Exercise 10.2

1. Compute the magnitude of the following vectors:

$$\vec{a}=\hat{i}+\hat{j}+\hat{k};\quad \vec{b}=2\hat{i}-7\hat{j}-3\hat{k};\qquad \vec{c}=\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}-\frac{1}{\sqrt{3}}\hat{k}$$

Solution:

Given vectors are:

$$\vec{a}=\hat{i}+\hat{j}+\hat{k};\quad \vec{b}=2\hat{i}-7\hat{j}-3\hat{k};\qquad \vec{c}=\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}-\frac{1}{\sqrt{3}}\hat{k}$$

$$|\vec{a}|=\sqrt{(1)^2+(1)^2+(1)^2}=\sqrt{3}$$

$$|\vec{b}|=\sqrt{(2)^2+(-7)^2+(-3)^2}$$

$$=\sqrt{4+49+9}$$

$$=\sqrt{62}$$

$$|\vec{c}|=\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2+\left(-\frac{1}{\sqrt{3}}\right)^2}$$

$$=\sqrt{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}=1$$

2. Write two different vectors having same magnitude.
Solution:

Consider $\vec{a}=\left(\hat{i}-2\hat{j}+4\hat{k}\right)$ and $\vec{b}=\left(2\hat{i}+\hat{j}-4\hat{k}\right)$.

It can be observed that $|\vec{a}|=\sqrt{1^2+(-2)^2+4^2}=\sqrt{1+4+16}=\sqrt{21}$ and

$|\vec{b}|=\sqrt{2^2+1^2+(-4)^2}=\sqrt{4+1+16}=\sqrt{21}$

Thus, $\vec{a}$ and $\vec{b}$ are two different vectors having the same magnitude. Here, the vectors are different as they have different directions.

3. Write two different vectors having same direction.
Solution:

Consider $\vec{p}=\left(\hat{i}+\hat{j}+\hat{k}\right)$ and $\vec{q}=\left(2\hat{i}+2\hat{j}+2\hat{k}\right)$.

The direction cosines of $\vec{p}$ are given by,

$$l=\frac{1}{\sqrt{1^2+1^2+1^2}}=\frac{1}{\sqrt{3}},\ m=\frac{1}{\sqrt{1^2+1^2+1^2}}=\frac{1}{\sqrt{3}},\ \text{and } n=\frac{1}{\sqrt{1^2+1^2+1^2}}=\frac{1}{\sqrt{3}}$$

The direction cosines of $\vec{q}$ are given by

$$l = \frac{2}{\sqrt{2^2+2^2+2^2}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}},\ m = \frac{2}{\sqrt{2^2+2^2+2^2}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}},$$

and $n = \frac{2}{\sqrt{2^2+2^2+2^2}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$.

4. Find the values of *x* and *y* so that the vectors $2\hat{i}+3\hat{j}$ and $x\hat{i}+y\hat{j}$ are equal
Solution:

Given vectors $2\hat{i}+3\hat{j}$ and $x\hat{i}+y\hat{j}$ will be equal only if their corresponding components are equal.
Thus, the required values of *x* and *y* are 2 and 3 respectively.

5. Find the scalar and vector components of the vector with initial point (2, 1) and terminal point (–5, 7).
Solution:

The vector with initial point P (2, 1) and terminal point Q (–5, 7) can be shown as,

$$\overrightarrow{PQ} = (-5-2)\hat{i} + (7-1)\hat{j}$$

$$\overrightarrow{PQ} = -7\hat{i} + 6\hat{j}$$

Thus, the required scalar components are –7 and 6 while the vector components are $-7\hat{i}$ and $6\hat{j}$.

6. Find the sum of the vectors $\vec{a} = \hat{i}-2\hat{j}+\hat{k},\ \vec{b} = -2\hat{i}+4\hat{j}+5\hat{k}$ and $\vec{c} = \hat{i}-6\hat{j}-7\hat{k}$
Solution:

The given vectors are $\vec{a} = \hat{i}-2\hat{j}+\hat{k},\ \vec{b} = -2\hat{i}+4\hat{j}+5\hat{k}$ and $\vec{c} = \hat{i}-6\hat{j}-7\hat{k}$

Hence,

$$\begin{aligned}\vec{a}+\vec{b}+\vec{c} &= (1-2+1)\hat{i} + (-2+4-6)\hat{j} + (1+5-7)\hat{k} \\ &= 0\cdot\hat{i} - 4\hat{j} - 1\cdot\hat{k} \\ &= -4\hat{j} - \hat{k}\end{aligned}$$

7. Find the unit vector in the direction of the vector $\vec{a} = \hat{i}+\hat{j}+2\hat{k}$
Solution:

The unit vector $\hat{a}$ in the direction of vector $\vec{a} = \hat{i}+\hat{j}+2\hat{k}$ is given by $\hat{a} = \frac{\vec{a}}{|a|}$.

