

**EXERCISE 19.5**
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**1. If  $1/a, 1/b, 1/c$  are in A.P., prove that:**

**(i)  $(b+c)/a, (c+a)/b, (a+b)/c$  are in A.P.**

**(ii)  $a(b+c), b(c+a), c(a+b)$  are in A.P.**

**Solution:**

**(i)**  $(b+c)/a, (c+a)/b, (a+b)/c$  are in A.P.

We know that, if  $a, b, c$  are in AP, then  $b - a = c - b$

If,  $1/a, 1/b, 1/c$  are in AP

Then,  $1/b - 1/a = 1/c - 1/b$

If  $(b+c)/a, (c+a)/b, (a+b)/c$  are in AP

Then,  $(c+a)/b - (b+c)/a = (a+b)/c - (c+a)/b$

Let us take LCM

$$\frac{ca + a^2 - b^2 - cb}{ab} = \frac{ab + b^2 - c^2 - ac}{bc}$$

Now let us consider LHS:

$$\frac{ca + a^2 - b^2 - cb}{ab}$$

Multiply both numerator and denominator by 'c', we get,

$$\begin{aligned} \frac{ca + a^2 - b^2 - cb}{ab} &= \frac{c^2a + ca^2 - cb^2 - c^2b}{abc} \\ &= \frac{C(b-a)(a+b+c)}{abc} \end{aligned}$$

Now let us consider RHS:

$$\frac{ab + b^2 - c^2 - ac}{bc}$$

Multiply both numerator and denominator by 'a', we get,

$$\begin{aligned} \frac{ab + b^2 - c^2 - ac}{bc} &= \frac{a^2b + ab^2 - ac^2 - a^2c}{abc} \\ &= \frac{a(b-c)(a+b+c)}{abc} \end{aligned}$$

**LHS = RHS**

$$\frac{C(b-a)(a+b+c)}{abc} = \frac{a(b-c)(a+b+c)}{abc}$$

Since,  $1/a, 1/b, 1/c$  are in AP

$$1/b - 1/a = 1/c - 1/b$$

$$C(b-a) = a(b-c)$$

Hence, the given terms are in AP.

(ii)  $a(b + c)$ ,  $b(c + a)$ ,  $c(a + b)$  are in A.P.

We know that if,  $b(c + a) - a(b+c) = c(a+b) - b(c+a)$

Consider LHS:

$$b(c + a) - a(b+c)$$

Upon simplification we get,

$$\begin{aligned} b(c + a) - a(b+c) &= bc + ba - ab - ac \\ &= c(b-a) \end{aligned}$$

Now,

$$\begin{aligned} c(a+b) - b(c+a) &= ca + cb - bc - ba \\ &= a(c-b) \end{aligned}$$

We know,

$1/a$ ,  $1/b$ ,  $1/c$  are in AP

So,  $1/a - 1/b = 1/b - 1/c$

Or  $c(b-a) = a(c-b)$

Hence, given terms are in AP.

**2. If  $a^2$ ,  $b^2$ ,  $c^2$  are in AP., prove that  $a/(b+c)$ ,  $b/(c+a)$ ,  $c/(a+b)$  are in AP.**

**Solution:**

If  $a^2$ ,  $b^2$ ,  $c^2$  are in AP then,  $b^2 - a^2 = c^2 - b^2$

If  $a/(b+c)$ ,  $b/(c+a)$ ,  $c/(a+b)$  are in AP then,

$$b/(c+a) - a/(b+c) = c/(a+b) - b/(c+a)$$

Let us take LCM on both the sides we get,

$$\frac{b^2 + bc - a^2 - ac}{(a + c)(b + c)} = \frac{ca + c^2 - b^2 - ab}{(a + b)(b + c)}$$

$$\frac{(b - a)(a + b + c)}{(a + c)(b + c)} = \frac{(c - b)(a + b + c)}{(a + b)(b + c)}$$

Since,  $b^2 - a^2 = c^2 - b^2$

Substituting  $b^2 - a^2 = c^2 - b^2$  in above, we get

LHS = RHS

Hence, given terms are in AP

**3. If  $a$ ,  $b$ ,  $c$  are in A.P., then show that:**

(i)  $a^2(b + c)$ ,  $b^2(c + a)$ ,  $c^2(a + b)$  are also in A.P.

(ii)  $b + c - a$ ,  $c + a - b$ ,  $a + b - c$  are in A.P.

(iii)  $bc - a^2$ ,  $ca - b^2$ ,  $ab - c^2$  are in A.P.

