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1. An urn contains 5 red and 5 black balls. A ball is drawn at random, its colour is noted and is returned to the urn. Moreover, 2 additional balls of the colour drawn are put in the urn and then a ball is drawn at random. What is the probability that the second ball is red?

Solution:

Given urn contains 5 red and 5 black balls.

Let in first attempt the ball drawn is of red colour.

⇒ P (probability of drawing a red ball) = 5/10 = ½

Now the two balls of same colour (red) are added to the urn then the urn contains 7 red and 5 black balls.

⇒ P (probability of drawing a red ball) = 7/12

Now let in first attempt the ball drawn is of black colour.

⇒ P (probability of drawing a black ball) = 5/10 = ½

Now the two balls of same colour (black) are added to the urn then the urn contains 5 red and 7 black balls.

⇒ P (probability of drawing a red ball) = 5/12

Therefore, the probability of drawing the second ball as of red colour is:

$$= \left(\frac{1}{2} \times \frac{7}{12}\right) + \left(\frac{1}{2} \times \frac{5}{12}\right) = \frac{1}{2}\left(\frac{7}{12} + \frac{5}{12}\right) = \frac{1}{2} \times 1 = \frac{1}{2}$$

2. A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.

Solution:

Let E_1 be the event of choosing the bag I, E_2 be the event of choosing the bag say bag II and A be the event of drawing a red ball.

Then P (E_1) = P (E_2) = 1/2

Also P ($A|E_1$) = P (drawing a red ball from bag I) = 4/8 = ½

And P ($A|E_2$) = P (drawing a red ball from bag II) = 2/8 = ¼

Now the probability of drawing a ball from bag I, being given that it is red, is P ($E_1|A$).

By using Bayes' theorem, we have:

$$P(E_1|A) = \frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2)}$$

$$= \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{4}}$$

$$= \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{8}}$$

$$= \frac{\frac{1}{4}}{\frac{3}{8}} = \frac{2}{3}$$

$$\Rightarrow P(E_1|A) = \frac{2}{3}$$

3. Of the students in a college, it is known that 60% reside in hostel and 40% are day scholars (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain A grade and 20% of day scholars attain A grade in their annual examination. At the end of the year, one student is chosen at random from the college and he has an A grade, what is the probability that the student is a hostlier?

Solution:

Let E_1 be the event that student is a hostler, E_2 be the event that student is a day scholar and A be the event of getting A grade.

Then $P(E_1) = 60\% = \frac{60}{100} = 0.6$

And $P(E_2) = 40\% = \frac{40}{100} = 0.4$

Also P $(A|E_1)$ = P (students who attain A grade reside in hostel) = 30% = 0.3

And P $(A|E_2)$ = P (students who attain A grade is day scholar) = 20% = 0.2

Now the probability of students who reside in hostel, being given he attain A grade, is P $(E_1|A)$.

By using Bayes' theorem, we have:

$$P(E_1|A) = \frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2)}$$

Substituting the values we get

$$= \frac{0.6 \times 0.3}{0.6 \times 0.3 + 0.4 \times 0.2}$$

$$= \frac{0.18}{0.18 + 0.08}$$

$$= \frac{0.18}{0.26} = \frac{18}{26} = \frac{9}{13}$$

$$\Rightarrow P(E_1|A) = \frac{9}{13}$$

4. In answering a question on a multiple choice test, a student either knows the answer or guesses. Let 3/4 be the probability that he knows the answer and 1/4 be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability 1/4. What is the probability that the student knows the answer given that he answered it correctly?

Solution:

Let E_1 be the event that the student knows the answer, E_2 be the event that the student guess the answer and A be the event that the answer is correct.

Then P (E_1) = ¾

And P (E_2) = ¼

Also P (A|E_1) = P (correct answer given that he knows) = 1

And P (A|E_2) = P (correct answer given that he guesses) = ¼

Now the probability that he knows the answer, being given that answer is correct, is P (E_1|A).

