{1}{2}} - i \left(\frac{0 + \sqrt{1}}{2} \right)^{\frac{1}{2}} \right]$$

$$= \pm \left[\left(\frac{0+1}{2} \right)^{\frac{1}{2}} - i \left(\frac{0+1}{2} \right)^{\frac{1}{2}} \right]$$

$$= \pm \left[\left(\frac{1}{2} \right)^{\frac{1}{2}} - i \left(\frac{1}{2} \right)^{\frac{1}{2}} \right]$$

$$= \pm \left[\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right]$$

$$= \pm \frac{1}{\sqrt{2}} (1 - i)$$

\therefore Square root of $-i$ is $\pm \frac{1}{\sqrt{2}} (1 - i)$

EXERCISE 13.4
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1. Find the modulus and argument of the following complex numbers and hence express each of them in the polar form:

(i) $1 + i$

(ii) $\sqrt{3} + i$

(iii) $1 - i$

(iv) $(1 - i) / (1 + i)$

(v) $1/(1 + i)$

(vi) $(1 + 2i) / (1 - 3i)$

(vii) $\sin 120^\circ - i \cos 120^\circ$

(viii) $-16 / (1 + i\sqrt{3})$

Solution:

We know that the polar form of a complex number $Z = x + iy$ is given by $Z = |Z| (\cos \theta + i \sin \theta)$

Where,

$$|Z| = \text{modulus of complex number} = \sqrt{(x^2 + y^2)}$$

$$\theta = \arg(z) = \text{argument of complex number} = \tan^{-1} (|y| / |x|)$$

(i) $1 + i$

Given: $Z = 1 + i$

So now,

$$|Z| = \sqrt{(x^2 + y^2)}$$

$$= \sqrt{(1^2 + 1^2)}$$

$$= \sqrt{(1 + 1)}$$

$$= \sqrt{2}$$

$$\theta = \tan^{-1} (|y| / |x|)$$

$$= \tan^{-1} (1 / 1)$$

$$= \tan^{-1} 1$$

Since $x > 0$, $y > 0$ complex number lies in 1st quadrant and the value of θ is $0^\circ \leq \theta \leq 90^\circ$.

$$\theta = \pi/4$$

$$Z = \sqrt{2} (\cos (\pi/4) + i \sin (\pi/4))$$

$$\therefore \text{Polar form of } (1 + i) \text{ is } \sqrt{2} (\cos (\pi/4) + i \sin (\pi/4))$$

(ii) $\sqrt{3} + i$

Given: $Z = \sqrt{3} + i$

So now,

$$|Z| = \sqrt{(x^2 + y^2)}$$

$$= \sqrt{((\sqrt{3})^2 + 1^2)}$$

$$\begin{aligned}
 &= \sqrt{3 + 1} \\
 &= \sqrt{4} \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \theta &= \tan^{-1} (|y| / |x|) \\
 &= \tan^{-1} (1 / \sqrt{3})
 \end{aligned}$$

Since $x > 0$, $y > 0$ complex number lies in 1st quadrant and the value of θ is $0^{\circ} \leq \theta \leq 90^{\circ}$.

$$\theta = \pi/6$$

$$Z = 2 (\cos (\pi/6) + i \sin (\pi/6))$$

\therefore Polar form of $(\sqrt{3} + i)$ is $2 (\cos (\pi/6) + i \sin (\pi/6))$

(iii) $1 - i$

$$\text{Given: } Z = 1 - i$$

So now,

$$\begin{aligned}
 |Z| &= \sqrt{(x^2 + y^2)} \\
 &= \sqrt{(1^2 + (-1)^2)} \\
 &= \sqrt{(1 + 1)} \\
 &= \sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \theta &= \tan^{-1} (|y| / |x|) \\
 &= \tan^{-1} (1 / 1) \\
 &= \tan^{-1} 1
 \end{aligned}$$

Since $x > 0$, $y > 0$ complex number lies in 1st quadrant and the value of θ is $0^{\circ} \leq \theta \leq 90^{\circ}$.

$$\theta = -\pi/4$$

$$\begin{aligned}
 Z &= \sqrt{2} (\cos (-\pi/4) + i \sin (-\pi/4)) \\
 &= \sqrt{2} (\cos (\pi/4) - i \sin (\pi/4))
 \end{aligned}$$

