EXERCISE 18.1 PAGE NO: 18.11

1. Using binomial theorem, write down the expressions of the following:

(i) $(2x + 3y)^5$

(ii) $(2x - 3y)^4$

(iii) $\left(x - \frac{1}{x}\right)^6$

(iv) $(1 - 3x)^7$

(v) $\left(ax - \frac{b}{x}\right)^6$

(vi) $\left(\sqrt{\frac{x}{a}} - \sqrt{\frac{a}{x}}\right)^6$

(vii) $\left(\sqrt[3]{x} - \sqrt[3]{a}\right)^6$

(viii) $(1 + 2x - 3x^2)^5$

(ix) $\left(x + 1 - \frac{1}{x}\right)^3$

(x) $(1 - 2x + 3x^2)^3$

Solution:

(i) $(2x + 3y)^5$

Let us solve the given expression:

$(2x + 3y)^5 = {}^5C_0 (2x)^5 (3y)^0 + {}^5C_1 (2x)^4 (3y)^1 + {}^5C_2 (2x)^3 (3y)^2 + {}^5C_3 (2x)^2 (3y)^3 + {}^5C_4 (2x)^1 (3y)^4 + {}^5C_5 (2x)^0 (3y)^5$

$= 32x^5 + 5 (16x^4) (3y) + 10 (8x^3) (9y)^2 + 10 (4x)^2 (27y)^3 + 5 (2x) (81y^4) + 243y^5$

$= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$

(ii) $(2x - 3y)^4$

Let us solve the given expression:

$(2x - 3y)^4 = {}^4C_0 (2x)^4 (3y)^0 - {}^4C_1 (2x)^3 (3y)^1 + {}^4C_2 (2x)^2 (3y)^2 - {}^4C_3 (2x)^1 (3y)^3 + {}^4C_4 (2x)^0 (3y)^4$

$= 16x^4 - 4 (8x^3) (3y) + 6 (4x^2) (9y^2) - 4 (2x) (27y^3) + 81y^4$

$= 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$

(iii) $\left(x - \frac{1}{x}\right)^6$

Let us solve the given expression:

$(x - \frac{1}{x})^6$

$$= {}^6C_0 x^6 \left(\frac{1}{x}\right)^0 - {}^6C_1 x^5 \left(\frac{1}{x}\right)^1 + {}^6C_2 x^4 \left(\frac{1}{x}\right)^2 - {}^6C_3 x^3 \left(\frac{1}{x}\right)^3$$

$$+ {}^6C_4 x^2 \left(\frac{1}{x}\right)^4 - {}^6C_5 x^1 \left(\frac{1}{x}\right)^5 + {}^6C_6 x^0 \left(\frac{1}{x}\right)^6$$

$$= x^6 - 6x^5 \times \frac{1}{x} + 15x^4 \times \frac{1}{x^2} - 20x^3 \times \frac{1}{x^3} + 15x^2 \times \frac{1}{x^4} - 6x \times \frac{1}{x^5} + \frac{1}{x^6}$$

$$= x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6}$$

(iv) $(1 - 3x)^7$

Let us solve the given expression:

$(1 - 3x)^7 = {}^7C_0 (3x)^0 - {}^7C_1 (3x)^1 + {}^7C_2 (3x)^2 - {}^7C_3 (3x)^3 + {}^7C_4 (3x)^4 - {}^7C_6 (3x)^6 - {}^7C_7 (3x)^7$

$= 1 - 7 (3x) + 21 (9x)^2 - 35 (27x^3) + 35 (81x^4) - 21 (243x^5) + 7 (729x^6) - 2187(x^7)$