By using Bayes' theorem, we have:

$$P(E_1|A) = \frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2)}$$

Substituting the values we get

$$= \frac{\frac{3}{4}.1}{\frac{3}{4}.1 + \frac{1}{4}.\frac{1}{4}}$$

$$= \frac{\frac{3}{4}}{\frac{3}{4} + \frac{1}{16}}$$

$$= \frac{\frac{3}{4}}{\frac{13}{16}} = \frac{12}{13}$$

$$\Rightarrow P(E_1|A) = \frac{12}{13}$$

5. A laboratory blood test is 99% effective in detecting a certain disease when it is in fact, present. However, the test also yields a false positive result for 0.5% of the healthy person tested (i.e. if a healthy person is tested, then, with probability 0.005, the test will imply he has the disease). If 0.1 percent of the population actually has the disease, what is the probability that a person has the disease given that his test result is positive?

Solution:

Let E_1 be the event that person has a disease, E_2 be the event that person don not have a disease and A be the event that blood test is positive.
As E_1 and E_2 are the events which are complimentary to each other.
Then P (E_1) + P (E_2) = 1
⇒ P (E_2) = 1 - P (E_1)
Then P (E_1) = 0.1% = 0.1/100 = 0.001 and P (E_2) = 1 – 0.001 = 0.999
Also P (A|E_1) = P (result is positive given that person has disease) = 99% = 0.99
And P (A|E_2) = P (result is positive given that person has no disease) = 0.5% = 0.005
Now the probability that person has a disease, give that his test result is positive is P (E$_1$|A).
By using Bayes' theorem, we have

$$P(E_1|A) = \frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2)}$$

Substituting the values we get

$$= \frac{0.001 \times 0.99}{0.001 \times 0.99 + 0.999 \times 0.005}$$

$$= \frac{0.00099}{0.00099 + 0.004995}$$

$$= \frac{0.00099}{0.005985} = \frac{990}{5985} = \frac{110}{665}$$

$$\Rightarrow P(E_1|A) = \frac{22}{133}$$

6. There are three coins. One is a two headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the time and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows heads, what is the probability that it was the two headed coin?

Solution:

Let E_1 be the event of choosing a two headed coin, E_2 be the event of choosing a biased coin and E_3 be the event of choosing an unbiased coin. Let A be the event that the coin shows head.

Then P (E_1) = P (E_2) = P (E_3) = 1/3

As we a headed coin has head on both sides so it will shows head.

Also P (A|E_1) = P (correct answer given that he knows) = 1

And P (A|E_2) = P (coin shows head given that the coin is biased) = 75% = 75/100 = ¾

And P (A|E_3) = P (coin shows head given that the coin is unbiased) = ½

Now the probability that the coin is two headed, being given that it shows head, is P (E_1|A).

By using Bayes' theorem, we have

$$P(E_1|A) = \frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2) + P(E_3).P(A|E_3)}$$

By substituting the values we get

$$= \frac{\frac{1}{3}.1}{\frac{1}{3}.1 + \frac{1}{3}.\frac{3}{4} + \frac{1}{3}.\frac{1}{2}}$$

$$= \frac{\frac{1}{3}}{\frac{1}{3}\left(1 + \frac{3}{4} + \frac{1}{2}\right)}$$

$$= \frac{1}{\frac{9}{4}} = \frac{4}{9}$$

$$\Rightarrow P(E_1|A) = \frac{4}{9}$$

7. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accidents are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?

Solution:

Let E_1 be the event that the driver is a scooter driver, E_2 be the event that the driver is a car driver and E_3 be the event that the driver is a truck driver. Let A be the event that the person meet with an accident.

Total number of drivers = 2000 + 4000 + 6000 = 12000

Then P (E_1) = 2000/12000 = 1/6

P (E_2) = 4000/12000 = 1/3

P (E_3) = 6000/12000 = ½

As we a headed coin has head on both sides so it will shows head.

Also P (A|E_1) = P (accident of a scooter driver) = 0.01 = 1/100

And P (A|E_2) = P (accident of a car driver) = 0.03 = 3/100

And P (A|E_3) = P (accident of a truck driver) = 0.15 = 15/100 = 3/20

Now the probability that the driver is a scooter driver, being given that he met with an accident, is P (E_1|A).