\therefore Polar form of $(1 - i)$ is $\sqrt{2} (\cos (\pi/4) - i \sin (\pi/4))$

(iv) $(1 - i) / (1 + i)$

$$\text{Given: } Z = (1 - i) / (1 + i)$$

Let us multiply and divide by $(1 - i)$, we get

$$\begin{aligned}
 z &= \frac{1-i}{1+i} \times \frac{1-i}{1-i} \\
 &= \frac{(1-i)^2}{1^2 - i^2} \\
 &= \frac{1^2 + i^2 - 2(1)(i)}{1 - (-1)} \\
 &= \frac{1 + (-1) - 2i}{2} \\
 &= \frac{-2i}{2} \\
 &= 0 - i
 \end{aligned}$$

So now,

$$\begin{aligned} |Z| &= \sqrt{x^2 + y^2} \\ &= \sqrt{0^2 + (-1)^2} \\ &= \sqrt{0 + 1} \\ &= \sqrt{1} \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1} (|y| / |x|) \\ &= \tan^{-1} (1 / 0) \\ &= \tan^{-1} \infty \end{aligned}$$

Since $x \geq 0$, $y < 0$ complex number lies in 4th quadrant and the value of θ is $-90^\circ \leq \theta \leq 0^\circ$.

$$\theta = -\pi/2$$

$$\begin{aligned} Z &= 1 (\cos (-\pi/2) + i \sin (-\pi/2)) \\ &= 1 (\cos (\pi/2) - i \sin (\pi/2)) \end{aligned}$$

\therefore Polar form of $(1 - i) / (1 + i)$ is $1 (\cos (\pi/2) - i \sin (\pi/2))$

(v) $1/(1 + i)$

Given: $Z = 1 / (1 + i)$

Let us multiply and divide by $(1 - i)$, we get

$$\begin{aligned} z &= \frac{1}{1+i} \times \frac{1-i}{1-i} \\ &= \frac{1-i}{1^2 - i^2} \\ &= \frac{1-i}{1 - (-1)} \\ &= \frac{1-i}{2} \end{aligned}$$

So now,

$$\begin{aligned} |Z| &= \sqrt{x^2 + y^2} \\ &= \sqrt{(1/2)^2 + (-1/2)^2} \\ &= \sqrt{1/4 + 1/4} \\ &= \sqrt{2/4} \\ &= 1/\sqrt{2} \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1} (|y| / |x|) \\ &= \tan^{-1} ((1/2) / (1/2)) \\ &= \tan^{-1} 1 \end{aligned}$$

Since $x > 0$, $y < 0$ complex number lies in 4th quadrant and the value of θ is $-90^\circ \leq \theta \leq 0^\circ$.

$$\theta = -\pi/4$$

$$\begin{aligned} Z &= 1/\sqrt{2} (\cos (-\pi/4) + i \sin (-\pi/4)) \\ &= 1/\sqrt{2} (\cos (\pi/4) - i \sin (\pi/4)) \end{aligned}$$

\therefore Polar form of $1/(1 + i)$ is $1/\sqrt{2} (\cos (\pi/4) - i \sin (\pi/4))$

(vi) $(1 + 2i) / (1 - 3i)$

Given: $Z = (1 + 2i) / (1 - 3i)$

Let us multiply and divide by $(1 + 3i)$, we get

$$\begin{aligned} Z &= \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i} \\ &= \frac{1(1+3i)+2i(1+3i)}{1^2-(3i)^2} \\ &= \frac{1+3i+2i+6i^2}{1-9i^2} \\ &= \frac{1+5i+6(-1)}{1-9(-1)} \\ &= \frac{-5+5i}{10} \\ &= \frac{-1+i}{2} \end{aligned}$$

So now,

$$\begin{aligned} |Z| &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-1/2)^2 + (1/2)^2} \\ &= \sqrt{1/4 + 1/4} \\ &= \sqrt{2/4} \\ &= 1/\sqrt{2} \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1} (|y| / |x|) \\ &= \tan^{-1} ((1/2) / (1/2)) \\ &= \tan^{-1} 1 \end{aligned}$$