**Solution:**

(i)  $a^2(b + c)$ ,  $b^2(c + a)$ ,  $c^2(a + b)$  are also in A.P.

If  $b^2(c + a) - a^2(b + c) = c^2(a + b) - b^2(c + a)$

$$b^2c + b^2a - a^2b - a^2c = c^2a + c^2b - b^2a - b^2c$$

Given,  $b - a = c - b$

And since  $a, b, c$  are in AP,

$$c(b^2 - a^2) + ab(b - a) = a(c^2 - b^2) + bc(c - b)$$

$$(b - a)(ab + bc + ca) = (c - b)(ab + bc + ca)$$

Upon cancelling,  $ab + bc + ca$  from both sides

$$b - a = c - b$$

$$2b = c + a \text{ [which is true]}$$

Hence, given terms are in AP

(ii)  $b + c - a$ ,  $c + a - b$ ,  $a + b - c$  are in A.P.

If  $(c + a - b) - (b + c - a) = (a + b - c) - (c + a - b)$

Then,  $b + c - a$ ,  $c + a - b$ ,  $a + b - c$  are in A.P.

Let us consider LHS and RHS

$$(c + a - b) - (b + c - a) = (a + b - c) - (c + a - b)$$

$$2a - 2b = 2b - 2c$$

$$b - a = c - b$$

And since  $a, b, c$  are in AP,

$$b - a = c - b$$

Hence, given terms are in AP.

(iii)  $bc - a^2$ ,  $ca - b^2$ ,  $ab - c^2$  are in A.P.

If  $(ca - b^2) - (bc - a^2) = (ab - c^2) - (ca - b^2)$

Then,  $bc - a^2$ ,  $ca - b^2$ ,  $ab - c^2$  are in A.P.

Let us consider LHS and RHS

$$(ca - b^2) - (bc - a^2) = (ab - c^2) - (ca - b^2)$$

$$(a - b^2 - bc + a^2) = (ab - c^2 - ca + b^2)$$

$$(a - b)(a + b + c) = (b - c)(a + b + c)$$

$$a - b = b - c$$

And since  $a, b, c$  are in AP,

$$b - c = a - b$$

Hence, given terms are in AP

**4. If  $(b+c)/a$ ,  $(c+a)/b$ ,  $(a+b)/c$  are in AP., prove that:**

(i)  $1/a$ ,  $1/b$ ,  $1/c$  are in AP

(ii)  $bc$ ,  $ca$ ,  $ab$  are in AP

**Solution:**

(i)  $1/a, 1/b, 1/c$  are in AP

If  $1/a, 1/b, 1/c$  are in AP then,

$$1/b - 1/a = 1/c - 1/b$$

Let us consider LHS:

$$1/b - 1/a = (a-b)/ab$$

$$= c(a-b)/abc \text{ [by multiplying with 'c' on both the numerator and denominator]}$$

Let us consider RHS:

$$1/c - 1/b = (b-c)/bc$$

$$= a(b-c)/bc \text{ [by multiplying with 'a' on both the numerator and denominator]}$$