By using Bayes' theorem, we have

$$P(E_1|A) = \frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2) + P(E_3).P(A|E_3)}$$

Now by substituting the values we get

$$=\frac{\frac{1}{6}\cdot\frac{1}{100}}{\frac{1}{6}\cdot\frac{1}{100}+\frac{1}{3}\cdot\frac{3}{100}+\frac{1}{2}\cdot\frac{3}{20}}$$

$$=\frac{\frac{1}{600}}{\frac{1}{20}\left(\frac{1}{30}+\frac{1}{5}+\frac{3}{2}\right)}$$

$$=\frac{\frac{1}{30}}{\frac{52}{30}}=\frac{1}{52}$$

$$\Rightarrow P(E_1|A)=\frac{1}{52}$$

8. A factory has two machines A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Further, 2% of the items produced by machine A and 1% produced by machine B were defective. All the items are put into one stockpile and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by machine B?

Solution:

Let E_1 be the event that item is produced by A, E_2 be the event that item is produced by B and X be the event that produced product is found to be defective.

Then P (E_1) = 60% = 60/100 = 3/5

P (E_1) = 40% = 40/100 = 2/5

Also P (X|E_1) = P (item is defective given that it is produced by machine A) = 2% = 2/100 = 1/50

And P (X|E_2) = P (item is defective given that it is produced by machine B) = 1% = 1/100

Now the probability that item is produced by B, being given that item is defective, is P (E_2|A).

By using Bayes' theorem, we have

$$P(E_2|A)=\frac{P(E_2).P(X|E_2)}{P(E_1).P(X|E_1)+P(E_2).P(X|E_2)}$$

By substituting the values we get

$$= \frac{\frac{2}{5} \times \frac{1}{100}}{\frac{3}{5} \times \frac{2}{100} + \frac{2}{5} \times \frac{1}{100}}$$

$$= \frac{\frac{2}{5} \times \frac{1}{100}}{\frac{1}{500}(6+2)} = \frac{2}{8}$$

$$\Rightarrow P(E_2|A) = \frac{1}{4}$$

9. Two groups are competing for the position on the Board of directors of a corporation. The probabilities that the first and the second groups will win are 0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.7 and the corresponding probability is 0.3 if the second group wins. Find the probability that the new product introduced was by the second group.

Solution:

Let E_1 be the event that first group wins the competition, E_2 be the event that that second group wins the competition and A be the event of introducing a new product.
Then P (E_1) = 0.6 and P (E_2) = 0.4
Also P (A|E_1) = P (introducing a new product given that first group wins) = 0.7
And P (A|E_2) = P (introducing a new product given that second group wins) = 0.3
Now the probability of that new product introduced was by the second group, being given that a new product was introduced, is P (E_2|A).
By using Bayes' theorem, we have

$$P(E_2|A) = \frac{P(E_2).P(A|E_2)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2)}$$

Now by substituting the values we get

$$= \frac{0.4 \times 0.3}{0.6 \times 0.7 + 0.4 \times 0.3}$$

$$= \frac{0.12}{0.42 + 0.12}$$

$$= \frac{0.12}{0.54} = \frac{12}{54} = \frac{2}{9}$$

P $(E_2|A) = 2/9$

10. Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die?

Solution:

let E_1 be the event that the outcome on the die is 5 or 6, E_2 be the event that the outcome on the die is 1, 2, 3 or 4 and A be the event getting exactly head.

Then P $(E_1) = 2/6 = 1/3$

P $(E_2) = 4/6 = 2/3$

As in throwing a coin three times we get 8 possibilities.

{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}

⇒ P $(A|E_1)$ = P (obtaining exactly one head by tossing the coin three times if she get 5 or 6) = 3/8

And P $(A|E_2)$ = P (obtaining exactly one head by tossing the coin three times if she get 1, 2, 3 or 4) = ½

Now the probability that the girl threw 1, 2, 3 or 4 with a die, being given that she obtained exactly one head, is P $(E_2|A)$.