Since $x < 0$, $y > 0$ complex number lies in 2nd quadrant and the value of θ is $90^\circ \leq \theta \leq 180^\circ$.

$$\theta = 3\pi/4$$

$$Z = 1/\sqrt{2} (\cos (3\pi/4) + i \sin (3\pi/4))$$

\therefore Polar form of $(1 + 2i) / (1 - 3i)$ is $1/\sqrt{2} (\cos (3\pi/4) + i \sin (3\pi/4))$

(vii) $\sin 120^\circ - i \cos 120^\circ$

$$\begin{aligned} \text{Given: } Z &= \sin 120^\circ - i \cos 120^\circ \\ &= \sqrt{3}/2 - i (-1/2) \\ &= \sqrt{3}/2 + i (1/2) \end{aligned}$$

So now,

$$\begin{aligned} |Z| &= \sqrt{x^2 + y^2} \\ &= \sqrt{(\sqrt{3}/2)^2 + (1/2)^2} \\ &= \sqrt{3/4 + 1/4} \\ &= \sqrt{4/4} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

$$\begin{aligned}\theta &= \tan^{-1} (|y| / |x|) \\ &= \tan^{-1} ((1/2) / (\sqrt{3}/2)) \\ &= \tan^{-1} (1/\sqrt{3})\end{aligned}$$

Since $x > 0$, $y > 0$ complex number lies in 1st quadrant and the value of θ is $0^0 \leq \theta \leq 90^0$.

$$\theta = \pi/6$$

$$Z = 1 (\cos (\pi/6) + i \sin (\pi/6))$$

\therefore Polar form of $\sqrt{3}/2 + i (1/2)$ is $1 (\cos (\pi/6) + i \sin (\pi/6))$

(viii) $-16 / (1 + i\sqrt{3})$

Given: $Z = -16 / (1 + i\sqrt{3})$

Let us multiply and divide by $(1 - i\sqrt{3})$, we get

$$\begin{aligned}Z &= \frac{-16}{1+i\sqrt{3}} \times \frac{1-i\sqrt{3}}{1-i\sqrt{3}} \\ &= \frac{-16+i16\sqrt{3}}{1^2-(i\sqrt{3})^2} \\ &= \frac{-16+i16\sqrt{3}}{1-3i^2} \\ &= \frac{-16+i16\sqrt{3}}{1-3(-1)} \\ &= \frac{-16+i16\sqrt{3}}{4} \\ &= -4 + i4\sqrt{3}\end{aligned}$$

So now,

$$\begin{aligned}|Z| &= \sqrt{(x^2 + y^2)} \\ &= \sqrt{(-4)^2 + (4\sqrt{3})^2} \\ &= \sqrt{16 + 48} \\ &= \sqrt{64} \\ &= 8\end{aligned}$$

$$\begin{aligned}\theta &= \tan^{-1} (|y| / |x|) \\ &= \tan^{-1} ((4\sqrt{3}) / 4) \\ &= \tan^{-1} (\sqrt{3})\end{aligned}$$

Since $x < 0$, $y > 0$ complex number lies in 2nd quadrant and the value of θ is $90^0 \leq \theta \leq 180^0$.

$$\theta = 2\pi/3$$

$$Z = 8 (\cos (2\pi/3) + i \sin (2\pi/3))$$

\therefore Polar form of $-16 / (1 + i\sqrt{3})$ is $8 (\cos (2\pi/3) + i \sin (2\pi/3))$

2. Write $(i^{25})^3$ in polar form.

Solution:

$$\begin{aligned}\text{Given: } Z &= (i^{25})^3 \\ &= i^{75}\end{aligned}$$

$$\begin{aligned}
 &= i^{74} \cdot i \\
 &= (i^2)^{37} \cdot i \\
 &= (-1)^{37} \cdot i \\
 &= (-1) \cdot i \\
 &= -i \\
 &= 0 - i
 \end{aligned}$$