So,

$$|\vec{a}| = \sqrt{1^2+1^2+2^2} = \sqrt{1+1+4} = \sqrt{6}$$

Thus,

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i}+\hat{j}+2\hat{k}}{\sqrt{6}} = \frac{1}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j} + \frac{2}{\sqrt{6}}\hat{k}$$

8. Find the unit vector in the direction of vector $\overrightarrow{PQ}$, where P and Q are the points

(1, 2, 3) and (4, 5, 6), respectively
Solution:

Given points are P (1, 2, 3) and Q (4, 5, 6).

So, $\overrightarrow{PQ} = (4-1)\hat{i} + (5-2)\hat{j} + (6-3)\hat{k} = 3\hat{i} + 3\hat{j} + 3\hat{k}$

$\left|\overrightarrow{PQ}\right| = \sqrt{3^2 + 3^2 + 3^2} = \sqrt{9+9+9} = \sqrt{27} = 3\sqrt{3}$

Thus, the unit vector in the direction of $\overrightarrow{PQ}$ is

$$\frac{\overrightarrow{PQ}}{\left|\overrightarrow{PQ}\right|} = \frac{3\hat{i} + 3\hat{j} + 3\hat{k}}{3\sqrt{3}} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$$

9. For given vectors, $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$, find the unit vector in the direction of the vector $\vec{a} + \vec{b}$
Solution:

Given vectors are $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$

$\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$

$\vec{b} = -\hat{i} + \hat{j} - \hat{k}$

$\therefore \vec{a} + \vec{b} = (2-1)\hat{i} + (-1+1)\hat{j} + (2-1)\hat{k} = 1\hat{i} + 0\hat{j} + 1\hat{k} = \hat{i} + \hat{k}$

$\left|\vec{a} + \vec{b}\right| = \sqrt{1^2 + 1^2} = \sqrt{2}$

Thus, the unit vector in the direction of $(\vec{a} + \vec{b})$ is

$$\frac{\left(\vec{a} + \vec{b}\right)}{\left|\vec{a} + \vec{b}\right|} = \frac{\hat{i} + \hat{k}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}.$$

10. Find a vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units.
Solution:

Let $\vec{a} = 5\hat{i} - \hat{j} + 2\hat{k}$.

So,

$|\vec{a}| = \sqrt{5^2 + (-1)^2 + 2^2} = \sqrt{25+1+4} = \sqrt{30}$

$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{5\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{30}}$

Thus, the vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units is given by,

$$8\hat{a} = 8\left(\frac{5\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{30}}\right) = \frac{40}{\sqrt{30}}\hat{i} - \frac{8}{\sqrt{30}}\hat{j} + \frac{16}{\sqrt{30}}\hat{k}$$

$$= 8\left(\frac{5\vec{i} - \vec{j} + 2\vec{k}}{\sqrt{30}}\right)$$

$$= \frac{40}{\sqrt{30}}\vec{i} - \frac{8}{\sqrt{30}}\vec{j} + \frac{16}{\sqrt{30}}\vec{k}$$

11. Show that the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $-4\hat{i} + 6\hat{j} - 8\hat{k}$ **are collinear.**
Solution:

Let $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = -4\hat{i} + 6\hat{j} - 8\hat{k}$.

It is seen that $\vec{b} = -4\hat{i} + 6\hat{j} - 8\hat{k} = -2\left(2\hat{i} - 3\hat{j} + 4\hat{k}\right) = -2\vec{a}$

$\therefore \vec{b} = \lambda\vec{a}$

where,

$\lambda = -2$

Therefore, the given vectors are collinear.

12. Find the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$
Solution:

Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$.

The modulus is given by,

$|\vec{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$

Thus, the direction cosines of $\vec{a}$ are $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$.