By using Bayes' theorem, we have

$$P(E_2|A) = \frac{P(E_2).P(A|E_2)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2)}$$

Now by substituting the values we get

$$= \frac{\frac{2}{3}\cdot\frac{1}{2}}{\frac{1}{3}\cdot\frac{3}{8} + \frac{2}{3}\cdot\frac{1}{2}}$$

$$= \frac{\frac{1}{3}}{\frac{1}{8} + \frac{1}{3}}$$

$$= \frac{\frac{1}{3}}{\frac{3+8}{24}} = \frac{8}{11}$$

P $(E_2|A) = 8/11$

11. A manufacturer has three machine operators A, B and C. The first operator A produces 1% defective items, whereas the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B is on the job for 30% of the time and C is on the job for 20% of the time. A defective item is produced, what is the probability that it was produced by A?

Solution:

Let E_1 be the event of time consumed by machine A, E_2 be the event of time consumed by machine B and E_3 be the event of time consumed by machine C. Let X be the event of producing defective items.

Then P (E_1) = 50% = 50/100 = ½

P (E_2) = 30% = 30/100 = 3/10

P (E_3) = 20% = 20/100 = 1/5

As we a headed coin has head on both sides so it will shows head.

Also P $(X|E_1)$ = P (defective item produced by A) = 1% = 1/100

And P $(X|E_2)$ = P (defective item produced by B) = 5% = 5/100

And P $(X|E_3)$ = P (defective item produced by C) = 7% = 7/100

Now the probability that item produced by machine A, being given that defective item is produced, is P $(E_1|A)$.

By using Bayes' theorem, we have

$$P(E_1|X) = \frac{P(E_1).P(X|E_1)}{P(E_1).P(X|E_1) + P(E_2).P(X|E_2) + P(E_3).P(X|E_3)}$$

Now by substituting the values we get

$$= \frac{\frac{1}{2}.\frac{1}{100}}{\frac{1}{2}.\frac{1}{100} + \frac{3}{10}.\frac{5}{100} + \frac{1}{5}.\frac{7}{100}}$$

$$= \frac{\frac{1}{2}.\frac{1}{100}}{\frac{1}{100}\left(\frac{1}{2} + \frac{3}{2} + \frac{7}{5}\right)}$$

$$= \frac{\frac{1}{2}}{\frac{17}{5}} = \frac{5}{34}$$

P $(E_1|A)$ = 5/34

12. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both diamonds. Find the probability of the lost card being a diamond.

Solution:

Let E1 be the event that the drawn card is a diamond, E2 be the event that the drawn card is not a diamond and A be the event that the card is lost.

As we know, out of 52 cards, 13 cards are diamond and 39 cards are not diamond.

Then P (E_1) = 13/52 and P (E_2) = 39/52

Now, when a diamond card is lost then there are 12 diamond cards out of total 51 cards.

Two diamond cards can be drawn out of 12 diamond cards in ${}^{12}C_2$ ways.

Similarly, two diamond cards can be drawn out of total 51 cards in ${}^{51}C_2$ ways.

Then probability of getting two cards, when one diamond card is lost, is P ($A|E_1$).

Also P ($A|E_1$) $= {}^{12}C_2 / {}^{51}C_2$

Also P ($A|E_1$) $= {}^{12}C_2 / {}^{51}C_2$

$$= \frac{12!}{2! \times 10!} \times \frac{2! \times 49!}{51!}$$

$$= \frac{12 \times 11 \times 10!}{2 \times 1 \times 10!} \times \frac{2 \times 1 \times 49!}{51 \times 50 \times 49!}$$

$$= \frac{12 \times 11}{51 \times 50} = \frac{22}{425}$$

Now, when not a diamond card is lost then there are 13 diamond cards out of total 51 cards.

Two diamond cards can be drawn out of 13 diamond cards in ${}^{13}C_2$ ways.

Similarly, two diamond cards can be drawn out of total 51 cards in ${}^{51}C_2$ ways.

Then probability of getting two cards, when card is lost which is not diamond, is P ($A|E_2$).