So now,

$$\begin{aligned}
 |Z| &= \sqrt{x^2 + y^2} \\
 &= \sqrt{0^2 + (-1)^2} \\
 &= \sqrt{0 + 1} \\
 &= \sqrt{1}
 \end{aligned}$$

$$\begin{aligned}
 \theta &= \tan^{-1} (|y| / |x|) \\
 &= \tan^{-1} (1 / 0) \\
 &= \tan^{-1} \infty
 \end{aligned}$$

Since $x \geq 0$, $y < 0$ complex number lies in 4th quadrant and the value of θ is $-90^\circ \leq \theta \leq 0^\circ$.

$$\theta = -\pi/2$$

$$\begin{aligned}
 Z &= 1 (\cos (-\pi/2) + i \sin (-\pi/2)) \\
 &= 1 (\cos (\pi/2) - i \sin (\pi/2))
 \end{aligned}$$

$$\therefore \text{Polar form of } (i^{25})^3 \text{ is } 1 (\cos (\pi/2) - i \sin (\pi/2))$$

3. Express the following complex numbers in the form $r (\cos \theta + i \sin \theta)$:

(i) $1 + i \tan \alpha$

(ii) $\tan \alpha - i$

(iii) $1 - \sin \alpha + i \cos \alpha$

(iv) $(1 - i) / (\cos \pi/3 + i \sin \pi/3)$

Solution:

(i) $1 + i \tan \alpha$

Given: $Z = 1 + i \tan \alpha$

We know that the polar form of a complex number $Z = x + iy$ is given by $Z = |Z| (\cos \theta + i \sin \theta)$

Where,

$$|Z| = \text{modulus of complex number} = \sqrt{x^2 + y^2}$$

$$\theta = \arg (z) = \text{argument of complex number} = \tan^{-1} (|y| / |x|)$$

We also know that $\tan \alpha$ is a periodic function with period π .

So α is lying in the interval $[0, \pi/2) \cup (\pi/2, \pi]$.

Let us consider case 1:

$$\alpha \in [0, \pi/2)$$

So now,

$$\begin{aligned}
 |Z| = r &= \sqrt{x^2 + y^2} \\
 &= \sqrt{1^2 + \tan^2 \alpha} \\
 &= \sqrt{\sec^2 \alpha} \\
 &= |\sec \alpha| \text{ since, } \sec \alpha \text{ is positive in the interval } [0, \pi/2) \\
 \theta &= \tan^{-1} (|y| / |x|) \\
 &= \tan^{-1} (\tan \alpha / 1) \\
 &= \tan^{-1} (\tan \alpha) \\
 &= \alpha \text{ since, } \tan \alpha \text{ is positive in the interval } [0, \pi/2) \\
 \therefore \text{ Polar form is } Z &= \sec \alpha (\cos \alpha + i \sin \alpha)
 \end{aligned}$$

Let us consider case 2:

$$\alpha \in (\pi/2, \pi]$$

So now,

$$\begin{aligned}
 |Z| = r &= \sqrt{x^2 + y^2} \\
 &= \sqrt{1^2 + \tan^2 \alpha} \\
 &= \sqrt{\sec^2 \alpha} \\
 &= |\sec \alpha| \\
 &= -\sec \alpha \text{ since, } \sec \alpha \text{ is negative in the interval } (\pi/2, \pi]
 \end{aligned}$$

$$\begin{aligned}
 \theta &= \tan^{-1} (|y| / |x|) \\
 &= \tan^{-1} (\tan \alpha / 1) \\
 &= \tan^{-1} (\tan \alpha) \\
 &= -\pi + \alpha \text{ since, } \tan \alpha \text{ is negative in the interval } (\pi/2, \pi]
 \end{aligned}$$

$$\theta = -\pi + \alpha \text{ [since, } \theta \text{ lies in 4}^{\text{th}} \text{ quadrant]}$$

$$Z = -\sec \alpha (\cos (\alpha - \pi) + i \sin (\alpha - \pi))$$

$$\therefore \text{ Polar form is } Z = -\sec \alpha (\cos (\alpha - \pi) + i \sin (\alpha - \pi))$$

(ii) $\tan \alpha - i$

$$\text{Given: } Z = \tan \alpha - i$$

We know that the polar form of a complex number $Z = x + iy$ is given by $Z = |Z| (\cos \theta + i \sin \theta)$

Where,

$$|Z| = \text{modulus of complex number} = \sqrt{x^2 + y^2}$$

$$\theta = \arg (z) = \text{argument of complex number} = \tan^{-1} (|y| / |x|)$$

We also know that $\tan \alpha$ is a periodic function with period π .