13. Find the direction cosines of the vector joining the points A (1, 2, –3) and B (–1, –2, 1) directed from A to B.
Solution:

Given points are A (1, 2, –3) and B (–1, –2, 1).
Now,

$\overrightarrow{AB} = (-1-1)\hat{i} + (-2-2)\hat{j} + \{1-(-3)\}\hat{k}$

$\overrightarrow{AB} = -2\hat{i} - 4\hat{j} + 4\hat{k}$

$|\overrightarrow{AB}| = \sqrt{(-2)^2 + (-4)^2 + 4^2} = \sqrt{4+16+16} = \sqrt{36} = 6$

Therefore, the direction cosines of $\overrightarrow{AB}$ are $\left(-\frac{2}{6}, -\frac{4}{6}, \frac{4}{6}\right) = \left(-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$.

14. Show that the vector $\hat{i} + \hat{j} + \hat{k}$ **is equally inclined to the axes OX, OY, and OZ.**

Solution:

Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$.
Then,
$|\vec{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

Hence, the direction cosines of $\vec{a}$ are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

Now, let α, β, and γ be the angles formed by $\vec{a}$ with the positive directions of x, y, and z axes.

So, we have $\cos\alpha = \frac{1}{\sqrt{3}}, \cos\beta = \frac{1}{\sqrt{3}}, \cos\gamma = \frac{1}{\sqrt{3}}$.

Therefore, the given vector is equally inclined to axes OX, OY, and OZ.

15. Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively, in the ration 2:1
(i) internally
(ii) externally
Solution:

The position vector of point R dividing the line segment joining two points
P and Q in the ratio *m: n* is given by:

(i) Internally: $\frac{m\vec{b} + n\vec{a}}{m+n}$

(ii) Externally: $\frac{m\vec{b} - n\vec{a}}{m-n}$

$\overrightarrow{OP} = \hat{i} + 2\hat{j} - \hat{k}$ and $\overrightarrow{OQ} = -\hat{i} + \hat{j} + \hat{k}$

(i) The position vector of point R which divides the line joining two points P and Q internally in the ratio 2:1 is given by,

$$\overrightarrow{OR} = \frac{2\left(-\hat{i} + \hat{j} + \hat{k}\right) + 1\left(\hat{i} + 2\hat{j} - \hat{k}\right)}{2+1} = \frac{\left(-2\hat{i} + 2\hat{j} + 2\hat{k}\right) + \left(\hat{i} + 2\hat{j} - \hat{k}\right)}{3}$$

$$= \frac{-\hat{i} + 4\hat{j} + \hat{k}}{3} = -\frac{1}{3}\hat{i} + \frac{4}{3}\hat{j} + \frac{1}{3}\hat{k}$$

(ii) The position vector of point R which divides the line joining two points P and Q externally in the ratio 2:1 is given by,

$$\overrightarrow{OR} = \frac{2\left(-\hat{i} + \hat{j} + \hat{k}\right) - 1\left(\hat{i} + 2\hat{j} - \hat{k}\right)}{2-1} = \left(-2\hat{i} + 2\hat{j} + 2\hat{k}\right) - \left(\hat{i} + 2\hat{j} - \hat{k}\right)$$

$$= -3\hat{i} + 3\hat{k}$$

16. Find the position vector of the mid point of the vector joining the points P (2, 3, 4) and Q (4, 1, – 2).
Solution:

The position vector of mid-point R of the vector joining points P (2, 3, 4) and Q (4, 1, – 2) is given by,

$$\overline{\mathrm{OR}} = \frac{\left(2\hat{i}+3\hat{j}+4\hat{k}\right)+\left(4\hat{i}+\hat{j}-2\hat{k}\right)}{2} = \frac{(2+4)\hat{i}+(3+1)\hat{j}+(4-2)\hat{k}}{2}$$
$$= \frac{6\hat{i}+4\hat{j}+2\hat{k}}{2} = 3\hat{i}+2\hat{j}+\hat{k}$$

17. Show that the points A, B and C with position vectors, $\vec{a} = 3\hat{i}-4\hat{j}-4\hat{k}$, $\vec{b} = 2\hat{i}-\hat{j}+\hat{k}$ and $\vec{c} = \hat{i}-3\hat{j}-5\hat{k}$, respectively form the vertices of a right angled triangle.