Also P ($A|E_2$) $= {}^{13}C_2 / {}^{51}C_2$

$$= \frac{13!}{2! \times 11!} \times \frac{2! \times 49!}{51!}$$

$$= \frac{13 \times 12 \times 11!}{2 \times 1 \times 10!} \times \frac{2 \times 1 \times 49!}{51 \times 50 \times 49!}$$

$$= \frac{13 \times 12}{51 \times 50} = \frac{26}{425}$$

Now the probability that the lost card is diamond, being given that the card is lost, is P (E_1|A).

By using Bayes' theorem, we have:

$$P(E_1|A) = \frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2)}$$

Now by substituting the values we get

$$= \frac{\frac{1}{4}\cdot\frac{22}{425}}{\frac{1}{4}\cdot\frac{22}{425} + \frac{3}{4}\cdot\frac{26}{425}}$$

$$= \frac{\frac{1}{425}\cdot\frac{22}{4}}{\frac{1}{425}\left(\frac{22}{4} + \frac{26\times 3}{4}\right)}$$

$$= \frac{\frac{11}{2}}{\frac{100}{4}} = \frac{11}{50}$$

$$\Rightarrow P(E_1|A) = \frac{11}{50}$$

13. Probability that A speaks truth is 4/5. A coin is tossed. A reports that a head appears. The probability that actually there was head is
A. 4/5
B. 1/2
C. 1/5
D. 2/5

Solution:
A. 4/5

Explanation:
Let E_1 be the event that A speaks truth, E_2 be the event that A lies and X be the event that it appears head.
Therefore, P (E_1) = 4/5

As E_1 and E_2 are the events which are complimentary to each other.
Then P (E_1) + P (E_2) = 1
$\Rightarrow$ P (E_2) = 1 - P (E_1)
$\Rightarrow$ P (E_2) = 1 – 4/5 = 1/5
If a coin is tossed it may show head or tail.
Hence the probability of getting head is 1/2 whether A speaks a truth or A lies.
P $(X|E_1)$ = P $(X|E_2)$ = ½
Now the probability that actually there was head, give that A speaks a truth is P $(E_1|X)$.
By using Bayes' theorem, we have

$$P(E_1|X) = \frac{P(E_1).P(X|E_1)}{P(E_1).P(X|E_1) + P(E_2).P(X|E_2)}$$

Now substituting the values we get

$$= \frac{\frac{4}{5}\cdot\frac{1}{2}}{\frac{4}{5}\cdot\frac{1}{2} + \frac{1}{5}\cdot\frac{1}{2}}$$

$$= \frac{\frac{2}{5}}{\frac{2}{5} + \frac{1}{10}} = \frac{\frac{2}{5}}{\frac{5}{10}} = \frac{4}{5}$$

$$\Rightarrow P(E_1|X) = \frac{4}{5}$$

Therefore correct answer is (A).

14. If A and B are two events such that A ⊂ B and P(B) ≠ 0, then which of the following is correct?
A. P (A| B) = P (B)/ P (A)
B. P(A|B) < P(A)
C. P(A|B) ≥ P(A)
D. None of these

Solution:
C. P (A|B) ≥ P (A)

Explanation:

A and B are two events such that A ⊂ B and P (B) ≠ 0

As A ⊂ B ⇒ A ∩ B = A

⇒ P (A ∩ B) = P (A)

As A ⊂ B ⇒ P (A) < P (B)

As we know $P\ (A|B) = \frac{P\ (A \cap B)}{P(B)} = \frac{P(A)}{P(B)} \neq \frac{P(B)}{P(A)}\ ..(1)$

Consider

$$P\ (A|B) = \frac{P\ (A\ \cap\ B)}{P(B)} = \frac{P(A)}{P(B)}\ ..(2)$$

It is also known that P (B) ≤ 1

$$\Rightarrow \frac{1}{P(B)} \geq 1$$

$$\Rightarrow \frac{P(A)}{P(B)} \geq P(A)$$

⇒ P (A|B) ≥ P (A) ... (3)

Hence, the correct answer is (C).