So α is lying in the interval $[0, \pi/2) \cup (\pi/2, \pi]$.

Let us consider case 1:

$$\alpha \in [0, \pi/2)$$

So now,

$$\begin{aligned}
 |Z| = r &= \sqrt{(x^2 + y^2)} \\
 &= \sqrt{(\tan^2 \alpha + 1^2)} \\
 &= \sqrt{(\sec^2 \alpha)} \\
 &= |\sec \alpha| \text{ since, } \sec \alpha \text{ is positive in the interval } [0, \pi/2) \\
 &= \sec \alpha
 \end{aligned}$$

$$\begin{aligned}
 \theta &= \tan^{-1} (|y| / |x|) \\
 &= \tan^{-1} (1/\tan \alpha) \\
 &= \tan^{-1} (\cot \alpha) \text{ since, } \cot \alpha \text{ is positive in the interval } [0, \pi/2) \\
 &= \alpha - \pi/2 \text{ [since, } \theta \text{ lies in 4}^{\text{th}} \text{ quadrant]}
 \end{aligned}$$

$$Z = \sec \alpha (\cos (\alpha - \pi/2) + i \sin (\alpha - \pi/2))$$

$$\therefore \text{Polar form is } Z = \sec \alpha (\cos (\alpha - \pi/2) + i \sin (\alpha - \pi/2))$$

Let us consider case 2:

$$\alpha \in (\pi/2, \pi]$$

So now,

$$\begin{aligned}
 |Z| = r &= \sqrt{(x^2 + y^2)} \\
 &= \sqrt{(\tan^2 \alpha + 1^2)} \\
 &= \sqrt{(\sec^2 \alpha)} \\
 &= |\sec \alpha| \\
 &= -\sec \alpha \text{ since, } \sec \alpha \text{ is negative in the interval } (\pi/2, \pi]
 \end{aligned}$$

$$\begin{aligned}
 \theta &= \tan^{-1} (|y| / |x|) \\
 &= \tan^{-1} (1/\tan \alpha) \\
 &= \tan^{-1} (\cot \alpha) \\
 &= \pi/2 + \alpha \text{ since, } \cot \alpha \text{ is negative in the interval } (\pi/2, \pi]
 \end{aligned}$$

$$\theta = \pi/2 + \alpha \text{ [since, } \theta \text{ lies in 3}^{\text{th}} \text{ quadrant]}$$

$$Z = -\sec \alpha (\cos (\pi/2 + \alpha) + i \sin (\pi/2 + \alpha))$$

$$\therefore \text{Polar form is } Z = -\sec \alpha (\cos (\pi/2 + \alpha) + i \sin (\pi/2 + \alpha))$$

(iii) $1 - \sin \alpha + i \cos \alpha$

$$\text{Given: } Z = 1 - \sin \alpha + i \cos \alpha$$

By using the formulas,

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

So,

$$\begin{aligned}
 z &= \left(\sin^2 \left(\frac{\theta}{2} \right) + \cos^2 \left(\frac{\theta}{2} \right) - 2 \sin \left(\frac{\theta}{2} \right) \cos \left(\frac{\theta}{2} \right) \right) + i \left(\cos^2 \left(\frac{\theta}{2} \right) - \sin^2 \left(\frac{\theta}{2} \right) \right) \\
 &= \left(\cos \left(\frac{\theta}{2} \right) - \sin \left(\frac{\theta}{2} \right) \right)^2 + i \left(\cos^2 \left(\frac{\theta}{2} \right) - \sin^2 \left(\frac{\theta}{2} \right) \right)
 \end{aligned}$$

We know that the polar form of a complex number $Z = x + iy$ is given by $Z = |Z| (\cos \theta + i \sin \theta)$