$= 1 - 21x + 189x^2 - 945x^3 + 2835x^4 - 5103x^5 + 5103x^6 - 2187x^7$

(v) $\left(ax - \frac{b}{x}\right)^6$

Let us solve the given expression:

$$= {}^6C_0 (ax)^6 (\frac{b}{x})^0 - {}^6C_1 (ax)^5 (\frac{b}{x})^1 + {}^6C_2 (ax)^4 (\frac{b}{x})^2 - {}^6C_3 (ax)^3 (\frac{b}{x})^3$$

$$+ {}^6C_4 (ax)^2 (\frac{b}{x})^4 - {}^6C_5 (ax)^1 (\frac{b}{x})^5 + {}^6C_6 (ax)^0 (\frac{b}{x})^6$$

$$= a^6x^6 - 6a^5x^5 \times \frac{b}{x} + 15a^4x^4 \times \frac{b^2}{x^2} - 20a^3b^3 \times \frac{b^3}{x^3} + 15a^2x^2 \times \frac{b^4}{x^4} - 6ax \times \frac{b^5}{x^5} + \frac{b^6}{x^6}$$

$$= a^6x^6 - 6a^5x^4b + 15a^4x^2b^2 - 20a^3b^3 + 15\frac{a^2b^4}{x^2} - 6\frac{ab^5}{x^4} + \frac{b^6}{x^6}$$

(vi) $\left(\sqrt{\frac{x}{a}}-\sqrt{\frac{a}{x}}\right)^6$

Let us solve the given expression:

$$={}^6C_0\left(\sqrt{\frac{x}{a}}\right)^6\left(\sqrt{\frac{a}{x}}\right)^0-{}^6C_1\left(\sqrt{\frac{x}{a}}\right)^5\left(\sqrt{\frac{a}{x}}\right)^1+{}^6C_2\left(\sqrt{\frac{x}{a}}\right)^4\left(\sqrt{\frac{a}{x}}\right)^2-{}^6C_3\left(\sqrt{\frac{x}{a}}\right)^3\left(\sqrt{\frac{a}{x}}\right)^3$$

$$+{}^6C_4\left(\sqrt{\frac{x}{a}}\right)^2\left(\sqrt{\frac{a}{x}}\right)^4-{}^6C_5\left(\sqrt{\frac{x}{a}}\right)^1\left(\sqrt{\frac{a}{x}}\right)^5+{}^6C_6\left(\sqrt{\frac{x}{a}}\right)^0\left(\sqrt{\frac{a}{x}}\right)^6$$

$$=\frac{x^3}{a^3}-6\frac{x^2}{a^2}+15\frac{x}{a}-20+15\frac{a}{x}-6\frac{a^2}{x^2}+\frac{a^3}{x^3}$$

(vii) $\left(\sqrt[3]{x}-\sqrt[3]{a}\right)^6$

Let us solve the given expression:

$$={}^6C_0(\sqrt[3]{x})^6(\sqrt[3]{a})^0-{}^6C_1(\sqrt[3]{x})^5(\sqrt[3]{a})^1+{}^6C_2(\sqrt[3]{x})^4(\sqrt[3]{a})^2-{}^6C_3(\sqrt[3]{x})^3(\sqrt[3]{a})^3$$

$$+{}^6C_4(\sqrt[3]{x})^2(\sqrt[3]{a})^4-{}^6C_5(\sqrt[3]{x})^1(\sqrt[3]{a})^5+{}^6C_6(\sqrt[3]{x})^0(\sqrt[3]{a})^6$$

$$=x^2-6x^{5/3}a^{1/3}+15x^{4/3}a^{2/3}-20xa+15x^{2/3}a^{4/3}-6x^{1/3}a^{5/3}+a^2$$

(viii) $(1 + 2x - 3x^2)^5$

Let us solve the given expression:

Let us consider $(1 + 2x)$ and $3x^2$ as two different entities and apply the binomial theorem.

$(1 + 2x - 3x^2)^5 = {}^5C_0 (1 + 2x)^5 (3x^2)^0 - {}^5C_1 (1 + 2x)^4 (3x^2)^1 + {}^5C_2 (1 + 2x)^3 (3x^2)^2 - {}^5C_3 (1 + 2x)^2 (3x^2)^3 + {}^5C_4 (1 + 2x)^1 (3x^2)^4 - {}^5C_5 (1 + 2x)^0 (3x^2)^5$

$= (1 + 2x)^5 - 5(1 + 2x)^4 (3x^2) + 10 (1 + 2x)^3 (9x^4) - 10 (1 + 2x)^2 (27x^6) + 5 (1 + 2x) (81x^8) - 243x^{10}$