Since,  $(b+c)/a, (c+a)/b, (a+b)/c$  are in AP

$$\frac{c+a}{b} - \frac{b+c}{a} = \frac{a+b}{c} - \frac{c+a}{b}$$

$$\frac{b^2 + bc - a^2 - ac}{ab} = \frac{ca + c^2 - b^2 - ab}{bc}$$

$$\frac{(b-a)(a+b+c)}{ab} = \frac{(c-b)(a+b+c)}{bc}$$

$$\frac{a(b-c)}{abc} = \frac{c(a-b)}{abc}$$

$$a(b-c) = c(a-b)$$

$$\text{LHS} = \text{RHS}$$

Hence, the given terms are in AP

(ii)  $bc, ca, ab$  are in AP

If  $bc, ca, ab$  are in AP then,

$$ca - bc = ab - ca$$

$$c(a-b) = a(b-c)$$

If  $1/a, 1/b, 1/c$  are in AP then,

$$1/b - 1/a = 1/c - 1/b$$

$$c(a-b) = a(b-c)$$

Hence, the given terms are in AP

**5. If  $a, b, c$  are in A.P., prove that:**

(i)  $(a - c)^2 = 4(a - b)(b - c)$

(ii)  $a^2 + c^2 + 4ac = 2(ab + bc + ca)$

(iii)  $a^3 + c^3 + 6abc = 8b^3$

**Solution:**

**(i)**  $(a - c)^2 = 4(a - b)(b - c)$

Let us expand the above expression

$$a^2 + c^2 - 2ac = 4(ab - ac - b^2 + bc)$$

$$a^2 + 4c^2b^2 + 2ac - 4ab - 4bc = 0$$

$$(a + c - 2b)^2 = 0$$

$$a + c - 2b = 0$$

Since a, b, c are in AP

$$b - a = c - b$$

$$a + c - 2b = 0$$

$$a + c = 2b$$

Hence,  $(a - c)^2 = 4(a - b)(b - c)$

**(ii)**  $a^2 + c^2 + 4ac = 2(ab + bc + ca)$

Let us expand the above expression

$$a^2 + c^2 + 4ac = 2(ab + bc + ca)$$

$$a^2 + c^2 + 2ac - 2ab - 2bc = 0$$

$$(a + c - b)^2 - b^2 = 0$$

$$a + c - b = b$$

$$a + c - 2b = 0$$

$$2b = a + c$$

$$b = (a + c)/2$$

Since a, b, c are in AP

$$b - a = c - b$$

$$b = (a + c)/2$$

Hence,  $a^2 + c^2 + 4ac = 2(ab + bc + ca)$

**(iii)**  $a^3 + c^3 + 6abc = 8b^3$

Let us expand the above expression

$$a^3 + c^3 + 6abc = 8b^3$$

$$a^3 + c^3 - (2b)^3 + 6abc = 0$$

$$a^3 + (-2b)^3 + c^3 + 3a(-2b)c = 0$$

Since, if  $a + b + c = 0$ ,  $a^3 + b^3 + c^3 = 3abc$

$$(a - 2b + c)^3 = 0$$

$$a - 2b + c = 0$$

$$a + c = 2b$$

$$b = (a + c)/2$$

Since a, b, c are in AP

$$a - b = c - b$$

$$b = (a+c)/2$$

$$\text{Hence, } a^3 + c^3 + 6abc = 8b^3$$

**6. If  $a(1/b + 1/c)$ ,  $b(1/c + 1/a)$ ,  $c(1/a + 1/b)$  are in AP., prove that  $a$ ,  $b$ ,  $c$  are in AP.**

**Solution:**

Here, we know  $a(1/b + 1/c)$ ,  $b(1/c + 1/a)$ ,  $c(1/a + 1/b)$  are in AP

Also,  $a(1/b + 1/c) + 1$ ,  $b(1/c + 1/a) + 1$ ,  $c(1/a + 1/b) + 1$  are in AP

Let us take LCM for each expression then we get,

$(ac+ab+bc)/bc$ ,  $(ab+bc+ac)/ac$ ,  $(cb+ac+ab)/ab$  are in AP

$1/bc$ ,  $1/ac$ ,  $1/ab$  are in AP

Let us multiply numerator with 'abc', we get

$abc/bc$ ,  $abc/ac$ ,  $abc/ab$  are in AP

$\therefore a$ ,  $b$ ,  $c$  are in AP.

Hence proved.

**7. Show that  $x^2 + xy + y^2$ ,  $z^2 + zx + x^2$  and  $y^2 + yz + z^2$  are in consecutive terms of an A.P., if  $x$ ,  $y$  and  $z$  are in A.P.**

**Solution:**

$x$ ,  $y$ ,  $z$  are in AP

Given,  $x^2 + xy + y^2$ ,  $z^2 + zx + x^2$  and  $y^2 + yz + z^2$  are in AP

$$(z^2 + zx + x^2) - (x^2 + xy + y^2) = (y^2 + yz + z^2) - (z^2 + zx + x^2)$$

Let  $d$  = common difference,

So,  $Y = x + d$  and  $x = x + 2d$

Let us consider the LHS:

$$(z^2 + zx + x^2) - (x^2 + xy + y^2)$$

$$z^2 + zx - xy - y^2$$

$$(x + 2d)^2 + (x + 2d)x - x(x + d) - (x + d)^2$$

$$x^2 + 4xd + 4d^2 + x^2 + 2xd - x^2 - xd - x^2 - 2xd - d^2$$

$$3xd + 3d^2$$

Now, let us consider RHS:

$$(y^2 + yz + z^2) - (z^2 + zx + x^2)$$

$$y^2 + yz - zx - x^2$$

$$(x + d)^2 + (x + d)(x + 2d) - (x + 2d)x - x^2$$

$$x^2 + 2dx + d^2 + x^2 + 2dx + xd + 2d^2 - x^2 - 2dx - x^2$$

$$3xd + 3d^2$$

LHS = RHS

$\therefore x^2 + xy + y^2$ ,  $z^2 + zx + x^2$  and  $y^2 + yz + z^2$  are in consecutive terms of A.P

Hence proved.