Where,

$$|Z| = \text{modulus of complex number} = \sqrt{x^2 + y^2}$$

$$\theta = \arg(z) = \text{argument of complex number} = \tan^{-1} (|y| / |x|)$$

Now,

$$\begin{aligned} |z| &= \sqrt{(1 - \sin \alpha)^2 + \cos^2 \alpha} \\ &= \sqrt{1 + \sin^2 \alpha - 2 \sin \alpha + \cos^2 \alpha} \\ &= \sqrt{1 + 1 - 2 \sin \alpha} \\ &= \sqrt{(2)(1 - \sin \alpha)} \\ &= \sqrt{(2) \left(\sin^2 \left(\frac{\alpha}{2} \right) + \cos^2 \left(\frac{\alpha}{2} \right) - 2 \sin \left(\frac{\alpha}{2} \right) \cos \left(\frac{\alpha}{2} \right) \right)} \\ &= \sqrt{(2) \left(\cos \left(\frac{\alpha}{2} \right) - \sin \left(\frac{\alpha}{2} \right) \right)^2} \\ &= \left| \sqrt{2} \left(\cos \left(\frac{\alpha}{2} \right) - \sin \left(\frac{\alpha}{2} \right) \right) \right| \\ \theta &= \tan^{-1} \left(\frac{\cos^2 \left(\frac{\alpha}{2} \right) - \sin^2 \left(\frac{\alpha}{2} \right)}{\left(\cos \left(\frac{\alpha}{2} \right) - \sin \left(\frac{\alpha}{2} \right) \right)^2} \right) \\ &= \tan^{-1} \left(\frac{\left(\cos \left(\frac{\alpha}{2} \right) - \sin \left(\frac{\alpha}{2} \right) \right) \left(\cos \left(\frac{\alpha}{2} \right) + \sin \left(\frac{\alpha}{2} \right) \right)}{\left(\cos \left(\frac{\alpha}{2} \right) - \sin \left(\frac{\alpha}{2} \right) \right)^2} \right) \\ &= \tan^{-1} \left(\frac{\cos \left(\frac{\alpha}{2} \right) + \sin \left(\frac{\alpha}{2} \right)}{\cos \left(\frac{\alpha}{2} \right) - \sin \left(\frac{\alpha}{2} \right)} \right) \\ &= \tan^{-1} \left(\frac{\cos \left(\frac{\alpha}{2} \right) (1 + \tan \left(\frac{\alpha}{2} \right))}{\cos \left(\frac{\alpha}{2} \right) (1 - \tan \left(\frac{\alpha}{2} \right))} \right) \\ &= \tan^{-1} \left(\frac{\tan \left(\frac{\pi}{4} \right) + \tan \left(\frac{\alpha}{2} \right)}{1 - \tan \left(\frac{\pi}{4} \right) \tan \left(\frac{\alpha}{2} \right)} \right) \\ &= \tan^{-1} \left(\tan \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) \right) \end{aligned}$$

We know that sine and cosine functions are periodic with period 2π

Here we have 3 intervals:

$$0 \leq \alpha \leq \pi/2$$

$$\pi/2 \leq \alpha \leq 3\pi/2$$

$$3\pi/2 \leq \alpha \leq 2\pi$$

Let us consider case 1:

In the interval $0 \leq \alpha \leq \pi/2$

$\cos(\alpha/2) > \sin(\alpha/2)$ and also $0 < \pi/4 + \alpha/2 < \pi/2$

So,

$$\begin{aligned} |z| &= \left| \sqrt{2} \left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right) \right) \right| \\ &= \sqrt{2} \left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right) \right) \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1} \left(\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) \right) \\ &= \pi/4 + \alpha/2 \text{ [since, } \theta \text{ lies in 1}^{\text{st}} \text{ quadrant]} \end{aligned}$$

\therefore Polar form is $Z = \sqrt{2} (\cos(\alpha/2) - \sin(\alpha/2)) (\cos(\pi/4 + \alpha/2) + i \sin(\pi/4 + \alpha/2))$

Let us consider case 2:

In the interval $\pi/2 \leq \alpha \leq 3\pi/2$

$\cos(\alpha/2) < \sin(\alpha/2)$ and also $\pi/2 < \pi/4 + \alpha/2 < \pi$

So,

$$\begin{aligned} |z| &= \left| \sqrt{2} \left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right) \right) \right| \\ &= -\sqrt{2} \left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right) \right) \\ &= \sqrt{2} \left(\sin\left(\frac{\alpha}{2}\right) - \cos\left(\frac{\alpha}{2}\right) \right) \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1} \left(\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) \right) \\ &= \pi - [\pi/4 + \alpha/2] \text{ [since, } \theta \text{ lies in 4}^{\text{th}} \text{ quadrant]} \\ &= 3\pi/4 - \alpha/2 \end{aligned}$$

Since, $(1 - \sin \alpha) > 0$ and $\cos \alpha < 0$ [Z lies in 4th quadrant]

$$= \alpha/2 - 3\pi/4$$

\therefore Polar form is $Z = -\sqrt{2} (\cos(\alpha/2) - \sin(\alpha/2)) (\cos(\alpha/2 - 3\pi/4) + i \sin(\alpha/2 - 3\pi/4))$

Let us consider case 3:

In the interval $3\pi/2 \leq \alpha \leq 2\pi$

$\cos(\alpha/2) < \sin(\alpha/2)$ and also $\pi < \pi/4 + \alpha/2 < 5\pi/4$

So,

$$\begin{aligned}
 |z| &= \left| \sqrt{2} \left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right) \right) \right| \\
 &= -\sqrt{2} \left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right) \right) \\
 &= \sqrt{2} \left(\sin\left(\frac{\alpha}{2}\right) - \cos\left(\frac{\alpha}{2}\right) \right)
 \end{aligned}$$

$$\begin{aligned}
 \theta &= \tan^{-1} (\tan (\pi/4 + \alpha/2)) \\
 &= \pi - (\pi/4 + \alpha/2) \text{ [since, } \theta \text{ lies in 1st quadrant and tan's period is } \pi] \\
 &= \alpha/2 - 3\pi/4
 \end{aligned}$$

$$\therefore \text{ Polar form is } Z = -\sqrt{2} (\cos (\alpha/2) - \sin (\alpha/2)) (\cos (\alpha/2 - 3\pi/4) + i \sin (\alpha/2 - 3\pi/4))$$

(iv) $(1 - i) / (\cos \pi/3 + i \sin \pi/3)$

Given: $Z = (1 - i) / (\cos \pi/3 + i \sin \pi/3)$

Let us multiply and divide by $(1 - i\sqrt{3})$, we get

$$\begin{aligned}
 Z &= \frac{1-i}{\frac{1}{2} + \frac{i\sqrt{3}}{2}} \\
 &= 2 \times \frac{1-i}{1+i\sqrt{3}} \times \frac{1-i\sqrt{3}}{1-i\sqrt{3}} \\
 &= 2 \times \frac{1+i^2\sqrt{3}-i(1+\sqrt{3})}{1-i^2 3} \\
 &= 2 \times \frac{(1+(-\sqrt{3})-i(1+\sqrt{3}))}{1-(-3)} \\
 &= 2 \times \frac{(1-\sqrt{3})-i(1+\sqrt{3})}{4} \\
 &= \frac{(1-\sqrt{3})-i(1+\sqrt{3})}{2}
 \end{aligned}$$

We know that the polar form of a complex number $Z = x + iy$ is given by $Z = |Z| (\cos \theta + i \sin \theta)$

Where,

$$|Z| = \text{modulus of complex number} = \sqrt{(x^2 + y^2)}$$

$$\theta = \arg (z) = \text{argument of complex number} = \tan^{-1} (|y| / |x|)$$

Now,

$$\begin{aligned}
 |z| &= \sqrt{\left(\frac{1-\sqrt{3}}{2}\right)^2 + \left(\frac{-1-\sqrt{3}}{2}\right)^2} \\
 &= \sqrt{\frac{1+3-2\sqrt{3}+1+2+2\sqrt{3}}{4}} \\
 &= \sqrt{\frac{8}{4}} \\
 &= \sqrt{2}
 \end{aligned}$$

$$\begin{aligned}\theta &= \tan^{-1} \left(\left| \frac{\frac{1+\sqrt{3}}{2}}{\frac{1-\sqrt{3}}{2}} \right| \right) \\ &= \tan^{-1} \left(\left| \frac{1+\sqrt{3}}{1-\sqrt{3}} \right| \right) \\ &= \tan^{-1} \left(\left| \frac{(1+\sqrt{3})(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})} \right| \right) \\ &= \tan^{-1} \left(\left| \frac{1+3+2\sqrt{3}}{1-3} \right| \right) \\ &= \tan^{-1} \left(\frac{4+2\sqrt{3}}{2} \right)\end{aligned}$$