$= {}^5C_0 (2x)^0 + {}^5C_1 (2x)^1 + {}^5C_2 (2x)^2 + {}^5C_3 (2x)^3 + {}^5C_4 (2x)^4 + {}^5C_5 (2x)^5 - 15x^2 [{}^4C_0 (2x)^0 + {}^4C_1 (2x)^1 + {}^4C_2 (2x)^2 + {}^4C_3 (2x)^3 + {}^4C_4 (2x)^4] + 90x^4 [1 + 8x^3 + 6x + 12x^2] - 270x^6(1 + 4x^2 + 4x) + 405x^8 + 810x^9 - 243x^{10}$

$= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5 - 15x^2 - 120x^3 - 360^4 - 480x^5 - 240x^6 + 90x^4 + 720x^7 + 540x^5 + 1080x^6 - 270x^6 - 1080x^8 - 1080x^7 + 405x^8 + 810x^9 - 243x^{10}$

$= 1 + 10x + 25x^2 - 40x^3 - 190x^4 + 92x^5 + 570x^6 - 360x^7 - 675x^8 + 810x^9 - 243x^{10}$

(ix) $\left(x+1-\frac{1}{x}\right)^3$

Let us solve the given expression:

$$={}^3C_0(x+1)^3(\frac{1}{x})^0-{}^3C_1(x+1)^2(\frac{1}{x})^1+{}^3C_2(x+1)^1(\frac{1}{x})^2-{}^3C_3(x+1)^0(\frac{1}{x})^3$$

$$=(x+1)^3-3(x+1)^2\times\frac{1}{x}+3\frac{x+1}{x^2}-\frac{1}{x^3}$$

$$=x^3+1+3x+3x^2-\frac{3x^2+3+6x}{x}+3\frac{x+1}{x^2}-\frac{1}{x^3}$$

$$=x^3+1+3x+3x^2-3x-\frac{3}{x}-6+\frac{3}{x}+\frac{3}{x^2}-\frac{1}{x^3}$$

$$=x^3+3x^2-5+\frac{3}{x^2}-\frac{1}{x^3}$$

(x) $(1-2x+3x^2)^3$

Let us solve the given expression:

$$={}^3C_0(1-2x)^3+{}^3C_1(1-2x)^2(3x^2)+{}^3C_2(1-2x)(3x^2)^2+{}^3C_3(3x^2)^3$$

$$=(1-2x)^3+9x^2(1-2x)^2+27x^4(1-2x)+27x^6$$

$$=1-8x^3+12x^2-6x+9x^2(1+4x^2-4x)+27x^4-54x^5+27x^6$$

$$=1-8x^3+12x^2-6x+9x^2+36x^4-36x^3+27x^4-54x^5+27x^6$$

$$=1-6x+21x^2-44x^3+63x^4-54x^5+27x^6$$

2. Evaluate the following:

(i) $\left(\sqrt{x+1}+\sqrt{x-1}\right)^6+\left(\sqrt{x+1}-\sqrt{x-1}\right)^6$

(ii) $\left(x+\sqrt{x^2-1}\right)^6+\left(x-\sqrt{x^2-1}\right)^6$

(iii) $\left(1+2\sqrt{x}\right)^5+\left(1-2\sqrt{x}\right)^5$

(iv) $\left(\sqrt{2}+1\right)^6+\left(\sqrt{2}-1\right)^6$

(v) $\left(3+\sqrt{2}\right)^5-\left(3-\sqrt{2}\right)^5$

(vi) $\left(2+\sqrt{3}\right)^7+\left(2-\sqrt{3}\right)^7$

(vii) $\left(\sqrt{3}+1\right)^5-\left(\sqrt{3}-1\right)^5$

(viii) $(0.99)^5+(1.01)^5$

(ix) $\left(\sqrt{3}+\sqrt{2}\right)^6-\left(\sqrt{3}-\sqrt{2}\right)^6$

(x) $\left\{a^2+\sqrt{a^2-1}\right\}^4+\left\{a^2-\sqrt{a^2-1}\right\}^4$

Solution:

(i) $\left(\sqrt{x+1}+\sqrt{x-1}\right)^6+\left(\sqrt{x+1}-\sqrt{x-1}\right)^6$

Let us solve the given expression:

$= 2[{}^6C_0\ (\sqrt{x+1})^6(\sqrt{x-1})^0 + {}^6C_2\ (\sqrt{x+1})^4(\sqrt{x-1})^2$

$+ {}^6C_4\ (\sqrt{x+1})^2(\sqrt{x-1})^4 + {}^6C_6\ (\sqrt{x+1})^0(\sqrt{x-1})^6]$

$= 2[(x+1)^3 + 15(x+1)^2(x-1) + 15(x+1)(x-1)^2 + (x-1)^3$

$= 2[x^3+1+3x+3x^2+15(x^2+2x+1)(x-1)+15(x+1)(x^2+1-2x)+x^3-1+3x-3x^2]$

$= 2[2x^3+6x+15x^3-15x^2+30x^2-30x+15x-15+15x^3+15x^2-30x^2-30x+15x+15]$

$= 2[32x^3-24x]$

$= 16x[4x^2-3]$

(ii) $\left(x+\sqrt{x^2-1}\right)^6+\left(x-\sqrt{x^2-1}\right)^6$

Let us solve the given expression:

$= 2\left[{}^6C_0x^6\left(\sqrt{x^2-1}\right)^0 + {}^6C_2x^4\left(\sqrt{x^2-1}\right)^2 + {}^6C_4x^2\left(\sqrt{x^2-1}\right)^4 +\right.$

$\left.{}^6C_6x^0\left(\sqrt{x^2-1}\right)^6\right]$

$= 2\left[x^6 + 15x^4\left(x^2-1\right) + 15x^2\left(x^2-1\right)^2 + \left(x^2-1\right)^3\right]$

$= 2\left[x^6 + 15x^6 - 15x^4 + 15x^2\left(x^4 - 2x^2 + 1\right) + \left(x^6 - 1 + 3x^2 - 3x^4\right)\right]$

$= 2\left[x^6 + 15x^6 - 15x^4 + 15x^6 - 30x^4 + 15x^2 + x^6 - 1 + 3x^2 - 3x^4\right]$

$= 64x^6 - 96x^4 + 36x^2 - 2$

(iii) $\left(1 + 2\sqrt{x}\right)^5 + \left(1 - 2\sqrt{x}\right)^5$

Let us solve the given expression:
$= 2\ [{}^5C_0\ (2\sqrt{x})^0 + {}^5C_2\ (2\sqrt{x})^2 + {}^5C_4\ (2\sqrt{x})^4]$
$= 2\ [1 + 10\ (4x) + 5\ (16x^2)]$
$= 2\ [1 + 40x + 80x^2]$

(iv) $\left(\sqrt{2} + 1\right)^6 + \left(\sqrt{2} - 1\right)^6$

Let us solve the given expression:
$= 2\ [{}^6C_0\ (\sqrt{2})^6 + {}^6C_2\ (\sqrt{2})^4 + {}^6C_4\ (\sqrt{2})^2 + {}^6C_6\ (\sqrt{2})^0]$
$= 2\ [8 + 15\ (4) + 15\ (2) + 1]$
$= 2\ [99]$
$= 198$

(v) $\left(3 + \sqrt{2}\right)^5 - \left(3 - \sqrt{2}\right)^5$

Let us solve the given expression:
$= 2\ [{}^5C_1\ (3^4)\ (\sqrt{2})^1 + {}^5C_3\ (3^2)\ (\sqrt{2})^3 + {}^5C_5\ (3^0)\ (\sqrt{2})^5]$
$= 2\ [5\ (81)\ (\sqrt{2}) + 10\ (9)\ (2\sqrt{2}) + 4\sqrt{2}]$
$= 2\sqrt{2}\ (405 + 180 + 4)$
$= 1178\sqrt{2}$

(vi) $\left(2 + \sqrt{3}\right)^7 + \left(2 - \sqrt{3}\right)^7$

Let us solve the given expression:
$= 2\ [{}^7C_0\ (2^7)\ (\sqrt{3})^0 + {}^7C_2\ (2^5)\ (\sqrt{3})^2 + {}^7C_4\ (2^3)\ (\sqrt{3})^4 + {}^7C_6\ (2^1)\ (\sqrt{3})^6]$
$= 2\ [128 + 21\ (32)(3) + 35(8)(9) + 7(2)(27)]$
$= 2\ [128 + 2016 + 2520 + 378]$
$= 2\ [5042]$
$= 10084$