Solution:

Given position vectors of points A, B, and C are:

$\vec{a} = 3\hat{i}-4\hat{j}-4\hat{k}$, $\vec{b} = 2\hat{i}-\hat{j}+\hat{k}$ and $\vec{c} = \hat{i}-3\hat{j}-5\hat{k}$

$\therefore \overrightarrow{\mathrm{AB}} = \vec{b}-\vec{a} = (2-3)\hat{i}+(-1+4)\hat{j}+(1+4)\hat{k} = -\hat{i}+3\hat{j}+5\hat{k}$

$\overrightarrow{\mathrm{BC}} = \vec{c}-\vec{b} = (1-2)\hat{i}+(-3+1)\hat{j}+(-5-1)\hat{k} = -\hat{i}-2\hat{j}-6\hat{k}$

$\overrightarrow{\mathrm{CA}} = \vec{a}-\vec{c} = (3-1)\hat{i}+(-4+3)\hat{j}+(-4+5)\hat{k} = 2\hat{i}-\hat{j}+\hat{k}$

Now,

$\left|\overrightarrow{\mathrm{AB}}\right|^2 = (-1)^2+3^2+5^2 = 1+9+25 = 35$

$\left|\overrightarrow{\mathrm{BC}}\right|^2 = (-1)^2+(-2)^2+(-6)^2 = 1+4+36 = 41$

$\left|\overrightarrow{\mathrm{CA}}\right|^2 = 2^2+(-1)^2+1^2 = 4+1+1 = 6$

Hence,

$$\left|\overrightarrow{\mathrm{AB}}\right|^2 + \left|\overrightarrow{\mathrm{CA}}\right|^2 = 35+6 = 41 = \left|\overrightarrow{\mathrm{BC}}\right|^2$$

18. In triangle ABC (Fig 10.18) which of the following is not true:

(A) $\overrightarrow{\mathrm{AB}}+\overrightarrow{\mathrm{BC}}+\overrightarrow{\mathrm{CA}} = \vec{0}$

(B) $\overrightarrow{\mathrm{AB}}+\overrightarrow{\mathrm{BC}}-\overrightarrow{\mathrm{AC}} = \vec{0}$

(C) $\overrightarrow{\mathrm{AB}}+\overrightarrow{\mathrm{BC}}-\overrightarrow{\mathrm{AC}} = \vec{0}$

(D) $\overrightarrow{\mathrm{AB}}-\overrightarrow{\mathrm{CB}}+\overrightarrow{\mathrm{CA}} = \vec{0}$

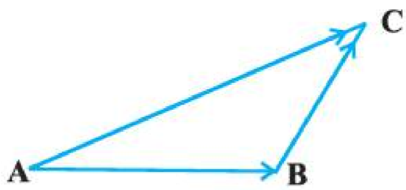

Fig 10.18

Solution:

Applying the triangle law of addition in the given triangle, we get:

$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ ...(1)

$\overrightarrow{AB}+\overrightarrow{BC}=-\overrightarrow{CA}$

$\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\vec{0}$ ...(2)

$\therefore$ The equation given in alternative A is true.

$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{AC}=\vec{0}$

$\therefore$ The equation given in alternative B is true.

From equation (2), we have:

$\overrightarrow{AB}-\overrightarrow{CB}+\overrightarrow{CA}=\vec{0}$

$\therefore$ The equation given in alternative D is true.

Now, consider the equation given in alternative C:

$\overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{CA}=\vec{0}$

$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{CA}$ ...(3)

From equations (1) and (3), we get:

$\overrightarrow{AC}=\overrightarrow{CA}$

$\overrightarrow{AC}=-\overrightarrow{AC}$

$\overrightarrow{AC}+\overrightarrow{AC}=\vec{0}$

$2\overrightarrow{AC}=\vec{0}$

$\overrightarrow{AC}=\vec{0}$, which is not true.

Thus, the equation given in alternative C is **incorrect**.

The correct answer is **C.**

19. If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then which of the following are incorrect:

A. $\vec{b}=\lambda\vec{a}$, for some scalar λ

B. $\vec{a}=\pm\vec{b}$

C. the respective components of $\vec{a}$ and $\vec{b}$ are proportional

D. both the vectors $\vec{a}$ and $\vec{b}$ have same direction, but different magnitudes

Solution:

If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then they are parallel.

So, we have:

$\vec{b}=\lambda\vec{a}$ (For some scalar λ)

If $\lambda=\pm1$, then $\vec{a}=\pm\vec{b}$.

If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then

$$\vec{b} = \lambda\vec{a}.$$
$$b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = \lambda\left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)$$
$$b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$$
$$b_1 = \lambda a_1, b_2 = \lambda a_2, b_3 = \lambda a_3$$
$$\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$$

Hence, the respective components of $\vec{a}$ and $\vec{b}$ are proportional.

But, vectors $\vec{a}$ and $\vec{b}$ can have different directions.

Thus, the statement given in **D** is **incorrect**.

The correct answer is **D**.