Since $x < 0$, $y < 0$ complex number lies in 3rd quadrant and the value of θ is $180^\circ \leq \theta \leq -90^\circ$.
 $= \tan^{-1} (2 + \sqrt{3})$
 $= -7\pi/12$

$$\begin{aligned}Z &= \sqrt{2} (\cos (-7\pi/12) + i \sin (-7\pi/12)) \\ &= \sqrt{2} (\cos (7\pi/12) - i \sin (7\pi/12))\end{aligned}$$

\therefore Polar form of $(1 - i) / (\cos \pi/3 + i \sin \pi/3)$ is $\sqrt{2} (\cos (7\pi/12) - i \sin (7\pi/12))$

4. If z_1 and z_2 are two complex number such that $|z_1| = |z_2|$ and $\arg (z_1) + \arg (z_2) = \pi$, then show that $z_1 = -\bar{z}_2$

Solution:

Given:

$$|z_1| = |z_2| \text{ and } \arg (z_1) + \arg (z_2) = \pi$$

Let us assume $\arg (z_1) = \theta$

$$\arg (z_2) = \pi - \theta$$

We know that in the polar form, $z = |z| (\cos \theta + i \sin \theta)$

$$z_1 = |z_1| (\cos \theta + i \sin \theta) \dots\dots\dots (i)$$

$$z_2 = |z_2| (\cos (\pi - \theta) + i \sin (\pi - \theta))$$

$$= |z_2| (-\cos \theta + i \sin \theta)$$

$$= -|z_2| (\cos \theta - i \sin \theta)$$

Now let us find the conjugate of

$$\bar{z}_2 = -|z_2| (\cos \theta + i \sin \theta) \dots\dots (ii) \text{ (since, } |\bar{z}_2| = |z_2| \text{)}$$

Now,

$$z_1 / \bar{z}_2 = [|z_1| (\cos \theta + i \sin \theta)] / [-|z_2| (\cos \theta + i \sin \theta)]$$

$$= -|z_1| / |z_2| \text{ [since, } |z_1| = |z_2| \text{]}$$

$$= -1$$

When we cross multiply we get,

$$z_1 = -\bar{z}_2$$

Hence proved.

5. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, prove that $\arg(z_1/z_4) + \arg(z_2/z_3) = 0$

Solution:

Given:

$$z_1 = \bar{z}_2$$

$$z_3 = \bar{z}_4$$

We know that $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$

So,

$$\begin{aligned} \arg(z_1/z_4) + \arg(z_2/z_3) &= \arg(z_1) - \arg(z_4) + \arg(z_2) - \arg(z_3) \\ &= \arg(\bar{z}_2) - \arg(z_4) + \arg(z_2) - \arg(\bar{z}_4) \\ &= [\arg(z_2) + \arg(\bar{z}_2)] - [\arg(z_4) + \arg(\bar{z}_4)] \\ &= 0 - 0 \text{ [since, } \arg(z) + \arg(\bar{z}) = 0] \\ &= 0 \end{aligned}$$

Hence proved.

6. Express $\sin \pi/5 + i(1 - \cos \pi/5)$ in polar form.

Solution:

Given:

$$Z = \sin \pi/5 + i(1 - \cos \pi/5)$$

By using the formula,

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$1 - \cos 2\theta = 2 \sin^2 \theta$$

So,

$$Z = 2 \sin \pi/10 \cos \pi/10 + i(2 \sin^2 \pi/10)$$

$$= 2 \sin \pi/10 (\cos \pi/10 + i \sin \pi/10)$$

\therefore The polar form of $\sin \pi/5 + i(1 - \cos \pi/5)$ is $2 \sin \pi/10 (\cos \pi/10 + i \sin \pi/10)$