(vii) $\left(\sqrt{3}+1\right)^5-\left(\sqrt{3}-1\right)^5$

Let us solve the given expression:

$= 2\,[{}^5C_1\,(\sqrt{3})^4 + {}^5C_3\,(\sqrt{3})^2 + {}^5C_5\,(\sqrt{3})^0]$

$= 2\,[5\,(9) + 10\,(3) + 1]$

$= 2\,[76]$

$= 152$

(viii) $(0.99)^5+(1.01)^5$

Let us solve the given expression:

$= (1 - 0.01)^5 + (1 + 0.01)^5$

$= 2\,[{}^5C_0\,(0.01)^0 + {}^5C_2\,(0.01)^2 + {}^5C_4\,(0.01)^4]$

$= 2\,[1 + 10\,(0.0001) + 5\,(0.00000001)]$

$= 2\,[1.00100005]$

$= 2.0020001$

(ix) $\left(\sqrt{3}+\sqrt{2}\right)^6-\left(\sqrt{3}-\sqrt{2}\right)^6$

Let us solve the given expression:

$= 2\,[{}^6C_1\,(\sqrt{3})^5\,(\sqrt{2})^1 + {}^6C_3\,(\sqrt{3})^3\,(\sqrt{2})^3 + {}^6C_5\,(\sqrt{3})^1\,(\sqrt{2})^5]$

$= 2\,[6\,(9\sqrt{3})\,(\sqrt{2}) + 20\,(3\sqrt{3})\,(2\sqrt{2}) + 6\,(\sqrt{3})\,(4\sqrt{2})]$

$= 2\,[\sqrt{6}\,(54 + 120 + 24)]$

$= 396\,\sqrt{6}$

(x) $\left\{a^2+\sqrt{a^2-1}\right\}^4+\left\{a^2-\sqrt{a^2-1}\right\}^4$

Let us solve the given expression:

$$= 2\left[{}^4C_0\left(a^2\right)^4\left(\sqrt{a^2-1}\right)^0 + {}^4C_2\left(a^2\right)^2\left(\sqrt{a^2-1}\right)^2 + {}^4C_4\left(a^2\right)^0\left(\sqrt{a^2-1}\right)^4\right]$$

$$= 2\left[a^8 + 6a^4\left(a^2-1\right) + \left(a^2-1\right)^2\right]$$

$= 2\,[a^8 + 6a^6 - 6a^4 + a^4 + 1 - 2a^2]$

$= 2a^8 + 12a^6 - 10a^4 - 4a^2 + 2$

3. Find $(a + b)^4 - (a - b)^4$. Hence, evaluate $(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4$.

Solution:

Firstly, let us solve the given expression:

$(a + b)^4 - (a - b)^4$

The above expression can be expressed as,

$(a + b)^4 – (a – b)^4 = 2 [{}^4C_1\, a^3b^1 + {}^4C_3\, a^1b^3]$

$= 2 [4a^3b + 4ab^3]$

$= 8 (a^3b + ab^3)$

Now,

Let us evaluate the expression:

$(\sqrt{3} + \sqrt{2})^4 – (\sqrt{3} - \sqrt{2})^4$

So consider, $a = \sqrt{3}$ and $b = \sqrt{2}$ we get,

$(\sqrt{3} + \sqrt{2})^4 – (\sqrt{3} - \sqrt{2})^4 = 8 (a^3b + ab^3)$

$= 8 [(\sqrt{3})^3 (\sqrt{2}) + (\sqrt{3}) (\sqrt{2})^3]$

$= 8 [(3\sqrt{6}) + (2\sqrt{6})]$

$= 8 (5\sqrt{6})$

$= 40\sqrt{6}$

4. Find $(x + 1)^6 + (x – 1)^6$. Hence, or otherwise evaluate $(\sqrt{2} + 1)^6 + (\sqrt{2} – 1)^6$.

Solution:

Firstly, let us solve the given expression:

$(x + 1)^6 + (x – 1)^6$

The above expression can be expressed as,

$(x + 1)^6 + (x – 1)^6 = 2 [{}^6C_0\, x^6 + {}^6C_2\, x^4 + {}^6C_4\, x^2 + {}^6C_6\, x^0]$

$= 2 [x^6 + 15x^4 + 15x^2 + 1]$

Now,

Let us evaluate the expression:

$(\sqrt{2} + 1)^6 + (\sqrt{2} – 1)^6$

So consider, $x = \sqrt{2}$ then we get,

$(\sqrt{2} + 1)^6 + (\sqrt{2} – 1)^6 = 2 [x^6 + 15x^4 + 15x^2 + 1]$

$= 2 [(\sqrt{2})^6 + 15 (\sqrt{2})^4 + 15 (\sqrt{2})^2 + 1]$

$= 2 [8 + 15 (4) + 15 (2) + 1]$

$= 2 [8 + 60 + 30 + 1]$

$= 198$

5. Using binomial theorem evaluate each of the following:

(i) $(96)^3$

(ii) $(102)^5$

(iii) $(101)^4$

(iv) $(98)^5$

Solution:

(i) $(96)^3$

We have,

$(96)^3$

Let us express the given expression as two different entities and apply the binomial theorem.

$(96)^3 = (100 - 4)^3$

$= {}^3C_0 (100)^3 (4)^0 - {}^3C_1 (100)^2 (4)^1 + {}^3C_2 (100)^1 (4)^2 - {}^3C_3 (100)^0 (4)^3$

$= 1000000 – 120000 + 4800 – 64$

$= 884736$

(ii) $(102)^5$

We have,

$(102)^5$

Let us express the given expression as two different entities and apply the binomial theorem.

$(102)^5 = (100 + 2)^5$

$= {}^5C_0 (100)^5 (2)^0 + {}^5C_1 (100)^4 (2)^1 + {}^5C_2 (100)^3 (2)^2 + {}^5C_3 (100)^2 (2)^3 + {}^5C_4 (100)^1 (2)^4 + {}^5C_5 (100)^0 (2)^5$

$= 10000000000 + 1000000000 + 40000000 + 800000 + 8000 + 32$

$= 11040808032$

(iii) $(101)^4$

We have,

$(101)^4$

Let us express the given expression as two different entities and apply the binomial theorem.

$(101)^4 = (100 + 1)^4$

$= {}^4C_0 (100)^4 + {}^4C_1 (100)^3 + {}^4C_2 (100)^2 + {}^4C_3 (100)^1 + {}^4C_4 (100)^0$

$= 100000000 + 4000000 + 60000 + 400 + 1$

$= 104060401$

(iv) $(98)^5$

We have,

$(98)^5$

Let us express the given expression as two different entities and apply the binomial theorem.

$(98)^5 = (100 - 2)^5$

$= {}^5C_0 (100)^5 (2)^0 - {}^5C_1 (100)^4 (2)^1 + {}^5C_2 (100)^3 (2)^2 - {}^5C_3 (100)^2 (2)^3 + {}^5C_4 (100)^1 (2)^4 - {}^5C_5 (100)^0 (2)^5$

$= 10000000000 - 1000000000 + 40000000 - 800000 + 8000 – 32$

$= 9039207968$

6. Using binomial theorem, prove that $2^{3n} – 7n – 1$ is divisible by 49, where n ∈ N.

Solution:

Given:

$2^{3n} – 7n – 1$

So, $2^{3n} – 7n – 1 = 8^n – 7n – 1$

Now,

$8^n – 7n – 1$

$8^n = 7n + 1$

$= (1 + 7)^n$

$= {}^nC_0 + {}^nC_1 (7)^1 + {}^nC_2 (7)^2 + {}^nC_3 (7)^3 + {}^nC_4 (7)^2 + {}^nC_5 (7)^1 + … + {}^nC_n (7)^n$

$8^n = 1 + 7n + 49 [{}^nC_2 + {}^nC_3 (7^1) + {}^nC_4 (7^2) + … + {}^nC_n (7)^{n-2}]$

$8^n – 1 – 7n = 49$ (integer)

So now,

$8^n – 1 – 7n$ is divisible by 49

Or

$2^{3n} – 1 – 7n$ is divisible by 49.

Hence proved.